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MODELS OF DIFFERENTIAL ALGEBRA IN THE CONTEXT OF SYNTHETIC DIFFERENTIAL GEOMETRY

by Marta BUNGE *

Differential Algebra (cf. [13, 10]) has as its main goal the development of a theory of differential equations from an algebraic point of view. Thus, a differential (ordinary, in one variable y) equation is a polynomial equation (in the variables y, dy, $d^2 y, \ldots$, etc.) over some differential ring.

In order to develop this sort of DA (short, from now on, for «Differential Algebra») in a topos, all one needs is to be given a differential ring object in the topos. Yet an interesting kind of DA exists already within the context of Synthetic Differential Geometry (cf. [11, 6]) in which the derivation process is not an added structure, but is intrinsic and arises from the «line-type» property of the ring object considered.

It is my aim in this paper to conciliate these two points of view. The key is the interplay, first pointed out by F.W. Lawvere in [11], between derivations on an algebra and vector fields on the *Spec* of the algebra. I wish to thank F.W. Lawvere for several good suggestions concerning the work contained here. Useful conversations with R. Diaconescu, D. Dubrovsky, E. Dubuc, G. Reyes, and R. Rosebrugh, are also gratefully acknowledged.

If $\mathcal{E}$ is a topos (with natural numbers object, N), a differential ring S in $\mathcal{E}$ is a ring object in $\mathcal{E}$ (commutative, with 1) equipped with

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a «derivation», i. e., a morphism $d: S \to S$ for which the statements

$$d(a+b) = da + db$$

and

$$d(a.b) = da.b + a.db \quad \text{(Leibniz rule)}$$

hold for S in $\mathcal{E}$. Denote by d^n the n^{th} iterate of d ; i. e.,

$$d^0 a = a \quad \text{and} \quad d^{n+1} a = d(d^n a) \quad \text{for } n \geq 0 .$$

For each $r, m \geq 0$, let E_{rm} be the subobject of S defined by the formula:

$$d^{r+1}y = 0 \wedge (\forall (i_0, i_1, \dots, i_m) \in N^{m+1} \, [d^{i_0}y . d^{i_1}y. \dots . d^{i_m}y = 0]).$$

DEFINITION. A differential ring S in $\mathcal{E}$ is said to be *of differential line type of order* r $(r \geq 0)$ if, for any $m \geq 0$, the canonical morphism

$$\underbrace{S \times S \times \dots\dots \times S}_{\sigma(r,m) \text{ times}} \xrightarrow{\alpha_{rm}} S^{E_{rm}}$$

given by the rule (which also makes explicit the cardinality $\sigma(r,m) > 0$):

$$(a, (a_i)_{0 \leq i \leq r}, (a_{i_1 i_2})_{0 \leq i_1 \leq i_2 \leq r}, \dots, (a_{i_1 i_2 \dots i_m})_{0 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq r})$$

$$\longmapsto$$

$$a + \sum_{0 \leq i \leq r} a_i . d^i y + \sum_{0 \leq i_1 \leq i_2 \leq r} a_{i_1} a_{i_2} . d^{i_1} y . d^{i_2} y + \dots$$

$$\dots + \sum_{0 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq r} a_{i_1 i_2 \dots i_m} . d^{i_1} y . d^{i_2} y . \dots . d^{i_m} y$$

is an isomorphism.

Whenever this property holds for S, the exponential $S^{E_{rm}}$ may be thought of as «the object of differential equations over S of order $\leq r$ and degree $\leq m$». It may also be regarded as an object of «differential jets» of order $\leq r$ and degree $\leq m$ (cf. [9]); this is correct in some models to be considered below.

DEFINITION. By a *model of Differential Algebra (in the context of Synthetic Differential Geometry)* we mean a differential ringed topos $(\mathcal{E}, S)$, where S is of differential line type of order r, for any $r \geq 0$, and for any $r, m \geq 0$, E_{rm} is an atom (i. e., $(-)^{E_{rm}}$ has a coadjoint).

