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SOME RELATIONS BETWEEN SHAPE CONSTRUCTIONS

by Friedrich W. BAUER

0. INTRODUCTION.

The intention of this paper is to display the elementary properties of the shape construction $\bar{K}$ of [1, 2] in more detail than this was done in these original papers (where some of the proofs were either omitted or only sketched since we put all emphasis on more involved theorems). Simultaneously we present as a new application of these results a proof of the following assertion (Theorem 6.1):

Two compact metric spaces X and Y are homotopy equivalent in the shape category $\bar{K}$ if and only if they are equivalent in the Borsuk-Mardesič shape category $\bar{H}$.

This result does not imply that both categories ($\bar{K}_h$ and $\bar{H}$) are equivalent. In [2] we provided an example of two shape mappings

$$\bar{f}_i \in \bar{K}(X, Y), \quad i = 0, 1 \quad (X, Y \text{ compact metric})$$

such that $\bar{f}_0 \neq \bar{f}_1$ in $\bar{K}_h$ but with $\eta[\bar{f}_0] = [\bar{f}_1]$, where $\eta : \bar{K}_h \to \bar{H}$ is the canonical functor (which forgets the additional structure which $\bar{K}_h$ carries in comparison to $\bar{H}$). This example is essentially due to N. Steenrod.

Judging on the basis of this counterexample, the previous result comes rather unexpectedly because at first glance the equivalence of two spaces in $\bar{H}$, the Borsuk-Mardesič category, seems to be a much weaker property compared with the equivalence of two spaces in $\bar{K}_h$: Let

$$\alpha \in \bar{H}(X, Y), \quad \beta \in \bar{H}(Y, X)$$

be two Borsuk-Mardesič shape mappings with $\beta\alpha = 1_X$, $\alpha\beta = 1_Y$, then one is supposed to find mappings $\bar{a} \in \bar{K}(X, Y)$, $\bar{b} \in \bar{K}(Y, X)$ such that

$$\eta\bar{a} = \alpha, \quad \eta\bar{b} = \beta \quad \text{and} \quad \bar{b}\bar{a} \approx 1_X, \quad \bar{a}\bar{b} \approx 1_Y \quad \text{in } \bar{K}.$$

That this ultimately can be managed is a consequence of the very pleasant properties of compact metric spaces.

On the other hand there is in fact a good geometric argument in favour of this result:

According to a famous theorem by T.A. Chapman [4] two compact metric spaces X and Y embedded in the pseudo-interior of the Hilbert cube Q are equivalent in $\bar{\mathrm{H}}$ if and only if the complementary spaces $Q-X$, $Q-Y$ are homeomorphic.

This fact signalizes that the classification of metric compacta by means of $\bar{\mathrm{H}}$ is optimal from a geometric point of view. Since the category $\bar{\mathrm{K}}$ allows much more geometric constructions (as for example a singular complex $\bar{S}(X)$, fibrations and cofibrations) compared to $\bar{\mathrm{H}}$, it would be highly unlikely to suppose that for $\bar{\mathrm{K}}$ T.A. Chapman's Theorem is not true.

For spaces which are not compact metric the situation changes completely because we have to use for a proof of Theorem 6.1 the entire machinery which is developed in Section 4.

This theorem does not really affect the conviction that the category $\bar{\mathrm{K}}$ (defined in Section 1) is superior to $\bar{\mathrm{H}}$ (defined in Section 5): Although one must work much more to give detailed proofs of elementary, though well-expected facts, one can go much further with $\bar{\mathrm{H}}$ (in comparison with $\bar{\mathrm{K}}$) as the results about $\bar{\mathrm{K}}$ show [1,2]. One can treat arbitrary topological spaces in $\bar{\mathrm{K}}$ as one treated CW-spaces in ordinary homotopy theory, because, due to the richer structure of $\bar{\mathrm{K}}$, one can simply imitate most constructions of ordinary homotopy theory.

