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ET GEOMETRIE DIFFERENTIELLE

STACK COMPLETIONS AND MORITA EQUIVALENCE FOR CATEGORIES IN A TOPOS

by Marta BUNGE 1)

0. INTRODUCTION.

The origin of this paper can be traced back to one of a series of lectures given by F.W. Lawvere [10, Lecture V]. In it, Lawvere dealt for the case of an arbitrary topos $\underline{S}$, with the notion of stack, a notion which, for Grothendieck toposes, had been considered by J. Giraud [6], and which is given relative to a site.

A topos $\underline{S}$ may always be regarded as a site with the regular epimorphism topology, and the notion of stack over $\underline{S}$ is then defined with respect to this particular topology.

Special as it may be, this notion of stack over a topos $\underline{S}$ plays an important role in the development of Category Theory over a base topos $\underline{S}$. In particular, it is the purpose of this paper to establish the following conjecture of Lawvere [10, Lecture V]. Two category objects C and D in $\underline{S}$ are Morita equivalent iff Karoubian envelopes of C and D (i.e., «the closures of C and D under splitting of idempotents» in P. Freyd's [13] terminology) have equivalent stack completions.

In order to carry out this program, it has been necessary to study stacks in the context of Indexed Categories, as well as to clear up the various notions of equivalence of indexed categories which arise when the base category $\underline{S}$ does not satisfy the axiom of choice. This has been done in [4], a paper to which the present work is to be considered a sequel.

The contents of this paper are as follows. In Section 1, we define

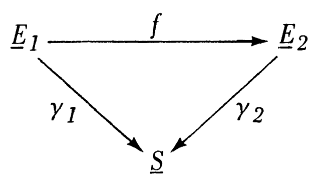
1) This work has been partially supported by grants from the National Research Council of Canada and from the Ministère de l'Education du Québec.

the $\underline{S}$-indexed category of $\underline{S}$-essential points of an $\underline{S}$-topos $\underline{E}$, and prove that it is a stack over $\underline{S}$. In Section 2, we construct for a category object C in $\underline{S}$, the $\underline{S}$-indexed category $\tilde{C}$ of locally representables (in $\underline{S}^{C^o}$) and prove that $\tilde{C}$, together with a canonical indexed functor $F: [C] \to \tilde{C}$, is the stack completion of C. This provides numerous examples of stacks, to be added to those studied in [4] by direct methods. In Section 3, we construct internally the Karoubian envelope $\hat{C}$ of a category object C in $\underline{S}$, equipped with a canonical internal functor $U: C \to \hat{C}$. In Section 4 we are led to consider $\underline{S}$-atomic families of objects of $\underline{S}^{C^o}$, as being the necessary link between the two previous constructions associated with a given C. In this context we find useful the Special Adjoint Functor Theorem for indexed categories [11]. It is shown that *Pointess*$(\underline{S}^{C^o})$, the $\underline{S}$-indexed category of $\underline{S}$-essential points of $\underline{S}^{C^o})$, is the stack completion of the Karoubian envelope of C, which then leads us, in Section 5, to our ultimate aim.

Remarks by John Gray, André Joyal, Bill Lawvere, Bob Paré and Myles Tierney in connection with this work, are gratefully acknowledged.

1. THE $\underline{S}$-INDEXED CATEGORY OF $\underline{S}$-ESSENTIAL POINTS OF AN $\underline{S}$-TOPOS.

Let $\underline{S}$ be an elementary topos. An $\underline{S}$-geometric morphism of $\underline{S}$-toposes



is said to be *essential* if there exists $f_! \dashv f^*$, where $f = (f^*, f_*)$ with $f^* \dashv f_*$. The geometric morphism f is called $\underline{S}$-*essential* (cf. [1, 12]) if it is essential and further $f_!$ is an indexed left adjoint to f^*. From $f_! \dashv f^*$ we can define

$$f_!{}^I(Y \xrightarrow{b} \gamma_1^* I) = (f_! Y \xrightarrow{\hat{b}} \gamma_2^* I),$$

where $\hat{b}$ is the adjoint transpose of

$$Y \xrightarrow{\ b\ } \gamma_1^* I \approx f^* \gamma_2^* I,$$

and see that $f_!^{\,I} \dashv f^{*I}$ for any $I \in |\underline{S}|$. Hence the only non trivial addition here is the condition of stability under pullbacks. This says: given $a: I' \to I$ in S, and given a pullback

$$\begin{array}{ccc}
Y' & \xrightarrow{\quad a' \quad} & Y \\
{\scriptstyle b'}\downarrow & & \downarrow{\scriptstyle b} \\
\gamma_1^* I' & \xrightarrow[\gamma_1^* a]{\qquad} & \gamma_1^* I \\
\| & & \| \\
f^*\gamma_2^* I' & \xrightarrow[f^*\gamma_2^* a]{\qquad} & f^*\gamma_2^* I
\end{array}$$

it should follow that the diagram

$$\begin{array}{ccc}
f_! Y' & \xrightarrow{\quad f_! a' \quad} & f_! Y \\
{\scriptstyle \hat{b}'}\downarrow & & \downarrow{\scriptstyle \hat{b}} \\
\gamma_2^* I' & \xrightarrow[\gamma_2^* a]{\qquad} & \gamma_2^* I
\end{array}$$

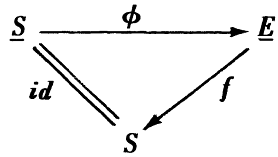
be a pullback.

There are other ways of expressing that $f_!$ is an indexed left adjoint to f^*. Briefly, one of them says that $f_!^{\,I}$ is a strong left adjoint relative to f^{*I}, where $\underline{E}_1$ and $\underline{E}_2$ are regarded as categories relative to $\underline{S}$ by means of $hom_{\underline{E}}(Y, Z) = \gamma_*(Z^Y)$. Another equivalent condition says that

$$I \otimes f_! Y \approx f_! (I \otimes Y)$$

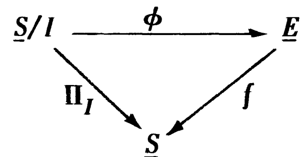
stated for families, where, in general, for an object Y of $\gamma: \underline{E} \to \underline{S}$ and $I \in |\underline{S}|$, $I \otimes Y = \gamma^* I \times Y$ by definition, and if $b: Y \to \gamma^* I$ is an I-indexed family and $(a: I' \to I) \in |\underline{S}/I|$, by $I' \otimes Y$ meaning $a \otimes b$ one understands $\gamma^* I' \underset{\gamma^* I}{\times} Y$. These notions are discussed in [12].

An $\underline{S}$-geometric morphism of the type



where $f: \underline{E} \to \underline{S}$ is an $\underline{S}$-topos, and where ϕ is $\underline{S}$-essential, is called an *$\underline{S}$-essential point of $\underline{E}$.*

For any $\underline{S}$-topos $f: \underline{E} \to \underline{S}$, we shall define here an $\underline{S}$-indexed category, called *$Pointess_{\underline{S}}(\underline{E})$* (using a terminology employed in [7]), of $\underline{S}$-essential points of $\underline{E}$ or rather, of families of them in the following sense. By an *I-indexed family of $\underline{S}$-essential points of $\underline{E}$* we shall understand an $\underline{S}$-geometric morphism

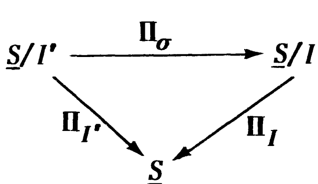


where ϕ is $\underline{S}$-essential. (We write Π_I meaning the geometric morphism given by $I^* \dashv \Pi_I$.) Such will then be the objects of the category

$$(Pointess_{\underline{S}}(\underline{E}))^I, \quad \text{for } I \in |\underline{S}|.$$

The morphisms will be the morphisms between geometric maps, in the usual way.

$Pointess_{\underline{S}}(\underline{E})$ is $\underline{S}$-indexed; given $\sigma: I' \to I$ in $\underline{S}$, this induces an $\underline{S}$-essential geometric morphism



Hence, composing with it induces the required functor

$$(Pointess_{\underline{S}}(\underline{E}))^I \xrightarrow{\sigma^*} (Pointess_{\underline{S}}(\underline{E}))^{I'}$$

and the coherence conditions then come from the fact that $\Pi_\sigma \Pi_{\sigma'} = \Pi_{\sigma\sigma'}$

given $\sigma' : I'' \to I'$.

There are two «essential» facts about $Pointess_{\underline{S}}(\underline{E})$. One says that it is a stack. The other is the indexed version of the property which left continuous *Set*-valued functors possess ; that of being representable.

(1.1) THEOREM. *For any $\underline{S}$-topos $f: \underline{E} \to \underline{S}$, the $\underline{S}$-indexed category $Pointess_{\underline{S}}(\underline{E})$ is a stack.*

PROOF. Consider a diagram

$$J'' \;\begin{array}{c} \xrightarrow{\;\Pi_{01}\;} \\ \xrightarrow[\;\Pi_{02}\;]{} \\ \xrightarrow[\;\Pi_{12}\;]{} \end{array}\; J' \;\begin{array}{c} \xrightarrow{\;\Pi_0\;} \\ \xrightarrow[\;\Pi_1\;]{} \end{array}\; J \xrightarrow{\;\alpha\;} I$$

as in Section 2 of [4]. Let $\phi \in (Pointess_{\underline{S}}(\underline{E}))^J$, i.e., let ϕ be a J-indexed family of $\underline{S}$-essential points of $\underline{E}$, or an $\underline{S}$-essential morphism

$$\begin{array}{ccc} \underline{S}/J & \xrightarrow{\;\phi\;} & \underline{E} \\ & {\scriptstyle \Pi_J}\searrow \quad \swarrow{\scriptstyle f} & \\ & \underline{S} & \end{array}$$

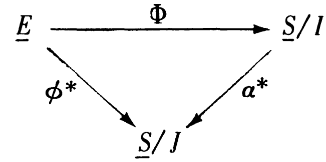
Suppose, further, that ϕ has descent data, with an isomorphism

$$\theta : \Pi_0^*(\phi) \xrightarrow{\sim} \Pi_1^*(\phi) \quad \text{in} \quad (Pointess_{\underline{S}}(\underline{E}))^{J'}.$$

A morphism $f \to g$ of geometric maps is a natural transformation $f^* \to g^*$. Hence, θ is a natural isomorphism as in

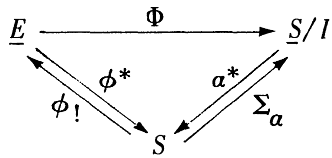
$$\begin{array}{ccc} \underline{E} & \xrightarrow{\;\phi^*\;} & \underline{S}/J \\ {\scriptstyle \phi^*}\big\downarrow & \overset{\theta}{\swarrow} & \big\downarrow{\scriptstyle \Pi_0^*} \\ \underline{S}/J & \xrightarrow[\;\Pi_1^*\;]{} & \underline{S}/J' \end{array}$$

and θ satisfies the appropriate coherence conditions. This means, in particular, and by (2.2) and (2.3) of [4], that for each object X of $\underline{E}$, $\phi^* X$, together with $\theta_X : \Pi_0^* \phi^* X \to \Pi_1^* \phi^* X$, is an algebra for the triple T induced by the adjoint pair $\Sigma_\alpha \dashv \alpha^*$. Moreover, the latter being tripleable (since α is epi), we have a functor Φ as in

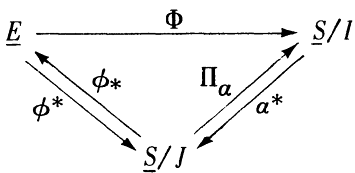


commutative, after a similar verification that compatible morphisms for descent are T-algebra maps.

