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LOCALLY INJECTIVE G-SHEAVES OF ABELIAN GROUPS

by Roswitha HARTING

The problem of the existence of enough injective abelian group objects in an elementary topos with a natural number object leads to the construction of the internal (parametrized) coproduct of abelian group objects [4]. From certain properties of this parametrized coproduct we earlier derived some further consequences [5], among them the surprising result that all internal notions of injectivity for abelian group objects are equivalent.

In the following summary we shall apply this result to $Shv(X)^{G^{op}}$, the topos of set-valued sheaves on a topological space X with a left action of a group-valued sheaf G.

We require the following results and definitions [4, 5] (where \mathcal{E} denotes an elementary topos with natural number object and $Ab(\mathcal{E})$ the category of abelian group objects in \mathcal{E}).

(0.1) THEOREM AND DEFINITION. For any object X in \mathcal{E} the functor X^* : $Ab(\mathcal{E}) \to Ab(\mathcal{E}/X)$ has a left adjoint Θ_X : $Ab(\mathcal{E}/X) \to Ab(\mathcal{E})$ which respects monomorphisms and is faithful.

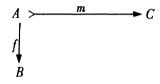
For $A(x) \in Ob \ Ab(\mathcal{E}/X)$ the abelian group object $\bigoplus_X A(x)$ in \mathcal{E} is called parametrized coproduct of A(x). (We use «parametrized» to emphasize that the indexing object is in general not just a set but for example a set with an action of a group on it.)

A consequence of this theorem is the following proposition [5]: (0.2) PROPOSITION. If $Ab(\mathcal{E})$ has enough injectives, then so does $Ab(\mathcal{E}^{A})$ for any internal category \underline{A} in \mathcal{E} .

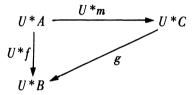
In the following the internal Hom-functor $Ab(\mathcal{E})^{op} \times Ab(\mathcal{E}) \rightarrow Ab(\mathcal{E})$

is denoted by Hom(-,-). For A, B abelian group objects in \mathfrak{E} , Hom(A,B) is the abelian group object in \mathfrak{E} that internalizes the abelian group of group-morphisms from A to B.

- (0.3) DEFINITION. An abelian group object B in & is called
- (i) internally injective if for every monomorphism $A \rightarrowtail C$ in $Ab(\mathcal{E})$, $Hom(C, B) \nrightarrow Hom(A, B)$ is an epimorphism in \mathcal{E} .
 - (ii) locally injective if for every diagram of the form

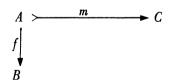


in $Ab(\mathfrak{S})$ there exists a cover $U \to 1$ of \mathfrak{S} and a morphism $g: U^*C \to U^*B$ in $Ab(\mathfrak{S}/U)$ such that



commute s.

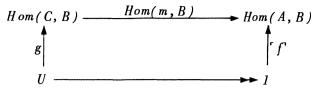
- (0.4) PROPOSITION. The following conditions on an abelian group object B in \mathcal{E} are equivalent (for (iii) we suppose that $Ab(\mathcal{E})$ has enough injectives):
 - (i) B is locally injective.
 - (ii) For every diagram of the form



in $Ab(\mathcal{E})$ there exist a cover $U \rightarrow 1$ of \mathcal{E} and a morphism

$$g: U \rightarrow Hom(C, B)$$
 in &

such that the diagram

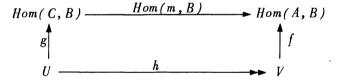


commutes in &.

(iii) There exists a cover $U \rightarrow 1$ of \mathfrak{E} such that $\eta_B \colon B \rightarrowtail B^U$ is an injective effacement [2]. Here $\eta \colon \operatorname{id} Ab(\mathfrak{E}) \rightarrow \pi_U . U^*$ denotes the unit of the adjunction $U^* \dashv \pi_U$, and the abelian group object structure of B^U is induced by that of B.