The category $\bar{\mathrm{K}}$ belongs to the class of «strong» shape theories, where one starts with a category K of topological spaces, equipped with a concept of homotopy, homotopies between homotopies and so on; in other words, one recognizes K as a n-category for arbitrary n. This fact imposes on the shape mappings (cf. Section 1) very restrictive coherence conditions. As an example we give a proof of Theorem 2.4 which becomes rather long whenever one has the ambition to display all the details. For $\bar{\mathrm{H}}$ there is an analogous assertion which is in fact a triviality. T. Porter is developing other models of strong shape theories for topological spaces [8].

The Borsuk-Mardesič category $\overline{H}$ provides an example of a «weak» shape category where one first forms the homotopy category thereby turning all higher n-morphisms ($n > 1$) into identities and performing the shape construction afterwards.

In section 3 we give an axiomatic characterization of $\overline{K}$ which resembles the corresponding theorem in [6] for $\overline{H}$.

Although we could totally avoid the concept of a n-category by simply refering to well-known properties of, for example, homotopies as 2-morphisms, we found it helpful to have some notion of a n-category at hand, which is described in the Appendix. Furthermore, there are collected some facts about point set topology, which are elementary, but nevertheless necessary for the course of our consideration.

1. THE SHAPE CONSTRUCTION.

Let K be a suitable category of topological spaces as for example K = Top or Top_0 and $P \subset K$ a full subcategory. We are going to construct a *shape category* $\overline{K}$ which has the same objects as K but with different mappings. This construction depends on the subcategory P .

For this purpose we define for any object $X \in K$ the following category P_X :

The objects in P_X are morphisms $g: X \to P \in P$. A morphism

$$(r, \omega): g_1 \to g_2, \quad g_i: X \to P_i ,$$

is a pair, where $r: P_1 \to P_2$ is a mapping and $\omega: rg_1 \approx g_2$ a prescribed homotopy. Composition is defined as follows:

$$(1) \qquad (r_2, \omega_2)(r_1, \omega_1) = (r_2 r_1, \omega_2 \circ r_2 \omega_1).$$

Concerning homotopies and compositions of homotopies we refer to Section 7. On the basis of Section 7 it is immediate that this composition is associative and equipped with a unit, hence gives a category P_X .

Furthermore P_X has the structure of a 2-category: We define a 2-morphism

$$(\nu,\xi):(r_1,\omega_1)\approx(r_2,\omega_2): g_1\to g_2$$

where $\nu: r_1\approx r_2$ is a homotopy and $\xi:\omega_2\circ\nu g_1\approx\omega_1$ a homotopy of homotopies. Let

$$(\nu_1,\xi_1):(r_1,\omega_1)\approx(r_2,\omega_2),\quad (\nu_2,\xi_2):(r_2,\omega_2)\approx(r_3,\omega_3)$$

be two morphisms for fixed g_1, g_2, then we set

(2) $$(\nu_2,\xi_2)(\nu_1,\xi_1)=(\nu_2\nu_1,\ \xi_1\circ\xi_2(\nu_1 g_1)).$$

1.1. These definitions provide us with a 2-category P_X. The verification of all necessary relations which hold in a 2-category is technical and left to the reader.

1.2. All 2-morphisms in P_X are isomorphisms.

PROOF. Let $(\nu,\xi):(r_1,\omega_1)\approx(r_2,\omega_2)$ be a 2-morphism, then

$$(\nu^{-1},\ \xi^{-1}(\nu^{-1}g_1))$$

is its inverse. Here ν^{-1} for example is the inverse homotopy for ν as it was defined in Section 7.

REMARK. Notwithstanding the fact that every category carries the structure of a (trivial) 3-, 4-, ..., category, we can use the n-category structure of K (through homotopies of homotopies of homotopies ...) to equip P_X with the structure of an n-category for arbitrary n. We will briefly return to this point a little later.

Now we are ready to define the morphisms $\bar{f}: X\to Y$ in $\bar{K}$: These are 2-functors $\bar{f}: P_Y\to P_X$ such that the following three conditions are fulfilled:

a) $g\in P_Y,\ g: Y\to P \implies \bar{f}(g): X\to P$.

b) $(r,\omega)\in P_Y(g_1,g_2) \implies \bar{f}(r,\omega)=(r,\omega_1)$ for a suitable homotopy ω_1.

c) $(\nu,\xi):(r_1,\omega_1)\approx(r_2,\omega_2) \implies \bar{f}(\nu,\xi)=(\nu,\xi_1)$ for a suitable 3-morphism ξ_1 in K.