THEOREM. *Let k be a commutative ring with 1, and let* T *be the theory of differential k-algebras. Then, the classifying topos* $(\mathcal{E}[\mathrm{T}], G[\mathrm{T}])$ *is a model of DA.*

PROOF. $\mathcal{E}[\mathrm{T}] = \mathcal{S}^{\mathbf{C}}$, where C is the category of finitely presented differential k-algebras, and $G[\mathrm{T}] = S$, the underlying set functor. S is represented by $k\{y\}$, the k-algebra of differential polynomials in one differential variable over k. Denote by $k_{rm}\{\epsilon\}$ the k-algebra of differential polynomials in a new element ϵ, with k-algebra operations and derivation determined by the conditions:

$d^{r+1}\epsilon = 0$ and any product of $m+1$ or more of the $d^i\epsilon$, $i \geq 0$, is zero.

(It may alternatively be given as the quotient of $k\{y\}$ by a suitable differential ideal.) Similarly, one defines $C_{rm}\{\epsilon\}$, given any object C of C. It is the case that $C_{rm}\{\epsilon\} = C \otimes_k k_{rm}\{\epsilon\}$ and one has the canonical bijections:

$$\frac{\dfrac{\dfrac{hC \longrightarrow hk\{y\}^{h\,k_{rm}\{\epsilon\}}}{hC \times h\,k_{rm}\{\epsilon\} \longrightarrow hk\{y\}}}{h(C \otimes_k k_{rm}\{\epsilon\}) \longrightarrow hk\{y\}}}{k\{y\} \longrightarrow C_{rm}\{\epsilon\}},$$

where $h: \mathbf{C}^{op} \to \mathcal{S}^{\mathbf{C}}$ is the Yoneda embedding. Therefore, for any $C \in |\mathbf{C}|$,

$$S^{E_{rm}}(C) \approx |C_{rm}\{\epsilon\}| \approx |C| \times \ldots \times |C| \quad (\sigma(r,m) \text{ times})$$
$$\approx S(C) \times \ldots \times S(C) \approx (S \times S \times \ldots \times S)(C)$$

which says that S is of differential line type of any order. In $\mathcal{S}^{\mathbf{C}}$, the objects E_{rm} are atoms because they are representable. □

REMARK. In fact, each $k_{rm}\{\epsilon\}$ is exponentiable in $\mathbf{C}^{op}$ and the exponentials of the form $\overline{C}^{k_{rm}\{\epsilon\}}$ are jet bundles (cf. [5]). As an example, let $C = k\{y\} \bmod f\{y\}$, where $f\{y\}$ is a differential polynomial of degree $s > 0$. For $r = 1$, $m = 2$, calculating the exponential gives a quotient of $k\{x_0, x_1, x_2\}$ by the differential ideal generated by the differential polynomials

$$f\{x_0\}, \quad \sum_{i=0}^{s} \frac{\partial f}{\partial d^i y}\{x_0\}.\, d^i x_1 \quad \text{and} \quad \sum_{i=0}^{s} \frac{\partial f}{\partial d^i y}\{x_0\}[\binom{i}{2} d^{i-1} x_1 + d^i x_2].$$

Also, the full subcategory of C consisting of the $k_{rm}\{\epsilon\}$ weakly generates (cf. [11]); such is the content of the differential algebraic version of Hilbert's Nullstellensatz (cf. [14]).

REMARK. A model of DA need not become a model of SDG (meaning «Synthetic Differential Geometry» in the restricted sense of having a ring object R satisfying the Kock-Lawvere axiom, or «line-type» axiom, and such that the

$$D_m = \| x \in R \mid x^{m+1} = 0 \|, \quad m > 0,$$

are atoms) by simply forgetting the derivation. For example, for S in $\mathcal{S}^{\mathrm{C}}$, the exponential S^D, with D the subobject of S defined by the formula « $y^2 = 0$ », is isomorphic to the cartesian product of a countable number of copies of S.

We need several notions from SDG in order to state the next result. By «*ring object R of line type*» and «*Euclidean R-module S*», we mean always «*in the extended sense*» (cf. [6]). Also, letting $D_\infty \subset R$ be the colimit of $D_1 \subset D_2 \subset \dots$, where R is a ring object of line type in $\mathcal{E}$, by *vector field* we shall understand a pair $\langle X, \xi \rangle$ where X is an object of $\mathcal{E}$ and $\xi: X \to X^{D_\infty}$ is such that $\check{\xi}: X \times D_\infty \to X$ is an action of the additive monoid D_∞ on X. In this context we can define the *intrinsic n^{th} derivative of a morphism* $g: X \to Y$ (where X is a vector field with ξ, and Y is a Euclidean R-module) to be $n!\, \delta^n_\xi(g)$, where $\delta^n_\xi(g)$ is the composite

$$X \xrightarrow{\xi} X^{D_\infty} \xrightarrow{X^{j_n}} X^{D_n} \xrightarrow{g^{D_n}} Y^{D_n} \xrightarrow{\alpha_n^{-1}} Y^{n+1} \xrightarrow{\pi_{n+1}} Y.$$

