

# CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES

RONALD BROWN

PETER NICKOLAS

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*Cahiers de topologie et géométrie différentielle catégoriques*, tome  
20, n° 2 (1979), p. 179-198

[http://www.numdam.org/item?id=CTGDC\\_1979\\_\\_20\\_2\\_179\\_0](http://www.numdam.org/item?id=CTGDC_1979__20_2_179_0)

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**EXPONENTIAL LAWS FOR TOPOLOGICAL CATEGORIES, GROUPOIDS  
AND GROUPS, AND MAPPING SPACES OF COLIMITS**

*by Ronald BROWN and Peter NICKOLAS*

**INTRODUCTION.**

One aim of this paper is to give a topological version of the well known exponential law for categories, which states that for categories  $A$  and  $B$  there is a functor category  $(A, B)$  such that for any  $C, D$  and  $E$ , there is a natural isomorphism of categories

$$(1) \quad \theta: (C \times D, E) \rightarrow (C, (D, E)).$$

For our topological version we use as underlying category of spaces the category  $\mathcal{K}$  of topological spaces  $X$  and  $k$ -continuous maps, by which we mean functions  $f: X \rightarrow Y$  such that  $fa: C \rightarrow Y$  is continuous for all continuous maps  $a: C \rightarrow X$  of compact Hausdorff spaces  $C$  into  $X$ . We give the set  $\mathcal{K}(X, Y)$  of  $k$ -maps  $X \rightarrow Y$  a topology with a subbase of sets  $\mathcal{W}(a, U)$  of those functions  $f$  such that  $fa(C) \subset U$ , with  $a: C \rightarrow X$  as above and  $U$  open in  $Y$ . Then  $\mathcal{K}$  has an exponential law generalizing to non-Hausdorff spaces the exponential law of [4]. (The corresponding category of  $k$ -spaces and continuous maps is well known - see for example [10] - but gives less precise results than ours.)

Our topological version of (1) is for  $k$ -categories, that is, category objects in  $\mathcal{K}$ , and this brings us to our second expository point - that we must prove  $(D, E)$  a  $k$ -category if  $D$  and  $E$  are. This means we must prove that the structure maps of  $(D, E)$  are  $k$ -continuous, and to do this it is convenient to have these maps defined as induced maps. We therefore use methods of A. and C. Ehresmann [2] involving the double  $k$ -category  $\square E$  of commuting squares of  $E$ , with its horizontal and vertical compositions  $\boxplus$  and  $\boxminus$ . M. and M<sup>me</sup> Ehresmann have pointed out to us that our Theorem 2 is very close to a special case of Proposition 4.10

of [2], which proves that the category of category (or groupoid) objects in a cartesian closed category is itself cartesian closed. However we hope that our exposition of this special case and of its applications will show the simplicity and utility of these results and methods.

The exponential isomorphism (1) is also valid for  $k$ -groupoids  $C, D, E$  and hence also for  $k$ -groups (though  $(D, E)$  is of course a groupoid and not a group). Our main application of the exponential law is to the space  $M(D, E)$  of morphisms of  $k$ -groups with the compact-open topology. We prove that if  $D$  is a colimit  $\varinjlim_{\lambda} D_{\lambda}$  of  $k$ -groups, then the natural bijection

$$(2) \quad \Phi: M(\varinjlim_{\lambda} D_{\lambda}, E) \rightarrow \varinjlim_{\lambda} M(D_{\lambda}, E)$$

is a  $k$ -homeomorphism. Also, if  $D$  is a  $k$ -groupoid,  $UD$  is its universal  $k$ -group and  $E$  is any  $k$ -group, then the natural map

$$(3) \quad \Phi: M(UD, E) \rightarrow M(D, E)$$

is a  $k$ -homeomorphism. As a consequence we obtain results on free  $k$ -products and free  $k$ -groups. These results are in Section 2.

In Section 3 we prove that if  $D$  is a Hausdorff  $k_{\omega}$ -groupoid, then the map  $\Phi$  of (3) is a homeomorphism. The approach here is more direct than that of previous sections, and makes use of the explicit construction of  $UD$  given in [5].

For further references to the theory and applications of topological categories and groupoids we refer the reader to the 80 papers listed in the bibliography of [6].

## 1. AN EXPONENTIAL LAW FOR $k$ -CATEGORIES.

Our object in this section is to set up an exponential law for topological categories. To do this we need to start with a cartesian closed category of topological spaces. A number of such categories are available, but for our purposes it is convenient to use a modification of the  $k$ -continuous maps of [4] to allow for non-Hausdorff spaces.

Let  $X$  and  $Y$  be topological spaces and  $f: X \rightarrow Y$  a function. We say  $f$  is *k-continuous* if for all compact Hausdorff spaces  $C$  and continuous maps  $a: C \rightarrow X$  the composite  $fa: C \rightarrow Y$  is continuous. These spaces and functions form a category which we denote by  $\mathbb{K}$ , so that  $\mathbb{K}(X, Y)$  is the set of *k-continuous* maps  $X \rightarrow Y$ . The usual product and sum of spaces give the product and sum in  $\mathbb{K}$ . Although the analogous theory of *k-spaces* and continuous maps has been considered by a number of writers, the reader is warned that the literature contains many references to «*k-spaces*» and «*k-continuous functions*» defined in senses different from those used here.

