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Generalized manifolds

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ET GEOMETRIE DIFFERENTIELLE

GENERALIZED MANIFOLDS

by Eckhard LOHRE

INTRODUCTION.

In order to get manifolds one has to glue together local objects along special morphisms (e.g. [1,3,4,6,8,9,14,20]). S. Mac Lane already has pointed out the importance of a categorical description of manifolds in [16]. Refering to this one finds quite different methods for this in the following papers.

C. Ehresmann descibes in [7] the complete enlargement of local functors and gets in this way for instance a construction for differential manifolds starting with the groupoid of diffeomorphisms between open parts of locally convex spaces. J. Bosch and M. Sagastume develop in [2] abstract varieties using Grothendieck topologies and generalized functors and natural transformations in the sense of C. Ehresmann. In [1] H. Appelgate and M. Tierney identify manifolds with coalgebras over a model induced cotriple. With this one gets exactly the correct objects but not enough morphisms. R. Ouzilou gives a functorial desciption of the process of gluing in [18] using Grothendieck topologies and transition isomorphisms in detail. The compatibility is formulated as in [2]. Y. Kawahara describes in [13] the topological context by a structure functor gluing together saturated subcategories.

H. Holmann and D. Pumplün work in [11] with categories of subobjects as models. Manifolds are locally represented by special functors. Defining proper morphisms between such functors one gets a diagram category which leads directly to the definition of a category of manifolds.

The following paper starts with categories with subobjects and coverings. Manifolds are locally defined by functors as in [11]. But the definition of cards, coverings and compatibility of cards allows to take also the global objects, in a natural manner, into consideration. First coverings are introduced as special colimits depending on a category with subobjects. Then one deduces some lemmas which are important for the following considerations. The definition of generalized manifolds starts with categories with subobjects $(\underline{K}, \underline{U})$, $(\underline{K}', \underline{U}')$ and a functor $V: \underline{K}' \to \underline{K}$ which can be restricted to the subcategories $\underline{U}$, $\underline{U}'$. If this functor has suitable properties it is possible to define generalized atlases, manifolds and smooth morphisms so that the classical cases of topological, differentiable, complex, etc... manifolds are described as special cases (further examples are in (4.10)). The generalized manifolds with smooth morphisms form a category in a natural way.

I would like to thank Prof. Dr. Pumplün for his encouragement.

1. COVERINGS.

For a category $\underline{K}$, $Ob\underline{K}$ denotes the class of objects of $\underline{K}$. Objects and identities in $\underline{K}$ are identified and marked by capital letters. The (co)domain of a morphism $h \in \underline{K}$ is noted as $Dom(h)$ ($Cod(h)$). $Mono(\underline{K})$ is the subcategory of monomorphisms, $Retr(\underline{K})$ the subcategory of retractions in $\underline{K}$. The dual subclasses are $Epi(\underline{K})$ and $Coretr(\underline{K})$. $Iso(\underline{K})$ is the class of isomorphisms. Functor always means covariant functor.

If $f, k \in \underline{K}$ have the same codomain and (l, g) is a pullback of (f,k) in $\underline{K}$, one often writes

$$f^{-1}(k) := l \quad \text{and} \quad k^{-1}(f) := g.$$

By f and k, $f^{-1}(k)$ ($k^{-1}(f)$) is only fixed up to isomorphy in the comma category $\underline{K}/Dom(f)$ ($\underline{K}/Dom(k)$).

From now on $(\underline{K}, \underline{U})$ denotes a category with subobjects, i.e., $\underline{K}$ is a category, $\underline{U}$ a subcategory of $\underline{K}$ with

$$Iso(\underline{K}) \subset \underline{U} \subset Mono(\underline{K}),$$

and $\underline{K}$ has inverse $\underline{U}$-images. This last property means that there exists a pullback (v, g) in $\underline{K}$ with $v \in \underline{U}$ of (f, u) for all $f \in \underline{K}$, $u \in \underline{U}$ with the same codomain.

For instance the category of topological spaces *Top* together with the subcategory of open embeddings *Op* is a category with subobjects. In this case the comma category Op/D describes the topology of a topological space D. For an object A of $\underline{K}$ a subclass $\underline{B}$ of the class $Ob(\underline{U}/A)$ is therefore called a basis of $\underline{U}/A$ if for all $u \in Ob(\underline{U}/A)$ there is a class $\underline{B}' \subset \underline{B}$ so that u is the $\underline{U}$-union of $\underline{B}'$ (one writes: $u \approx \cup \underline{B}'$), i.e., a colimit (supremum) of $\underline{B}'$ in the directed category $\underline{U}/A$. $\underline{U}$-intersections ($\cap$) are introduced dually.

O-categories are defined as in [11], i.e., as directed categories with fixed finite products. In this paper we always claim that O-categories are small and not empty. The morphisms in an O-category $\underline{I}$ are written

$$(j, i) : i \to j \quad \text{for} \quad i, j \in Ob\underline{I}.$$

$i \cap j$ is the product of i and j. In the following $(\underline{K}, \underline{U})$ will always be a category with subobjects. Starting with [11] one gets a functorial description of coverings.

(1.1) DEFINITION. γ is called a *$(\underline{K}, \underline{U})$-covering* of an object A of $\underline{K}$ if the following are fulfilled:

(C 0) There is an O-category $\underline{I}$ and a functor $C : \underline{I} \to \underline{K}$ so that $\gamma : C \to \Delta(A)$ is a natural transformation pointwise in $\underline{U}$.

(C 1) (γ, A) is a universal colimit of C in $\underline{K}$.

(C 2) For all $i, j \in Ob\underline{I}$, $(C(i, i \cap j), C(j, i \cap j))$ is a pullback of $(\gamma(i), \gamma(j))$ in $\underline{K}$.