PROPOSITION. *Let $\mathcal{E}$ be a topos, and S a Ritt algebra in $\mathcal{E}$ (i.e., a differential ring which is also an algebra over the rationals), such that S is of differential line type of order 0. Then, there exists a ring object R in $\mathcal{E}$, which is of line type, and relative to which S is a Euclidean R-module and a vector field with ξ, and such that, if d is the internal der-*

ivation of S, d^n *is also the intrinsic* n^{th} *derivative of the identity on* S *in the direction of* ξ.

PROOF. Define

$$R = [\![\, y \in S \mid dy = 0 \,]\!] .$$

Then, as subobjects of R, $E_{0m} \approx D_m$, from which it follows readily that R is of line type and S Euclidean. Define $\xi : S \to S^{D_\infty}$ so that

$$\check{\xi}(a, b) = \sum_{n \geq 0} \frac{d^n a}{n!} . b^n \quad \text{for } a \in S, \ b \in D_\infty$$

(hence, b nilpotent). The first equation $\check{\xi}(a, 0) = a$ is clear, and the second, which says that

$$\check{\xi}(a, b_1 + b_2) = \check{\xi}(\check{\xi}(a, b_1), b_2) \quad \text{for all } a \in S, \ b_1, b_2 \in D_\infty ,$$

follows from Leibniz rule for higher derivatives (using that every $b \in D_\infty$ is a constant) and the binomial theorem. □

REMARK. The above process for deriving a model of SDG given one of DA can be reversed under certain conditions. The object R^R has a canonical vector field induced by the sum $+ : R \times D_\infty \to R$, hence there is an intrinsic derivation $' : R^R \to R^R$ which is, in fact, internal, as one has the equations (cf. [6]):

$$(f+g)' = f' + g' \quad \text{and} \quad (f . g)' = f' . g + f . g'$$

valid for R^R in $\mathcal{E}$. Suppose that R is recoverable as

$$R \approx [\![\, f \in R^R \mid f' = 0 \,]\!] .$$

Then, R^R is at least of differential line type of order 0.

In connection with this, we point out that, although the sum $+ :$ $R \times D_\infty \to R$ also induces a vector field $\hat{+} : R \to R^{D_\infty}$, hence an intrinsic derivative $' : R \to R$, the latter is not, in general, an internal derivation. For example, let R be the generic k-algebra, represented by the k-algebra $k[x]$ of ordinary polynomials over k, in one variable. The intrinsic derivative on R is induced by ordinary derivative of polynomials, but a simple calculation shows that already the equation $(a+b)' = a' + b'$ fails for R in $\mathcal{S}^{\mathbf{A}}$, the k-algebra classifier, i. e., with $\mathbf{A}$ the category of finitely presented k-algebras.

There is also a trivial way of regarding any model $(\mathcal{E}, R)$ of SDG as a model of DA, namely, by letting R be a differential ring with trivial derivation, i.e., with $d = 0$. This reflects the idea, current in DA (cf. [10]) that DA should contain ordinary algebra as being trivially differential.

Other models of DA will be derived as suitable subtoposes of the differential k-algebra classifier, after stating the differential analogue of a theorem of M.-F. Coste, M. Coste and A. Kock (cf. [3], also [4]).

DEFINITION. Let T^* be a geometric quotient of the theory T of differential k-algebras. Say that T^* is *differentially ϵ-stable* if, for each $r, m \geq 0$, if A is a model of T^*, so is $A_{rm}\{\epsilon\}$.