There is a functor  $k: \mathbb{K} \rightarrow \mathbb{K}$  where, for  $X$  a topological space,  $k(X)$  has the final topology with respect to all continuous maps  $a: C \rightarrow X$  for  $C$  compact and Hausdorff. Then  $k(X)$  is an identification space of a locally compact space - namely, the sum of spaces  $C_A$  obtained by choosing for each non-open set  $A$  of  $k(X)$  a compact Hausdorff  $C_A$  and a map  $a_A: C_A \rightarrow X$  such that  $a_A^{-1}(A)$  is not open in  $C_A$  (cf. [10]). A function  $f: X \rightarrow Y$  is *k-continuous* if and only if  $f: k(X) \rightarrow Y$  is continuous.

We now define a *compact-open topology* on  $\mathbb{K}(X, Y)$  by taking as sub-base of open sets the sets  $W(a, U)$  for  $U$  open in  $Y$  and  $a: C \rightarrow X$  a continuous map from a compact Hausdorff space  $C$  to  $X$ ; here  $W(a, U)$  consists of the *k-continuous* maps  $f: X \rightarrow Y$  such that  $fa(C) \subset U$ . (This topology is considered in [10].) The exponential law stated in [4] for Hausdorff spaces and extended to some non-Hausdorff spaces in [3], Section 5.6, can now be stated for all spaces as follows:

(1.1) Exponential law in  $\mathbb{K}$ . The exponential correspondence

$$\theta: \mathbb{K}(X \times Y, Z) \rightarrow \mathbb{K}(X, \mathbb{K}(Y, Z)), \quad \theta(f)(x)(y) = f(x, y),$$

is well-defined, is a bijection and is a homeomorphism.

The function space  $\mathbb{K}(X, Y)$  is functorial in the sense that if

$$f: W \rightarrow X, \quad g: Y \rightarrow Z$$

are *k-continuous*, then the induced function  $f^*: \mathbb{K}(X, Y) \rightarrow \mathbb{K}(W, Y)$  is continuous and the induced function  $g_*: \mathbb{K}(X, Y) \rightarrow \mathbb{K}(X, Z)$  is *k-con-*

tinuous. Further, if  $g$  is continuous so also is  $g_*$ . (The proofs that  $f^*$ ,  $g_*$  are continuous in the given circumstances are easy; that  $g_*$  is always  $k$ -continuous follows from the exponential law (1.1) as in [4].) It is also easy to prove that if  $g: Y \rightarrow Z$  is a homeomorphism into, so also is  $g_*: \mathcal{K}(X, Y) \rightarrow \mathcal{K}(X, Z)$ .

Another result proved in [4] is that if  $f: W \rightarrow X$  is a  $k$ -identification map (that is, if  $f: k(W) \rightarrow X$  is an identification map), then

$$f_*: \mathcal{K}(X, Y) \rightarrow \mathcal{K}(W, Y)$$

is a  $k$ -homeomorphism into (that is,  $f^*$  is injective and its inverse is  $k$ -continuous on its domain).

It can be shown that, if  $\mathcal{U}$  is a sub-base for the open sets of  $Y$ , then the sets  $W(a, U)$  for  $a: C \rightarrow X$  continuous,  $C$  compact Hausdorff and  $U \in \mathcal{U}$ , form a sub-base for the open sets of  $\mathcal{K}(X, Y)$ . From this we deduce in a standard way:

(1.2) The natural map

$$\alpha: \mathcal{K}(X, Y \times Z) \rightarrow \mathcal{K}(X, Y) \times \mathcal{K}(X, Z)$$

is a homeomorphism.

Suppose further that  $Y \tilde{\times} Z$  is a pull-back as in the diagram

$$\begin{array}{ccc} Y \tilde{\times} Z & \longrightarrow & Z \\ \downarrow & & \downarrow h \\ Y & \xrightarrow{g} & W \end{array}$$

and  $Y \tilde{\times} Z$  has its topology as a subspace of  $Y \times Z$ . Then we have:

(1.3) The natural map

$$\alpha: \mathcal{K}(X, Y \tilde{\times} Z) \rightarrow \mathcal{K}(X, Y) \tilde{\times} \mathcal{K}(X, Z)$$

is a homeomorphism where the latter space is the pull-back of  $g_*$  and  $h_*$ .

Our objective now might be to extend the exponential law (1.1) to the cases of topological categories and groupoids. Since the morphisms are only to be  $k$ -continuous rather than continuous, however it seems reasonable to deal instead with  $k$ -categories and  $k$ -groupoids, in which the

structure maps are  $k$ -continuous.

To fix the notation, we first recall that a (small) category consists of a set  $C$  of arrows and a set  $Ob(C)$  of objects, together with functions  $\partial', \partial: C \rightarrow Ob(C)$  (the final and initial maps, respectively),  $u: Ob(C) \rightarrow C$  (the unit map) and

the composition  $m: C \tilde{\times} C \rightarrow C$ , where  $C \tilde{\times} C$  is the subset of  $C \times C$ , of pairs  $(p, q)$  such that  $\partial p = \partial' q$ , and where  $m(p, q)$  is written  $pq$ ; these functions must satisfy the usual axioms for a category. It is usual to confuse the category with the set  $C$  of arrows.

A  $k$ -category is such a category  $C$  in which  $C$  and  $Ob(C)$  are spaces and  $\partial', \partial, u$  and  $m$  are  $k$ -continuous. Further,  $C$  is a  $k$ -groupoid if in addition it is a groupoid and the inverse map  $p \mapsto p^{-1}$  is  $k$ -continuous.

A morphism  $f: C \rightarrow D$  of  $k$ -categories consists of a pair of  $k$ -continuous functions

$$f: C \rightarrow D, \quad Ob(f): Ob(C) \rightarrow Ob(D)$$

which commute with the category structure. The set of these morphisms is written  $M(C, D)$ . This set can be identified with a subset of  $K(C, D)$ , the space of  $k$ -continuous functions between the spaces of arrows, and we give  $M(C, D)$  the compact-open topology.

