CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

MANFRED B. WISCHNEWSKY

Topologically-algebraic structure functors full reflective or coreflective restrictions of semitopological functors

Cahiers de topologie et géométrie différentielle catégoriques, tome 20, nº 3 (1979), p. 311-330

<http://www.numdam.org/item?id=CTGDC_1979__20_3_311_0>

CAHIERS DE TOPOLOGIE *Vol. XX-3 (1979)*
ET GEOMETRIE DIFFERENTIELLE

TOPOLOGICALLY-ALGEBRAIC STRUCTURE FUNCTORS FULL REFLECTIVE OR COREFLECTIVE RESTRICTIONS OF SEMITOPOLOGICAL FUNCTORS

by Manfred B. WISCHNEWSKY

INTRODUCTION.

The aim of this paper is to develop a new concept which is fit for describing at the same time all full reflective or coreflective subcategories of locally presentable categories (Gabriel and Ulmer [4]) - hence in particular of algebraic categories (over sets) - as well as of topological categories. In fact the notion introduced here describes more generally all full reflective and coreflective restrictions of semitopological functors. Semitopological functors which represent all reflective restrictions of topological functors were investigated by Börger & Tholen [3], Herrlich & Strecker [7], Hoffmann [8, 9, 10], Tholen [13, 16], Tholen & Wischnewsky [15, 14] and Wischnewsky [17]. The results presented here generalize fundamental results for semitopological functors.

In the language of points and arrows the generalizations presented here are obtained by adding new arrows to old ones, and not by erasing arrows in old notions (as one might expect).

Hence from a diagrammatic point of view the characterizations given here look more complicated than e.g. for topological or semitopological functors. But the importance and usefullness lie in the theorems which one obtains.

The main techniques used here (beside categorical routine methods) are a «Cantor's diagonal Lemma for categories» (Börger & Tholen [2]) and applications of the notion «connectedness with respect to a sequence of functors» (Wischnewsky [18]). Finally, I would like to thank R. Börger and W. Tholen for several helpful discussions. Furthermore I am indebted to Fernuniversität Hagen for the support in typing the manuscript.

0. NOTATIONS.

0.1. Let $S: \underline{A} \to \underline{X}$ be a functor. A *S-cone* is a triple $(X, \psi, D(\underline{A}))$, where X is an $\underline{X}$-object, $D(\underline{A}): \underline{D} \to \underline{A}$ is an $\underline{A}$-diagram ($\underline{D}$ may be void or large) and $\psi: \Delta X \to SD(\underline{A})$ is a functorial morphism (Δ denotes the «constant» functor into the functor category). We shall abbreviate often $(X, \psi, D(\underline{A}))$ by ψ. $Cone(S)$ denotes the class of all S-cones. If $\underline{D} = 1$ (i. e., the one-point category), then ψ is called a *S-morphism*, denoted by (A, a) where A is an $\underline{A}$-object and $a: X \to SA$ is an X-morphism. The dual notions are *S-cocone* and *S-comorphism*. The corresponding classes of S-morphisms, S-cocones and S-comorphisms are denoted by

$$Mor(S), \quad Co\text{-}Cone(S), \quad Co\text{-}Mor(S).$$

$Epi(S)$ denotes the class of all S-epimorphisms $(A, e: X \to SA)$, i.e., the class of all S-morphisms (A, e) with the property: for all $\underline{A}$-morphisms $p, q: A \rightrightarrows B$, the equation

$$(Sp)e = (Sq)e \quad \text{implies} \quad p = q.$$

The dual notion is *S-monomorphism*. The class of all S-monomorphisms is denoted by $Mono(S)$. $Iso(S)$ denotes the class of all *S-isomorphisms*, i.e., of all objects (A, a) in $Mor(S)$ with a an isomorphism in $\underline{X}$.

$Init(S)$ denotes the class of all *S-initial cones*, i. e., of all $\underline{A}$-cones $\alpha: \Delta A \to D(\underline{A})$ such that for any $\underline{A}$-cone $\beta: \Delta B \to D(\underline{A})$ and any S-morphism $x: SB \to SA$ with $(S\beta) = (S\alpha)(\Delta x)$ there exists a unique $\underline{A}$-morphism $a: B \to A$ with

$$\beta = \alpha(\Delta a) \quad \text{and} \quad Sa = x.$$

$Lim(\underline{D})$, the class of all *limit-cones (over $\underline{D}$) in $\underline{A}$* is the class of cones $(A, \psi, D(\underline{A}))$ over $\underline{A}$ such that for all cones $(B, \phi, D(\underline{A}))$ there exists a unique $\underline{A}$-morphism $t: B \to A$ with $\phi = \psi(\Delta t)$.

0.2. Let $S: \underline{A} \to \underline{X}$ be a functor and

$$S = (\underline{A} \xrightarrow{\hat{Q}} \underline{B} \xrightarrow{Q} \underline{X})$$

be a factorization of S by the functors $\hat{Q}: \underline{A} \to \underline{B}$ and $Q: \underline{B} \to \underline{X}$.

A $(S, \hat{Q}, Q)$-*double-cocone* is a quintuple $(D(\underline{A}), \phi, D(\underline{B}), \psi, X)$, where $X \in Ob(\underline{X})$, $D(\underline{A})$ is a diagram in $\underline{A}$, $D(\underline{B})$ is a diagram in $\underline{B}$ with $dom\, D(\underline{A}) = dom\, D(\underline{B})$ and where ϕ, ψ are functorial morphisms with $\phi : D(\underline{B}) \to \hat{Q} D(\underline{A})$ and $\psi : QD(\underline{B}) \to \Delta X$,

$$SD(\underline{A}) \xleftarrow{Q\phi} QD(\underline{B}) \xrightarrow{\psi} \Delta X.$$

The class of all $(S, \hat{Q}, Q)$-double-cocones is denoted by

$$Co\text{-}Cone^2(S, \hat{Q}, Q).$$

If $dom\, D(\underline{A}) = 1$, then a $(S, \hat{Q}, Q)$-double-cocone is called a $(S, \hat{Q}, Q)$-*double-comorphism*. The class of all $(S, \hat{Q}, Q)$-double-comorphisms is denoted by $Co\text{-}Mor^2(S, \hat{Q}, Q)$.