PROPOSITION. *If* T^* *is a differentially ϵ-stable geometric quotient of* T, *then the classifying topos* $(\mathcal{E}[T^*], G[T^*])$ *is a model of DA.*

PROOF. Along the lines of [4], using that the $k_{rm}\{\epsilon\}$ are of finite (linear) dimension $\sigma(r, m)$ over k and their duals are exponentiable in C^{op}, and also that $A_{rm}\{\epsilon\} \approx A \otimes_k k_{rm}\{\epsilon\}$. But it can also be derived directly from a general result proven in [3]. □

Consider the theory T_{dl} of *differential local k-algebras*, a geometric quotient of T with the sequents

$$a + b \in U \Rightarrow a \in U \vee b \in U \quad \text{and} \quad da \in U \Rightarrow a \in U,$$

where U means «units». The differential Zariski topology (described in [2] in order to define the differential local spectrum of a differential ring) makes C^{op} into a site, where a cocovering of C is a family of localizations

$$(C \longrightarrow C[a_i^{-1}])_{i \in I}$$

provided, given any differential prime ideal P of C, there exists $i_0 \in I$ for which $a_{i_0} \notin P$. Let us call the topos $\widetilde{C^{op}_{dZar}}$ of sheaves for the differential Zariski topology on C^{op}, the *differential Zariski topos*. Let S_{dZar} be the associated sheaf of $S \in \mathcal{S}^C$.

PROPOSITION. *The differential Zariski topos, together with* S_{dZar}, *classifies models of* T_{dl} *in* $\mathcal{S}$*-based toposes.*

PROOF. A simple argument shows that, if $C \in |\mathbb{C}|$ is a differential local ring, then $[-,C] : \mathbb{C}^{op} \to \mathcal{S}$ is continuous for the differential Zariski topology, since the inverse image of a differential prime ideal under a differential homomorphism is a differential prime ideal. Conversely, assume that $[-,C] : \mathbb{C}^{op} \to \mathcal{S}$ is continuous. The family

$$\{ k\{y\} \to k\{y\}[y^{-1}],\ k\{y\} \to k\{y\}[(1-dy)^{-1}] \}$$

is a cocovering of $k\{y\}$; this shows that C (assumed to be differential, and proven local in the usual way) is differential local. □

COROLLARY. *The pair* $(\widetilde{\mathbb{C}^{op}_{dZar}}, S_{dZar})$ *is a model of DA.*

PROOF. Let A be a differential local k-algebra, and let $r, m \geq 0$. Then, $A_{rm}\{\epsilon\}$ is also differential local because an element of the form

$$(a + \Sigma a_i . d^i y + \Sigma a_{i_1 i_2} . d^{i_1} y d^{i_2} y + \ldots$$
$$\ldots + \Sigma a_{i_1 i_2 \ldots i_m} . d^{i_1} y . d^{i_2} y \ldots d^{i_m} y)$$

is a unit if and only if a is a unit. □

Consider next the geometric quotient of T given by the theory T_{fin} of *finitary differential k-algebras.* We define this notion by adding the (infinitary) geometric sequent:

$$\text{true} \Rightarrow \bigvee_{r \in N} d^{r+1} y = 0 .$$

to T.

By means of general results on classifying toposes (cf. [12]) the classifying topos for T_{fin}-models in $\mathcal{S}$-based toposes is easily described. Make $\mathbb{C}^{op}$ into a site where the topology is generated by the cocovering of $k\{y\}$ given by the family

$$(k\{y\} \xrightarrow{a} A)_{a \in A,\ A \in |\mathbb{B}|} ,$$

where $\mathbb{B}$ is the category of finitely presented finitary differential k-algebras. Denote by S_{fin} the associated sheaf of S in this topology.

COROLLARY. *The pair* $(\widetilde{\mathbb{C}^{op}_{fin}}, S_{fin})$ *is a model of DA.*

PROOF. If A is a finitary differential k-algebra, so is $A_{rm}\{\epsilon\}$ for any $r, m \geq 0$. □

We have introduced the small category B above, in order to describe the finitary topology on C^{op}. We now have:

THEOREM. *Let $\bar{S} \in |\mathcal{S}^{B}|$ be the underlying set functor. Then, there exists an equivalence $\mathcal{S}^{B} \approx \widetilde{C^{op}_{fin}}$ of $\mathcal{S}$-based toposes, under which $\bar{S}$ is mapped onto S_{fin}.*

PROOF. Let $\phi : \mathcal{S}^{B} \to \widetilde{C^{op}_{fin}}$ be the geometric morphism for which $\phi^* S_{fin} \approx \bar{S}$, which exists since $\bar{S}$ is a model of T_{fin} in $\mathcal{S}^{B}$. The left exact functor

$$B^{op} \xrightarrow{i} C^{op} \xrightarrow{\epsilon} \widetilde{C^{op}_{fin}}$$