We now wish to construct a  $k$ -category  $(C, D)$  having  $M(C, D)$  as object space and with  $k$ -continuous natural transformations as arrows. For this purpose it is convenient to follow [2] defining first the space  $\square D$  of commuting squares in  $D$  to be the subspace of  $D^4$  of quadruples

$$\begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix}$$

of arrows of  $D$  such that  $pq, rs$  are defined and equal. This space is the arrow space of two  $k$ -categories with object space  $D$ . One of these, the horizontal category  $\boxtimes D$ , has initial and final maps

$$\begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix} \mapsto p, \quad \begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix} \mapsto s$$

respectively, unit map

$$p \mapsto \begin{pmatrix} & u\partial'p & \\ p & & p \\ & u\partial p & \end{pmatrix}$$

and composition

$$\begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix} \boxplus \begin{pmatrix} & u & \\ s & & v \\ & t & \end{pmatrix} = \begin{pmatrix} & ru & \\ p & & v \\ & qt & \end{pmatrix}.$$

The vertical category  $\boxplus D$  has initial and final maps

$$\begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix} \mapsto r, \quad \begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix} \mapsto q,$$

unit map

$$p \mapsto \begin{pmatrix} & p & \\ u\partial'p & & u\partial p \\ & p & \end{pmatrix}$$

and vertical composition

$$\begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix} \boxplus \begin{pmatrix} & q & \\ x & & z \\ & y & \end{pmatrix} = \begin{pmatrix} & r & \\ px & & sz \\ & y & \end{pmatrix}.$$

Now let  $X$  be the domain of the composition  $\boxplus$ , with its topology as a subspace of  $\square D \times \square D$ . Then  $X$  is the arrow space of a  $k$ -category with object space  $D \tilde{\times} D$  and with composition

$$(\alpha, \beta)(\gamma, \delta) = (\alpha \boxplus \gamma, \beta \boxplus \delta).$$

The vertical composition  $\boxplus: X \rightarrow \square D$  is then a morphism of  $k$ -categories. In effect, we are expressing the statement that  $\square D$  is a double  $k$ -category.

PROPOSITION 1. For  $k$ -categories  $C, D$  there is a  $k$ -category  $(C, D)$  with object space  $M(C, D)$  and arrow space  $M(C, \boxplus D)$ , and with structure maps induced by the vertical category structure on  $\square D$ . If  $D$  is a groupoid, so also is  $(C, D)$  with the induced inverse map. Finally, if  $D$  is a topological category or topological groupoid, so also is  $(C, D)$ .

PROOF. The initial, final and unit maps of the vertical category structure  $\boxplus D$  on  $\square D$  are morphisms between the horizontal category  $\square D$  and

$D$  itself. They therefore induce  $k$ -continuous maps between  $M(C, \boxplus D)$  and  $M(C, D)$ , and these we take as the initial, final and unit maps of  $(C, D)$ . The composition in  $(C, D)$  is the  $k$ -continuous map

$$M(C, \boxplus D) \tilde{\times} M(C, \boxplus D) \xrightarrow{\alpha^{-1}} M(C, \boxplus D \tilde{\times} \boxplus D) \xrightarrow{\boxplus_*} M(C, \boxplus D)$$

where  $\alpha$  is the homeomorphism given by (1.3). The axioms for a  $k$ -category are easily verified. Similarly, if  $D$  is a  $k$ -groupoid the inverse map on  $\boxplus D$  is a morphism for  $\boxplus D$ , and induces a  $k$ -continuous map

$$M(C, \boxplus D) \rightarrow M(C, \boxplus D)$$

making  $(C, D)$  a  $k$ -groupoid.

If the structure functions of  $D$  are continuous so also are those of  $(C, D)$ . This proves Proposition 1.

**THEOREM 2** (*The exponential law for  $k$ -categories*). *If  $C, D$  and  $E$ , are  $k$ -categories, there is a natural isomorphism of  $k$ -categories*

$$\Theta: (C \times D, E) \rightarrow (C, (D, E))$$

*which is continuous and has  $k$ -continuous inverse. Further, if  $E$  is a topological category, then  $\Theta^{-1}$  is continuous.*

**PROOF.** A complete proof of the theorem is quite lengthy, and we therefore omit a number of straightforward details.

For  $k$ -categories  $A, B$  an application of (1.2) shows that we may regard the space  $(A, B)$  as a subspace of  $\mathbb{K}(A, B)^4$ . In particular, we can regard  $(C, (D, E))$  as a subspace of  $\mathbb{K}(C, \mathbb{K}(D, E)^4)^4$  and hence of  $\mathbb{K}(C, \mathbb{K}(D, E))^{16}$ .