By reversing the arrows one obtains the notions $(S, \hat{Q}, Q)$-*double cone*, resp. $(S, \hat{Q}, Q)$-*double-morphism*. The corresponding classes are

$$Cone^2(S, \hat{Q}, Q), \text{ resp. } Mor^2(S, \hat{Q}, Q).$$

If $Q = Id : \underline{X} \to \underline{X}$ and $\hat{Q} = S$, then a (S, S, Id)-double-cocone (-cone, -morphism, -comorphism) is called a *S-double-cocone (-cone, -morphism, -comorphism)*.

1. TOPOLOGICALLY-ALGEBRAIC STRUCTURE FUNCTORS. BASIC DEFINITIONS.

1.1. DEFINITION. Let $S : \underline{A} \to \underline{X}$ be a functor and let

$$S = (\underline{A} \xrightarrow{\hat{Q}} \underline{B} \xrightarrow{Q} \underline{X})$$

be an arbitrary factorization. Let

$$Id(Q) \subset \Phi \subset Mor(Q), \quad Id(\hat{Q}) \subset \Gamma \subset Co\text{-}Mor(\hat{Q}), \quad Id(\hat{Q}) \subset \Pi \subset Co\text{-}Mor(\hat{Q})$$

be classes of Q-morphisms, resp. $\hat{Q}$-comorphisms.

1. Let

$$SD(\underline{A}) \xrightarrow{Q\pi} QD(\underline{B}) \xleftarrow{\phi} D(\underline{X}) \xrightarrow{\psi} \Delta X$$

be a functorial chain with $(D(\underline{A}), \pi : \hat{Q} D(\underline{A}) \to D(\underline{B}))$ being pointwise in Π and $(D(\underline{B}), \phi : D(\underline{X}) \to QD(\underline{B}))$ being pointwise in Φ. We call a functorial chain of this type a (Π,Φ)-functorial chain. The class of all (Π,Φ)-functorial chains is denoted by $Chs(\Pi,\Phi)$.

Let (A,g,B,e,X) be a $(S,\hat{Q},Q)$-double morphism with

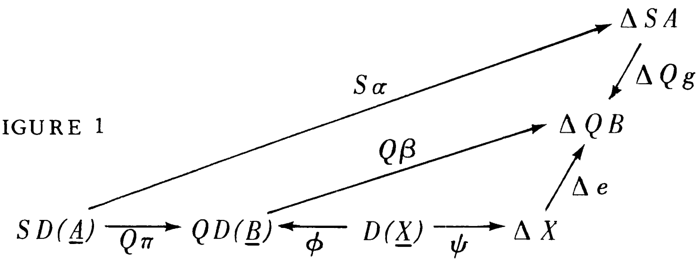
$$(A,\ g: \hat{Q}A \to B) \in \Gamma .$$

A pair of functorial morphisms $\alpha: D(\underline{A}) \to \Delta A$ and $\beta: D(\underline{B}) \to \Delta B$ with

$$\beta\pi = (\Delta g)(\hat{Q}\alpha) \quad \text{and} \quad (\Delta e)\psi = (Q\beta)\phi$$

is called a (Π,Φ,Γ)-*extension of* $(D(\underline{A}),\pi,D(\underline{B}),\phi,\psi)$ *by* (A,g,B,e,X).

FIGURE 1



If there is no confusion, we will abbreviate the notation by

$$(\alpha,\beta):(\pi,\phi,\psi) \to (g,e).$$

Let

$$(\alpha,\beta):(\pi,\phi,\psi) \to (g,e) \quad \text{and} \quad (\alpha',\beta'):(\pi,\phi,\psi) \to (g',e')$$

be (Π,Φ,Γ)-extensions. A morphism from (α,β) to (α',β') is a pair of morphisms $a: A \to A'$ and $b: B \to B'$ such that

$$\alpha' = (\Delta a)\alpha, \quad \beta' = (\Delta b)\beta, \quad e' = (Qb)e \quad \text{and} \quad bg = g'(\hat{Q}a).$$

This defines the category of all (Π,Φ,Γ)-extensions over (π,ϕ,ψ).

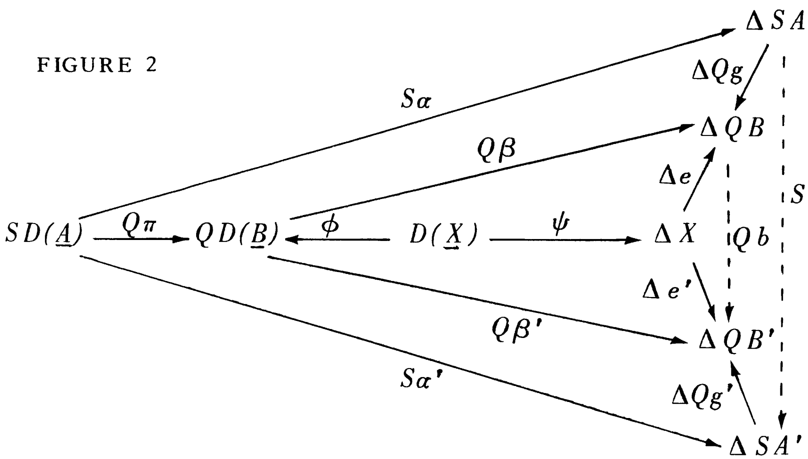
2. An initial object $(\alpha,\beta):(\pi,\phi,\psi) \to (g,e)$ in the category of all the (Π,Φ,Γ)-extensions is called a *semi-final* (Π,Φ,Γ)-*extension.*

3. A functor $S: \underline{A} \to \underline{X}$ is called a *semifinal topologically-algebraic structure functor* (or for short a semifinal Top-algebraic structure functor) if there exist a factorization $S = Q\hat{Q}$, classes Π, Φ, Γ and a subclass $S(\Pi,\Phi)$ of the class of all (Π,Φ)-chains containing all chains of the type

$$SA \xrightarrow{Qg} QB \xrightarrow{x} X$$

with $(A,g) \in \Gamma$ and $(B,x) \in \text{Co-Mor}(Q)$ such that every (Π,Φ)-functorial chain in $\Phi(S(\Pi,\Phi))$ [1] has a semifinal (Π,Φ,Γ)-extension. In this case we say that S *has semifinal* $(S(\Pi,\Phi),\Gamma)$*-extensions.* It is obvious that a semifinal $(S(\Pi,\Phi),\Gamma)$-extension is unique up to isomorphisms.