(where i is the inclusion, and ϵ is the composite of Yoneda and the associated sheaf functor) induces (by letting ψ^* be the Kan extension of the above along Yoneda) a geometric morphism $\psi : \widetilde{C^{op}_{fin}} \to \mathcal{S}^{B}$. We now prove that $\psi_* \bar{S} \approx S_{fin}$. Indeed, compare the colimit diagram

$$(\psi^* h\bar{A} \xrightarrow{\psi^* a} \psi^* \bar{S})_{a \in A,\ A \in |B|},$$

which is the image, under ψ^*, of the canonical colimit ending in $\bar{S}$, with

$$(\epsilon i \bar{A} \xrightarrow{\epsilon i a} \epsilon i\, k\{y\})_{a \in A,\ A \in |B|},$$

which is obtained by the continuity of ϵi. By definition, $\psi^* h \approx \epsilon i$; hence

$$\psi^* \bar{S} \approx \epsilon i\, k\{y\} = S_{fin}.$$

The two geometric morphisms are easily shown to be inverse to each other. □

We shall now strengthen an observation of Lawvere (cf. [11]) concerning a relationship which exists between derivations and vector fields. We shall make the assumption that k *is a field of characteristic 0.*

PROPOSITION. *Let* A *be the category of finitely presented k-algebras. For $A \in |A|$, the following structures are in bijective correspondence:*

(i) a derivation d on A;

(ii) a vector field ξ on $X = hA \in |\mathcal{S}^{A}|$.

PROOF. The bijection is given by the canonical correspondences:

$$hA \xrightarrow{\xi_d} hA^{D_\infty}$$

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{hA \times D_\infty \longrightarrow hA}{hA \times \varinjlim_{m\in N} D_m \longrightarrow hA}}{\varinjlim_{m \in N}(hA \times D_m) \longrightarrow hA}}{\varinjlim_{m\in N} hA_m[\epsilon] \longrightarrow hA}}{\{hA_m[\epsilon] \longrightarrow hA\}_{m\in N}}}{\{A \xrightarrow{a_m} A_m[\epsilon]\}_{m\in N}}}{A \xrightarrow{d} A}$$

where the correspondence between $\{a_m\}$ and d is: given d (hence all higher order derivations d^m as well) define

$$a_m(a) = a + da.\epsilon + \frac{d^2a}{2!}.\epsilon^2 + \dots + \frac{d^m a}{m!}.\epsilon^m \ ;$$

conversely, given $\{a_m\}$, let $d(a)$ be the second projection of the element of $A \times A$ corresponding to $a_1(a) \epsilon A[\epsilon]$ under the canonical isomorphism $A[\epsilon] \approx A \times A$. □

This leads us to a deeper connection between vector fields and derivations, below. But note first that the vector fields arising from Ritt algebras are always vector fields in the «strong» sense, i. e., D_∞-«sets». A weaker notion (cf. [11, 6]) is usually considered. Also note that, in the strong version of vector field we use here, a certain «integrability condition» (involving infinitesimals) is added to the usual notion of vector field; a condition satisfied by all integrable vector fields, i. e., by the restrictions of total flows (cf. [11]).

Denote by $Vect(\mathcal{S}^{\bar{A}})$ the category of (strong) vector fields in $\mathcal{S}^{A}$, the latter regarded as a model of SDG with the generic k-algebra R as ring of line type. Let $\bar{A}$ be the category of differential k-algebras which are finitely presented as ordinary k-algebras, and consider the functor

$$F: \bar{A}^{op} \to Vect(\mathcal{S}^{A}) \quad \text{given by} \quad F(A, d) = \langle hA, \xi_d \rangle .$$