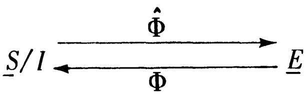
The data



is now seen to satisfy exactly the conditions of Theorem 1 of E. Dubuc [5], where Φ is seen to have a «derivable adjoint triangle» because $\Sigma_a \dashv a^*$ is tripleable. Thus, Φ has a left adjoint $\check{\Phi}$ which can be computed as the coequalizer of some pair involving $\phi_!$, ϕ^*, a^* and Σ_a. Taking opposites everywhere, the resulting version of Dubuc's theorem guarantees also the existence of a right adjoint $\hat{\Phi}$ to Φ , using this time the «coadjoint» triangle



and the fact that $a^* \dashv \Pi_a$ is cotripleable, again because a is epi, so that a^* is faithful. In fact, since $\check{\Phi}$ and $\hat{\Phi}$ are given by formulas involving indexed functors, they are indexed themselves, hence the geometric morphism



is $\underline{S}$-essential and so, defines an I-indexed family of $\underline{S}$-essential points of $\underline{E}$, i.e., an object of $(Pointess_{\underline{S}}(\underline{E}))^I$, unique (by the uniqueness of comparison functors) with the property that $a^*\Phi = \phi^*$. □

(1.2) PROPOSITION. *Let* $f: \underline{E} \to \underline{S}$ *be an* $\underline{S}$*-topos.*

(i) Let $\phi: \underline{S} \to \underline{E}$ *(over* $\underline{S}$*) be an* $\underline{S}$*-essential point of* $\underline{E}$*. Then, there exists* $X \in |\underline{E}|$ *and a natural isomorphism* $\phi^* \xrightarrow{\approx} hom_{\underline{E}}(X, -)$*.*

(ii) Let $\phi: \underline{S}/I \to \underline{E}$ *(over* $\underline{S}$*) be an I-indexed family of* $\underline{S}$*-essential points of* $\underline{E}$*, i.e., an* $\underline{S}$*-essential geometric morphism as indicated. Then, there exist an I-indexed family* $\zeta: X \to f^*I$ *in* $\underline{E}/f^*I$ *and a natural isomorphism*

$$\phi^* \xrightarrow{\approx} hom_{\underline{E}/f^*I}(\zeta, \Delta_I(-)),$$

where $\Delta_I: \underline{E} \to \underline{E}/f^*I$ *takes* $Y \in |\underline{E}|$ *to* $Y \times f^*I \xrightarrow{\pi} f^*I$*.*

PROOF. Although (i) is a special case of (ii), we give its proof first, for the sake of clarity.

(i) Let $X = \phi_! 1$. To show: for every $E \in |\underline{E}|$, there is a natural isomorphism $\phi^*E \xrightarrow{\approx} f_*(E^{\phi_! 1})$. This will come out of the bijections for any $K \in |\underline{S}|$, as follows:

$$\frac{\frac{\frac{\frac{K \longrightarrow f_*(E^{\phi_! 1})}{f^*K \longrightarrow E^{\phi_! 1}}}{f^*K \times \phi_! 1 \longrightarrow E}}{\phi_! K \longrightarrow E}}{K \longrightarrow \phi^*E\,.} \qquad (*)$$

Notice that the bijection (*) comes from the fact that $\phi_!$ is an indexed left adjoint to ϕ^*, as follows: Since ϕ is an essential point of $\underline{E}$, $f\phi \approx id_{\underline{S}}$, so we have a pullback

$$\begin{array}{ccc} K \times J & \xrightarrow{\pi_J} & J \\ {\scriptstyle \pi_K}\downarrow & & \downarrow \\ K & \longrightarrow & 1 \\ \wr| & & |\wr \\ \phi^*f^*K & \longrightarrow & \phi^*f^*1, \end{array}$$

thus

$$\begin{array}{ccc} \phi_!(K\times J) & \longrightarrow & \phi_! J \\ \downarrow & & \downarrow \\ f^*K & \longrightarrow & f^*1 \approx 1 \end{array}$$

is also a pullback, i.e., $f^*K \times \phi_! J \approx \phi_! (K\times J)$. If $J = 1$, we get

$$f^*K \times \phi_! 1 \approx \phi_! K.$$

(ii) Let

$$X \xrightarrow{\zeta} f^*I = \phi_!^I (I \xrightarrow{\delta} I\times I),$$

where

$$(\underline{S}/I)^I \approx \underline{S}/I\times I \xrightarrow{\phi_!^I} \underline{E}/f^*I.$$

To show: for every $E \in |\underline{E}|$, there is a natural isomorphism

$$\phi^*E \xrightarrow{\approx} f_*^I((\Delta_I E)^{(\phi_!^I)(\delta)}).$$

This will be obtained from the following bijections, where $a: J \to I$ is any object of $\underline{S}/I$:

$$\begin{array}{c} a \longrightarrow f_*^I(\Delta_I E^{\phi_!^I \delta}) \\ \hline (f^*)^I a \longrightarrow (\Delta_I E^{\phi_!^I \delta}) \\ \hline \psi = (f^*)^I a \times (\phi_!)^I(\delta) \longrightarrow \Delta_I E \\ \hline \Sigma_I \psi \longrightarrow E \\ \hline \phi_! a \longrightarrow E \\ \hline a \longrightarrow \phi^* E, \end{array} \qquad \begin{array}{c} \\ \\ \\ (*) \\ \\ (**) \\ \\ \end{array}$$

where (*) and (**) are justified as follows. By the indexness of $\phi_! \dashv \phi^*$ applied to the pullback

$$\begin{array}{ccc} J & \xrightarrow{a} & I \\ {\scriptstyle (a,J)}\downarrow & & \downarrow{\scriptstyle \delta} \\ I\times J & \xrightarrow{I\times a} & I\times I \end{array}$$

or rather, to the pullback of which the above is the image under Σ_I, i.e., (where «a» is used both as an object of $\underline{S}/I$ as well as a map, the terminal map) to the pullback

$$\begin{array}{ccc} a & \xrightarrow{\quad a \quad} & 1 \\ \Big\downarrow{\scriptstyle (a,J)} & & \Big\downarrow{\scriptstyle \delta} \\ \Delta_I J & \xrightarrow[\Delta_I a]{\qquad} & \Delta_I I \end{array} ,$$

one gets a pullback

$$\begin{array}{ccc} \phi_! a & \xrightarrow{\quad \phi_! a \quad} & \phi_! 1 = 1 \\ \Big\downarrow & \searrow^{\psi} & \Big\downarrow \\ f^* J & \xrightarrow[f^* a]{\qquad} & f^* I \end{array}$$

hence the object ψ of $\underline{E}/f^*I$ whose image under Σ_I is then $\Sigma_I \psi = \phi_! a$.□

By (1.1), $Pointess_{\underline{S}}(\underline{S}^{C^o})$ is a stack but need not be the stack completion of C although there is a canonical indexed functor

$$H: [C] \longrightarrow Pointess_{\underline{S}}(\underline{S}^{C^o}).$$

Define it as follows: for $I \in |\underline{S}|$, given $c: I \to C_0$, let $H^I(c)$ be the composite of $\underline{S}$-essential morphisms

$$\underline{S}/I \xrightarrow{\ \Pi_c\ } \underline{S}/C_0 \xrightarrow{\ G\ } \underline{S}^{C^o} ,$$

where the first arises from $\Sigma_c \dashv c^* \dashv \Pi_c$, and the second by the canonical functors $F \dashv U \dashv G$ whereby $F \dashv U$ makes $\underline{S}^{C^o}$ tripleable over $\underline{S}/C_0$, while $U \dashv G$ makes it cotripleable over it. Both are $\underline{S}$-essential. $\underline{S}$-essential maps compose. The action of H^I on a map $\gamma: I \to C_1$ with

$$\partial_0 \gamma = c_1 \quad \text{and} \quad \partial_1 \gamma = c_2$$

is clear, as γ induces a morphism of geometric morphisms $\gamma: c_1^* \to c_2^*$, which then gives

$$\gamma U: c_1^* U \to c_2^* U , \ \text{ i.e., a map } \ H^I(c_1) \to H^I(c_2).$$

This is clearly functorial. For the property of being indexed we must check the commutativity (up to natural isomorphisms) of the following diagrams, for each $\sigma: J \to I$ in $\underline{S}$. Analyzing what this means, we see that the condition is equivalent to having

$$\begin{array}{ccc} [I, \mathbb{C}] & \xrightarrow{H^I} & (Pointess_{\underline{S}}(\underline{S}^{\mathbb{C}_0}))^I \\ \sigma^* \downarrow & & \downarrow \sigma^* \\ [J, \mathbb{C}] & \xrightarrow{H^J} & (Pointess_{\underline{S}}(\underline{S}^{\mathbb{C}^0}))^J \end{array}$$

$$\Pi_c \Pi_\sigma \approx \Pi_{c\sigma}, \quad \text{for each} \quad c: I \to C_0 ,$$

which, of course, is true.