To define  $\Theta$ , note that an arrow  $f$  of  $(C \times D, E)$  is a morphism:  $C \times D \rightarrow \boxplus E$ , so that for each arrow  $(c, d): (x, y) \rightarrow (w, z)$  of  $C \times D$  we may write

$$(1.4) \quad f(c, d) = \begin{pmatrix} f(x, y) & r(c, d) & f(w, z) \\ & q(c, d) & \end{pmatrix}.$$

Note that  $q, r$  are morphisms  $C \times D \rightarrow E$  since they are the composites of  $f$  with  $\partial_{\boxplus}, \partial'_{\boxplus}: \boxplus E \rightarrow E$  respectively. We require that  $\Theta f$  be a mor-



phism  $C \rightarrow \mathbb{M}(D, E)$ , so for any given  $c: x \rightarrow w$  of  $C$  define

$$\Theta f(c) = \begin{pmatrix} k_1(c) & k_3(c) & \\ & k_2(c) & k_4(c) \end{pmatrix},$$

a commuting square in  $(D, E)$  where the  $k_i(c)$  in  $M(D, \mathbb{M}E)$  are defined, for  $d: y \rightarrow z$  an arrow of  $D$ , by

$$(1.5) \quad \begin{aligned} k_1(c)(d) &= f(x, d), & k_4(c)(d) &= f(w, d), \\ k_2(c)(d) &= \begin{pmatrix} q(x, d) & & \\ & q(w, d) & \\ & & q(c, z) \end{pmatrix}, \\ k_3(c)(d) &= \begin{pmatrix} r(c, y) & & \\ & r(x, d) & \\ & & r(c, z) \end{pmatrix} \end{aligned}$$

Certainly each  $k_i(c)(d)$  is in  $\square E$  since  $q, r$  are morphisms and  $f$  maps into  $\square E$ . To show that (say)  $k_2(c) \in M(D, \mathbb{M}E)$  we note that the four components of  $k_2(c)$  are  $k$ -continuous (since  $q$  is  $k$ -continuous) and so  $k_2(c)$  is  $k$ -continuous. Also, if  $d = d_1 d_2$  in  $D$ , then we have

$$u \partial' d_1 = u \partial' d, \quad u \partial d_2 = u \partial d \quad \text{and} \quad u \partial d_1 = u \partial' d_2,$$

and so

$$k_2(c)(d_1) \mathbb{M} k_2(c)(d_2) = k_2(c)(d)$$

as required. As for the verifications that  $k_1(c), k_3(c), k_4(c)$  belong to  $M(D, \mathbb{M}E)$ , that for  $k_3(c)$  is similar to that for  $k_2(c)$ , while those for  $k_1(c), k_4(c)$ , though a little different, are equally straightforward.

Again, the proofs that  $\Theta f \in (C, (D, E))$ , and that

$$\Theta(f) = \Theta(f_1) \Theta(f_2) \quad \text{if} \quad f = f_1 f_2$$

in  $(C \times D, E)$ , are routine though rather lengthy, and we omit details. To check continuity (rather than just  $k$ -continuity) of  $\Theta$ , note that by (1.2) the map

$$f \mapsto \begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix}$$

of (1.4) embeds  $M(C \times D, \mathbb{M}E)$  as a subspace of  $K(C \times D, E)^4$ ; then

$\Theta$ , as a map into  $\mathcal{K}(C, \mathcal{K}(D, E))^{16}$ , can be written as  $\Theta_1 \times \Theta_2 \times \Theta_3 \times \Theta_4$  where the four components of each  $\Theta_i$  are the composites of the exponential map  $\theta$  of (1.1) and the maps on  $\mathcal{K}(C, \mathcal{K}(D, E))$  induced by  $u\partial'$  or  $u\partial$  on  $C$  or  $D$ . Since each  $(u\partial')^*$ ,  $(u\partial)^*$  is continuous, the continuity of  $\Theta$  follows.

To construct  $\Phi = \Theta^{-1}$ , first define  $\pi, \sigma, \gamma: \square E \rightarrow E$  to be the maps which take

$$\begin{pmatrix} & r & \\ p & & s \\ & q & \end{pmatrix} \text{ to } p, s, pq$$

respectively. Thus  $\pi$  and  $\sigma$  are the initial and final maps of  $\square E$  and  $\gamma$  is  $k$ -continuous. Now given an arrow  $g$  of  $(C, (D, E))$  and an arrow  $c: x \rightarrow w$  of  $C$ , write  $g(c)$  as a commuting square

$$\begin{pmatrix} g(x) & g_3(c) & \\ & & g(w) \\ & g_2(c) & \end{pmatrix}$$

in  $(D, E)$ , and define for  $(c, d): (x, y) \rightarrow (w, z)$  an arrow in  $C \times D$ :

$$(\Phi g)(c, d) = \begin{pmatrix} & \gamma(g_3(c)(d)) & \\ \pi(g(x)(d)) & & \sigma(g(w)(d)) \\ & \gamma(g_2(c)(d)) & \end{pmatrix}.$$

It is straightforward to check that  $\Phi$  is the inverse of  $\Theta$ . The  $k$ -continuity of  $\Phi$  follows from the continuity of  $\pi, \sigma, \theta^{-1}$  and the  $k$ -continuity of  $\gamma$ ; and if  $E$  is a topological category, then  $\gamma$  is continuous, and hence so is  $\Phi$ .

COROLLARY 1. For any  $k$ -categories  $C, D, E$  there is a natural bijection  $M(C, (D, E)) \rightarrow M(D, (C, E))$ .

From Theorem 2 and the facts about groupoids in Proposition 1, we deduce:

COROLLARY 2 (The exponential law for  $k$ -groupoids). If  $C, D, E$  are  $k$ -groupoids, there is a natural isomorphism of  $k$ -groupoids

$$\Theta: (C \times D, E) \rightarrow (C, (D, E))$$

which is continuous and has  $k$ -continuous inverse. Further, if  $E$  is a to-

*topological groupoid, then  $\Theta$  is a homeomorphism.*

In particular, we obtain an exponential law for the case when  $C$ ,  $D$ ,  $E$  are topological groups, although of course  $(D, E)$  remains a topological groupoid and not a group.