This induces an adjoint pair

$$\begin{tikzcd} Vect(\mathcal{S}^{\mathbf{A}}) \arrow[r, bend left, "Hom(F,-)"] & \mathcal{S}^{\bar{\mathbf{A}}} \arrow[l, bend left, "F\otimes -"] \end{tikzcd}$$

where $Hom(F,-)$ assigns, to a vector field $<X,\xi>$ in $\mathcal{S}^{\mathbf{A}}$, the functor $\bar{\mathbf{A}} \to \mathcal{S}$ which, to (A,d) assigns

$$HOM(<hA, \xi_d>, <X,\xi>) \quad (\text{as vector fields}),$$

and where $F\otimes -$ assigns, to an object Y of $\mathcal{S}^{\bar{\mathbf{A}}}$ the vector field

$$\varinjlim_{\eta \in (\bar{\mathbf{A}}^{op}/Y)} <hA, \xi_d>_\eta .$$

Denote by θ the counit of this adjointness. For a vector field $<X,\xi>$ in $\mathcal{S}^{\mathbf{A}}$,

$$\theta_{<X,\xi>}: \varinjlim_{\eta \in HOM(<hA, \xi_d>, <X,\xi>)} <hA, \xi_d>_\eta \longrightarrow <X,\xi>$$

is the canonical isomorphism such that $\theta_{<X,\xi>} \cdot u_\eta = \eta$, in $Vect\,(\mathcal{S}^{\mathbf{A}})$ (where u_η is the injection corresponding to η).

DEFINITION. A vector field $<X,\xi>$ in $\mathcal{S}^{\mathbf{A}}$ is said to be *algebraic* if $\theta_{<X,\xi>}$ is an isomorphism.

THEOREM. *Denoting by* $Vect^{alg}(\mathcal{S}^{\mathbf{A}})$ *the full subcategory of* $Vect(\mathcal{S}^{\mathbf{A}})$ *consisting of the algebraic vector fields, there exists an equivalence of categories* $Vect^{alg}(\mathcal{S}^{\mathbf{A}}) \approx \mathcal{S}^{\bar{\mathbf{A}}}$.

PROOF. The composite

$$Hom(F,-) \circ F\otimes - : \mathcal{S}^{\bar{\mathbf{A}}} \to \mathcal{S}^{\bar{\mathbf{A}}}$$

is naturally equivalent to the identity. The image of $F\otimes -$ is

$$Vect^{alg}(\mathcal{S}^{\mathbf{A}}). \quad \square$$

DEFINITION. Let $\phi : \mathfrak{X} \to \mathcal{S}^{\mathbf{A}}$ be a geometric morphism. By a *ϕ-vector field in* $\mathfrak{X}$ we mean a pair $<X,\xi>$ where X is an object of $\mathfrak{X}$ and where $\check{\xi}: X \times \phi^* D_\infty \to X$ is an action of the monoid object $\phi^* D_\infty$ in $\mathfrak{X}$. Denote by $Vect_\phi(\mathfrak{X})$ the category of ϕ-vector fields and their morphisms

(it is the topos $\mathfrak{X}^{\phi^* D_\infty}$). A ϕ-vector field is said to be *algebraic* if the canonical morphism of ϕ-vector fields

$$(\varinjlim_{\eta \in HOM(\phi^*<h A, \xi_d>, <X, \xi>)} \phi^*<h A, \xi_d>_\eta) \xrightarrow{\theta^\phi_{<X, \xi>}} <X, \xi>$$

defined by

$$\theta^\phi_{<X, \xi>} \cdot u_\eta = \eta$$

is an isomorphism.

THEOREM. *The pair* $<\mathcal{S}^{\bar{A}}, \bar{S}>$, *where* $\mathcal{S}^{\bar{A}}$ *is an* $\mathcal{S}^A$*-based topos with the geometric morphism* $\gamma : \mathcal{S}^{\bar{A}} \to \mathcal{S}^A$ *induced by the functor* $\bar{A} \xrightarrow{u} A$ *which forgets the derivation, classifies algebraic* ϕ*-vector fields in* $\mathcal{S}^A$*-based toposes* $\phi : \mathfrak{X} \to \mathcal{S}^A$.

PROOF. $\bar{S}$ is a γ-vector field with

$$\xi : \bar{S} \to \bar{S}^{(\gamma^* D_\infty)} \quad \text{given, for } (A, d) \text{ in } \bar{A}, \text{ by:}$$

$$\check{\xi}_{(A, d)} : A \times [\![b \in A \mid b \text{ nilpotent}]\!] \to A,$$

defined by

$$\check{\xi}_{(A, d)}(a, b) = \sum_{n \geq 0} \frac{d^n a}{n!} \cdot b^n .$$

Hence, if $\bar{\phi} : \mathfrak{X} \to \mathcal{S}^{\bar{A}}$ is a geometric morphism over $\mathcal{S}^A$, $\bar{\phi}^* \bar{S}$ is a ϕ-vector field in $\mathfrak{X}$. Since $\bar{S}$ is algebraic, so is $\bar{\phi}^* \bar{S}$. Conversely an algebraic ϕ-vector field X is also a finitary differential k-algebra in $\mathfrak{X}$ and therefore induces a geometric morphism $\bar{\phi}$ over $\mathcal{S}$, also over $\mathcal{S}^A$. □