Shown in [12] is the fact that a map between internal presheaf toposes induced by an internal functor $F: \mathbb{C} \to \mathbb{D}$ is $\underline{S}$-essential. Hence, composition with one such gives

$$(Pointess_{\underline{S}}(\underline{S}^{\mathbb{C}^0}))^I \to (Pointess_{\underline{S}}(\underline{S}^{\mathbb{D}^0}))^I$$

which is indexed. Moreover, the functors

$$H_{\mathbb{C}}: [\mathbb{C}] \to Pointess_{\underline{S}}(\underline{S}^{\mathbb{C}^0})$$

defined above (noting the dependence on $\mathbb{C}$) constitute a natural transformation

$$H: [-] \to Pointess_{\underline{S}}(\underline{S}^{(-)^0}),$$

where $[-]: Cat\underline{S} \to \underline{S}\text{-}ind.\,cat$ is the externalization functor.

2. STACK COMPLETIONS OF LOCALLY INTERNAL INDEXED CATEGORIES.

Let $\underline{A}$ be an $\underline{S}$-indexed category with small homs, so that there is defined the Yoneda embedding. $Yon: \underline{A} \to \underline{S}^{A^0}$ is an indexed functor. For such an $\underline{A}$, we shall construct its stack completion (cf. [4], Definition (2.10)). In particular, this will give us the stack completion of any category object $\mathbb{C}$ of $\underline{S}$, by regarding $\mathbb{C}$ as an indexed category via its externalization.

(2.1) DEFINITION. An object X of $\underline{S}^{A^0}$ is called *locally representable* if there exists an object K of $\underline{S}$, with global support, as well as an object a of $\underline{A}^K$ such that there exists an isomorphism $Yon^K(a) \cong K^*X$.

More generally, an I-indexed family X of $\underline{S}^{A^0}$, i.e., an object X of $(\underline{S}^{A^0})^I$, is said to be *locally representable* if there exists a regular epi $\beta : K \twoheadrightarrow I$ in $\underline{S}$, as well as $a \in \underline{A}^K$, such that there exists an isomorphism $Yon^K(a) \cong \beta^*(X)$.

REMARK 1. If $\underline{A} = [\mathrm{C}]$, for $\mathrm{C} \in cat\underline{S}$, an I-indexed family $\zeta : X \to \Delta I$ of

$$\underline{S}^{\mathrm{C}^\circ} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \underline{S}$$

is locally representable iff there exists a regular epi $\beta : K \twoheadrightarrow I$ in $\underline{S}$ and a morphism $c : K \to C_0$, as well as a morphism $f : X \underset{\Delta I}{\times} \Delta K \to C_1$ such that the diagram

$$\begin{array}{ccc} X \underset{\Delta I}{\times} \Delta K & \xrightarrow{\ f\ } & C_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle <\partial_1, \partial_0>} \\ \Delta K & \xrightarrow{\ \Delta c\ } & C_0 \times C_0 = \Delta C_0 \end{array}$$

is a pullback.

(2.2) PROPOSITION. *Letting* $\bar{\underline{A}}^I$ *be the category whose objects are those* $X \in |(\underline{S}^{\underline{A}^\circ})^I|$ *which are locally representable and its morphisms all morphisms between such objects in* $(\underline{S}^{\underline{A}^\circ})^I$ *defines an* $\underline{S}$*-indexed category* $\bar{\underline{A}}$*, an indexed full subcategory of* $\underline{S}^{\underline{A}^\circ}$.

PROOF. We must check stability under substitution functors. If

$$(I' \xrightarrow{\ \gamma\ } I) \in S \quad \text{and} \quad X \in \bar{\underline{A}}^I,$$

check that $\gamma^* X \in \bar{\underline{A}}^{I'}$. Indeed, say that there is given $\beta : K \twoheadrightarrow I$ and $a \in \underline{A}^K$ as well as an isomorphism $Yon^K(a) \cong \beta^*(X)$. Consider the pullback:

$$\begin{array}{ccc} K' & \xrightarrow{\ \gamma'\ } & K \\ {\scriptstyle \beta'}\downarrow & & \downarrow{\scriptstyle \beta} \\ I' & \xrightarrow{\ \gamma\ } & I \end{array}$$

We have now $Yon^K(a) \approx \beta^* X$ and so, also

$$Yon^{K'}(\gamma'^*(a)) \approx \gamma'^* Yon^K(a) \approx \gamma'^* \beta^* X \approx$$

$$\approx (\beta\gamma')^*X \approx (\gamma\beta')^*X \approx \beta'^*(\gamma^*X)\ ,$$

thus showing that γ^*X is also locally representable, with $\beta' : K' \twoheadrightarrow I'$ and $\gamma'^*(a) \in \underline{A}^{K'}$. □

(2.3) PROPOSITION. *For a locally internal $\underline{S}$-indexed category $\underline{A}$, the $\underline{S}$-indexed category $\bar{\underline{A}}$ of (2.2) is the stack completion of $\underline{A}$.*

PROOF. Firstly, we show that $\bar{\underline{A}}$ is a stack. Let $X \in |\bar{\underline{A}}^J|$ be an object satisfying *descent data* relative to the regular epimorphism

$$J \overset{a}{\twoheadrightarrow} I \in \underline{S} \quad (\text{cf. [4], 2}).$$

Since X is an object of $(\underline{S}^{\underline{A}^\circ})^J$ which is locally representable, there is some regular epi $\beta : K \twoheadrightarrow J$ as well as some $a \in \underline{A}^K$ such that there is an isomorphism $Yon^K(a) \overset{\approx}{\to} \beta^*(X)$. By ([4], (2.6)), $\underline{A}$ is an $\underline{S}$-stack and so, by ([4], (2.9)), $\underline{S}^{\underline{A}^\circ}$ is also a stack. Hence, there exists $\bar{X} \in (\underline{S}^{\underline{A}^\circ})^I$ such that $X \approx a^*\bar{X}$. To show: $\bar{X} \in \bar{\underline{A}}^I$. Consider the regular epi $a\beta$:

$$K \overset{\beta}{\twoheadrightarrow} J \overset{a}{\twoheadrightarrow} I$$

and note that

$$Yon^K(a) \approx \beta^*(X) \approx \beta^*(a^*\bar{X}) \approx (a\beta)^*(\bar{X}),$$

which shows that $\bar{X}$, too, is locally representable.

According to ([4], (2.11)), in order to show that $\bar{\underline{A}}$ is the stack completion of $\underline{A}$, it is enough to find some weak equivalence functor $F : \underline{A} \to \bar{\underline{A}}$, once we know that $\bar{\underline{A}}$ is a stack. We claim here that such a functor is the Yoneda functor itself, or, to be more precise, the functor F in the factorization

$$\begin{array}{ccc} \underline{A} & \xrightarrow{\quad Yon \quad} & \underline{S}^{\underline{A}^\circ} \\ & {\scriptstyle F}\searrow \quad \nearrow & \\ & \bar{\underline{A}} & \end{array}$$

By the very definition of $\bar{\underline{A}}$, F (i.e., Yoneda) is a weak equivalence since, being fully faithful in each component, one needs only remark that if $X \in |\bar{\underline{A}}^I|$, then X is locally representable, so there are

$\beta : K \twoheadrightarrow I$ and $a \in |\underline{A}^K|$ as well as an isomorphism $F^K(a) \cong \beta^*(X)$ which is the second requirement in Definition (1.1) of [4] for a weak equivalence functor. This completes the proof. □

(2.4) PROPOSITION. *For* $\mathrm{C} \in Cat\underline{S}$, *there exists an* $\underline{S}$*-indexed functor*

$$L : \tilde{\mathrm{C}} \to Pointess_{\underline{S}}(\underline{S}^{\mathrm{C}^o})$$

which is fully faithful in each component, and such that L *fits into a commutative diagram*

$$\begin{array}{ccc} \mathrm{C} & \xrightarrow{H} & Pointess_{\underline{S}}(\underline{S}^{\mathrm{C}^o}) \\ & {\scriptstyle F}\searrow & \uparrow{\scriptstyle L} \\ & & \tilde{\mathrm{C}} \; . \end{array}$$

PROOF. By (1.1), the $\underline{S}$-topos

$$\underline{S}^{\mathrm{C}^o} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \underline{S}$$

gives rise to $Pointess_{\underline{S}}(\underline{S}^{\mathrm{C}^o})$, and the latter is a stack (over $\underline{S}$). By (2.3), we have the stack completion of C (i.e., of $[\mathrm{C}]$) given by: $F : \mathrm{C} \to \tilde{\mathrm{C}}$. Hence, by universal property of the stack completion, L exists as required and is unique up to natural isomorphism. □

REMARK 2. In view of Remark 1, note that an alternative and more direct definition of $F : \mathrm{C} \to \tilde{\mathrm{C}}$ can be given as follows. Given $c : I \to C_0$, let $F^I(c)$ be the left vertical arrow in the pullback

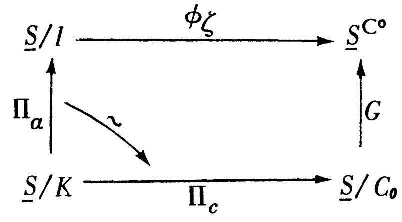
$$\begin{array}{ccc} X & \xrightarrow{f} & C_1 \\ {\scriptstyle \zeta}\downarrow & & \downarrow{\scriptstyle <\partial_1, \partial_0>} \\ \Delta I & \xrightarrow{\Delta c} & \Delta C_0 \end{array}$$

The rest of the structure of F is now clear, as it is just the Yoneda embedding extended to families.

The definition of L can also be made explicit. If $\zeta : X \to \Delta I$ is a locally representable I-indexed family of $\underline{S}^{\mathrm{C}^o}$, ζ induces an $\underline{S}$-essential morphism $\phi_\zeta : \underline{S}/I \to \underline{S}^{C^o}$ by letting

$$\phi_\zeta^* = hom_{\underline{S}^{C^o}/\Delta I}(\zeta, \Delta_I(-)), \quad \text{where } \Delta_I : \underline{S} \to \underline{S}/\Delta I.$$

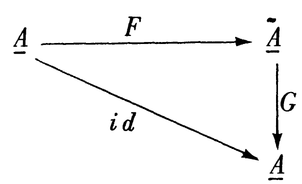
Notice that ϕ_ζ fits into the diagram



where $a : K \twoheadrightarrow I$ and $c : K \to C_0$ render $\zeta : X \to \Delta I$ locally representable. The above suggests an alternative way of defining $\bar{C}$: as the full $\underline{S}$-indexed category of $Pointess_{\underline{S}}(\underline{S}^{C^o})$ whose objects are those $\underline{S}$-essential families of points which are «locally representable» in the sense of their fitting into a diagram as the one above for ϕ_ζ. Using *this* embedding it is equally simple to show that $\bar{C}$ is a stack, because $Pointess_{\underline{S}}(\underline{S}^{C^o})$ is one.