A colimit (γ, A) of C in $\underline{K}$ is called universal if, for all $f : B \to A$, $(f^{-1}(\gamma), B)$ is a colimit of $f^{-1}(C)$. One gets the functor $f^{-1}(C) : \underline{I} \to \underline{K}$ and the natural transformation $f^{-1}(\gamma) : f^{-1}(C) \to \Delta(B)$ in a canonical way by choosing a pullback of $(f, \gamma(i))$ for all $i \in Ob\underline{I}$ with

$$f^{-1}(\gamma)(i) := f^{-1}(\gamma(i)) \quad \text{and} \quad f^{-1}(C)(i) := f^{-1}(C(i)).$$

(Δ denotes the canonical embedding of $\underline{K}$ into the functor category $[\underline{I}, \underline{K}]$.) One gets immediately that γ is a $(\underline{K}, \underline{U})$-covering of A iff for all $f \in \underline{K}$ with $Cod(f) = A$, $f^{-1}(\gamma)$ is a $(\underline{K}, \underline{U})$-covering of $Dom(f)$. If $\gamma: C \to \Delta(A)$ is a $(\underline{K}, \underline{U})$-covering one has

$$C(\underline{I}) \subset \underline{U}, \quad \gamma(Ob\underline{I}) \subset Ob(\underline{U}/A)$$

is closed under finite $\underline{U}$-intersections in $\underline{U}/A$ (because of

$$\gamma(i \cap j) \approx \gamma(i) \cap \gamma(j), \quad i, j \in Ob\underline{I}),$$

C preserves pullbacks and $A \approx \bigcup_{i \in Ob\underline{I}} \gamma(i)$.

As coverings are special colimits of functors whose domain is an O-category we want to describe how one gets such colimits if $\underline{K}$ fulfills additional assumptions (The proofs of (1.2), (1.3) are similar to [19] IV (6.1), (6.2)).

(1.2) PROPOSITION. *If $\underline{K}$ has coproducts and $C: \underline{I} \to \underline{K}$ is a functor, $\underline{I}$ an O-category, the coproducts $(j(i), i \in Ob\underline{I} ; \coprod_{i \in Ob\underline{I}} C(i))$ and*

$$(j(i,k), (i,k) \in Ob\underline{I}^2 ; \quad \coprod_{(i,k) \in Ob\underline{I}^2} C(i \cap k))$$

exist, and one gets morphisms f, g, $d \in \underline{K}$ which are uniquely determined by:

$$fj(i,k) = j(i) C(i, i \cap k) \quad \text{for all } (i,k) \in Ob\underline{I}^2,$$

$$gj(i,k) = j(k) C(k, i \cap k) \quad \text{for all } (i,k) \in Ob\underline{I}^2,$$

$$dj(i) = j(i,i) \quad \text{for all } i \in Ob\underline{I}.$$

f and g are retractions with common coretraction d.

(1) If h is a coequalizer of (f,g), (γ, A) yields a colimit of C by taking $A := Cod(h)$ and $\gamma(i) := hj(i)$ for all $i \in Ob\underline{I}$.

(2) Conversely, a colimit (γ, A) of C defines by $hj(i) = \gamma(i)$, $i \in Ob\underline{I}$, a unique morphism $h \in \underline{K}$ which is a coequalizer of (f,g).

(1.3) PROPOSITION. *f, g, $d \in \underline{K}$ are defined as in (1.2) and for all $(i,k) \in Ob\underline{I}^2$, $d(i,k)$ is to be a coequalizer of*

$$(j(i) C(i, i \cap k), j(k) C(k, i \cap k)).$$

(1) If $(q(i,k))_{(i,k)\in Ob\underline{I}^2}$ *is a pushout of* $(d(i,k))_{(i,k)\in Ob\underline{I}^2}$, *then*

$$h := q(i,k)\, d(i,k),\ (i,k)\in Ob\underline{I}^2,$$

is a coequalizer of (f,g).

(2) In case that h *is a coequalizer of* (f,g) *there exists exactly one* $q(i,k)\in \underline{K}$ *with*

$$q(i,k)d(i,k) = h \quad \text{for all} \quad (i,k)\in Ob\underline{I}^2$$

and $(q(i,k))_{(i,k)\in Ob\underline{I}^2}$ *is a pushout of* $(d(i,k))_{(i,k)\in Ob\underline{I}^2}$.

The process of gluing functors according to (1.2) and (1.3) yields in a canonical way coverings in the sense of (1.1) as the following example shows:

$\underline{S}$ denotes the category of sets, $V: Top \to \underline{S}$ the canonical forgetful functor and $\underline{I}$ an O-category. For a functor $C: \underline{I} \to \underline{S}$ one defines the following relation on the coproduct

$(j(i),\ i\in Ob\underline{I}\,;\ M_C := \bigcup_{i\in Ob\underline{I}} C(i)\times\{i\})$:

$(x,i) \sim (z,k)$ for $(x,i), (z,k)\in M_C$ iff there exists a

$y\in C(i\cap k)$ with $x = C(i,i\cap k)(y)$ and $z = C(k,i\cap k)(y)$.

Now C preserves pullbacks iff $C(\underline{I})\subset Mono(\underline{S})$ and $\sim$ is an equivalence relation on M_C. If C preserves pullbacks, $\gamma: C\to \Delta(A)$ is an $(\underline{S}, Mono(\underline{S}))$-covering of $A := M_C/\sim$ with $\gamma(i) := \pi_C \circ j(i),\ i\in Ob\underline{I}$ and the canonical projection $\pi_C : M_C \to A$.

In case that C is given in the form $V\circ F$ with a functor

$$F: \underline{I} \to Top, \quad F(\underline{I}) \subset Op,$$

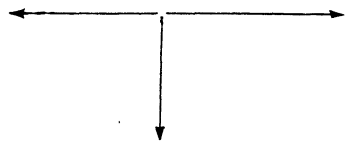
a colimit (δ, B) of F exists with $V(B) = A$ and $V\circ\delta = \gamma$ so that δ induces a (Top, Op)-covering of B. One gets A, respectively B, by gluing the functor C, respectively F (cp. [3]).