Denote by $\mathring{A}$ and $\mathring{\bar{A}}$, respectively, the completions of A and $\bar{A}$ with respect to inverse limits of countable chains. Thus, for example, together with $A \in |\mathring{A}|$, one also has

$$A^*[\![t]\!] \in |\mathring{A}| , \quad \text{where} \quad A^*[\![t]\!] = \varprojlim_{m \in N} A_m [\epsilon] ;$$

alternatively (cf. [1]), $A^*[\![t]\!]$ is a power series ring over A , in which there are only finitely many terms of any given order.

Let $\mathring{A}^{op}$ be made into a site taking as cocoverings the families

$$(A^*[\![t]\!] \to A_m [\epsilon])_{m \in N} .$$

It follows that $X \in |\mathcal{S}^{\mathring{A}}|$ is a sheaf if and only if

$$(X^{D_\infty})(A) \approx X(A^*[\![t]\!]), \quad \text{for each } A \in |\overset{\circ}{\mathrm{A}}| .$$

Notice that $\mathcal{S}^{\overset{\circ}{\mathrm{A}}^{op}}$ as well as the category $\widetilde{\overset{\circ}{\mathrm{A}}^{op}}$ of sheaves for this topology, can both be regarded as models of SDG with $R = h\,\overline{k\{x\}}$.

We end with the following result:

THEOREM. *There exists an equivalence of categories :* $Vect(\widetilde{\overset{\circ}{\mathrm{A}}^{op}}) \approx \mathcal{S}^{\overset{\circ}{\mathrm{B}}}$.

PROOF. As in the proof of the previous theorem, we consider the functor:

$$\overset{\circ}{F} : \overset{\Omega}{\mathrm{A}}^{op} \to Vect(\widetilde{\overset{\circ}{\mathrm{A}}^{op}}) \quad \text{given by} \quad \overset{\circ}{F}(A, d) = \langle hA, \xi d \rangle$$

using that there is a functor $u : \overset{\Omega}{\mathrm{A}} \to \overset{\circ}{\mathrm{A}}$ forgetting the derivation, which we take for granted. Here, too, there is an adjoint pair with one of the composites being naturally equivalent to the identity on $\mathcal{S}^{\overset{\Omega}{\mathrm{A}}}$. The other composite is also naturally equivalent to the identity, since a vector field $\langle X, \xi \rangle$, with X a sheaf, must be algebraic. Indeed, the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "\xi"] \arrow[d, "\xi"'] & X^{D_\infty} \arrow[d, "X^+"] \\ X^{D_\infty} \arrow[r, "\xi^{D_\infty}"] & (X^{D_\infty})^{D_\infty} \end{tikzcd}$$

(on account of the vector field structure), when evaluated at an object A, gives

$$\begin{tikzcd} X(A) \arrow[r, "\xi_A"] \arrow[d, "\xi_A"'] & X(A^*[\![t]\!]) \arrow[d, "X(a(d/dt))"] \\ X(A^*[\![t]\!]) \arrow[r, "\xi_{A^*[\![t]\!]}"] & X(A^*[\![t]\!][\![t']\!]) \end{tikzcd}$$

commutative. It follows that, for each $a \in X(A)$, there exists a differential k-algebra (C, d), together with a map $f: C \to A$, as well as

$$b \in X(C) \quad \text{such that} \quad X(f)(b) = a,$$

and such that

$$X(a_d)(b) = \xi_C(b), \quad \text{where} \quad a_d : C \to C^*[\![t]\!]$$

is defined using d («Taylor series»). This says that the category

$$(\overset{\circ}{F}/<X,\xi>) \textit{ is cofinal in } (h/X) \;;$$

hence that the canonical morphism

$$\varinjlim_{\eta \in HOM(<hA,\xi_d>,<X,\xi>)} hA_\eta \xrightarrow{\quad\theta\quad} \varinjlim_{\eta \in HOM(hA,X)} hA_\eta \approx X$$

induced by forgetting the derivation is an isomorphism. □

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