(2.5) PROPOSITION. *Given a locally internal $\underline{S}$-indexed category $\underline{A}$, $\underline{A}$ is a stack iff the canonical embedding $F : \underline{A} \to \bar{\underline{A}}$ into its stack completion is an equivalence of $\underline{S}$-indexed categories.*

PROOF. The condition being obviously sufficient, let us see that it is necessary. Let



be a commutative diagram where G is the unique $\underline{S}$-indexed functor given by universality. Since $GF \approx id_{\underline{A}}$ and since F^I is fully faithful for each I, F is an equivalence. □

In view of (2.3) above, it is easy to find examples of stacks and of stacks completions. We shall look at some examples of category objects in a topos and their stack completions.

(2.6) EXAMPLES.

(1) In [9, Lesson 3], Lawvere discusses several examples which illustrate well how the notion of stack completion unifies the facts that a scheme is locally affine, a vector bundle is locally trivial, and Azumaya algebra is locally a matrix algebra. For instance, if X is a manifold admitting partitions of unity, then, as a category object in $sh(X)$, the category of vector bundles of finite type is the stack completion of that of trivial vector bundles. The other examples take place in the Zariski topos.

(2) For a topos $\underline{S}$ with natural number object, $\underline{S}_{fin}$, the internal category of finite sets, has a stack completion: the $\underline{S}$-indexed category of locally finite sets. In general, a topos $\underline{S}$ need not have stack completions for its category objects; an example due to A. Joyal is mentioned in [10]. However, it follows from Lemma 8.35 of P. Johnstone [8], that Grothendieck toposes, because of the existence of a generating set, do have small stack completions for their category objects.

(3) Let $\underline{S}$ be a topos, G a group object in $\underline{S}$. We may regard G as a category object in $\underline{S}$, and as such, describe its stack completion. A right G-set X with action $\xi: X\times G \to X$ is called a right G-torsor (cf. [8], Section 8.3) if $X \twoheadrightarrow 1$ and $<\pi_1, \xi>: X\times G \to X\times X$ is an isomorphism. G itself, with action $m: G\times G \to G$ multiplication, is a right G-torsor, the trivial one. By definition then, a right G-torsor is locally isomorphic to the trivial G-torsor. Hence, G-torsors form the stack completion of G. Torsors play an important role in non-abelian cohomology (cf. Giraud [6]) and are the key to understanding stacks.

Now that we have everything we need about one of the ingredients entering a Morita Theorem for category objects in a topos, namely, the stack completion, we turn to the other.

3. KAROUBIAN ENVELOPES OF INTERNAL CATEGORIES.

The well-known construction (cf. [7, 2 and 13]) of the Karoubian envelope of a small category can be internalized. That is, given a category C in a topos $\underline{S}$, we can construct an internal category $\hat{C}$ in $\underline{S}$, and an internal functor $U : C \to \hat{C}$, fully faithful, and such that, in $\hat{C}$, idempotents coming from C via U (the exact meaning of which is to be given below) split, and U is universal with this property. $\hat{C}$ is called *the Karoubian envelope of* C (in the terminology of [7]), or the *closure of* C *under splitting idempotents* (in [2]). It is unique to within equivalence of internal categories.

Let C be concretely given by the diagram

$$C_1 \underset{C_0}{\times} C_0 \;\underset{\pi_1}{\overset{\pi_0}{\underset{m}{\rightrightarrows}}}\; C_1 \;\underset{\partial_1}{\overset{\partial_0}{\underset{\xleftarrow{u}}{\rightrightarrows}}}\; C_0$$

satisfying the usual identities. Let $\hat{C}_0$ be given as in the equalizer

$$\hat{C}_0 \overset{a}{\rightarrowtail} Endo\, C_1 \;\underset{i}{\overset{\langle i, i\rangle}{\rightrightarrows}}\; \begin{matrix} C_1 \underset{C_0}{\times} C_1 \xrightarrow{m} \\ \end{matrix} \; C_1$$

where

$$Endo\, C_1 \overset{i}{\rightarrowtail} C_1 \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} C_0$$

is an equalizer diagram. This says, using the internal language of $\underline{S}$, that for $j = ia$,

$$\hat{C}_0 \overset{j}{\rightarrowtail} C_1 = \| \partial_0 f = \partial_1 f \wedge f^2 = f \|$$

in the notation employed in [8, 5.4], i.e., $\hat{C}_0$ is the object of idempotents of C_1 (above, $f^2 = \langle f, f\rangle m$).

Let $\hat{C}_1$ be given, in turn, by the equalizer

$$\hat{C}_1 \overset{v}{\rightarrowtail} \hat{C}_0 \underset{C_0}{\times} C_1 \underset{C_0}{\times} \hat{C}_0 \;\underset{\pi_1}{\overset{j\times 1\times j}{\rightrightarrows}}\; \begin{matrix} C_1 \underset{C_0}{\times} C_1 \underset{C_0}{\times} C_1 \xrightarrow{m\times 1} C_1 \underset{C_0}{\times} C_1 \xrightarrow{m} \end{matrix} \; C_1$$

This says that

$$\hat{C}_1 = \| \partial_0 k = \partial_0 e \wedge \partial_1 k = \partial_1 e' \wedge e' k e = k \|$$

where e and e' are variables of type $\hat{C}_0$ and k is a variable of type

C_1 . Then, let

$$\hat{C}_1 \underset{\hat{\partial}_1}{\overset{\hat{\partial}_0}{\rightrightarrows}} \hat{C}_0$$

be given by the pair

$$\hat{C}_1 \overset{v}{\rightarrowtail} \hat{C}_0 \underset{C_0}{\times} C_1 \underset{C_0}{\times} \hat{C}_0 \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} \hat{C}_0 \times \hat{C}_0 \,.$$

Identities: a unique map $\hat{u}: \hat{C}_0 \to \hat{C}_1$ is given, such that

$$\begin{array}{ccc} \hat{C}_0 & \xrightarrow{\langle 1, j, 1 \rangle} & \hat{C}_0 \underset{C_0}{\times} C_1 \underset{C_0}{\times} \hat{C}_0 \\ & {\scriptstyle \hat{u}} \searrow & \uparrow {\scriptstyle v} \\ & & \hat{C}_1 \end{array}$$

commutes; it exists because

$$\| \, e^2 = e \, \| \leq \| \, e^3 = e \, \|$$

as subobjects of C_1 , i.e., where e is a variable of type C_1.

Composition: a unique map $\hat{m}: \hat{C}_1 \underset{C_0}{\times} \hat{C}_1 \to \hat{C}_1$ is given, such that

$$\begin{array}{ccccc} \hat{C}_1 \underset{\hat{C}_0}{\times} \hat{C}_1 & \xrightarrow{\pi_{01} v \times \pi_{12} v} & \hat{C}_0 \underset{C_0}{\times} C_1 \underset{C_0}{\times} C_1 \underset{C_0}{\times} \hat{C}_0 & \xrightarrow{1 \times m \times 1} & \hat{C}_0 \underset{C_0}{\times} C_1 \underset{C_0}{\times} \hat{C}_0 \\ & & {\scriptstyle \hat{m}} \searrow & & \uparrow {\scriptstyle v} \\ & & & & \hat{C}_1 \end{array}$$

commutes, and exists because

$$\| \, e' k e = k \wedge e'' k' e' = k' \, \| \leq \| \, e'' k' k e = k' k \, \| \,.$$

The embeddings $U_0: C_0 \to \hat{C}_0$ and $U_1: C_1 \to \hat{C}_1$ are given as follows. First,

$$\begin{array}{ccc} C_0 & \overset{U_0}{\rightarrowtail} & \hat{C}_0 \\ & {\scriptstyle u} \searrow & \downarrow {\scriptstyle j} \\ & & C_1 \end{array}$$

exists because of the equations $\partial_0 u = \partial_1 u = 1_{C_0}$, and

$$\begin{array}{ccc} C_1 & \overset{U_1}{\rightarrowtail} & \hat{C}_1 \\ & \searrow^{<U_0\hat{\partial}_0, U_0\hat{\partial}_1>} & \downarrow v \\ & & \hat{C}_0 \underset{C_0}{\times} C_1 \underset{C_0}{\times} \hat{C}_0 \end{array}$$

because

$$m(1 \times u) = m(u \times 1) = 1_{C_1} ,$$

which are part of the structure on C .

Finally

$$\begin{array}{ccc} C_1 & \overset{U_1}{\rightarrowtail} & \hat{C}_1 \\ {\scriptstyle <\partial_0, \partial_1>} \downarrow & & \downarrow {\scriptstyle <\hat{\partial}_0, \hat{\partial}_1>} \\ C_0 \times C_0 & \underset{\hat{U}_0 \times \hat{U}_0}{\rightarrowtail} & \hat{C}_0 \times \hat{C}_0 \end{array}$$

is a pullback, because of the bijections, for any $K \in |\underline{S}|$, between the data

$$K \xrightarrow{<e, f, e'>} \hat{C}_1, \quad K \xrightarrow{<X, Y>} C_0 \times C_0$$

such that $e = uX$ and $e' = uY$, and the data

$$K \xrightarrow{\ f\ } C_1 \quad \text{with } \partial_0 f = X \text{ and } \partial_1 f = Y.$$

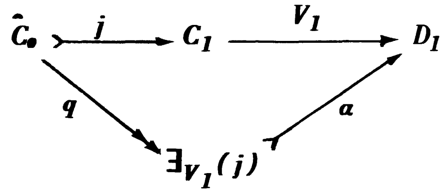
Thus, U given by U_0, U_1 , is internally fully faithful.

Our next task will be to express the property of an internal functor $V: \mathrm{C} \to \mathrm{D}$ of splitting in D the idempotents of C .