The last two conditions are very closely related. Formally condition b implies a.

The concept of a 2-functor in our sense is defined in the Appendix: A 2-functor is an ordinary functor which for fixed $g_1, g_2 \in \mathrm{P}_Y$ maps

$$\mathrm{P}_Y(g_1, g_2) \text{ into } \mathrm{P}_X(\bar{f}(g_1), \bar{f}(g_2))$$

functorially. Composition of functors defines composition of morphisms in $\bar{\mathrm{K}}$.

The category $\bar{\mathrm{K}}$ will be equipped with the structure of a 2-category or alternatively with the structure of a homotopy (which amounts to the same). We proceed analogously as in ordinary topology:

Two morphisms $\bar{f}_0, \bar{f}_1 \in \bar{\mathrm{K}}(X, Y)$ *are homotopic* whenever there exists a $\bar{F} \in \bar{\mathrm{K}}(X \times I_n, Y)$ such that

$$\bar{F} i_0 = \bar{f}_0, \quad \bar{F} i_n = \bar{f}_1,$$

where $i_t: X \to X \times I_n$ are the shape mappings defined by

$$(3) \qquad i_t(g)(x) = g(x, t), \quad (x, t) \in X \times I_n .$$

Concerning the definition of I_n and all details about homotopies, see Section 7. It is immediate that $i_t \in \bar{\mathrm{K}}(X, X \times I_n)$. Recall that we agreed to denote I_1 by I.

The map $\bar{F} \in \bar{\mathrm{K}}(X \times I_n, Y)$ is called a homotopy between $\bar{f}_0$ and $\bar{f}_1$. Correspondingly we define elementary homotopies. Occasionally we write

$$(4) \qquad \bar{F}: \bar{f}_0 \approx \bar{f}_1 .$$

One could equally well propose a different definition for homotopies in $\bar{\mathrm{K}}$ (as we did in [1]). A homotopy between two maps $\bar{f}_0, \bar{f}_1 \in \bar{\mathrm{K}}(X, Y)$ is defined as a family

$$\bar{\nu}(g): \bar{f}_0(g) \approx \bar{f}_1(g), \quad g \in \mathrm{P}_Y ,$$

of homotopies in K. In order to formulate the necessary compatibility conditions, we need for example 3-morphisms

$$\bar{\nu}(r, \omega) = \mu : \omega_1 \circ r \bar{\nu}(g_1) \approx \bar{\nu}(g_2) \circ \omega_0$$

where

$$(r, \omega) \in \mathrm{P}_Y(g_1, g_2) \text{ and } \bar{f}_i(r, \omega) = (r, \omega_i).$$

One could easily write down all details obtaining thereby a definition of a homotopy $\bar{\nu}: \bar{f}_0 \approx \bar{f}_1$ which turns out to be equivalent to the previous one. The explicit details are rather lengthy and left to the reader. Moreover the next section contains an example of such an explicit construction of a homotopy $\bar{F}: \bar{f}_0 \approx \bar{f}_1$ (proof of Theorem 2.4) originating from such a family of $\bar{\nu}(g)$, $\bar{\nu}(r, \omega), \dots$

In our definition of $\bar{\mathrm{K}}$ we have used the category P_X as a 2-category. One can however equip P_X with the structure of a 3-, 4-, etc... category, because the category K is in fact a n-category for arbitrary n. We have confined ourselves with a 2-category P_X (and consequently with a 2-functor $\bar{f}: \mathrm{P}_Y \to \mathrm{P}_X$ as a morphism in $\bar{\mathrm{K}}$) because this turned out to be sufficient for our geometric purposes (cf. [1, 2]). However we can prove that every such $\bar{f}$ can be given the structure of a 3-, 4-, etc... functor with respect to 3-, 4-, etc... structures of P_X, P_Y which can be defined analogously to the 2-structure of P_X, P_Y in a natural way. In order to accomplish that we will prove the following fact. Let P be the category of spaces having the homotopy type of CW-spaces.