(A monomorphism $m:A \longrightarrow C$ in $Ab(\mathcal{E})$ is an injective effacement iff for every monomorphism $f:A \longrightarrow B$ in $Ab(\mathcal{E})$ there exists a morphism $g:B \to C$ in $Ab(\mathcal{E})$ such that g,f=m.)

(0.5) LEMMA. An abelian group object B in \mathcal{E} is intermally injective iff for every monomorphism $m: A \longrightarrow C$ in $Ab(\mathcal{E})$ and every generalized element $f: V \to Hom(A, B)$ in \mathcal{E} there exist an object U, an epimorphism $h: U \to V$ and a morphism $g: U \to Hom(C, B)$ in \mathcal{E} such that



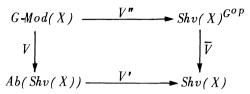
commutes.

(0.6) THEOREM. An abelian group object B in \mathcal{E} is locally injective iff B is internally injective.

In the following we shall study the meaning of this result in the topos $\mathcal{E} = Shv(X)^{G^{op}}$, where X denotes a topological space (resp. a locale [7,9]) and G a group-valued sheaf on X. Then $Ab(Shv(X)^{G^{op}})$ is the category of abelian group-valued sheaves on X equipped with a left action of G compatible with the abelian group structure. So $Ab(Shv(X)^{G^{op}})$ is the category of G-modules on X, and will be denoted from now on by G-Mod(X).

(1.1) SOME REMARKS.

(i) The following diagram of forgetful functors commutes:



All these forgetful functors create epimorphisms and monomorphisms, they all have a left adjoint, and they respect injectives [10].

(ii) In Ab(Shv(X)) the notions of injectivity and internal injectivity coincide [5]. For A, B in Ab(Shv(X)) the internal Hom is obtained as follows: for U open in X, Hom(A, B)(U) is defined to be

$$Hom_{A\ b\ (Sh\ v\ (U\))}(\ A/U\ ,\ B/U\).$$

- (iii) G-Mod(X) has enough injectives. (This follows immediately from (0.2).)
- (iv) In $G ext{-Mod}(X)$ the internal Hom is obtained as follows: For A, B $G ext{-modules}$ on X, Hom(A,B) is defined to be

$$Hom(VA, VB)$$
 in $Ab(Shv(X))$

equipped with the following action of G: for U open in X,

is defined by:

$$(s \circ h)_{W}(x) := (s/W).h_{W}((s^{-1}/W).x),$$

where $W \subset U$, W open in X and $x \in AW$.

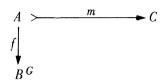
- (1.2) PROPOSITION. Let B be a G-module on X.
 - (i) B is intemally injective iff VB is injective in Ab(Shv(X)).
 - (ii) If B is internally injective, then BG is injective.

PROOF. (i): Suppose B to be internally injective. To show that B is injective in Ab(Shv(X)), it is sufficient to show that B is internally injective in Ab(Shv(X)) (cf. (1.1)(ii)). So let $m: A \rightarrowtail C$ be a mono-

morphism in Ab(Shv(X)). Let G operate trivially on A and C; then $m:A \rightarrow C$ becomes a monomorphism in G-Mod(X). Since B is internally injective it follows that $Hom(C,B) \rightarrow Hom(A,B)$ is an epimorphism in $Shv(X)^{G^{op}}$, and hence an epimorphism in Shv(X) (cf. (1.1)(i) and (iv)). So B is injective in Ab(Shv(X)).

The other implication is equally easy.