Denote by $\phi: Split D_1 \rightarrowtail D_1$ the subobject given by

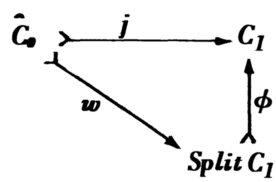
$$Split\ D_1 =$$
$$\| \exists <g, h> \in D_1 \underset{D_0}{\times} D_1 \ (f = hg \wedge gh = 1) \wedge \forall <g', h'> \in D_1 \underset{D_0}{\times} D_1$$
$$(f = h'g' \wedge g'h' = 1 \Rightarrow \exists <\alpha, \beta> (\alpha g = g' \wedge h\beta = h')) \| .$$

Then, we can also look at the subobject of D_1 given by the following diagram, which is the image factorization of $V_1 j$.



(3.1) DEFINITION. Say that *idempotents of* $\mathbb{C}$ *split in* $\mathbb{D}$ *via* $V: \mathbb{C} \to \mathbb{D}$ whenever $\exists_{V_1}(j) \leq \mathit{Split}\, D_1$.

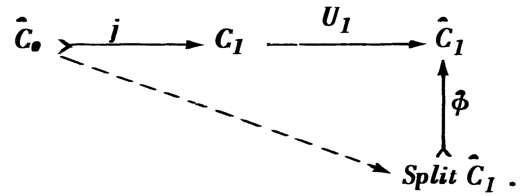
In particular, *idempotents of* $\mathbb{C}$ *split in* $\mathbb{C}$ (via the identity $\mathbb{C} \to \mathbb{C}$) iff $j \leq \mathit{Split}\, C_1$, i.e., iff there exists $w: \hat{C}_0 \to \mathit{Split}\, C_1$ such that



commutes. Later on we shall give an alternative formulation of this, in terms of $U: \mathbb{C} \to \hat{\mathbb{C}}$.

(3.2) PROPOSITION. *For any internal category* $\mathbb{C}$, *the internal category* $\hat{\mathbb{C}}$, *together with the embedding* $U: \mathbb{C} \to \hat{\mathbb{C}}$, *satisfies the properties required of the Karoubian envelope of* $\mathbb{C}$.

PROOF. We show first that idempotents of $\mathbb{C}$ split in $\hat{\mathbb{C}}$ via $U: \mathbb{C} \to \hat{\mathbb{C}}$ in the sense of (3.1). In this case, since $U_1 : C_1 \rightarrowtail \hat{C}_1$ is mono, the condition that we must verify says that there should be a factorization



We claim that, as subobjects of $\hat{C}_1$,

$$\| \partial_0 e = \partial_1 e = A \wedge e^2 = e \| \leq \| \exists <(1_A, e, e), (e, e, 1_A)> $$
$$((1_A, e, 1_A) = (e, e, 1_A)(1_A, e, e) \wedge (1_A, e, e)(e, e, 1_A) = 1_e = (e, e, e))\|$$

This is easily verified. Furthermore, uniqueness up to isomorphism holds since, given $<(1_A, k, e'), (e', k', 1_A)>$ such that

$$(1_A, e, 1_A) = (e', k', 1_A)(1_A, k, e)$$

and

$$(1_A, k, e')(e', k', 1_A) = 1_{e'} = (e', e', e'),$$

then $k: e \to e'$ and $k': e' \to e$ are inverses and satisfy

$$ke = k \quad \text{and} \quad ek' = 1.$$

Given any $V: \mathrm{C} \to \mathrm{D}$, with the property that idempotents of C split in D via V, we show that there exists a unique $\hat{V}: \hat{\mathrm{C}} \to \mathrm{D}$ such that

$$\mathrm{C} \xrightarrow{U} \hat{\mathrm{C}} \xrightarrow{\hat{V}} \mathrm{D} = \mathrm{C} \xrightarrow{V} \mathrm{D},$$

as follows: let $e: K \to \hat{C}_0$ be given, and consider its image in D_1 via

$$\hat{C}_0 \rightarrowtail^{j} C_1 \xrightarrow{V_1} D_1 .$$

Since $\exists_{V_1}(j) \leq Split\, D_1$, there exists

$$<g, h>: K \to D_1 \times_{D_0} D_1 \quad \text{such that} \quad V_1 j e = hg,$$

and

$$K \xrightarrow{<g,h>} D_1 \times_{D_0} D_1 \xrightarrow{m} D_1 = K \xrightarrow{X_e} D_0 \xrightarrow{u} D_1 ,$$

where

$$X_e = K \xrightarrow{g} D_1 \xrightarrow{\partial_1} D_0 = K \xrightarrow{h} D_1 \xrightarrow{\partial_0} D_0 .$$

Moreover, X_e is unique up to isomorphism satisfying these properties. We let

$$K \xrightarrow{e} \hat{C}_0 \xrightarrow{\hat{V}_0} D_0 = K \xrightarrow{X_e} D_0 .$$

We verify immediately the commutativity of

$$\begin{array}{ccc} C_0 & \xrightarrow{U_0} & \hat{C}_0 \\ & {\scriptstyle V_0} \searrow & \downarrow {\scriptstyle \hat{V}_0} \\ & & D_0 \end{array}$$

since, given $K \xrightarrow{A} C_0 \xrightarrow{U_0} \hat{C}_0$, look at

$$K \xrightarrow{A} C_0 \xrightarrow{U_0} \hat{C}_0 \xrightarrow{i} C_1 \xrightarrow{V_1} D_1 \;=\; K \xrightarrow{A} C_0 \xrightarrow{u} C_1 \xrightarrow{V_1} D_1 \,.$$

Since V is a functor, the latter is equal to

$$K \xrightarrow{A} C_0 \xrightarrow{V_0} D_0 \xrightarrow{u} D_1 \,,$$

and $1_{V_0 A}: V_0 A \to V_0 A$ splits by means of $< 1_{V_0 A}, 1_{V_0 A} >$, i.e., as required, $X_{U_0 A} = V_0 A$.

We now define $\hat{V}_1 : \hat{C}_1 \to D_1$ as follows: given $< e, f, e' > : K \to \hat{C}_1$ the uniqueness of the factorizations defining X_e and $X_{e'}$ and the relation $e'f = e$ imply the existence of a unique map $\phi : K \to D_1$ with

$$\partial_0 \phi = X_e, \quad \partial_1 \phi = X_{e'} \quad \text{and} \quad fh = h'\phi \,.$$

Indeed, $h: X_e \rightarrowtail A$ and $h': X_{e'} \rightarrowtail A'$ appear as equalizers of e with the identities and e' with the identity, respectively. Then, let

$$K \xrightarrow{< e, f, e' >} \hat{C}_1 \xrightarrow{\hat{V}_1} D_1 \;=\; K \xrightarrow{\phi} D_1 \,.$$

Clearly, this is functorial, and

$$C_1 \xrightarrow{U_1} \hat{C}_1 \xrightarrow{\hat{V}_1} D_1 \;=\; C_1 \xrightarrow{\hat{V}_1} D_1 \,.$$

the latter since, if

$$X_{U_0 A} = V_0 A \quad \text{and} \quad X_{U_0 A'} = V_0 A' \quad \text{and} \quad f: A \to A',$$

the unique map in question is $V_1 f$. □

(3.3) PROPOSITION. *For any internal category* $\mathbf{C}$ *in a topos* $\underline{S}$, *the embedding* $U: \mathbf{C} \to \hat{\mathbf{C}}$ *induces an equivalence* $\underline{S}^U : \underline{S}^{\hat{\mathbf{C}}^o} \to \underline{S}^{\mathbf{C}^o}$ *of categories.*

PROOF. $\underline{S}$ has regular image factorizations, since it is a topos. Hence, given an internal contravariant presheaf (X, ξ) on $\mathbf{C}$, we can get one on $\hat{\mathbf{C}}$ by the following device. Let $q: Y \to \hat{C}_0$ be given so that, for

$$e \in |\hat{C}_0|^I \quad \text{with} \quad \partial_0 e = \partial_1 e = A,$$

let $Y_e \rightarrowtail X_A$ be such that

$$Y_e = \| \; \exists x' \,(x = \xi < e, x' >) \| \,.$$

Let $\theta : \hat{C}_1 \underset{\hat{C}_0}{\times} Y \to Y$ be given by letting

$$\theta(<e, f, e'>, <e, x>) = <e', \xi(f, x)> .$$

Notice then that

$$\begin{array}{ccc} \hat{C}_1 \underset{\hat{C}_0}{\times} Y & \xrightarrow{\ \theta\ } & Y \\ {\scriptstyle \Pi_0} \downarrow & & \downarrow {\scriptstyle q} \\ \hat{C}_1 & \xrightarrow{\ \hat{\partial}_1\ } & \hat{C}_0 \end{array}$$

is commutative. This definition of (Y, θ) is nothing but the internal version of the assignment of $\hat{V} : \hat{C} \to D$ to $V : C \to D$ in (3.2).

Letting $\psi(X, \xi) = (Y, \theta)$, we notice that $\underline{S}^U(Y, \theta) = (X, \xi)$. Indeed, if $\underline{S}^U(Y, \theta) = (\hat{X}, \hat{\xi})$, we have, by definition, that

$$\begin{array}{ccc} \hat{X} & \longrightarrow & Y \\ {\scriptstyle \hat{p}} \downarrow & & \downarrow {\scriptstyle q} \\ C_0 & \xrightarrow{\ U_0\ } & \hat{C}_0 \end{array}$$

is a pullback, hence

$$\hat{X}_A = Y_{1_A} = \| \exists x' (x = \xi < 1_A, x' >) \| = X_A .$$

Conversely, if $\underline{S}^U(Y, \theta) = (X, \xi)$ and $\psi(X, \xi) = (\hat{Y}, \hat{\theta})$, then

$$\hat{Y}_e = \| \exists x' (x = \xi < e, x' >) \| \rightarrowtail X_A$$

is isomorphic to

$$Y_e = \| \exists x' (x = \theta < e, x' >) \| \rightarrowtail Y_{1_A} ,$$

the latter because $< 1_A, e, 1_A >$ splits in $\hat{C}$ (cf. (3.1)) and its image is e, and this state of affairs is preserved by any functor. Thus,

$$\underline{S}^U \psi = id \quad \text{and} \quad \psi \underline{S}^U \approx id. \quad \square$$

(3.4) REMARKS. Denote by $Split\, C$ the internal category with

$$(Split\, C)_0 = Split\, C_1$$

and with

$$(Split\, C)_1 \overset{w}{\rightarrowtail} Split\, C_1 \underset{C_0}{\times} C_1 \underset{C_0}{\times} Split\, C_1$$

being defined by the condition

$$(\mathit{Split}\,\mathbb{C})_1 = \| f'kf = k \| .$$

Let

$$(\mathit{Split}\,\mathbb{C})_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} (\mathit{Split}\,\mathbb{C})_0$$

be given by first and third projections. Notice then that there is a factorization

$$\begin{array}{ccc} (\mathit{Split}\,\mathbb{C})_0 = \mathit{Split}\, C_1 & \overset{\phi}{\rightarrowtail} & C_1 \\ & \searrow^{\gamma_0} & \uparrow j \\ & & \hat{C}_0 \end{array}$$

on account of the valid implication

$$\exists \langle g,h \rangle f = hg \wedge gh = 1 \Rightarrow f^2 = f,$$

for f a variable of type C_1 .