REMARK. The existence of the isomorphism  $\Theta$  of Theorem 2 and its  $k$ -continuity could be proved a little more easily by first constructing a natural bijection  $M(C \times D, E) \rightarrow M(C, (D, E))$  and applying a standard argument using associativity of the product. Such an argument does not easily give continuity of  $\Theta$  or (for the case when  $E$  is a topological groupoid) of  $\Theta^{-1}$ .

## 2. APPLICATIONS TO COLIMITS.

Our main kind of application is to determine up to  $k$ -homeomorphism the space  $M(D, E)$ , where  $E$  is a  $k$ -group and  $D$  is a colimit of  $k$ -groups. The use of  $k$ -groupoids rather than just  $k$ -groups, however, is not only required by our method of proof, but has the advantage of easily giving results on free  $k$ -groups. By generalising still further to  $k$ -categories we obtain results on  $k$ -monoids as well as  $k$ -groups.

The existence of arbitrary colimits of  $k$ -categories,  $k$ -groupoids or  $k$ -groups may be proved in a manner similar to that for the topological case of [6]. However, the results of this section do not require knowledge of the co-completeness of these categories.

We first need:

PROPOSITION 3. *The functor  $Ob$  from the category of  $k$ -categories, or from the category of  $k$ -groupoids, to the category  $\mathcal{K}$  preserves limits and colimits.*

PROOF. There are functors  $P$ ,  $T$ , respectively left and right adjoints to  $Ob$  - namely:

the point-like functor  $P: X \mapsto X$  (where the topological groupoid  $X$  has object space  $X$  and only identity arrows),

and the tree functor  $T: X \mapsto X \times X$  (where the topological groupoid

$X \times X$  has object space  $X$ , arrow space  $X \times X$  and  $\partial', \partial$  are the projections).

Hence  $Ob$  preserves limits and colimits.

**THEOREM 4.** *Let  $D, E$  be  $k$ -categories and suppose that  $D$  is a colimit  $\lim_{\lambda} D_{\lambda}$  of a diagram of  $k$ -categories  $D_{\lambda}$ . Then the natural map*

$$\Phi: (D, E) \rightarrow \lim_{\lambda} (D_{\lambda}, E)$$

*is an isomorphism of categories, is continuous and has  $k$ -continuous inverse. The analogous result also holds for colimits of  $k$ -groupoids.*

**PROOF.** The  $k$ -continuous morphisms  $D_{\lambda} \rightarrow D$  given by the colimit induce continuous morphisms  $(D, E) \rightarrow (D_{\lambda}, E)$  and hence a continuous morphism

$$\Phi: (D, E) \rightarrow \lim_{\lambda} (D_{\lambda}, E).$$

To prove that  $\Phi$  is an isomorphism of categories with  $k$ -continuous inverse we use Corollary 1 to Theorem 2, and standard arguments, to obtain for any  $k$ -category  $C$  natural bijections

$$\begin{aligned} M(C, (D, E)) &\rightarrow M(D, (C, E)) \rightarrow \lim_{\lambda} M(D_{\lambda}, (C, E)) \\ &\rightarrow \lim_{\lambda} M(C, (D_{\lambda}, E)) \rightarrow M(C, \lim_{\lambda} (D_{\lambda}, E)). \end{aligned}$$

The result follows from the Yoneda Lemma.

Theorem 4 and Proposition 3 now yield:

**COROLLARY 1.** *Let the  $k$ -category  $D$  be a colimit  $\lim_{\lambda} D_{\lambda}$  of a diagram of  $k$ -categories  $D_{\lambda}$ . Then for any  $k$ -category  $E$  the natural map*

$$\Phi: M(D, E) \rightarrow \lim_{\lambda} M(D_{\lambda}, E)$$

*is a continuous bijection with  $k$ -continuous inverse. The analogous result also holds for colimits of  $k$ -groupoids.*

For our next corollary, we rephrase the definition of the universal topological category [6] for the case of  $k$ -categories.

Let  $D$  be a  $k$ -category, and  $\sigma: Ob(D) \rightarrow Y$  a  $k$ -continuous func-

tion. The *universal k-category*  $U_\sigma(D)$  is defined to be the pushout in the category of k-categories as in the diagram

$$\begin{array}{ccc} Ob(D) & \xrightarrow{\sigma} & Y \\ \downarrow i & & \downarrow \\ D & \xrightarrow{\bar{\sigma}} & U_\sigma(D) \end{array}$$

COROLLARY 2. If  $D$  is a k-category, and  $\sigma: Ob(D) \rightarrow Y$  is a k-identification map, then for any k-category  $E$  the induced map

$$\bar{\sigma}^*: M(U_\sigma(D), E) \rightarrow M(D, E)$$

is a continuous injection whose inverse is k-continuous on its domain, which is the set of morphisms  $f: D \rightarrow E$  for which  $Ob(f)$  factors k-continuously through  $\sigma$ .

PROOF. Since  $\sigma$  is a k-identification map,

$$\sigma^*: M(Y, E) \rightarrow M(Ob(D), E)$$

is a continuous injection with k-continuous inverse. Let  $V$  be the pull-back of  $\sigma^*$  and

$$i^*: M(D, E) \rightarrow M(Ob(D), E).$$

Then the projection  $V \rightarrow M(D, E)$  is also a continuous injection with k-continuous inverse. Since  $\bar{\sigma}^*$  is the composite of this projection with the natural map  $\Phi: M(U_\sigma(D), E) \rightarrow V$ , the result follows from Corollary 1.

When the above space  $Y$  is a singleton,  $U_\sigma(D)$  is a k-monoid, called the *universal k-monoid of  $D$* , and is denoted by  $U(D)$ .