(ii): Let



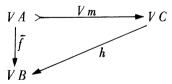
be a diagram in G-Mod(X) and suppose B to be internally injective. We have a sequence of natural bijections between the following sets:

$$\frac{A \to B^G \text{ in } G\text{-}Mod(X)}{G \to V''(Hom(A, B)) \text{ in } Shv(X)^{G^{op}}} \quad \overline{F} + \overline{V}, \quad \overline{F}(1) = G$$

$$\frac{1 \to \overline{V}V''(Hom(A, B)) \text{ in } Shv(X)}{I \to V'(Hom(VA, VB)) \text{ in } Shv(X)} \quad \overline{V}V'' = V'V, \quad (1.1) \text{ (iv)}$$

$$VA \to VB \text{ in } Ab(Shv(X)).$$

So $f: A \to B^G$ determines, and is determined by, a morphism $\bar{f}: VA \to VB$ in Ab(Shv(X)). B is supposed to be internally injective, so, by (i), VB is injective in Ab(Shv(X)). Hence there is a morphism h in Ab(Shv(X)) such that



commutes. As above h determines a morphism $\hat{h}: C \to B^G$ in G-Mod(X), and it is easy to verify that $\hat{h}m = f$. So B^G is injective in G-Mod(X).

(1.3) REMARK. Let ΔZ be the ring-valued sheaf on X associated to the

constant presheaf with value Z. Then $U \mapsto \Delta Z(U)[GU]$ defines an abelian group-valued presheaf with the usual left action of G, where we denote by $\Delta Z(U)[GU]$ the group-ring over GU. The associated sheaf is a G-module on X and is denoted by Z[G]. Some obvious calculations show that there is a natural isomorphism

$$Hom(Z[G], A) \approx A^G$$
.

In the following the composite

$$A > \frac{\eta_A}{\longrightarrow} A^G \xrightarrow{\approx} Hom(Z[G], A)$$

is again denoted by η_A (cf. (0.4)(iii)).

- (1.4) PROPOSITION. Let B be a G-module on X. The following conditions on B are equivalent:
 - (i) Hom(Z[G], B) is an injective G-module on X.
- (ii) There exists an epimorphism $D \rightarrow 1$ in $Shv(X)^{G^{op}}$ such that B^D is an injective G-module on X.
- (iii) $\eta_B: B \longrightarrow Hom(Z[G], B)$ is an injective effacement in G-Mod(X).
- (iv) There exists an epimorphism $D \to 1$ in $Shv(X)^{G^{op}}$ such that $\eta_B \colon B \rightarrowtail B^D$ is an injective effacement in $G\operatorname{-Mod}(X)$.
 - (v) B is injective in Ab(Shv(X)).
 - (vi) B is intermally injective in G-Mod(X).
- (vii) There exists an open cover of X, $X = \bigcup_{i \in I} U_i$, such that B/U_i is injective in $Ab(Shv(U_i))$ for every $i \in I$.

PROOF. (iii)
$$\Longrightarrow$$
 (iv) $\overbrace{(1.1)(iii), (0.4)(iii), (0.6)}$ (vi) $\overline{(1.2)(ii)}$ \Longrightarrow (i) \Longrightarrow (iii).

(ii)
$$\Longrightarrow$$
 (iv) \Longrightarrow (ii) \Longrightarrow (ii).

$$(vi) \iff (v) \iff (vii).$$

So they are all equivalent.

(1.5) REMARK. If X is reduced to a point, then $Shv(X)^{G^{op}}$ is the topos

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of G-sets, $Ab(Shv(X)^{G^{op}})$ is the category of left Z[G]-modules, and Ab(Shv(X)) is the category of abelian groups. For two Z[G]-modules A, B the internal Hom, Hom(A,B), is the abelian group of all Z-linear maps from A to B equipped with the following G-action:

$$(s \circ f)(x) = sf(s^{-1}x) \text{ for } s \in G, f \in Hom_{\mathbb{Z}}(A,B), x \in A.$$

$$\eta_B \colon B \rightarrowtail Hom(\mathbb{Z}[G],B) \text{ is here defined by}$$

$$\eta_B(b)(s) = b$$
 for every $s \in G$.

We remark that in the above case the Proposition (1.4) remains true if G is replaced by a monoid M and consequently Hom(Z[G], B) by B^{M} (cf. [3]).

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