FIGURE 2



4. Any 6-tuple $(\hat{Q}, Q, \Pi, \Phi, \Gamma, S(\Phi,\Gamma))$ such that S is a semifinal Top-algebraic structure functor with respect to it is called *S-(semifinal)-compatible.*

1.2. DEFINITION. Let $S: \underline{A} \to \underline{X}$ be a functor,

$$S = Q\hat{Q}: A \xrightarrow{\hat{Q}} B \xrightarrow{Q} X$$

be a factorization,

$$Id(Q) \subset \Phi \subset Mor(Q), \quad Id(\hat{Q}) \subset \Gamma \subset \text{Co-Mor}(\hat{Q}),$$
$$Id(\hat{Q}) \subset \Pi \subset \text{Co-Mor}(\hat{Q}) \text{ and } S(\Pi,\Phi)$$

be as in Definition 1.1.

1. Let $(D(\underline{A}), \gamma, D(\underline{B}), \rho, X)$ be a $(S, \hat{Q}, Q)$-double-cone with

$$(D(\underline{A}),\ \gamma: \hat{Q}D(\underline{A}) \to D(\underline{B}))$$

being pointwise in Γ. Let (A, p, B, f, x) be a $S(\Pi,\Phi)$-chain

1) $\Phi(S(\Pi,\Phi))$ denotes the «class» of all functorial chains being pointwise in $S(\Pi,\Phi)$.

$$((A,\ p: \hat{Q}A \to B) \in \Pi \quad \text{and} \quad (B,\ f: Y \to QB) \in \Phi\ ).$$

A pair of cones $\alpha : \Delta A \to D(\underline{A})$ and $\beta : \Delta B \to D(\underline{B})$ with

$$\rho(\Delta x) = (Q\beta)(\Delta f), \quad \beta(\Delta p) = \gamma(\hat{Q}\alpha)$$

is called a $(S(\Pi, \Phi), \Gamma)$-*coextension of* (γ, ρ).

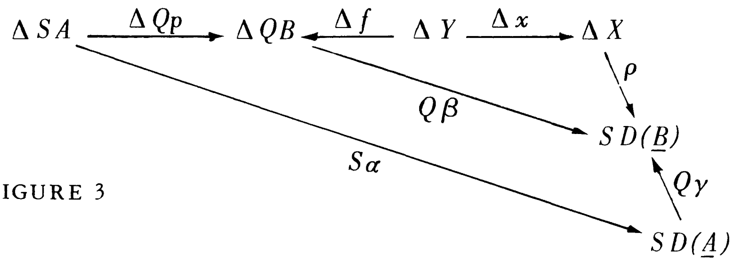


FIGURE 3

2. Let $(D(\underline{A}), \gamma, D(\underline{B}), \rho, X)$ be a $(S, \hat{Q}, Q)$-double-cone with

$$(D(\underline{A}),\ \gamma : \hat{Q}D(\underline{A}) \to D(\underline{B}))$$

being pointwise in Γ. A $(S, \hat{Q}, Q)$-double-morphism (A, g, B, e, X) with $(A,\ g: \hat{Q}A \to B)$ in Γ together with a pair (α, β) consisting of an $\underline{A}$-cone $\alpha : \Delta A \to D(\underline{A})$ and a $\underline{B}$-cone $\beta : \Delta B \to D(\underline{B})$ is called a *semi-initial* $(S(\Pi, \Phi), \Gamma)$-*coextension of* (γ, ρ) if

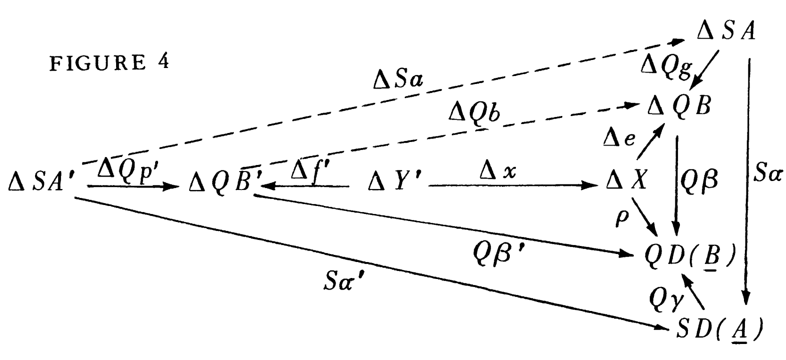
SI 1. For all $(S(\Pi, \Phi), \Gamma)$-coextensions

$$(\alpha', \beta') : (A', p', B', f', x) \to (\gamma, \rho)$$

of (γ, ρ) there exists a unique pair (a, b) consisting of an $\underline{A}$-morphism $a: A' \to A$ and a B-morphism $b: B' \to B$ such that

$$\alpha' = \alpha(\Delta a), \quad \beta' = \beta(\Delta b), \quad ex = (Qb)f' \quad \text{and} \quad bp' = g(\hat{Q}a).$$

FIGURE 4



SI 2. For any pair of morphisms $s: A \to A$ and $t: B \to B$ the equations

$$e = (Qt)e, \quad \beta = \beta(\Delta t), \quad \alpha = \alpha(\Delta s), \quad \text{and} \quad tg = g(\hat{Q}s)$$

imply $s = Id(A)$ and $t = Id(B)$.