We can extend γ_0 to morphisms by letting γ_1 as in

$$\begin{array}{ccc} (\mathit{Split}\,\mathbb{C})_1 & \overset{\gamma_1}{\rightarrowtail} & \hat{C}_1 \\ w \downarrow & & \downarrow v \\ \mathit{Split}\, C_1 \underset{C_0}{\times} C_1 \underset{C_0}{\times} \mathit{Split}\, C_1 & \underset{\gamma_0 \times 1 \times \gamma_0}{\rightarrowtail} & \hat{C}_0 \underset{C_0}{\times} C_1 \underset{C_0}{\times} \hat{C}_0 \end{array}$$

existing on account of the valid implication (trivially)

$$f'kf = k \Rightarrow f'kf = k.$$

The diagram also implies that we have an internal functor $\gamma : \mathit{Split}\,\mathbb{C} \to \hat{\mathbb{C}}$.

Notice now that, if idempotents split in $\mathbb{C}$, γ_0 is iso. This is so because, from $\phi w = j$ (by (5.1)) and $j\gamma_0 = \phi$ (above),

$$\phi w \gamma_0 = j \gamma_0 = \phi$$

gives, as ϕ is mono, that $w\gamma_0 = id$. Also, $j\gamma_0 w = \phi w = j$ says, since j is mono, that $\gamma_0 w = id$. So, γ_0 has an inverse w. Conversely, if γ_0 is iso, its inverse w must satisfy $\phi w = j$, hence idempotents split in C. We have shown:

(3.4.1) *Idempotents split in* C *iff the internal functor*

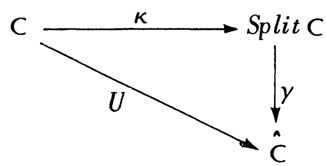
$$\gamma_0 : (\mathit{Split}\, C)_0 \longrightarrow \hat{C}_0$$

is an isomorphism.

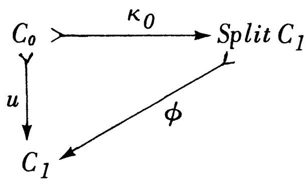
We can also define a functor $\kappa : C \to \mathit{Split}\, C$ such that κ is a weak equivalence functor and a cross section to an onto functor

$$\delta : \mathit{Split}\, C \longrightarrow\!\!\!\!\to C,$$

and such that



commutes. We let κ_0 be such that

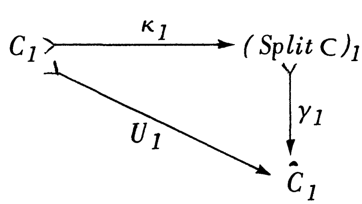


commutes; it exists because of the validity of the implication

$$f = 1 \Rightarrow \exists \langle g, h \rangle (\, 1 = hg \wedge gh = 1\,),$$

where f is a variable of type C_1.

Also, there is an inclusion of subobjects of $\hat{C}_1$,



as follows from the trivial implication:

$$e'ke = k \Rightarrow e'ke = k.$$

Now, κ is fully faithful and has the property: if

$$I \in |\underline{S}| \quad \text{and} \quad f: I \to Split\, C_1$$

is given, i.e., some $f: I \to C_1$ such that there exists

$$I \xrightarrow{<g,h>} C_1 \underset{C_0}{\times} C_1 \quad \text{with} \quad m<g,h> = 1 \text{ and } m<h,g> = f,$$

then, if $\partial_0 f = I \xrightarrow{A} C_0$, it follows that

$$I \xrightarrow{A} C_0 \xrightarrow{\kappa_0} Split\, C_1 \approx I \xrightarrow{f} Split\, C_1 .$$

This is so since there exists $I \xrightarrow{<a,\beta>} (Split\, \mathbb{C})_1$ with

$$\partial_0 a = \kappa_0 A, \quad \partial_1 a = f, \quad \partial_0 \beta = f, \quad \partial_1 \beta = \kappa_0 A$$

and $a\beta = 1_f$, $\beta a = 1_{\kappa_0 A}$. Indeed, let $a: I \to (Split\, \mathbb{C})_1$ be given by

$$I \xrightarrow{<1_A, g, f>} \hat{C}_0 \underset{C_0}{\times} C_1 \underset{C_0}{\times} \hat{C}_0$$

and $\beta: I \to (Split\, \mathbb{C})_1$ be given by

$$I \xrightarrow{<f, h, 1_A>} \hat{C}_0 \underset{C_0}{\times} C_1 \underset{C_0}{\times} \hat{C}_0 .$$

Thus, κ is a weak equivalence functor.

It has a cross section given by $\delta: Split\, \mathbb{C} \twoheadrightarrow \mathbb{C}$, where δ_0 is given as follows. Given $f: I \to Split\, C_1$, with g, h such that $f = hg$ and $gh = 1$, let

$$I \xrightarrow{f} Split\, C_1 \xrightarrow{\delta_0} C_0 =$$

$$I \xrightarrow{g} C_1 \xrightarrow{\partial_1} C_0 = I \xrightarrow{h} C_1 \xrightarrow{\partial_0} C_0 .$$

To define δ_1, we must use once more the fact that, if

$$f: A \to A \quad \text{and} \quad f = gh, \quad hg = 1_{X_f},$$

then $h: X_f \rightarrowtail A$ is the equalizer of f, 1_A. Then, any

$$I \xrightarrow{<f,k,f'>} (Split\, \mathbb{C})_1$$

will induce a unique $\lambda: I \to C_1$ with

$$\partial_0 \lambda = X_f, \quad \partial_1 \lambda = X_{f'} \quad \text{and} \quad h'\lambda = kh .$$

Then let

$$I \xrightarrow{<f,k,f'>} (Split\,\mathrm{C})_1 \xrightarrow{\delta_1} C_1 \;=\; I \xrightarrow{\lambda} C_1 .$$

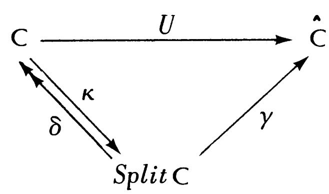
That δ is onto is clear: given any $A: I \to C_0$, there exists

$$1_A : I \to (Split\,\mathrm{C})_0 \quad \text{with} \quad \partial_0 1_A = A .$$

Also, κ is cross section to δ, for $\delta\kappa = id_{\mathrm{C}}$. And $\kappa\delta \approx id_{Split\,\mathrm{C}}$, so that, in fact, δ and κ set up a (super) equivalence between $Split\,\mathrm{C}$ and C. Now we prove (contrary to what has been suggested in [12], namely, that in the absence of the axiom of choice, idempotents split in C iff $U: \mathrm{C} \to \hat{\mathrm{C}}$ is locally an equivalence of categories) the following:

(3.4.2) PROPOSITION. *For any internal category* C *in* $\underline{S}$, *idempotents split in* C *iff* $U: \mathrm{C} \to \hat{\mathrm{C}}$ *is an equivalence of internal categories.*

PROOF. From the factorization



established above, follows that γ_0 is iso iff U is an equivalence. By (3.4.1) idempotents split in C iff γ_0 is iso. □

4. $\underline{S}$-ATOMIC FAMILIES OF PRESHEAVES.

We now wish to establish a certain relationship between $\hat{\mathrm{C}}$ and $Pointess_{\underline{S}}(\underline{S}^{\mathrm{C}^{\circ}})$. It is an exercise in [7] (cf. also [2]) that, if $\underline{S} = Set$, the canonical functor

$$H: \mathrm{C} \longrightarrow Pointess_{\underline{S}}(\underline{S}^{\mathrm{C}^{\circ}})$$

exhibits $Pointess_{\underline{S}}(\underline{S}^{\mathrm{C}^{\circ}})$ as the Karoubian envelope of C. This is no longer true if $\underline{S}$ is a topos which does not satisfy the axiom of choice. However, as we shall see in Section 5, $Pointess_{\underline{S}}(\underline{S}^{\mathrm{C}^{\circ}})$ plays, for an arbitrary topos $\underline{S}$, the same role it does for when $\underline{S}$ is Set, in the sense that it gives the proper way to retrieve C (for «good» internal categories

C anyway) from the presheaves category $\underline{S}^{C^o}$.