Some characteristic properties of coverings arise directly in the later chapters and are noted here. First we put down some aids which are often needed later on.

2. DIAGRAM LEMMAS.

If a limit (D.2) exists in $\underline{K}$ for a diagram of the form (D.1),

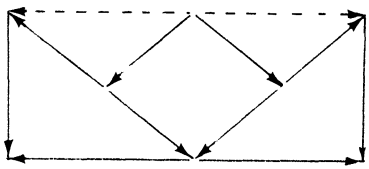
(D.1)



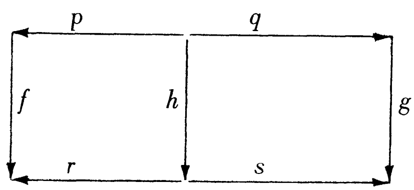
(D.2)

the composed diagram is called a *bipullback*.

Obviously the existence of bipullbacks in $\underline{K}$ is equivalent to the existence of pullbacks in $\underline{K}$. One gets a bipullback of (D.1) just by pulling back three times according to diagram (D.3) (cp. [11] (1.17)).



(D.3)



(D.4)

(2.1) LEMMA. Let (D.4) be a commutative diagram in $\underline{K}$, and let $(r, s; Dom(r))$ be a product. $(p, q; Dom(p))$ is a product iff (D.4) is a bipullback. In this case one writes $f \pi g = h$ and gets $f \pi g \in \underline{U}$ for f, g in $\underline{U}$ (notice (D.3)).

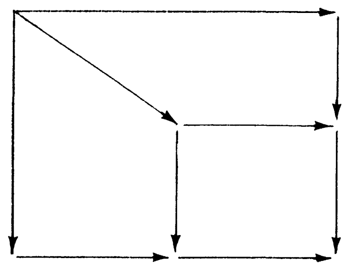
(2.2) LEMMA. (D.4) is to be a commutative diagram in $\underline{K}$.

(1) If the right square in (D.4) is a pullback and $f \in Mono(\underline{K})$, (D.4) is a bipullback.

(2) The right square in (D.4) is a pullback if (D.4) is a bipullback and $f \in Iso(\underline{K})$.

(3) If both squares in (D.4) are pullbacks, (D.4) is a bipullback if h is a monomorphism of $\underline{K}$.

(2.3) LEMMA. In case that the small square in the commutative diagram (D.5) is a pullback, the big square is a pullback iff the other two inner quadrangles are bipullbacks.

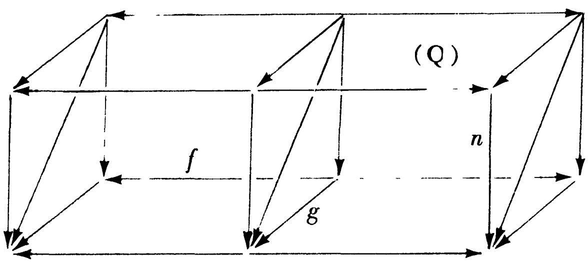


(D.5)

(2.4) LEMMA. (Q) is to denote the commutative right cube in (D.6).

(1) If two opposite sides and the base in (Q) are pullbacks, then the upper side is a pullback.

(2) Both vertical back squares in (Q) are a bipullback if base and upper side are pullbacks and $n \in Mono(\underline{K})$ (notice (2.3)).



(D.6)

(2.5) LEMMA. (D.6) is to be a commutative diagram in $\underline{K}$.

(1) If the base is a bipullback, then the back side is a bipullback iff the diagonal plane is a bipullback.

(2) In case that the diagonal plane is a bipullback and f or g are in $Mono(\underline{K})$, the back side turns out to be a bipullback.

(3) If the base is a bipullback and the left and right vertical sides are pullbacks, the vertical square in the middle is a pullback iff the upper side is a bipullback.
(Cp. also [11], (1.16), (1.19), (1.20).)

3. ATLASES.

In this section $(\underline{K}', \underline{U}')$ and $(\underline{K}, \underline{U})$ are to be categories with subobjects, and $V: \underline{K}' \to \underline{K}$ is to be a functor with $V(\underline{U}') \subset \underline{U}$. As an abbreviation for the pair of functors $(V, \hat{V})$ ($\hat{V}: \underline{U}' \to \underline{U}$ is the restriction of V to the subcategories) one writes $V: (\underline{K}', \underline{U}') \to (\underline{K}, \underline{U})$. For the description of atlases and manifolds relative to $V: (\underline{K}', \underline{U}') \to (\underline{K}, \underline{U})$ one has first to introduce some important properties of V.

(3.1) DEFINITION. (1) V *preserves inverse images* if for all

$$f' \in \underline{K}', \quad u' \in \underline{U}' \text{ with } Cod(f') = Cod(u')$$

and for every pullback (v', g') of (f', u') in $\underline{K}'$, $(V(v'), V(g'))$ is a pullback of $(V(f'), V(u'))$ in $\underline{K}$.

(2) V *reflects inverse images (finite intersections)* if for all

$$u', v' \in \underline{U}', \quad f', g' \in \underline{K}' \quad (f', g' \in \underline{U}') \text{ with } u'g' = f'v',$$

(v', g') is a pullback of (f', u') provided $(V(v'), V(g'))$ is a pullback of $(V(f'), V(u'))$.

(3) V *generates inverse images (finite intersections)* if for all

$$f': A' \to B' \text{ of } \underline{K}' \text{ (of } \underline{U}'), \quad u': C' \to B' \text{ of } \underline{U}'$$

and all pullbacks (v, g) of $(V(f'), V(u'))$, morphisms

$$v': D' \to A' \text{ of } \underline{U}', \quad g': D' \to C' \text{ of } \underline{K}' \text{ and } k \in Iso(\underline{K})$$

exist with

$$v = V(v')k \quad \text{and} \quad g = V(g')k,$$

so that (v', g') yields a pullback of (f', u').