1.3. PROPOSITION. *Let* $\Lambda: \mathrm{P}_Y \to \mathrm{P}_X$ *be a 1-functor such that*

$\Lambda 1)$ $g: Y \to P$ *in* $\mathrm{P}_Y \implies \Lambda(g): X \to P$.

$\Lambda 2)$ $\Lambda(r, \omega) \epsilon \mathrm{P}_Y(g_1, g_2) \implies \Lambda(r, \omega) = (r, \omega_1)$

hold, then there always exists a $\bar{f} \epsilon \bar{\mathrm{K}}(X, Y)$ *such that* $\bar{f} = \Lambda$ *as a 1-functor.*

PROOF. Let $(\nu, \xi): (r_0, \omega_0) \approx (r_1, \omega_1)$ be a 2-morphism in P_Y :

$$\xi: \omega_1 \circ \nu g_1 \approx \omega_0, \quad g_i: Y \to P_i, \quad (r_i, \omega_i): g_1 \to g_2.$$

We can assume without loss of generality that ν is an elementary homotopy which can therefore be represented by a mapping $R: P_1 \times I \to P_2$. Thus ξ can be interpreted as a homotopy (denoted by the same letter)

$$\xi: R(g_1 \times 1) \approx g_2 \rho_Y,$$

where $\rho = \rho_Y : Y \times I \to I$ denotes the projection. Now we are trying to obtain a homotopy between homotopies (hence a 3-morphism)

$$\xi^*: \omega_1' \circ \nu \Lambda(g_1) \approx \omega_0' \quad \text{where} \quad \Lambda(r_i, \omega_i) = (r_i, \omega_i').$$

To this end consider the subspace $q: L \subset (P_1 \times I)^I$ of all those

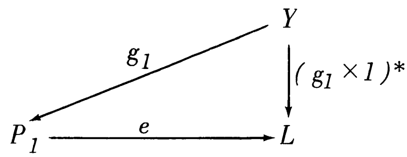
$$\sigma \epsilon (P_1 \times I)^I \quad \text{with} \quad \sigma(t) = (\rho\sigma(t))t)$$

(ρ = projection). The mapping $(g_1 \times 1)': Y \to (P_1 \times I)^I$ factorizes over L, thus we have a

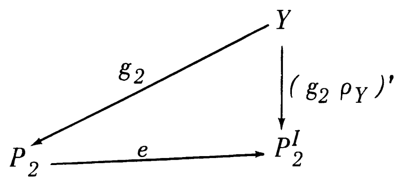
$$(g_1 \times 1)^*: Y \to L \quad \text{such that} \quad q(g_1 \times 1)^* = (g_1 \times 1)'.$$

We will adopt this $*$-notation for all kinds of maps into $(P_1 \times I)^I$ which happen to factorize over L.

We have the following commutative diagrams



with $e(x)(t) = (x, t)$, and



this time with $e(x)(t) = x$. Application of Λ yield homotopies

$$\eta_1 : (\Lambda(g_1) \times 1)^* = e\,\Lambda(g_1) \approx \Lambda(g_1 \times 1)^*,$$

$$\eta_2 : \Lambda(g_2)\,\rho_X = e\,\Lambda(g_2) \approx \Lambda(g_2 \rho_Y)'.$$

Moreover we have a homotopy

$$\zeta : S \circ \Lambda(g_1 \times 1)^* \approx \Lambda(g_2 \rho_Y)', \text{ where } S = R^I q.$$

Composing these homotopies yields

$$\xi' : S \circ (\Lambda(g_1) \times 1)^* \approx \Lambda(g_2)\,\rho_X .$$

Here f' denotes throughout the adjoint of a given map f. Let

$$\alpha_s : L \to P_1, \quad \beta_s : P_2^I \to P_2, \quad 0 \leqslant s \leqslant 1,$$

be the mappings

$$\alpha_s(\sigma) = \rho_{P_1}\sigma(s), \quad \beta_s(\sigma) = \sigma(s),$$

then we have

$$\alpha_s(g_1 \times 1)^* = g_1, \quad \alpha_s(\Lambda(g_1) \times 1)^* = \Lambda(g_1),$$