3. Notation as above. A functor $S: \underline{A} \to \underline{X}$ is called a *semiinitial Top-algebraic structure functor* if there exist a factorization $S = Q\hat{Q}$ and classes Π, Φ, Γ, $S(\Pi, \Phi)$ such that every $(S, \hat{Q}, Q)$-double-cone

$$(D(\underline{A}), \gamma, D(\underline{B}), \rho, X)$$

with $(D(\underline{A}), \gamma)$ being pointwise in Γ has a semiinitial $(S(\Pi, \Phi), \Gamma)$-coextension.

4. Any 6-tuple $(\hat{Q}, Q, \Pi, \Phi, \Gamma, S(\Pi, \Phi))$ such that S is a semiinitial Top-algebraic structure functor with respect to it is called *S-(semiinitial-)compatible.*

1.3. EXAMPLES.

1. *Semitopological functors* [3, 7, 8, 9, 10, 13, 14, 15, 16, 17] are (semifinal and semiinitial) topologically-algebraic structure functors. For this special instance we have the following characteristic datas:

$$\hat{Q} = Id: \underline{A} \to \underline{A}, \quad Q = S: \underline{A} \to \underline{X},$$
$$Id(\underline{X}) \subset \Phi \subset Mor(S), \quad \Pi = \Gamma = Id(\underline{A}).$$

$S(\Pi, \Phi)$ can be chosen in many different ways of which we will only note two of them:

(a) $S_1(\Pi, \Phi) = Co\text{-}Mor(S)$ (these datas give the well-known semifinal (resp. semiinitial) characterization of semi-topological functors).

(b) $S_2(\Pi, \Phi) \subset Co\text{-}Mor^2(S)$ where

$$(A, SA \xleftarrow{f} Y \xrightarrow{x} X) \in S_2(\Pi, \Phi) \quad \text{iff} \quad (A, f) \in \Phi .$$

In this case one has to demand in addition that the classes *Semi-Fin(S)* resp. *Semi-Init(S)* of all semifinal extensions $(A, e: X \to SA)$, resp. semi-initial coextensions, are subclasses of Φ. (These datas give the locally-orthogonal, resp. the left-extension (or Kan-coextension) characterization of semitopological functors.)

The most well-known examples for semitopological functors are the reflective restrictions of topological or monadic functors over the category of all sets.

2. *Full coreflective restrictions of semitopological functors* are topologically-algebraic structure functors. Let

$$(S: \underline{A} \to \underline{X}) = (\underline{A} \xrightarrow{E} \underline{B} \xrightarrow{T} \underline{X})$$

be a factorization and assume T is semitopological and E is a full coreflective embedding. Take:

$$\hat{Q} := E : \underline{A} \to \underline{B}, \quad Q := T : \underline{B} \to \underline{X}, \quad \Phi := Id(Q),$$

$\Pi = \Gamma =$ the class of all counits of the full coreflective embedding,

and $S(\Pi, \Phi) := Chs(\Pi, \Phi)$ (cf. Definition 1.1.1). Then it is easy to see that the 6-tuple $(E, T, \Phi, \Pi, \Gamma, Chs(\Pi, \Phi))$ is S-(semifinal and semi-initial)-compatible.

Hence in particular all reflective or coreflective restrictions of monadic functors over *Sets* are Top-algebraic structure functors.

In Section 4 we will show the converse, i. e. that every Top-algebraic structure functor is a reflective or coreflective restriction of a semitopological functor.

2. TOP-ALGEBRAIC STRUCTURE FUNCTORS ARE FAITHFUL.

The following results generalize theorems of Hoffmann [8], Herrlich [5], Tholen [13, 16]. The proof method used in the following goes back to R. Börger, resp. Börger & Tholen [2]. This method is Cantor's diagonal principle for categories.

Remember that an arbitrary category $\underline{C}$ is called *pointed* iff for all $\underline{C}$-objects A, B, there exists at most one $\underline{C}$-morphism $A \to B$.

Let $S: \underline{A} \to \underline{X}$ be a functor,

$$S = Q\hat{Q} : \underline{A} \xrightarrow{\hat{Q}} \underline{B} \xrightarrow{Q} \underline{X}$$

be an arbitrary factorization, Π, Φ, Γ, $S(\Pi, \Phi)$ be classes as in Definition 1.1. We call the pair $(S(\Pi, \Phi), \Gamma)$ *pointed* if, for any chain

$$(A, p, B, f, x, X) \text{ in } S(\Pi, \Phi)$$

and any $(S, \hat{Q}, Q)$-double-morphism (A', g', B', e, X) with

$$(A', g' : \hat{Q}A' \to B') \in \Gamma,$$

there exists at most one pair of morphisms

$$(a, b) : (A, p, B, f, x, X) \to (A', g', B', e, x)$$

with $a: A \to A'$, $b: B \to B'$ such that $g'(\hat{Q}a) = bp$ and $bf = ex$.

2.1. LEMMA. *Let $S: \underline{A} \to \underline{X}$ be a semifinal (semiinitial) Top-algebraic structure functor with respect to $(Q, Q, \Pi, \Phi, \Gamma, S(\Pi, \Phi))$. Then, the pair $(S(\Pi, \Phi), \Gamma)$ is pointed.*

PROOF. (i) Let S be a semifinal Top-algebraic structure functor with respect to $(\hat{Q}, Q, \Pi, \Phi, \Gamma, S(\Pi, \Phi))$. Let

$$SA' \xrightarrow{Qp} QB' \xleftarrow{f} Y \xrightarrow{x} X$$

be a $S(\Pi, \Phi)$-chain and

$$SA'' \xrightarrow{Qg''} QB'' \xleftarrow{e''} X$$

be a chain with $(A'', g'': \hat{Q}A'' \to B'') \in \Gamma$. Let

$$b, b' : B' \rightrightarrows B'' \quad \text{and} \quad a, a' : A' \rightrightarrows A''$$

be two pairs of morphisms with

$$g''(\hat{Q}a) = bp, \quad (Qb)f = e''x, \quad g''(\hat{Q}a') = b'p, \quad (Qb')f = e''x.$$