(4) V *generates local inverse images* if for all

$$f: V(B') \to A \text{ from } \underline{K} \text{ and } u: V(C') \to A \text{ of } \underline{U}$$

and all pullbacks (v, g) of (f, u) in $\underline{K}$, morphisms

$$v' \in \underline{U}', \quad g' \in \underline{K}' \text{ and } k \in Iso(\underline{K})$$

exist so that $v = V(v')k$ and $g = V(g')k$.

(5) V *partially generates local inverse images* if for all

$$f: V(B') \to A \text{ of } \underline{K} \text{ and } u: V(C') \to A \text{ of } \underline{U}$$

and all pullbacks (v, g) of (f, u), there exist morphisms

$k \in Iso(\underline{K})$, $v' \in \underline{U}'$ with $Cod(v') = B'$ and $v = V(v')k$.

(6) V *generates subobjects* if for all $u: A \to V(B')$ of $\underline{U}$, morphisms $k \in Iso(\underline{K})$, $u' \in \underline{U}'$ exist with $Cod(u') = B'$ and $u = V(u')k$.

There are a lot of connections between the notions introduced above (cp. [15], Section 2).

Here we only want to mention the following properties. Provided $\underline{U}' \subset Init_V(\underline{K}')$ (the subcategory of V-initial morphisms of $\underline{K}'$), V generates inverse images and finite intersections if V partially generates local inverse images. Moreover one gets the following obvious

(3.2) LEMMA. The statements (2), (3) and (4) are equivalent and (1) implies (2). If V preserves inverse images, then (2) implies (1):

(1) $\underline{U}' \subset Init_V(\underline{K}')$.

(2) V reflects inverse images.

(3) V reflects finite intersections.

(4) $V^{-1}(Iso(\underline{K})) \cap \underline{U}' = Iso(\underline{K}')$.

We put

$$\underline{K}(\underline{A}, \underline{B}) := \bigcup_{(A,B) \in \underline{A} \times \underline{B}} \underline{K}(A, B),$$

$$\underline{K}(A, \underline{A}) := \underline{K}(\{A\}, \underline{A}) \quad \text{and} \quad \underline{K}(\underline{A}, A) := \underline{K}(\underline{A}, \{A\})$$

for $\underline{A}$, $\underline{B} \subset Ob\underline{K}$ and $A \in Ob\underline{K}$. An empty object O in $\underline{K}$ is an initial object in $\underline{K}$ with

$$\underline{K}(Ob\underline{K}, O) \subset Iso(\underline{K}) \text{ - therefore } \underline{K}(O, Ob\underline{K}) \subset Mono(\underline{K}).$$

o_A denotes the unique morphism from the initial object O to A. An *empty object in* $(\underline{K}, \underline{U})$ is to be an empty object O in $\underline{K}$ with

$$\underline{K}(O, Ob\underline{K}) \subset \underline{U}.$$

Moreover one says that $V: (\underline{K}', \underline{U}') \to (\underline{K}, \underline{U})$ *generates empty objects* iff for every empty object O in $(\underline{K}, \underline{U})$ an empty object O' in $(\underline{K}', \underline{U}')$ exists with $V(O') \approx O$.

(3.3) DEFINITION. V is called *local* if the following properties hold for every natural transformation $\gamma': C \to \Delta(A')$ pointwise in $\underline{U}$ with

$A' \in Ob\underline{K}'$, $C: \underline{I} \to \underline{K}'$ a functor, $\underline{I}$ an O-category, for which $V' \circ \gamma'$ is a $(\underline{K}, \underline{U})$-covering of $V(A')$:

(LOC 1) For every natural transformation $\alpha': C \to \Delta(B')$,

$$B' \in Ob\underline{K}' \text{ and } f \in \underline{K} \text{ with } V \circ \alpha' = \Delta(f)(V \circ \gamma')$$

exists a unique $f' \in \underline{K}'$ with $\alpha' = \Delta(f')\gamma'$.

(LOC2) If α' in (LOC 1) is a natural transformation pointwise in $\underline{U}$, $f \in \underline{U}$ implies $f' \in \underline{U}'$.

From now on the following properties for $V: (\underline{K}', \underline{U}') \to (\underline{K}, \underline{U})$ are assumed:

(M 0) V generates empty objects.

(M 1) V is faithful.

(M 2) $\underline{U}' \subset Init_V(\underline{K}')$.

(M 3) V partially generates local inverse images.

(M 4) V is local.

V is then called a *Man(ifold)-functor.*

Now we want to introduce the notion of an atlas relative to a Man-functor showing the meaning of the conditions (M 0) to (M 4). A guiding example for further definitions is given by the following one: Let K denote the field of real or complex numbers (R or C) or a commutative field with a nontrivial ultrametric complete valuation. Adjoining 0, ω and ∞ to the set of natural numbers N one gets $\bar{\mathrm{N}}$ if $\mathrm{K} \neq \mathrm{R}$. If $\mathrm{K} = \mathrm{R}$ one defines $\bar{\mathrm{N}} := \{\omega\}$. Now C^r-Mor, $r \in \bar{\mathrm{N}}$ denotes the category of C^r-morphisms between open subsets of polynormed separated Banach spaces over K and $U(C^r\text{-}Mor)$ the subcategory with the same objects, whose morphisms are C^r-isomorphisms onto open subsets of the corresponding codomain. Then $(C^r\text{-}Mor, U(C^r\text{-}Mor))$ is a category with subobjects and the canonical functor

$$V^r: (C^r\text{-}Mor, U(C^r\text{-}Mor)) \to (Top, Op)$$

is a Man-functor (cp. [4, 14]).