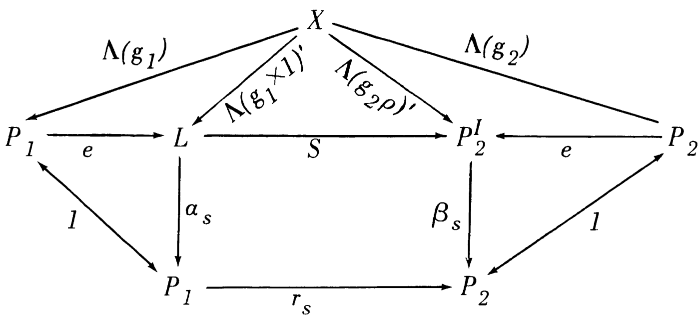
resp.

$$\beta_s(g_2 \rho_Y)' = g_2, \quad \beta_s(\Lambda(g_2)\rho_X)' = \Lambda(g_2).$$

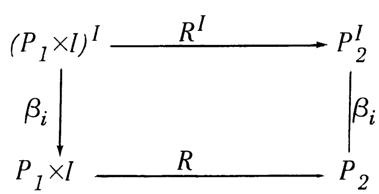
Furthermore one has

$$\alpha_s e = 1, \quad \beta_s e = 1, \quad \text{as well as } r_s \alpha_s = \beta_s S,$$

where $r_s(\ ) = R(\ ,s)$. So we get the following diagram:



The upper three triangles are homotopy commutative with the resp. homotopies η_1, ζ, η_2. The remaining triangles and the lower square are strictly commutative. Because Λ behaves functorially, we conclude that for $s = 0$ ($s = 1$) we get ω_0' (resp. ω_1'). Thus taking the adjoint of ξ', we receive the required ξ^*. The fact that this so constructed ξ^* fulfills all other requirements on a homotopy between homotopies (laid down in the Appendix) can be easily deduced from the following commutative square



(with $\beta_i(\sigma) = \sigma(i)$, $i = 0, 1$) which has to be used in the same way as the corresponding square in (5) by a completely analogous reasoning. This completes the proof of 1.3 because we can set

$$\bar{f}(\nu, \xi) = (\nu, \xi^*).$$

The proof of the functoriality of $\bar{f}$ is technical and left to the reader.

We can iterate this process and prove by the same method that for example every $\bar{f} \epsilon \bar{K}(X, Y)$ can be endowed with the structure of a 3-, 4-, etc... functor $\bar{f}: P_Y \to P_X$ provided P_X, P_Y inherit their 3-, 4-, etc... structure from K in an obvious way. This will give us the right to talk for example about $\bar{f}(\xi, \mu)$ for a 3-morphism $(\xi, \mu) \epsilon P_Y$ which can be defined analogously to the 2-morphisms. Now μ denotes of course a 4-morphism in K.

The category $\bar{K}$ is in our theory only a 2-category, but there is no reason which keeps us from giving $\bar{K}$ the structure of an n-category for any n, simply by taking homotopies of homotopies as 3-morphisms as we did in K (see Appendix) and iterating this process.

This observation makes sense regardless whether we use the 2-category structure of P_X which stems from the 3-category structure of K, or any higher n-category structure as well.

REMARK. It should be observed that the construction of f starting with a prescribed Λ in 1.3 is of course not unique: It may very well happen that there is already a 2-functor $\bar{f}_1: P_Y \to P_X$ at hand, such that $\Lambda = \bar{f}_1$ as a 1-functor, which differs from the $\bar{f}$ whose existence is established in 1.3.