We have to show that $a = a'$ and $b = b'$. Let $I = Mor(\underline{A})$ be the class of all $\underline{A}$-morphisms. Let

$$A_i = A', \quad B_i = B', \quad Y_i = Y, \quad \phi_i = f, \quad \pi_i = p \quad \text{and} \quad \psi_i = x$$

for all $i \in I$. Hence there exist a semifinal (Π, Φ, Γ)-extension

$$(a_i, b_i) : (\pi_i, \phi_i, \psi_i) \to (g, e)$$

with $(A, g) \in \Gamma$. Let

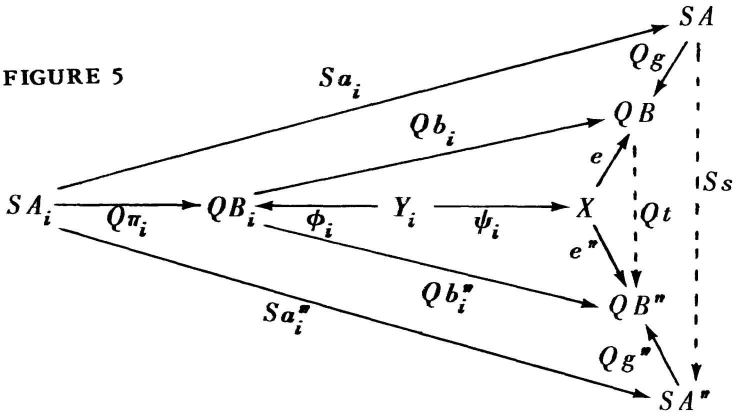
$$J := \{ s \in \underline{A}(A, A'') \mid \text{for all } i \in I,\ s a_i \in \{a, a'\}\}$$

and

$$\hat{J} := \{ t \in \underline{B}(B, B'') \mid \text{for all } i \in I,\ t b_i \in \{b, b'\}\}.$$

The classes J, $\hat{J}$ are nonvoid. Take for instance $a_i'' = a$ and $b_i'' = b$ for all $i \in I$. The universal property yields elements $s \in J$ and $t \in \hat{J}$: Choose

FIGURE 5



surjective mappings $\sigma : I \to J$ and $\bar{\sigma} : I \to \hat{J}$. Assume $a \neq a'$. Now define

$$a_i'' = \begin{cases} a & \text{if } \sigma(i)a_i = a' \\ a' & \text{if } \sigma(i)a_i = a \end{cases}$$

and

$$b_i'' = \begin{cases} b & \text{if } \sigma(i)a_i = a' \\ b' & \text{if } \sigma(i)a_i = a. \end{cases}$$

Then

$$(Qb_i'')\phi_i = e''\psi_i \quad \text{and} \quad b_i''\pi_i = g''(\hat{Q}a_i'').$$

Hence there exist morphisms t, s rendering commutative the above diagram. Let $i_0 \in I$ with $\sigma(i_0) = s$. Then

$$\sigma(i_0)a_{i_0} = a \iff \sigma(i_0)a_{i_0} = a'.$$

But this is a contradiction. Hence we have $a = a'$. Now define

$$b_i'' = \begin{cases} b & \text{if } \bar{\sigma}(i)b_i = b' \\ b' & \text{if } \bar{\sigma}(i)b_i = b \end{cases}$$

and $a_i'' := a = a'$. In the same way as above we obtain $j_0 \in I$ with

$$\bar{\sigma}(j_0)b_{j_0} = b \iff \bar{\sigma}(j_0)b_{j_0} = b'.$$

Hence $b = b'$.

(ii) If S is a semiinitial Top-algebraic structure functor the corresponding statement is proved in a similar way. This completes the proof.

2.2. Let S be a semifinal (semiinitial) Top-algebraic structure functor and $(\hat{Q}, Q, \Pi, \Phi, \Gamma, S(\Pi, \Phi))$ be a corresponding S-compatible 6-tuple. A pair of $\underline{B}$-morphisms $b,\ b': B \rightrightarrows B'$ is called (Π, Γ)-*compatible* if there exist

$$(A,\ p: \hat{Q}A \to B) \in \Pi, \quad (A',\ g: \hat{Q}A' \to B') \in \Gamma$$

and $\underline{A}$-morphisms $a,\ a': A \rightrightarrows A'$ with

$$bp = g(\hat{Q}a) \quad \text{and} \quad b'p = g(\hat{Q}a').$$

2.3. THEOREM. *Let S be a semifinal (semiinitial) Top-algebraic structure functor. Then:*

1. S is faithful.

2. Let $(\hat{Q}, Q, \Pi, \Phi, \Gamma, S(\Pi, \Phi))$ *be S-compatible. Then*

$$Id(\hat{Q}) \subset \Gamma \subset Mono(\hat{Q}).$$

The elements in Φ *are Q-epimorphisms with respect to* (Π, Γ)*-compatible pairs of B-morphisms.*

3. $\hat{Q}$ *is faithful and Q is faithful with respect to* (Π, Γ)*-compatible pairs of B-morphisms.*

PROOF. $(S(\Pi, \Phi), \Gamma)$ is pointed and $Id(\hat{Q}) \subset \Gamma, \Pi$ and $Id(\hat{Q}) \subset \Phi$.

2.4. REMARK. In many concrete cases we even have $\Phi \subset Epi(Q)$, or $\Phi \subset Epi(S)$. Take e.g.

$$\hat{Q} = S, \quad Q = Id: \underline{X} \to \underline{X} \quad \text{and} \quad \Pi = Id(S).$$

Then $\Phi \subset Epi(S)$. Functors with these special characterizing datas are just the (Π, Γ)-structure functors in the sense of [19], Part 1, resp. the (Φ, Γ)-concrete functors in the sense of W. Tholen [16].

3. THE DUALITY THEOREM FOR TOP-ALGEBRAIC STRUCTURE FUNCTORS.

The duality Theorem for Top-algebraic structure functors contains as special instances the duality theorems for topological functors (Antoine [1], Roberts [12]), for (E, M)-topological functors (Hoffmann [8]), for semitopological functors (Tholen [13]), for locally orthogonal Q-functors (op. Wolff [20]) and for cosemitopological functors (Tholen [16], Wischnewsky [18]). We omit here a proof since this theorem itself is a special instance of a much more general duality theorem for structure functor sequences (Wischnewsky [18]).