For an object A of $\underline{K}$, the morphisms of the comma category $<\hat{V}, A>$ are written as

$$(v, w', u) \text{ with } u, v \in \underline{U}, \ w' \in \underline{U}' \text{ and } u = v V(w').$$

Objects (u, A', u) of $<\hat{V}, A>$ are denoted as pairs

$$(u, A') \text{ for } u: V(A') \to A \text{ of } \underline{U}$$

and are called *V-cards* or simply cards on A if no confusion is possible. With the help of the coverings defined in (1.1) one gets now the following functorial description of an atlas.

(3.4) DEFINITION. (1) (γ, F) is called a *V-atlas* if an object A of $\underline{K}$, an O-category $\underline{I}$ and a functor $F: \underline{I} \to \underline{K}'$ exist with $F(\underline{I}) \subset \underline{U}'$ so that $\gamma: V \circ F \to \Delta(A)$ is a $(\underline{K}, \underline{U})$-covering of A. As A is uniquely determined by (γ, F), (γ, F) is also called V-atlas on A, and one often writes simply $\gamma: V \circ F \to \Delta(A)$. A arises from gluing together F relative to V. The cards of the V-atlas (γ, F) are $(\gamma(i), F(i))$, $i \in Ob\underline{I}$. (Notice that F preserves pullbacks, cp. the introduction of [9].)

(2) (δ, G) is to be another V-atlas on B, $G: \underline{J} \to \underline{K}'$ a functor, $\underline{J}$ an O-category. Then a morphism $f: A \to B$ of $\underline{K}$ is called a *morphism of V-atlases* or a *V-atlas morphism* if

$$\rho'(i,j) \in \underline{U}' \text{ and } \tau'(i,j) \in \underline{K}', \ (i,j) \in Ob\underline{I} \times \underline{J},$$

exist with the same domain $P(i,j)$ so that $(V(\rho'(i,j)), V(\tau'(i,j)))$ is a pullback of $(f\gamma(i), \delta(j))$ in $\underline{K}$. Therefore $V(\rho'(i, -))$ yields a $(\underline{K}, \underline{U})$-covering of $V(F(i))$, $i \in Ob\underline{I}$. One notes the V-atlas morphism by $f: (\gamma, F) \to (\delta, G)$.

(3) If there is another V-atlas morphism $g: (\delta, G) \to (\eta, H)$ the product $gf: (\gamma, F) \to (\eta, H)$ is also a V-atlas morphism. With this composition the class of V-atlas morphisms becomes a category $Atl(V)$, the *category of V-atlases*.

The objects of $Atl(V)$ are just the morphisms of the form

$$A: (\gamma, F) \to (\gamma, F), \quad A \in Ob\underline{K}$$

(cp. (C 2)) which are identified with the V-atlas (γ, F) on A. It is easily verified that Definition (3.4) (2) is independent of the choice of the pullback in the following sense (notations as in (3.4) (2)): f is a V-atlas morphism from (γ, F) to (δ, G) iff for all $(i,j) \in Ob(\underline{I} \times \underline{J})$ and for all pullbacks

$$(V(\rho'(i,j)), \tau(i,j)) \text{ of } (f\gamma(i), \delta(j)) \text{ with } \rho'(i,j) \in \underline{U}',$$

there exists $\tau'(i,j) \in \underline{K}'$ with

$$Dom(\tau'(i,j)) = Dom(\rho'(i,j)) \text{ and } V(\tau'(i,j)) = \tau(i,j).$$

V-atlas morphisms can therefore locally be described by morphisms of $\underline{K}'$. For the Man-functor V^r, $r \in \bar{\mathrm{N}}$, V^r-atlas morphisms induce C^r-morphisms between the manifolds represented by the atlases (cp. e.g. [20]).

$U(Atl(V))$ denotes the subclass of those morphisms

$$f: (\gamma, F) \to (\delta, G) \text{ of } Atl(V) \text{ with}$$
$$f \in \underline{U} \text{ and } \tau'(i,j) \in \underline{U}' \text{ for all } (i,j) \in Ob(\underline{I} \times \underline{J})$$

(notations as in (3.4)). The morphisms of $U(Atl(V))$ form a subcategory of $Atl(V)$ and can be described locally by $\underline{U}'$-morphisms. One easily proves that

$$Iso(Atl(V)) \subset U(Atl(V))$$

is valid because $f: (\gamma, F) \to (\delta, G)$ is an isomorphism of $Atl(V)$ iff $f \in Iso(\underline{K})$ and $f^{-1}: (\delta, G) \to (\gamma, F)$ is a V-atlas morphism. Before deducing further properties of $U(Atl(V))$ some remarks follow which can also be proved under slightly weaker conditions on $V: (\underline{K}', \underline{U}') \to (\underline{K}, \underline{U})$ (cp. [15]).

(3.5) LEMMA. *For all $(i,j) \in Ob(\underline{I} \times \underline{J})$ consider a bipullback in $\underline{K}$ of the form (D.7) with $u: X \to A$ from $\underline{U}$, $f: X \to B$ from $\underline{K}$, (γ, F) a V-atlas on A, (δ, G) a V-atlas on B, $F: \underline{I} \to \underline{K}'$, $G: \underline{J} \to \underline{K}'$, $\rho'(i,j) \in \underline{U}'$.*

$$\begin{array}{ccccc}
V(F(i)) & \xleftarrow{V(\rho'(i,j))} & V(P(i,j)) & \xrightarrow{\tau(i,j)} & V(G(j)) \\
\Big\downarrow{\scriptstyle \gamma(i)} & & \Big\downarrow{\scriptstyle \eta(i,j)} & & \Big\downarrow{\scriptstyle \delta(j)} \\
A & \xleftarrow[u]{} & X & \xrightarrow[f]{} & B
\end{array}$$

(D.7)