COROLLARY 3. If  $D$  is a k-category and  $E$  a k-monoid, then the map

$$M(U(D), E) \rightarrow M(D, E)$$

induced by the universal morphism  $D \rightarrow U(D)$  is a continuous bijection with k-continuous inverse.

This follows immediately from Corollary 2.

COROLLARY 4. Let  $X$  be a space and  $i: X \rightarrow F^+(X)$  the canonical map

from  $X$  to the free  $k$ -monoid on  $X$ . Then for any  $k$ -monoid  $E$  the induced map

$$i^*: M(F^+(X), E) \rightarrow \mathcal{K}(X, E)$$

is a continuous bijection with  $k$ -continuous inverse.

PROOF. If  $\mathcal{X}$  denotes the point-like category on  $X$  (cf. the proof of Proposition 3), and  $\mathcal{2}$  denotes the topological category with two objects  $0, 1$ , one non-identity arrow  $0 \rightarrow 1$ , and the discrete topologies on objects and arrows, then the composite

$$X \rightarrow X \times \mathcal{2} \rightarrow U(X \times \mathcal{2})$$

of the injection  $x \mapsto (x, 0)$  and the universal morphism defines the free  $k$ -monoid  $i: X \rightarrow F^+(X)$ . So the result follows from Corollary 3.

We now use the  $k$ -groupoid version of Corollary 1 to deduce some results on free  $k$ -groups and free  $k$ -products of  $k$ -groups. These results will be sharpened for  $k_\omega$ -groups in Section 3.

COROLLARY 5. *If  $G$  is a  $k$ -groupoid and  $K$  is a  $k$ -group, then the map*

$$M(U(G), K) \rightarrow M(G, K)$$

*induced by the universal morphism  $G \rightarrow U(G)$  is a continuous bijection with  $k$ -continuous inverse.*

This follows from Corollary 3 since the colimit as  $k$ -categories of  $k$ -groupoids is a  $k$ -groupoid (the analogous topological result is in [6]).

If  $\{G_\lambda\}_{\lambda \in \Lambda}$  is a family of  $k$ -groups, their free  $k$ -product is the coproduct  $G = \coprod_{\lambda} G_\lambda$  in the category of  $k$ -groups. This  $k$ -group can be taken as the universal group  $U(\bar{G})$  of the  $k$ -groupoid  $\bar{G}$ , which is the disjoint union  $\coprod_{\lambda} G_\lambda$  of the  $k$ -groups  $G_\lambda$ .

COROLLARY 6. *If  $\{G_\lambda\}_{\lambda \in \Lambda}$  is a family of  $k$ -groups, then for any  $k$ -group  $K$  the natural map*

$$\Phi: M(\coprod_{\lambda} G_\lambda, K) \rightarrow \prod_{\lambda \in \Lambda} M(G_\lambda, K)$$

*is a continuous bijection with  $k$ -continuous inverse.*

PROOF. This follows from Corollary 5 since the natural map

$$M\left(\prod_{\lambda} G_{\lambda}, K\right) \rightarrow \prod_{\lambda} M\left(G_{\lambda}, K\right)$$

is easily proved to be a homeomorphism.

Suppose now that  $X$  is a pointed space. The Graev free  $k$ -group on  $X$  consists of a  $k$ -group  $FGK(X)$  and a pointed map  $i: X \rightarrow FGK(X)$  in  $\mathcal{K}$  such that  $i$  is universal for pointed maps in  $\mathcal{K}$  from  $X$  to  $k$ -groups. It can be shown that if  $e$  is the base point of  $X$ ,  $i$  may be taken as the composite of the injection  $j: x \mapsto (x, e)$  of  $X$  into the tree topological groupoid  $X \times X$  with the universal morphism  $X \times X \rightarrow U(X \times X)$  (cf. [6]).

COROLLARY 7. For any space  $X$  and  $k$ -group  $K$  the map

$$i^*: M(FGK(X), K) \rightarrow \mathcal{K}_*(X, K)$$

into the space of pointed maps  $X \rightarrow K$ , with  $i^*$  induced by  $i: X \rightarrow FGK(X)$ , is a continuous bijection with  $k$ -continuous inverse.

PROOF. This follows from Corollary 5 once we have proved that

$$j^*: M(X \times X, K) \rightarrow \mathcal{K}_*(X, K)$$

is a continuous bijection with  $k$ -continuous inverse. Now  $j^*$  has inverse  $\psi$  where  $\psi(f): X \times X \rightarrow K$  is the morphism  $(x, y) \mapsto f(x)f(y)^{-1}$ . So  $j^*$  is a bijection, and is clearly continuous. The  $k$ -continuity of  $\psi$  comes from regarding it as the composite

$$\mathcal{K}_*(X, K) \xrightarrow{f \mapsto f \times f} \mathcal{K}_*(X \times X, K \times K) \xrightarrow{m'_*} \mathcal{K}_*(X \times X, K)$$

where  $m'_*$  is induced by the map  $m': (a, b) \mapsto ab^{-1}$  of  $K \times K$  into  $K$ .

The Graev free abelian  $k$ -group on a pointed space  $X$  consists of an abelian  $k$ -group  $AGK(X)$  and pointed map  $j: X \rightarrow AGK(X)$  in  $\mathcal{K}$  such that  $j$  is universal for pointed maps in  $\mathcal{K}$  from  $X$  to abelian  $k$ -groups. The map  $p: FGK(X) \rightarrow AGK(X)$  is a  $k$ -identification map.