Applying Theorem 4.2 in [18] we obtain the following

3.1. THEOREM *(Duality Theorem for Top-algebraic structure functors). Let $S: \underline{A} \to \underline{X}$ be a functor. Then the following assertions are equivalent:*

(i) S is a semiinitial Top-algebraic structure functor.

(ii) S is a semifinal Top-algebraic structure functor.

Furthermore a 6-tuple $(\hat{Q}, Q, \Pi, \Phi, \Gamma, S(\Pi, \Phi))$ is S-semiinitial compatible iff it is S-semifinal compatible. Hence we will say it is just S-compatible.

By varying the datas in $(\hat{Q}, Q, \Pi, \Phi, \Gamma, S(\Pi, \Phi))$ we obtain the duality theorems mentioned above as special instances. We will just state two of them.

But first we will introduce an abbreviation.

3.2. Let $S: \underline{A} \to \underline{X}$ be a functor and $C = (\hat{Q}, Q, \Pi, \Phi, \Gamma, S(\Pi, \Phi))$ be S-compatible. The class of all semifinal (semiinitial) chains of type

$$X \xrightarrow{e} QB \xleftarrow{Qg} SA \quad \text{with} \quad (A, g) \in \Gamma$$

is denoted by *Semi-Univ* (C).

3.3. *The duality theorem for topological functors*: Topological functors are Top-algebraic structure functors characterized by the following datas:

$$\hat{Q} = Id: \underline{A} \to \underline{A}, \quad Q = S, \quad \Pi = \Phi = \Gamma := Id(S)$$

and *Semi-Univ* $(C) \subset Iso(S)$, where

$$C = (Id, S, Id(S), Id(S), Id(S), Co\text{-}Mor(S)) \quad {}^{1)}.$$

Hence we obtain from Theorem 3.1:

3.4. COROLLARY *(Duality theorem for topological functors* [1, 12]*). The following assertions are equivalent for a functor* S:

(i) S *is topological.*

(ii) S^{op} *is topological.*

3.5. *The duality Theorem for semitopological functors.* Semitopological functors are exactly all full reflective restrictions of topological functors (cp. 1.3.1). They can be characterized as Top-algebraic structure functors in several ways. One possibility is given by the datas:

$$\hat{Q} = Id: \underline{A} \to \underline{A}, \quad Q = S: \underline{A} \to \underline{X}, \quad \Phi = Id(S),$$
$$\Pi = \Gamma = Id(\underline{A}), \quad S(\Pi, \Phi) = Co\text{-}Mor(S).$$

A semifinal (semiinitial) Top-algebraic structure functor with respect to these datas was called semifinal (semiinitial) functor in [13, 17].

Hence we obtain the extremely useful

3.6. COROLLARY *(Duality for semitopological functors - Tholen* [13]*). The following assertions are equivalent for a functor* S:

(i) S *is a semifinal functor.*

(ii) S *is a semiinitial functor.*

4. A REPRESENTATION THEOREM FOR TOP-ALGEBRAIC STRUCTURE FUNCTORS.

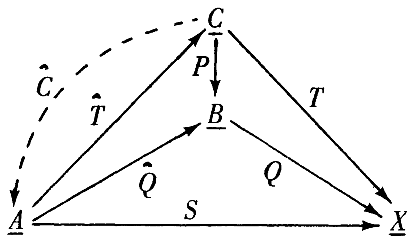
4.1. *The canonical representation* «induced by an arbitrary factorization». Let

$$S = Q\hat{Q}: \underline{A} \xrightarrow{\hat{Q}} \underline{B} \xrightarrow{Q} \underline{X}$$

be a factorization, $Id(\hat{Q}) \subset \Gamma \subset Co\text{-}Mor(\hat{Q})$ and $\hat{Q}$ be faithful. These datas induce in a canonical way a category $\underline{C}$ and several functors:

1) It is easy to see that $S(Id(S), Id(S)) = Co\text{-}Mor(S)$ is the only possibility in this case.

FIGURE 6



(1) The category $\underline{C}$:

$$Ob\underline{C} = \Gamma = \{(A,\ g: B \leftarrow \hat{Q}A) \in Co\text{-}Mor(Q) \mid (A,g) \in \Gamma\} .$$

Let (A,g) and (A',g') be $\underline{C}$-objects. A $\underline{C}$-morphism

$$(b,a): (A,g) \to (A',g')$$

is a pair consisting of a $\underline{B}$-morphism

$$b: B = codom(g) \to codom(g') = B'$$

and an $\underline{A}$-morphism $a: A \to A'$ with $g'(\hat{Q}a) = bg$.

(2) The functor $\hat{T}: \underline{A} \to \underline{C}$ is given by the assignments

$$A \mapsto (A,\ Id(\hat{Q}A)), \quad f \mapsto (\hat{Q}f, f).$$

$\hat{T}$ is obviously full and faithful ($\hat{Q}$ is faithful!).

(3) The functors

$$\hat{C}: \underline{C} \to \underline{A},\quad P: \underline{C} \to \underline{B},\quad T: \underline{C} \to \underline{X}$$

are given by the following assignments :

$$\hat{C}: \underline{C} \to \underline{A}: \quad (A,g) \mapsto A,$$
$$P: \underline{C} \to \underline{B}: \quad (A,g) \mapsto codom(g),$$
$$T: \underline{C} \to \underline{X}: \quad T := QP.$$

By these definitions we have $\hat{Q} = P\hat{T}$ and $S = T\hat{T}$. Furthermore we have a functorial morphism $\gamma: \hat{Q}\hat{C} \to P$ by the assignment

$$\gamma(A,g) := g: \hat{Q}\hat{C}(A,g) = \hat{Q}A \to codom(g).$$

The functorial morphism γ is pointwise in Γ.