Then the $P(i,j)$, $(i,j) \in Ob(\underline{I} \times \underline{J})$ *induce canonically a functor* $P: \underline{I} \times \underline{J} \to \underline{K}'$ *and the* $\eta(i,j)$ *a* V*-atlas* $\eta: V \circ P \to \Delta(X)$. *If* $P_{\underline{I}}: \underline{I} \times \underline{J} \to \underline{I}$ *and* $P_{\underline{J}}: \underline{I} \times \underline{J} \to \underline{J}$ *are the usual projections,*

$$\rho': P \to F \circ P_{\underline{I}} \quad and \quad \tau: V \circ P \to V \circ G \circ P_{\underline{J}}$$

are natural transformations. In case that $\tau'(i,j) \in \underline{K}'$ *exists with*

$$V(\tau'(i,j)) = \tau(i,j) \quad for\ all \quad (i,j) \in Ob(\underline{I} \times \underline{J}),$$

$\tau': P \to G \circ P_{\underline{J}}$ *induces also a natural transformation.*

PROOF. The existence of the functor P and the natural transformation η follows from the property of the bipullback using (M 2). In order to verify (C 1) for (η, X) one constructs the bipullback as in (D.3). Condition (C 2) follows immediately from (2.5) (1), (3). Defining

$$\eta(i,j) := \gamma(i)\, V(\rho'(i,j)), \quad (i,j) \in Ob(\underline{I} \times \underline{J}),$$

in (3.4) (2), $P: \underline{I} \times \underline{J} \to \underline{K}'$ induces a functor with

$$P(\underline{I} \times \underline{J}) \subset \underline{U}', \quad \rho': P \to F \circ P_{\underline{I}}, \quad \tau': P \to G \circ P_{\underline{J}}$$

natural transformations and $\eta: V \circ P \to \Delta(A)$ a V-atlas (which is finer than (γ, F) in the sense of (4.1) (3)). By (3.4) (1), $(G, \tau', P_{\underline{J}})$ is an element of the class $\bar{A}_{T'}$, $T' = (\underline{K}', \underline{U}')$, defined in [11] (2.1) and by which the local description of atlas morphisms is given in [11]. If $\gamma: C \to \Delta(A)$ and $\delta: D \to \Delta(A)$ are $(\underline{K}, \underline{U})$-coverings of $A \in Ob\underline{K}$, $C: \underline{I} \to \underline{K}$, $D: \underline{J} \to \underline{K}$, $\underline{I}$, $\underline{J}$ O-categories and $f, g \in \underline{K}$ with

$$f\gamma(i)\gamma(i)^{-1}(\delta(j)) = g\gamma(i)\gamma(i)^{-1}(\delta(j)), \quad (i,j) \in Ob(\underline{I} \times \underline{J}),$$

one gets $f = g$.

(3.6) COROLLARY. *Let* $\underline{I}$ *be an O-category,* $C: \underline{I} \to \underline{K}$ *a functor and* $\gamma: C \to \Delta(A)$ *a* $(\underline{K}, \underline{U})$*-covering of* $A \in Ob\underline{K}$. *Every natural transformation* $\alpha: C \to \Delta(B)$, $B \in Ob\underline{K}$, *induces a unique* $\underline{K}$*-morphism*

$$a: A \to B \quad with \quad \alpha = \Delta(a)\gamma.$$

a *is a monomorphism iff for all* $i, j \in Ob\underline{I}$, $(C(i, i \cap j), C(j, i \cap j))$ *is a pullback of* $(\alpha(i), \alpha(j))$.

For every V-atlas (γ, F) on A, $F: \underline{I} \to \underline{K}'$ and every V-card

(u, B') on A there is a V-atlas $(u^{-1}(\gamma), u^{-1}(F))$ on $Dom(u)$ and a natural transformation $\gamma_u : u^{-1}(F) \to F$ pointwise in $\underline{U}'$ so that for all $i \in Ob\underline{I}$, $(V(\gamma_u(i)), u^{-1}(\gamma)(i))$ is a pullback of $(\gamma(i), u)$.

Condition (C2) can be verified by (2.4) (1). If V generates subobjects one gets a V-atlas as above for all $u: B \to A$ of $\underline{U}$. In any case $u: (u^{-1}(\gamma), u^{-1}(F)) \to (\gamma, F)$ is an element of $U(Atl(V))$.

(3.7) PROPOSITION. *$f: A \to B$ is to be a K-morphism, (γ, F) a V-atlas on A and (δ, G) a V-atlas on B.*

(1) In case that $f: (\gamma, F) \to (\delta, G)$ is a V-atlas morphism,

$$f_{v,u} := v^{-1}(f) s_u : (u^{-1}(\gamma), u^{-1}(F)) \to (v^{-1}(\delta), v^{-1}(G))$$

is also a V-atlas morphism for all $u: V(A') \to A$ and all $v: V(B') \to B$ of $\underline{U}$ for which $s_u \in \underline{K}$ exists with $(f^{-1}(v), s_u, u) \in \underline{U}/A$ (notation $u \leq f^{-1}(v)$).

(2) If V generates subobjects the converse of (1) is true.

Obviously $U(Atl(V))$ is a subcategory of $Atl(V)$ with

$$Iso(Atl(V)) \subset U(Atl(V)) \subset Mono(Atl(V)).$$

Because of property (LOC1) and (3.5) one proves the existence of $U(Atl(V))$-inverse images in $Atl(V)$ (cp. [15]) and gets the following

(3.8) PROPOSITION. *$(Atl(V), U(Atl(V)))$ is a category with subobjects.*

At the end of this Chapter we introduce the canonical embedding

$$D(V): (\underline{K}', \underline{U}') \to (Atl(V), U(Atl(V)))$$

as follows:

1 denotes the one point category $\{1\}$. Then

$$F_{A'} : 1 \to \underline{K}' \quad \text{with} \quad F_{A'}(1) := A'$$

is a functor, $A' \in Ob\underline{K}'$, and

$$\gamma_{A'}: V \circ F_{A'} \to \Delta(V(A')) \quad \text{with} \quad \gamma_{A'}(1) := V(A')$$

is a natural transformation so that $(\gamma_{A'}, F_{A'})$ becomes a V-atlas on

$V(A')$. $f': A' \to B'$ of $\underline{K}'$ induces a V-atlas morphism

$$V(f'): (\gamma_{A'}, F_{A'}) \to (\gamma_{B'}, F_{B'})$$

which is noted $D(V)(f')$. $D(V): \underline{K}' \to Atl(V)$ is an embedding with $D(V)(\underline{U}') \subset U(Atl(V))$, therefore one abbreviates

$$D(V): (\underline{K}', \underline{U}') \to (Atl(V), U(Atl(V))).$$

On the other hand there is a canonical functor

$$S(V): Atl(V) \to \underline{K} \quad \text{with} \quad S(V)(U(Atl(V))) \subset \underline{U}$$

which attaches to every V-atlas morphism $f: (\gamma, F) \to (\delta, G)$ the corresponding morphism f of $\underline{K}$ so that the following diagram is commutative in both components.

$$\begin{array}{ccc} (\underline{K}', \underline{U}') & \xrightarrow{D(V)} & (Atl(V), U(Atl(V))) \\ & \searrow^{V} & \downarrow S(V) \\ & & (\underline{K}, \underline{U}) \end{array}$$

(D.8)

4. MANIFOLDS.

In order to get the notion of a V-manifold the relation of compatibility between cards is introduced as follows.

(4.1) DEFINITION. (u, A') and (v, B') are to be V-cards on $A \in Ob\underline{K}$, (γ, F) and (δ, G) V-atlases on A, $F: \underline{I} \to \underline{K}'$, $G: \underline{J} \to \underline{K}'$ functors.

(1) (u, A') and (v, B') are called *compatible* (notation

$$(u, A')\&(v, B')\ )$$

if $u', v' \in \underline{U}'$ exist with the same domain so that $(V(u'), V(v'))$ is a pullback of (u, v).

(2) (u, A') is *compatible with* (γ, F) ($(u, A')\&(\gamma, F)$) if (u, A') is compatible with each card of (γ, F). (γ, F) and (δ, G) are *compatible* ($(\gamma, F)\&(\delta, G)$) if every card of (γ, F) is compatible with (δ, G).

(3) (γ, F) is called *finer than* (δ, G) or a *refinement of* (δ, G) (notation $(\gamma, F) >> (\delta, G)$) if there is a mapping $\rho: Ob\underline{I} \to Ob\underline{J}$ with

$$<\hat{V}, A>((\gamma(i), F(i)), (\delta(\rho(i)), G(\rho(i)))) \neq \emptyset, \quad i \in Ob\underline{I}.$$

Because of the condition (C2) any two cards of a V-atlas are compatible.

If $(\underline{K}, \underline{U})$ has an empty object O and if $(o_{V(A')}, o_{V(B')})$ is a pullback of (u, v), (u, A'), (v, B') V-cards on an object A of $\underline{K}$, (u, A') and (v, B') are compatible, as V generates empty objects. Therefore (M0) guarantees just the compatibility of cards with empty intersections. If necessary one has to adjunct an empty object to the subobject categories, so that empty objects exist in the categories with subobjects.

In the example of the last chapter the compatibility of two V^r-cards on a topological space, $r \in \bar{\mathrm{N}}$, is exactly the C^r-compatibility in the sense of [4] 5.1.

(4.2) PROPOSITION. *The compatibility of V-atlases is an equivalence relation.*

This is an immediate consequence of the following

(4.3) LEMMA. *If* (γ, F) *is a V-atlas on* $A \in Ob\underline{K}$ *and if one has V-cards* (u, A') *and* (v, B') *on* A, *which are both compatible with* (γ, F), *they are already compatible.*

PROOF. $F: \underline{I} \to \underline{K}'$ is to be a functor, $\underline{I}$ an O-category. Then for all $i \in Ob\underline{K}$ one gets a diagram of the form (Q) in (D.6) in the following manner: Choose a pullback $(w, V(v'))$ of (u, v) with $v': D' \to B'$ from $\underline{U}'$ as base. Both vertical front sides are determined by pullbacks of $(u, \gamma(i))$ and $(\gamma(i), v)$ according to the condition of compatibility. Another pullback yields the upper side using the fact that V generates finite intersections. Then exactly one morphism of $\underline{U}$ exists as the last edge of (Q) so that both vertical back sides of (Q) commute and these form a pullback because of (2.4) (2). According to (M2) and (LOC2) one now gets a morphism $u': D' \to A'$ with $V(u') = w$.

If (γ, F) and (δ, G) are compatible V-atlases on A in $Ob\underline{K}$, (η, P) yields a V-atlas compatible with (γ, F) and (δ, G) in case that one takes the notations of (3.5) for

$$u = f = A = X = B$$

and writes

$$\tau(i,j) = V(\tau'(i,j)), \quad \tau'(i,j) \in \underline{U}', \quad (i,j) \in Ob(\underline{I} \times \underline{J}).$$

(η, P) is just a common refinement of (γ, F) and (δ, G). Conversely if a common refinement of the V-atlases (γ, F) and (δ, G) on $A \in Ob\,\underline{K}$ exists, then (γ, F) and (δ, G) are compatible with (η, H) and by (4.2) also (γ, F) and (δ, G). Therefore we get the following

(4.4) PROPOSITION. *V-atlases are compatible iff they have a common refinement.*

There is another characterization of the equivalence relation & by the following

(4.5) LEMMA. The V-atlases (γ, F) and (δ, G) on $A \in Ob\underline{K}$ are compatible iff

$$A : (\gamma, F) \to (\delta, G) \quad \text{and} \quad A : (\delta, G) \to (\gamma, F)$$

are V-atlas morphisms.

Now one verifies immediately that the relation & is compatible with the forming of V-atlases relative to $\underline{U}$-subobjects in the following sense.

(4.6) PROPOSITION. *(γ, F) and (δ, G) are to be V-atlases on the object $A \in Ob\underline{K}$.*

(1) $(\gamma, F)\,\&\,(\delta, G)$ implies

$$(u^{-1}(\gamma),\ u^{-1}(F))\,\&\,(u^{-1}(\delta),\ u^{-1}(G))$$

for every card (u, A') on A.