COROLLARY 8. Let  $X, Y$  be paracompact, Hausdorff pointed spaces such that  $AGK(X), AGK(Y)$  are isomorphic  $k$ -groups. Then, for any abelian group  $\pi$  and  $n \geq 0$ , the Čech reduced cohomology groups

$$H^n(X; \pi), \quad H^n(Y; \pi)$$

are isomorphic.

PROOF. Let  $K$  be an abelian  $k$ -group. The induced map

$$p^*: M(AGK(X), K) \rightarrow M(FGK(X), K)$$

is a  $k$ -homeomorphism (since  $p$  is a  $k$ -identification map). So by Corollary 7 a  $k$ -isomorphism  $AGK(X) \rightarrow AGK(Y)$  induces a  $k$ -isomorphism

$$\alpha: \mathcal{K}_*(X, K) \rightarrow \mathcal{K}_*(Y, K),$$

and hence a bijection of pointed homotopy classes  $[X, K] \rightarrow [Y, K]$ .

Let  $K$  be the Eilenberg-MacLane  $k$ -group  $K(\pi, n)$ . By [8] (cf. also [1] chapter 6) the abelian group  $[X, K]$  is isomorphic to  $H^n(X; \pi)$ , and the result follows.

EXAMPLE. Let  $X$  be the «Čech circle» - that is,  $X = A \cup B \cup C$ , where

$$A = \{ (x, \sin \pi/x) \mid 0 < x \leq 1 \}, \quad B = \{ 0 \} \times \{ -1, 1 \}$$

and  $C$  is an arc in  $R^2 \setminus (A \cup B)$  joining  $(0, 0)$  to  $(1, 0)$ . Then  $H^1(X; Z)$  is isomorphic to  $Z$ , and it follows that  $AGK(X)$  is not  $k$ -isomorphic to  $AGK([0, 1])$ . Similarly  $AGK(S^1)$  is not  $k$ -isomorphic to  $AGK([0, 1])$ . However this method does not distinguish between  $AGK(X)$  and  $AGK(S^1)$  and we do not know if these are  $k$ -isomorphic.

### 3. THE UNIVERSAL TOPOLOGICAL GROUP OF A HAUSDORFF $k_\omega$ -GROUP.

In this section we consider only topological groupoids and topological groups. Recall also that a  $k_\omega$ -space  $X$  is one which has the weak topology with respect to some increasing sequence  $\{X_n\}$  of compact subspaces with union  $X$ . A Hausdorff  $k_\omega$ -space  $X$  has the property that any  $k$ -continuous map  $X \rightarrow Y$  is continuous.

A topological groupoid which as a topological space is a  $k_\omega$ -space is called a  $k_\omega$ -groupoid. It is proved in [5] that if  $G$  is a Hausdorff  $k_\omega$ -groupoid, then its universal topological group  $U(G)$  is a Hausdorff  $k_\omega$ -group. We shall use the explicit description of the topology of  $U(G)$  given in [5] to prove:



THEOREM 5. *Let  $G$  be a Hausdorff  $k_\omega$ -groupoid and  $K$  a topological group. Then the map*

$$i^*: M(U(G), K) \rightarrow M(G, K)$$

*induced by the universal morphism  $i: G \rightarrow U(G)$  is a homeomorphism.*

PROOF. It is clear that  $i^*$  is a bijection and, as an induced map, is continuous, and we need only prove the continuity of  $(i^*)^{-1}$ .

Let  $f: U(G) \rightarrow K$  be a morphism and let

$$g = fi: G \rightarrow K.$$

Let  $W(A, U)$  be a sub-basic open neighbourhood of  $f$ , so that  $A$  is compact in  $U(G)$  and  $U$  is open in  $K$ .

Since  $G$  is a  $k_\omega$ -groupoid, it has the weak topology with respect to an increasing sequence  $\{G_n\}$  of compact subspaces. For each pair of integers  $m, n \geq 0$  there is a continuous map  $p: (G_n)^m \rightarrow U(G)$  sending each  $m$ -tuple of elements of  $G_n$  to its reduced form in  $U(G)$  ([5], page 432). The methods of [5] also show that the compact set  $A$  is contained in some  $p(G_n)^m$ , and the definition of multiplication in  $U(G)$  is such that

$$p(x_1, \dots, x_m) = i(x_1) \dots i(x_m) \text{ for } x_j \in G_n.$$

Setting  $V = f^{-1}(U)$ , put

$$N = p^{-1}(V \cap p(G_n)^m) \text{ and } C = p^{-1}(A),$$

so that  $N$  is an open set. Since  $C \subset N \subset (G_n)^m$  and  $(G_n)^m$  is compact Hausdorff, each  $x$  in  $C$  has a compact neighbourhood

$$C^x = C_1^x \times C_2^x \times \dots \times C_m^x$$

contained in  $N$ . Since  $C$  is closed and hence compact in  $(G_n)^m$  there is a finite set  $F$  in  $C$  such that  $C \subset \bigcup_{x \in F} C^x$ . Now  $g = fi: G \rightarrow K$  and we have

$$\begin{aligned} g(C_1^x) \dots g(C_m^x) &= fi(C_1^x) \dots fi(C_m^x) = fp(C^x) \\ &\subset fp(N) \subset f(V) \subset U. \end{aligned}$$