(4) $\hat{T}: \underline{A} \to \underline{C}$ is coreflective with coreflector $\hat{C}$. The counit

$$\sigma: \hat{T}\hat{C} \to Id(\underline{C}) \text{ is given by } \sigma(A,g) := (g, Id(A)).$$

Given $(\hat{Q}, Q, \Gamma)$, the factorization

$$S = (\underline{A} \xrightarrow{\hat{T}} \underline{C} \xrightarrow{T} \underline{X})$$

is called the *canonical factorization induced by* $(\hat{Q}, Q, \Gamma)$.

4.2. REMARKS. 1. Let

$$S = Q\hat{Q} : \underline{A} \xrightarrow{\hat{Q}} \underline{B} \xrightarrow{Q} \underline{X}$$

be a factorization and $Id(\hat{Q}) \subset \Gamma \subset Mor(\hat{Q})$. Then by dualizing 4.1 we obtain a canonical factorization

$$S = R\hat{R} : \underline{A} \xrightarrow{\hat{R}} \underline{D} \xrightarrow{R} \underline{X}$$

where $\hat{R} : \underline{A} \to \underline{D}$ is a full reflective embedding.

2. If $Q = Id : \underline{X} \to \underline{X}$, i.e., $\hat{Q} = S$, and $Id(S) \subset \Gamma \subset Mor(S)$, then the canonical «reflective» extension is just the classical Herrlich construction (Herrlich [5]).

Now we can prove the following fundamental theorem.

4.3. THEOREM *(Representation Theorem for Top-algebraic structure functors). Let* $S : \underline{A} \to \underline{X}$ *be a functor. Then the following assertions are equivalent:*

(i) S is a Top-algebraic structure functor.

(ii) S is a full coreflective restriction of a semitopological functor.

PROOF. (ii) ⇒ (i) by 1.3.2.

(i) ⇒ (ii) : Let S be a Top-algebraic structure functor with respect to $(\hat{Q}, Q, \Pi, \Phi, \Gamma, S(\Pi, \Phi))$. $\hat{Q}$ is faithful (Theorem 2.3.3) and

$$Id(\hat{Q}) \subset \Gamma \subset Co\text{-}Mor(\hat{Q}).$$

Hence by 4.1 we obtain a canonical factorization

$$\underline{A} \xrightarrow{\hat{T}} \underline{C} \xrightarrow{T} \underline{X}$$

of S where $\hat{T}$ is a full coreflective embedding. We have to show that T is semitopological. Let $\psi : \Delta X \to TD(\underline{C})$ be a T-cone. 4.1.3 delivers a functorial morphism $\gamma : \hat{Q}\hat{C} \to P$ (notation as in 4.1) being pointwise in Γ. This defines a $(\hat{Q}, Q, S)$-double-cone

$$\Delta X \xrightarrow{\psi} TD(\underline{C}) = QPD(\underline{C}) \xleftarrow{Q\gamma D(\underline{C})} S\hat{C}D(\underline{C})$$

with $\gamma D(\underline{C}): \hat{Q}\hat{C}D(\underline{C}) \to PD(\underline{C})$ being pointwise in Γ. Hence we obtain a semiinitial $(S(\Pi,\Phi),\Gamma)$-coextension

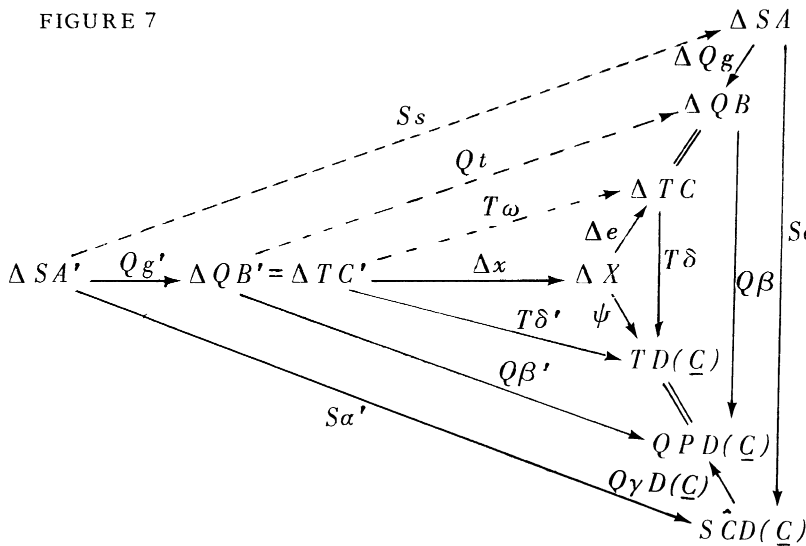
$$(\alpha,\beta):(A,g,B,e,X) \to (\hat{C}D(\underline{C}), \gamma D(\underline{C}), PD(\underline{C}), \psi, X).$$

Then

$$C := (A,g) \in \underline{C} \quad \text{and} \quad \delta := (\beta,\alpha): \Delta C \to D(\underline{C})$$

is a functorial morphism.

FIGURE 7



Obviously $\psi = (T\delta)(\Delta e)$. Let $(C', x: TC' \to X)$ be a T-comorphism with $C' = (A', g': \hat{Q}A' \to B') \in \Gamma$ and $\delta': \Delta C' \to D(\underline{C})$ be a functorial morphism with $T\delta' = \psi(\Delta x)$. Hence we have a pair of functorial morphisms $\alpha': \Delta A' \to \hat{C}D(\underline{C})$ and $\beta': \Delta B' \to PD(\underline{C})$ rendering the above diagram commutative. Since

$$(SA' \xrightarrow{Qg'} QB' \xrightarrow{x} X) \in S(\Pi,\Phi)$$

by assumption we get unique morphisms $s: A' \to A$ and $t: B' \to B$, and hence a unique $\omega: C' \to C$ such that

$$\delta' = \delta(\Delta\omega) \quad \text{and} \quad ex = T\omega .$$

The rest is clear. This completes the proof.