2. THE FUNCTOR $h: K \to \bar{K}$.

There exists a 2-functor $h: K \to \bar{K}$ which is the identity on the objects. On the 1-morphisms $f \epsilon K(X, Y)$ we define:

(1)
$$h(f)(g) = gf, \quad g \epsilon P_Y,$$
$$h(f)(r, \omega) = (r, \omega f), \quad (r, \omega) \epsilon P_Y(g_1, g_2),$$
$$h(f)(\nu, \xi) = (\nu, \xi f)$$
where $(\nu, \xi): (r_1, \omega_1) \approx (r_2, \omega_2)$ is a 2-morphism in P_Y and $\xi: \omega_2 \circ \nu\, g_1 \approx \omega_1$ a 3-morphism in K.

Let $\omega: f_0 \approx f_1$ be a 2-morphism in K (i. e. a homotopy) then we set:

$$h(\omega)(g) = g\omega : h(f_0)(g) \approx h(f_1)(g). \quad (2)$$

This process can easily be iterated, yielding a n-morphism

$$h(\mu_n): h(\zeta_0) \approx h(\zeta_1) \text{ in } \bar{K}$$

for any n-morphism $\mu_n : \zeta_0 \approx \zeta_1$, ζ_i an $(n\text{-}1)$-morphism in K. Concerning these concepts recall our explanations at the end of Section 1.

We can easily prove:

2.1. The assignment h is an n-functor $h: \mathrm{K} \to \bar{\mathrm{K}}$ for arbitrary n.

There exists some kind of inverse for h : Let $\bar{f} \in \bar{\mathrm{K}}(X, Y)$, $Y \in \mathrm{P}$ be given, then we define

$$h'(\bar{f}) = \bar{f}(1) \in \mathrm{K}(X, Y), \quad 1 = 1_Y : Y \to Y.$$

Analogously we define h' for any 2- and higher morphism in $\bar{\mathrm{K}}$ (in the sense of Section 1) whenever this makes sense.

2.2. The assignment h' becomes an n-functor for any n, whenever this makes sense. More precisely:

a) Let $\bar{f}_1 \in \bar{\mathrm{K}}(X, Y)$, $\bar{f}_2 \in \bar{\mathrm{K}}(Y, Z)$, $h'(\bar{f}_1)$, $h'(\bar{f}_2)$ be defined, then

$$h'(\bar{f}_2 \bar{f}_1) = h'(\bar{f}_2)\, h'(\bar{f}_1);$$

b) For $X \in \mathrm{P}$ we have $h'(1_X) = 1_X$.

c) Let analogously $\bar{\zeta}_0 \overset{\bar{\mu}_1}{\approx} \bar{\zeta}_1 \overset{\bar{\mu}_2}{\approx} \bar{\zeta}_2$ be any two n-morphisms such that $h'(\bar{\mu}_1)$, $h'(\bar{\mu}_2)$ are defined, then we have

$$h'(\bar{\mu}_2 \bar{\mu}_1) = h'(\bar{\mu}_2)\, h'(\bar{\mu}_1),$$

resp. $h'(1) = 1$ for the corresponding n-identities.

2.3. a) Let $f \in \mathrm{K}(X, Y)$, $Y \in \mathrm{P}$ be given, then we obtain $h'h(f) = f$, resp. for the homotopies and the higher n-morphisms.

b) Let $\bar{f}_1 \in \bar{\mathrm{K}}(X, Y)$, $\bar{f} \in \bar{\mathrm{K}}(Y, Z)$, $Z \in \mathrm{P}$, then one has

$$h'(h h'(\bar{f}) \bar{f}_1) = h'(\bar{f} \bar{f}_1), \tag{3}$$

correspondingly for $\bar{f}$ replaced by 2- and 3-morphisms in $\bar{\mathrm{K}}$.

PROOF. Ad a) We deduce

$$h'h(f) = h(f)(1) = f,$$

resp. for homotopies and higher n-morphisms.

Ad b) We have

$$h'(h h'(\bar{f}) \bar{f}_1) = (h h'(\bar{f})\, \bar{f}_1)(1_Z) = \bar{f}_1(h h'(\bar{f})(1_Z)) =$$
$$= \bar{f}_1(1_Z h'(\bar{f})) = \bar{f}_1(\bar{f}(1)) = h'(\bar{f} \bar{f}_1).$$

The proof for the higher n-morphisms is analogous.