From [3] we recall the internal version of the canonical epimorphism

$$p_X: \sum_{y \in E_X} h^y \twoheadrightarrow X \quad \text{for } X \in |\underline{S}^{C^o}|, \text{ where } E_X = \sum_{y \in C_0} hom(h^y, X).$$

Let

$$\underline{S}^{C^o} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \underline{S}$$

be the canonical geometric morphism into $\underline{S}$. Given $(X \xrightarrow{p} C_0, \xi)$ an object of $\underline{S}^{C^o}$, let

$$E_X \xrightarrow{e} C_0 = hom_{\underline{S}/C_0}(C_1, X),$$

where

$$C_1 \xrightarrow{\partial_1} C_0 \in \underline{S}/C_0 \quad \text{and} \quad E_X \xrightarrow{e} C_0 = \Gamma \Pi_{\partial_1} \partial_1^* \Delta X.$$

Then, for $<\partial_1, \partial_0>: C_1 \to \Delta C_0$, form

$$E_X \otimes C_1 = \Delta E_X \underset{\Delta C_0}{\times} C_1 = \Delta \Gamma \Pi_{\partial_1} \partial_1^* \Delta X \underset{\Delta C_0}{\times} C_1 .$$

The canonical map

$$\Delta \Gamma \Pi_{\partial_1} \partial_1^* \Delta X \underset{\Delta C_0}{\times} C_1 \xrightarrow{p_X} X$$

is explicitly defined in [3, page 21] by means of the given adjointness data. Also in [3, page 31], an I-indexed family $\xi: X \to \Delta I$ of $\underline{S}^{C^o}$ is called *$\underline{S}$-atomic* if $hom_{\underline{S}^{C^o}/\Delta I}(\xi, -)$ preserves coequalizers and $\underline{S}$-indexed coproducts. There is a precise way of stating this, by requiring that certain canonical maps be epi or iso as it be the case. Preserving coequalizers (or epis) says that, given $g: \eta \twoheadrightarrow \psi$ in $\underline{S}^{C^o}/\Delta I$ the induced

$$hom(\xi, g): hom(\xi, \eta) \to hom(\xi, \psi)$$

be epi in $\underline{S}/I$. Preserving $\underline{S}$-indexed coproducts says that, given any

$$E \xrightarrow{e} I \in \underline{S} \quad \text{and} \quad A \xrightarrow{a} \Delta I \in \underline{S}^{C^o}$$

the canonical map

$$\chi: ex\,hom(\xi, a) \to hom(\xi, \Delta\, exa)$$

(defined in [3, page 23]) be an iso.

It is also shown in [3, page 32] that, in $\underline{S}^{C^o}$,

$$C_1 \xrightarrow{<\partial_1, \partial_0>} \Delta C_0$$

is $\underline{S}$-atomic in the above sense, at least testing with respect to $\underline{S}$-coproducts of members of the family

$$A \xrightarrow{a} \Delta C_0 \;=\; C_1 \xrightarrow{<\partial_1, \partial_0>} \Delta C_0$$

itself. As we shall see below, this is all we shall need.

In the classical case ($\underline{S} = Set$), every atom is the retract of the representables. Here we can only have, in general, the local version of this result.

(4.1) DEFINITION. Say that an I-indexed family $\xi: X \to \Delta I$ of $\underline{S}^{C^o}$ is *locally the retract of representables* if there exist

$$K \xrightarrow{a} I, \quad K \xrightarrow{y} C_0$$

and maps

$$\Delta K \underset{\Delta C_0}{\times} C_1 \underset{g}{\overset{f}{\rightleftarrows}} \Delta K \underset{\Delta I}{\times} X$$

with $gf = 1_{\Delta K \underset{\Delta I}{\times} X}$.

(4.2) PROPOSITION. *In* $\underline{S}^{C^o}$, *every* $\underline{S}$*-atomic family is locally the retract of representables.*

PROOF. Let $\xi: X \to \Delta I$ be $\underline{S}$-atomic. For $\xi \in \underline{S}^{C^o}/\Delta I$ we also have the corresponding map p_ξ, or p_X (over X):

$$\Delta \Gamma \Pi_{\partial_1} \partial_1^* \Delta X \underset{\Delta C_0}{\times} C_1 \xrightarrow{p_X} X,$$

hence also an epi

$$hom(X, \Delta E_X \underset{\Delta C_0}{\times} C_1) \longrightarrow hom(X, X),$$

since $hom(\xi, -)$ preserves epis, as well as an iso

$$E_X \underset{C_0}{\times} hom(X, C_1) \xrightarrow{\approx} hom(X, \Delta E_X \underset{\Delta C_0}{\times} C_1)$$

since $hom(\xi, -)$ preserves I-indexed coproducts. Composing the two canonical maps, we obtain an epi

$$E_X \underset{C_0}{\times} hom(X, C_1) \xrightarrow{q_X} hom(X, X).$$

Now, let $\ulcorner 1_X \urcorner : I \to hom(X, X)$ be the identity map, i.e.,

$$\frac{\dfrac{I \xrightarrow{\ulcorner 1_X \urcorner} hom(X, X) = \Gamma(X^X)}{\Delta I \longrightarrow X^X}}{\Delta I \times X \xrightarrow{\pi_X} X\ .}$$

Taking the pullback of $\ulcorner 1_X \urcorner$ along the epi q_X gives

$$\begin{array}{ccc} K & \xrightarrow{\quad a \quad} & I \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle \ulcorner 1_X \urcorner} \\ E_X \underset{C_0}{\times} hom(\Delta X, C_1) & \xrightarrow{q_X} & hom(X, X) \end{array}$$

where k is determined by the data: maps

$$K \xrightarrow{y} C_0\ , \quad K \xrightarrow{f} (E_X)_y ,$$

where

$$\begin{array}{ccc} (E_X)_y & \longrightarrow & E_X \\ \big\downarrow & \text{PB} & \big\downarrow{\scriptstyle e} \\ K & \xrightarrow{\quad y \quad} & C_0 \end{array}$$

and a map $K \xrightarrow{g} hom_{\underline{S}^{C_0}}(\Delta X, (C_1)_y)$, where

$$\begin{array}{ccc} (C_1)_y & \longrightarrow & C_1 \\ \big\downarrow & \text{PB} & \big\downarrow{\scriptstyle <\partial_1, \partial_0>} \\ \Delta K & \xrightarrow[\Delta y]{} & \Delta C_0 \end{array} ,$$

such that

$$q_X <f, g> = \ulcorner 1_X \urcorner : K \longrightarrow hom(X, X).$$

Equivalently, k is given by a map $y: K \to C_0$, a map

$$\Delta K \underset{\Delta C_0}{\times} C_1 \xrightarrow{\ f\ } \Delta K \underset{\Delta I}{\times} X$$

and a map

$$\Delta K \underset{\Delta I}{\times} X \xrightarrow{\ g\ } \Delta K \underset{\Delta C_0}{\times} C_1$$

such that $gf = 1_{\Delta K \underset{\Delta I}{\times} X}$. Thus, $\xi: X \to \Delta I$ is locally (with

$$K \overset{a}{\twoheadrightarrow} I \quad \text{and} \quad K \xrightarrow{\ y\ } C_0 \ )$$

the retract of representables. □

(4.3) PROPOSITION. *There exists an* $\underline{S}$*-indexed functor*

$$M: \hat{\mathbb{C}} \longrightarrow Pointess_{\underline{S}}(\underline{S}^{\mathbb{C}^{\circ}}),$$

such that:

(i) the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\ H\ } & Pointess_{\underline{S}}(\underline{S}^{\mathbb{C}^{\circ}}) \\ {\scriptstyle U}\downarrow & \nearrow_{M} & \\ \hat{\mathbb{C}} & & \end{array}$$

commutes up to canonical isomorphisms;

(ii) M^I *is fully faithful for each* $I \in |\underline{S}|$ *and embeds* $\hat{\mathbb{C}}$ *into the full* $\underline{S}$*-indexed subcategory of the essential points represented by retracts of representables.*

(iii) M *is a weak equivalence functor.*

PROOF. If $\phi: \underline{S}/I \to \underline{S}^{\mathbb{C}^{\circ}}$ is given, with $\phi^* \approx hom(\xi, -)$, with $\xi: X \to \Delta I$ and ϕ is $\underline{S}$-essential, then, by definition, ξ is $\underline{S}$-atomic, as ϕ^* has an indexed (hence a strong, relative to $\underline{S}$) right adjoint ϕ_*. By (4.2), $\xi: X \to \Delta I$ is locally the retract of representables. Next, any family which is locally the retract of representables is $\underline{S}$-atomic by the following remarks. Firstly, the family of all the representables,

$$C_1 \xrightarrow{\ <\partial_0, \partial_1>\ } \Delta C_0$$

is $\underline{S}$-atomic, as remarked earlier. Secondly, any retract of an $\underline{S}$-atomic family is $\underline{S}$-atomic, as it is easy to see. Finally, if a family is locally $\underline{S}$-atomic, it must also be $\underline{S}$-atomic, since pulling back along an epi reflects epis and isos involved in stating the condition.

Also, if $\xi: X \to \Delta I$ is $\underline{S}$-atomic, then $hom(\xi, -): \underline{S}^{C^o}/\Delta I \to \underline{S}/I$ satisfies the conditions of the Special Adjoint Functor Theorem of [11, (3.2), page 107], as it preserves all $\underline{S}$-colimits and $\underline{S}^{C^o}/\Delta I$ has an internal cogenerator as is a topos. Hence, $\phi^* = hom(\xi, -)$ has an indexed right adjoint ϕ_*. But, ϕ^* always has an indexed left adjoint $\phi_!$ with $\phi_! \beta = \beta \otimes \xi = \Delta\beta \times \xi$ in $\underline{S}^{C^o}/\Delta I$, for any $\beta \in \underline{S}/I$. Hence, ξ defines an I-indexed family ϕ_ξ of $\underline{S}$-essential points of $\underline{S}^{C^o}$.

Denoting by $\underline{S}$-*Atomic*$(\underline{S}^{C^o})$ the $\underline{S}$-indexed category of $\underline{S}$-atomic families of $\underline{S}^{C^o}$, we have

(4.3.1) *There exists an equivalence*

$$\underline{S}\text{-}Atomic(\underline{S}^{C^o}) \equiv Pointess_{\underline{S}}(\underline{S}^{C^o})$$

of $\underline{S}$-indexed categories.

It is now easy to define M: given $I \in |\underline{S}|$ and $e: I \to \hat{C}_0$, letting $Y: C \to \underline{S}^{C^o}$ be the Yoneda embedding, then

$$I \xrightarrow{\;e\;} \hat{C}_0 \rightarrowtail^{i} C_1$$

gives rise to $Y^I(je) \in \underline{S}^{C^o}/\Delta I$, which is an idempotent map with domain (and codomain) $Y^I(y)$, where

$$I \xrightarrow{\;y\;} C_0 \;=\; I \xrightarrow{\;e\;} \hat{C}_0 \xrightarrow{\;i\;} C_1 \xrightarrow{\;\partial_0\;} C_0 .$$

Consider

$$\begin{array}{ccc} Y^I(y) & \xrightarrow{\quad Y^I(je) \quad} & Y^I(y) \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle g} & \\ & \gamma & \end{array}$$

a factorization through the image. Then, if $\gamma: Z \to \Delta I$ is the image, γ is a retract of $Y^I(y)$, hence also locally so, hence $\underline{S}$-atomic, hence

$$\gamma \in (\underline{S}\text{-}Atomic(\underline{S}^{C^o}))^I \equiv (Pointess_{\underline{S}}(\underline{S}^{C^o})^I .$$

This definition, $M^I(y) = Im_{Y^I(je)}$, extends to morphisms in a natural way by the universal property of the image. It is also clear from the construction that M^I is fully faithful (since Y^I is), and that the families which are images under M are retracts of representables. This gives (ii).