The (Φ,Γ)-structure functors [19], Part 1, resp. equivalently the (Φ,Γ)-concrete functors [16] are special Top-algebraic structure functors. Hence we obtain:

4.4. COROLLARY *(Tholen [16]). Let S be a (Φ,Γ)-structure functor in the sense of [19], resp. a (Φ,Γ)-concrete functor in the sense of [16]. Then S is a full coreflective restriction of a semitopological functor.*

The results in 1.3.2, resp. 4.2, give us the following corollary, containing several well-known theorems as special instances.

4.5. COROLLARY. Let

$$(S: \underline{A} \to \underline{X}) = (\underline{A} \xrightarrow{\hat{T}} \underline{B} \xrightarrow{T} \underline{X})$$

be a full coreflective restriction of a semitopological functor T, where $\hat{T}$ is the full coreflective embedding (i.e., S is a Top-algebraic structure functor). Let Γ be the class of all counits $g: \hat{T}A \to B$ of $\hat{T}$. If $T(\Gamma)$ is included in $Iso(\underline{X})$, we obtain the following assertions:

1. If T is topological, then S is topological (Wyler [21]).

2. If T is (E,M)-topological (in the sense of Herrlich [5]), then S is (E,M)-topological.

3. If T is topologically-algebraic in the sense of Y. Hong [11], resp. equivalently a M-functor in the sense of Tholen [13], then S is again a topologically-algebraic functor.

4.6. EXAMPLES. 1. Let *Top* be the category of all topological spaces (over the category of sets). Since every full coreflective subcategory of *Top* (except the trivial one) fulfills the assumption in 4.5, every non-trivial coreflective subcategory of *Top* is again a topological category (over *Sets*).

2. The category of compactly generated spaces is a full coreflective subcategory of the (E,M)-topological category of all Hausdorff spaces fulfilling the assumption in 4.5. Hence it is again an (E,M)-topological category.

By dualizing and specializing the results in 4.1 and 4.2 for semi-topological functors S, we obtain immediately the following result first proved by W. Tholen and M. Wischnewsky (Oberwolfach 1977).

4.7. COROLLARY *(Tholen & Wischnewsky - Representation Theorem for semitopological functors). The following assertions are equivalent for a functor S :*

(i) S is semitopological.

(ii) S is a full reflective restriction of a topological functor.

Together with 4.3, we obtain :

4.8. COROLLARY *(Representation Theorem). The following assertions are equivalent for a functor S :*

(i) S is a Top-algebraic structure functor.

(ii) There exists a factorization of S:

$$S: \underline{A} \to \underline{X} = (\underline{A} \xrightarrow{C} \underline{B} \xrightarrow{R} \underline{C} \xrightarrow{T} \underline{X})$$

where:

$C \equiv$ *full reflective embedding,*

$R \equiv$ *full reflective embedding, and*

$T \equiv$ *topological functor.*

REFERENCES.

1. ANTOINE, P., Etude élémentaire des catégories d'ensembles structurés, *Bull. Soc. Math. Belg.* 18 (1966).

2. BÖRGER, R. & THOLEN, W., *Cantors Diagonalprinzip für Kategorien*, Preprint, Fernuniversität Hagen, 1977.

3. BÖRGER, R. & THOLEN, W., *Is any semitopological functor topologically algebraic ?* Preprint, Fernuniversität Hagen, 1977.

4. GABRIEL, P. & ULMER, F., Lokal präsentierbare Kategorien, *Lecture Notes in Math.* 221, Springer (1971).

5. HERRLICH, H., Topological functors, *Gen. Topo. & Appl.* 4 (1974), 125 - 142.

6. HERRLICH, H., NAKAGAWA, R., STRECKER, E. & TITCOMB, T., *Topologically-algebraic and semitopological functors*, Preprint, Kansas Ste Univ. Manhattan 1977.

7. HERRLICH, H. & STRECKER, G., *Semi-universal maps and universal initial completions*, Preprint, Kansas Ste Univ., Manhattan, 1977.

8. HOFFMANN, R.-E., Semi-identifying lifts and a generalization of the duality theorem for topological functors, *Math. Nachr.* 74 (1976), 295-307.

9. HOFFMANN, R.-E., Topological functors admitting generalized Cauchy-completions, *Lecture Notes in Math.* 540, Springer (1976), 286-344.

10. HOFFMANN, R.-E., Full reflective restrictions of topological functors, Math. Coll. Univ. Cape Town, Preprint in *Math. Arbeitspapers* 7, Univ. Bremen (1976), 98 - 119.

11. HONG, Y. H., *Studies on categories of universal topological algebras*, Thesis, Mac Master Univ., Hamilton, 1974.

12. ROBERTS, J. E., A characterization of topological functors, *J. of Algebra* 8 (1968), 131- 193.

13. THOLEN, W., Semitopological functors, I, *J. Pure & Appl. Algebra* 15 (1979),53-73.

14. THOLEN, W. & WISCHNEWSKY, M. B., Semitopological functors II, Preprint Bremen, Hagen, 1977, *J. Pure & Appl. Algebra* (to appear).

15. THOLEN, W. & WISCHNEWSKY, M. B., Semitopological functors, III: Lifting of monads and adjoint functors, *Seminarberichte Fernuniv.* Hagen 4 (1978), 129.

16. THOLEN, W., *Konkrete Funktoren*, Habilitationschrift, Fernuniv. Hagen, 1978.

17. WISCHNEWSKY, M. B., A lifting theorem for right adjoints, Preprint Univ. Bremen 1976, *Cahiers Topo. et Géom. Diff.* (to appear).

18. WISCHNEWSKY, M. B., A generalized duality theorem for structure functors, Preprint, Univ. Bremen 1977.

19. WISCHNEWSKY, M. B., Existence and representation theorems for structure functors, Preprint, Univ. Bremen, 1977.

20. WOLFF, H., *External characterization of semitopological functors*, Handwritten manuscript, Univ. of Toledo, 1977.

21. WYLER, O., *Top* categories and categorical Topology, *Gen. Topo. & Appl.* 1 (1971), 17-28.

Fachbereich für Mathematik
Universität Bremen
Universitätsallee
D-2800 BREMEN 33. WEST GERMANY