A little more involved is the following fact:

2.4. THEOREM. *Let* P *be the category of spaces having the homotopy type of CW-spaces. There exists in* $\bar{\mathrm{K}}$ *a homotopy*

$$hh'(\bar{f}) \approx \bar{f}, \quad \bar{f} \in \bar{\mathrm{K}}(X, Y), \quad Y \in \mathrm{P}.$$

PROOF. We are going to construct a morphism $\bar{F} \in \bar{\mathrm{K}}(X \times I,\ Y)$ such that

$$\bar{F} i_0 = hh'(\bar{f}), \quad \bar{F} i_1 = \bar{f}.$$

Here $I = [0, 1]$ is the unit interval and i_t has to be understood as $h(i_t)$ ($i_t: X \to X \times I,\ t \in I$, the inclusion). This is evidently in accordance with Section 1 (3). For $g \in \mathrm{P}_Y$ we have a homotopy $\bar{\nu}(g)$ in $\bar{f}(g, 1) = (g, \bar{\nu}(g))$:

$$\bar{\nu}(g): g\bar{f}(1) = hh'(\bar{f})(g) \approx \bar{f}(g).$$

To this $\bar{\nu}(g)$ corresponds a map $F': X \times I_n \to P$ for a suitable n (cf. Section 7). We obtain a map $F: X \times I \to P$ by composing F' with a linear stretching $I \to I_n$ and define $\bar{F}(g) = F$. Observe that $\bar{\nu}(g)$ and F are related by a homotopy between homotopies (according to Lemma 7.2) which we are not going to mention in the future explicitly. One clearly has:

$$(\bar{F} i_0)(g) = hh'(\bar{f})(g), \quad (\bar{F} i_1)(g) = \bar{f}(g).$$

Let $(r, \omega): g_1 \to g_2$ be a map in P_Y, then we are supposed to define

$$\bar{F}(r, \omega): \bar{F}(g_1) \to \bar{F}(g_2), \quad \bar{F}(r, \omega) = (r, \tilde{\omega}), \quad \tilde{\omega}: r\bar{F}(g_1) \approx \bar{F}(g_2).$$

Thus $\tilde{\omega}$ is a homotopy between homotopies making the following diagram homotopy commutative:

$$(6) \qquad \begin{array}{ccc} rg_1\bar{f}(1) = rhh'(\bar{f})(g_1) & \xrightarrow{\ r\bar{F}(g_1)\ } & r\bar{f}(g_1) \\ \Big\downarrow{\scriptstyle \omega\bar{f}(1)} & & \Big\downarrow{\scriptstyle \omega'} \\ g_2 f(1) = hh'(\bar{f})(g_2) & \xrightarrow{\ \bar{F}(g_2)\ } & \bar{f}(g_2) \end{array}$$

where $\bar{f}(r, \omega) = (r, \omega')$. In order to accomplish this, we decompose

$$(7) \qquad (r, \omega) = (1, \omega)(r, 1),$$

and treat both factors separately.

1° Consider

$$(r,1): g_1 \to rg_1 \quad \text{and} \quad \bar f(r,1) = (r,\alpha),$$

as well as the diagram:

$$(8) \qquad \begin{array}{ccc} rg_1\bar f(1) = rhh'(\bar f)(g_1) & \xrightarrow{\;r\bar\nu(g_1)\;} & r\bar f(g_1) \\ \Vert & & \downarrow \alpha \\ rg_1\bar f(1) = hh'(\bar f)(rg_1) & \xrightarrow{\;\bar\nu(rg_1)\;} & \bar f(rg_1) \end{array}$$

We claim that (8) is strictly commutative: To this end consider the commutative diagram in P_Y:

$$(9) \qquad \begin{array}{ccc} 1 & \xrightarrow{\;(g_1,1)\;} & g_1 \\ & \searrow^{(rg_1,1)} & \downarrow (r,1) \\ & & rg_1 \end{array}$$

and apply $\bar f$:

$$\bar f(g_1,1) = (g_1,\bar\nu(g_1)), \quad \bar f(r,1) = (r,\alpha),$$
$$\bar f(rg_1,1) = (rg_1,\nu(rg_1)).$$

Since $\bar f$ is a functor, we have

$$\alpha \circ r\bar\nu(g_1) = \bar\nu(rg_1).$$

Now we know that $\bar F(\dots)$ and $\bar\nu(\dots)$ differ only by a homotopy between homotopies. Hence we obtain a 3-morphism

$$\zeta : \alpha \circ r\bar F(g_1) \approx \bar F(rg_1).$$

This settles the problems for morphisms of the form $(r,1)$.