Since  $U$  is open and each  $g(C_j^x)$  is compact, there are open sets  $U_1^x, \dots, U_m^x$  in  $K$  such that

$$g(C_j^x) \subset U_j^x \quad \text{and} \quad U_1^x \dots U_m^x \subset U.$$

Let  $W$  be the intersection of the sub-basic open sets  $W(C_j^x, U_j^x)$ , for  $j = 1, \dots, m$ , and  $x \in F$ . Then  $g \in W$ , and we prove that

$$(i^*)^{-1}(W) \subset W(A, U).$$

Let  $h \in W$ , and set  $h^* = (i^*)^{-1}(h)$ : we must prove  $h^*(A) \subset U$ . If we take  $y \in A$ , then  $y = p(c)$  for some  $c \in C$ , and then  $c \in C^x$  for an  $x \in F$ , so we have

$$\begin{aligned} h^*(y) &= h^*p(c) \in h^*p(C^x) = h^*p(C_1^x \times \dots \times C_m^x) = \\ &= h^*(i(C_1^x) \dots i(C_m^x)) = h(C_1^x) \dots h(C_m^x) \subset U_1^x \dots U_m^x \subset U. \end{aligned}$$

This completes the proof of the continuity of  $(i^*)^{-1}$ .

From Theorem 5 we can deduce results on countable free products of  $k_\omega$ -groups, and on Graev free topological groups on  $k_\omega$ -spaces [7], as in Corollaries 6 and 7 to Theorem 4. Rather than detail these, we shall pursue a different line of applications, namely to abelian topological groups.

**COROLLARY 1.** *If  $\{G_\lambda\}_{\lambda \in \Lambda}$  is a countable collection of abelian, Hausdorff  $k_\omega$ -groups, and  $G = \sum_\lambda G_\lambda$  is their sum in the category of topological abelian groups, then  $G$  is a Hausdorff  $k_\omega$ -group, and for any abelian topological group  $K$  the natural map*

$$\Phi: M(G, K) \rightarrow \prod_\lambda M(G_\lambda, K)$$

*is a topological isomorphism.*

**PROOF.** Clearly  $G$  is the quotient of  $G' = \sum_\lambda G_\lambda$  by its commutator subgroup and so is a Hausdorff  $k_\omega$ -space. But any quotient map of Hausdorff  $k_\omega$ -spaces is compact covering, and so the induced map

$$M(G, K) \rightarrow M(G', K)$$

is a homeomorphism, by the footnote to Proposition 3.5 of [4].

**COROLLARY 2.** *The dual group of a countable sum of abelian Hausdorff  $k_\omega$ -groups is naturally isomorphic to the product of their duals.*

This is the case  $K = S^1$  of Corollary 1. Similar results to this for countable sums of locally compact groups are due to Kaplan [9] (see also [11]).

Our remaining results are on free abelian topological groups. There are two such free groups ([14] and [7]). The Markov free abelian topological group is a continuous map  $i: X \rightarrow AM(X)$  into a topological abelian group such that  $i$  is universal for maps of  $X$  into topological abelian groups. The Graev free abelian topological group is a pointed continuous map  $j: X \rightarrow AG(X)$  universal for pointed maps into topological abelian groups. If  $X$  is a Hausdorff  $k_\omega$ -space, then so also are  $AM(X)$  and  $AG(X)$  [12].

If  $X, Y$  are spaces, then  $C(X, Y)$  will denote the space of continuous functions  $X \rightarrow Y$  with the compact open topology. If  $X, Y$  are pointed spaces, then  $C_*(X, Y)$  is the subspace of  $C(X, Y)$  of pointed maps.

COROLLARY 3. *If  $X$  is a pointed Hausdorff  $k_\omega$ -space, and  $K$  is an abelian topological group, then the maps*

$$i^*: M(AM(X), K) \rightarrow C(X, K), \quad j^*: M(AG(X), K) \rightarrow C_*(X, K)$$

*are homeomorphisms.*

PROOF. The proof is similar to that of Corollary 2, since the maps

$$FM(X) \rightarrow AM(X) \rightarrow AG(X)$$

are quotients by closed subgroups, and  $FM(X)$  is the universal topological group  $U(X \times 2^1)$ , where  $2^1$  is the tree groupoid on the discrete space  $\{0, 1\}$ .

EXAMPLE. Corollary 3 for  $i: X \rightarrow AM(X)$  suggests a class of counterexamples to Theorem 11 of [15], which states that under mild conditions on the spaces  $X, Y$  and the topological abelian group  $K$ , an algebraic isomorphism between  $C(X, K)$  and  $C(Y, K)$  which preserves constant functions induces a homeomorphism between  $X$  and  $Y$ . Let  $\epsilon_X: AM(X) \rightarrow Z$  (with  $Z$  the discrete group of integers) be the homomorphism determined

by the constant map  $X \rightarrow Z$  with value  $1$ . Let  $\Phi: AM(X) \rightarrow AM(Y)$  be a topological isomorphism such that  $\epsilon_Y \Phi = \epsilon_X$ . Then the composite

$$C(Y, K) \xrightarrow{(i^*)^{-1}} M(AM(Y), K) \xrightarrow{\Phi^*} M(AM(X), K) \xrightarrow{i^*} C(X, K)$$

is an algebraic isomorphism preserving constant functions, and is indeed a homeomorphism, by Corollary 3, if  $X$  and  $Y$  are  $k_\omega$ -spaces. However, it is easy to give examples of non-homeomorphic  $X, Y$  for which such a  $\Phi$  exists (cf. [7], Section 5) and  $X, Y$  and  $K$  satisfy the conditions of [15] (see the erratum to [15]).

School of Mathematics and Computer Science,  
University College of North Wales,  
BANGOR LL57 2UW, Gwynedd, U.K.

and

Department of Mathematics,  
University of Queensland,  
ST. LUCIA, Queensland 4067,  
AUSTRALIA

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