(2) In case that V generates subobjects, $(\gamma, F)\,\&\,(\delta, G)$ is true iff $(u^{-1}(\gamma), u^{-1}(F))\,\&\,(u^{-1}(\delta), u^{-1}(G))$ for all $u : D \to A$ of $\underline{U}$.

Obviously the relation & can be regarded as an equivalence relation on $ObAtl(V)$. In order to extend this relation to $Atl(V)$ the following general considerations are useful. Let $\underline{L}$ be a category, $\sim$ an equivalence relation on $\underline{L}$ and let $\underline{L}/\sim$ denote a full system of representatives of $\sim$-equivalence classes. Then $\sim$ is called *normal* if a partial composition on $\underline{L}/\sim$ exists so that $\underline{L}/\sim$ becomes a category (notation: *factor category*) and the canonical projection $P: \underline{L} \to \underline{L}/\sim$ a functor. The relation $\sim$ is called *compatible* if the following conditions are fulfilled:

(a) $Dom(f) \sim Dom(g)$ and $Cod(f) \sim Cod(g)$ for $f, g \in \underline{L}$ with $f \sim g$.

(b) $kh \sim gf$ for $f, g, h, k \in \underline{L}$ if the products are defined and if $k \sim g$, $h \sim f$ holds.

According to [19] I (7.8) (3), we get the

(4.7) LEMMA. A compatible equivalence relation $\sim$ on $\underline{L}$ is normal if for all $A, B \in Ob\underline{L}$ with $A \sim B$ a morphism $u \in \underline{L}(A, B)$ exists with $u \sim A$.

If $\underline{M}$ is another category and $W: \underline{L} \to \underline{M}$ a functor one shows the

(4.8) PROPOSITION. *Let $\sim$ denote an equivalence relation on $Ob\underline{L}$ which is contained in the equivalence relation on $Ob\underline{L}$ induced by W. Then $\sim$ can be extended to a compatible equivalence relation on $\underline{L}$ identifying $f, g \in \underline{L}$ iff there hold $Dom(f) \sim Dom(g)$, $Cod(f) \sim Cod(g)$ and $W(f) = W(g)$.*

Taking the functor $S(V): Atl(V) \to \underline{K}$ instead of W and the equivalence relation $\&$ on $Ob\,Atl(V)$ which is contained in the equivalence relation on $Ob\,Atl(V)$ induced by $S(V)$ one gets by (4.8) a compatible equivalence relation $\&$ on $Atl(V)$ that is contained in the equivalence relation induced by $S(V)$ on $Atl(V)$. As $\&$ is normal because of (4.5) and (4.7) the factor category is well defined.

(4.9) DEFINITION. The factor category $Man(V) := Atl(V)/\&$ is called *category of V-manifolds*, $P(V): Atl(V) \to Man(V)$ being the canonical projection. Objects $[\gamma, F] := P(V)(\gamma, F)$ of $Man(V)$ are called *V-manifolds*, $(\gamma, F) \in Ob\,Atl(V)$, morphisms are called *V-manifold morphisms*, and one writes $f: [\gamma, F] \to [\delta, G]$ for $f: (\gamma, F) \to (\delta, G)$ of

Atl(V).

Now are some examples of manifold extensions relative to a given functor.

(4.10) EXAMPLES. (1) The Man-functor V^r, $r \in \bar{\mathrm{N}}$, from Chapter 3 generates subobjects. The category of V^r-manifolds is just the category of C^r-morphisms between C^r-manifolds (cp. [4, 12, 14, 17]).

(2) For $r \in \bar{\mathrm{N}}$ let Γ^r denote any pseudogroup of transformations between open subsets of an r-dimensional euclidean space E^r (cp. [1]). Besides substitute in (1) $U(C^r\text{-}Mor)$ by the canonical subcategory $U(\Gamma^r)$ of Op induced by Γ^r. Then $Man(V^r)$ is the category of Γ^r-manifolds. For instance $U(\Gamma^r)$ is given by the following data:

(a) Objects are orientable open subsets of E^r, morphisms are orientation preserving homeomorphisms onto open subsets of the corresponding codomain.

(b) Objects are open subsets of E^r, morphisms are diffeomorphisms onto open subsets of the corresponding codomain whose functional matrix belongs to a fixed subgroup G of $GL(r, \mathrm{R})$ (cp. [5, 17]).

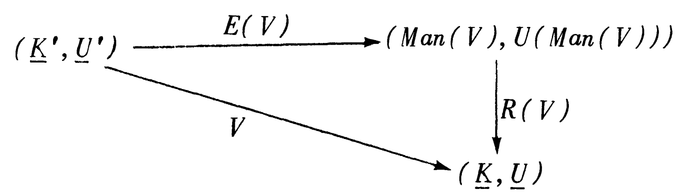
With $U(Man(V)) := P(V)(U(Atl(V)))$ one gets the following:

(4.11) PROPOSITION. *$(Man(V), U(Man(V)))$ is a category with subobjects.*

As & is contained in the equivalence relation induced on $Atl(V)$ by $S(V)$, there exists a unique functor

$$R(V): Man(V) \to K \quad \text{with} \quad S(V) = R(V) \circ P(V).$$

Obviously $R(V)(U(Man(V))) \subset \underline{U}$ and diagram (D.9) is commutative



(D.9)

in both components with $E(V) := P(V) \circ D(V)$.

Further properties concerning the functors defined in (D.9) especially the question which properties of V can be carried over to $R(V)$, the universality of the manifold extension and the properties of the category of V-manifolds will be examined in a continuation of this paper (cp. [15]).

Finally we note an obvious characterization of the situation presented in (D.9).

(4.12) PROPOSITION. *$(V, R(V))$ is a Kan co-extension of V along $E(V)$.*

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