The commutativity in (i) is immediate : taking the image factorization of the identity on some $Y^I(y)$ gives once more $Y^I(y)$ and H^I also takes y to $Y^I(y)$. And this is true for every $I \in |\underline{S}|$ and every $y: I \to C_0$.

(iii) Given an $\underline{S}$-essential morphism

$$\phi_\xi : \underline{S}/I \to \underline{S}^{C^o}, \quad \text{with} \quad \phi^* \approx hom(\xi, -),$$

then ξ is $\underline{S}$-atomic, hence locally the retract of representables, i.e., there is $\alpha : K \twoheadrightarrow I$ with $K^*(\xi)$ a retract of $K^*(<\partial_1, \partial_0>)$, i.e. of $(C_1)_y \to K$, where

$$\begin{array}{ccc}
(C_1)_y & \longrightarrow & C_1 \\
\downarrow & \text{PB} & \downarrow{\scriptstyle <\partial_1, \partial_0>} \\
K \times C_0 & \xrightarrow{\; y \times C_0 \;} & C_0 \times C_0 \\
\downarrow & \text{PB} & \downarrow{\scriptstyle \pi_0} \\
K & \longrightarrow & C_0
\end{array}$$

for some $y: K \to C_0$.

Now, it is easily seen that there is a bijection between K-indexed families of idempotents of C and retracts of K-indexed families of representables, as follows. Given any $e: K \to \hat{C}_0$, the composite

$$K \xrightarrow{\; e \;} \hat{C}_0 \rightarrowtail^{\; j \;} C_1 \xrightarrow{\; Y_1 \;} \underline{S}^{C^o}$$

gives an idempotent $Y^K(je)$ in $\underline{S}^{C^o}/\Delta K$, whose domain (andcodomain) is a family of representables indexed by K, i.e., some $Y^K(y)$ with

$$K \xrightarrow{\; y \;} C_0 \; = \; K \xrightarrow{\; e \;} \hat{C}_0 \rightarrowtail^{\; j \;} C_1 \xrightarrow{\; \partial_0 \;} C_0 .$$

Splitting this idempotent in $\underline{S}^{C^o}/\Delta K$ (by means of image factorizations) gives some $\gamma \in |\underline{S}^{C^o}/\Delta K|$, which is then a retract of $Y^K(y)$. Conversely, a retract of some $Y^K(y)$, say, some

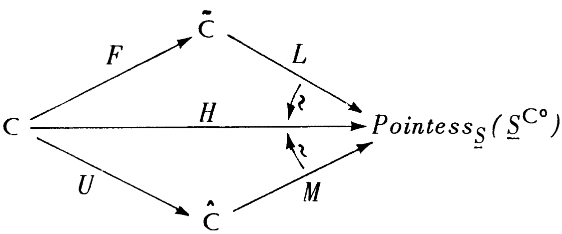
$$y \underset{g}{\overset{f}{\rightleftarrows}} Y^K(y) \quad \text{with} \quad gh = 1_y ,$$

gives rise to the idempotent $fg: Y^K(y) \to Y^K(y)$ in $\underline{S}^{C^o}/\Delta K$. But now since Y^K is full, there exists some $e: K \to C_1$ with $\partial_0 e = \partial_1 e = y$, such that $Y^K(e) = fg$. Hence, since Y^K is faithful, $e^2 = e$ and so y determines $e: K \to \hat{C}_0$.

Hence, given any I-indexed family of $\underline{S}$-essential points of $\underline{S}^{C^o}$, $\phi_\xi: \underline{S}/I \to \underline{S}^{C^o}$, there is $\alpha: K \twoheadrightarrow I$ and $y: K \to C_0$ such that $K^*(\xi)$ is a retract of $K \underset{C_0}{\times} C_1 \to K$. This says that M is a weak equivalence functor. □

5. MORITA EQUIVALENCE FOR CATEGORY OBJECTS IN A TOPOS.

We now put together the diagrams obtained in (2.2) and (4.3) as follows.



remembering that F and M are weak equivalence functors. If U is an equivalence functor, i.e., by (3.4) equivalently if idempotents split in C, then L is a weak equivalence functor since $LF \approx MU$ and MU is a weak equivalence functor in case idempotents split in C. Now, both $\tilde{C}$ and $Pointess_{\underline{S}}(\underline{S}^{C^o})$ are stacks; by the universal property of stack completions, or rather, by (2.11) of [4], since

$$C \xrightarrow{F} \tilde{C} \xrightarrow{L} Pointess_{\underline{S}}(\underline{S}^{C^o})$$

would be a weak equivalence functor, we must have L as equivalence of categories. In general, however, it is not the case that idempotents split in C and, in that case, $Pointess_{\underline{S}}(\underline{S}^{C^o})$ is «larger» than $\tilde{C}$.

(5.1) DEFINITION. For internal categories C, D in $\underline{S}$, say thac C and

D are *Morita equivalent* if $\underline{S}^{C^o}$ and $\underline{S}^{D^o}$ are equivalent $\underline{S}$-indexed categories.

(5.2) THEOREM. *Let* $\underline{S}$ *be a topos and let* C, D *be categories in* $\underline{S}$. *Then, the following are equivalent:*

(i) C *and* D *are Morita equivalent.*

(ii) $\hat{C}$ *and* $\hat{D}$ *are weakly equivalent.*

(iii) $\hat{\hat{C}}$ *and* $\hat{\hat{D}}$ *are equivalent.*

PROOF. (i) ⇒ (iii). Assume that $\underline{S}^{C^o} \equiv \underline{S}^{D^o}$. Then also

$$Pointess_{\underline{S}}(\underline{S}^{C^o}) \equiv Pointess_{\underline{S}}(\underline{S}^{D^o}) :$$

given an equivalence functor $\Phi : \underline{S}^{C^o} \to \underline{S}^{D^o}$, its indexed inverse guarantees that Φ is $\underline{S}$-essential so that, composing with Φ, induces the required equivalence between the $\underline{S}$-indexed categories of $\underline{S}$-essential points. By (4.3 (iii)), $M : \hat{C} \to Pointess_{\underline{S}}(\underline{S}^{C^o})$ is a weak equivalence functor and by (1.1) $Pointess_{\underline{S}}(\underline{S}^{C^o})$ is a stack. Hence, $Pointess_{\underline{S}}(\underline{S}^{\hat{C}^o})$ is the stack completion of $\hat{C}$. Similarly, $Pointess_{\underline{S}}(\underline{S}^{\hat{D}^o})$ is the stack completion of $\hat{D}$. Hence, $\hat{\hat{C}}$ and $\hat{\hat{D}}$ are indeed equivalent.

(ii) ⇒ (i). Assume that $\hat{C}$ and $\hat{D}$ are weakly equivalent. By (2.6), $\underline{S}^{\hat{C}^o}$ and $\underline{S}^{\hat{D}^o}$ are equivalent. But then, using (5.3) and the above, it follows that

$$\underline{S}^{C^o} \equiv \underline{S}^{\hat{C}^o} \equiv \underline{S}^{\hat{D}^o} \equiv \underline{S}^{D^o} .$$

Hence C and D are Morita equivalent.

(iii) ⇒ (ii). If $\hat{\hat{C}} \equiv \hat{\hat{D}}$, say, by means of some pair of indexed functors

$$\hat{\hat{C}} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \hat{\hat{D}}$$

inverse to each other, then Φ is a weak equivalence functor. The diagram

$$\hat{C} \xrightarrow{F_{\hat{C}}} \hat{\hat{C}} \xrightarrow{\Phi} \hat{\hat{D}} \xrightarrow{F_{\hat{D}}} \hat{D} ,$$

where $F_{\hat{C}}$ and $F_{\hat{D}}$ are the canonical weak equivalence functors of (2.3), can be completed to a 2-pullback (as in (1.4) of [4]) in order to produce an indexed category $\underline{E}$ and weak equivalence functors

$$\hat{C} \xleftarrow{G} \underline{E} \xrightarrow{H} \hat{D}$$

so that $\hat{C} \underset{w}{\equiv} \hat{D}$. □

(5.3) REMARKS. In the presence of the axiom of choice for $\underline{S}$, (5.2) says that C and D are Morita equivalent iff $\hat{C}$ and $\hat{D}$ are equivalent categories, as every category object is a stack. This result has been known for some time (cf. [7,2]), at least when $\underline{S}$ is *Set*.

Call an internal category C in $\underline{S}$ (any topos now) *good* when the following are true:

(i) $U: C \to \hat{C}$ is an equivalence.

(ii) $H: C \to \bar{C}$ is an equivalence.

Then, we have

If C *is a good internal category in* $\underline{S}$, *then as* $\underline{S}$*-indexed categories,* C *and* $Pointess_{\underline{S}}(\underline{S}^{C^o})$ *are equivalent.*

If C *and* D *are both good internal categories in* $\underline{S}$, *then* $\underline{S}^{C^o}$ *and* $\underline{S}^{D^o}$ *are equivalent iff* C *and* D *are equivalent.*

Finally, we point out that a conjecture stated by G. Wraith in [12] concerning Morita equivalence of internal categories (with «local equivalence» instead of «weak equivalence») is not true, as the following counterexample shows. Let $\underline{S}$ be a topos in which the internal axiom of choice does not hold, e.g., let $a: U \twoheadrightarrow I$ be an epi in $\underline{S}$ which does not split locally. As in the proof of (1.1) in [4], let $F_a : \mathrm{U}_a \to \mathrm{I}$ be the induced internal functor. It is a weak equivalence functor. By (2.6) and (2.7) of [4], $\underline{S}^{F_a}$ is an equivalence $\underline{S}^{\mathrm{I}} \to \underline{S}^{\mathrm{U}_a}$. According to [12], this would imply that F_a is a local equivalence, yet, since the object part of F_a is a itself, which does not split locally, this is impossible. For the exact relationship between the notions of weak and local equivalence, the reader may consult [4], especially Proposition (1.12).

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