2° Let $(1,\omega): rg_1 \to g_2$ and $\bar f(1,\omega) = (1,\beta)$; observe that $\beta\alpha = \omega'$. We have a morphism

$$(\omega,1): (rg_1,\omega) \approx (g_2,1) \quad \text{in } P_Y,$$

where $(rg_1,\omega), (g_2,1) \in P_Y(1_Y, g_2)$. After application of $\bar f$ we get

$$\bar f(\omega,1) = (\omega,\tilde\omega'): \bar f(rg_1,\omega) \approx \bar f(g_2,1).$$

We need a better insight into this situation:

$$(rg_1, \omega) = (1, \omega)(rg_1, 1),$$

$$\bar{f}(rg_1, \omega) = \bar{f}(1, \omega)\bar{f}(rg_1, 1) = (1, \beta(rg_1, \bar{\nu}(rg_1)),$$

$$\bar{f}(g_2, 1) = (g_2, \bar{\nu}(g_2)).$$

Hence we recognize $\tilde{\omega}'$ as a 3-morphism:

$$(10) \qquad \tilde{\omega}': \bar{\nu}(g_2) \circ \omega \bar{f}(1) \approx \beta \circ \bar{\nu}(rg_1).$$

Again we can replace $\bar{\nu}(\ldots)$ by $\bar{F}(\ldots)$ altering (10) by a homotopy between homotopies. This provides us with a

$$\tilde{\omega}: \bar{F}(g_2) \circ \omega \bar{f}(1) \approx \beta \circ \bar{F}(rg_1).$$

Thus we can define $\bar{F}(1, \omega) = (1, \tilde{\omega})$. Since the decomposition (7) is canonical, we have accomplished a functorial construction of

$$\bar{F}(r, \omega) = \bar{F}(1, \omega)\bar{F}(r, 1) = (r, \tilde{\omega} \circ \zeta).$$

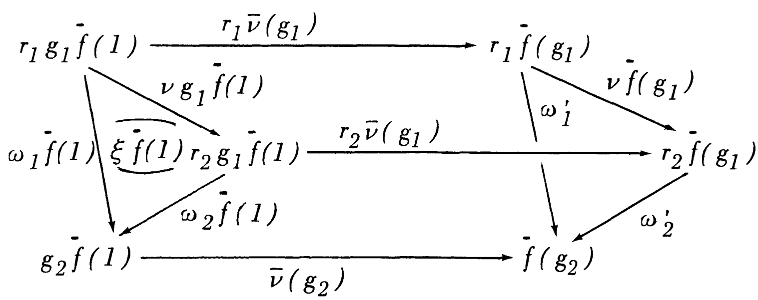
ATTENTION. The composition $\tilde{\omega} \circ \zeta$ has to be understood as composition of 2-morphisms, which is different from the composition of 3-morphisms. The reader is advised to draw the corresponding squares, which are rendered commutative by these 3-morphisms.

The next step in the construction of $\bar{F}$ is the establishment of a $\bar{F}(\nu, \xi)$ for given

$$(\nu, \xi): (r_1, \omega_1) \approx (r_2, \omega_2) \quad \text{where } (r_i, \omega_i): g_1 \to g_2, \; i = 1, 2.$$

Instead of (6) we deal with the following 3-dimensional diagram:

(11)



where the encircled $\xi \bar{f}(1)$ indicates a 3-morphism in the corresponding triangle. The opposite triangle represents the 3-morphism

$$\tilde{\xi} \text{ in } \bar{f}(\nu, \xi) = (\nu, \tilde{\xi}).$$

We clearly have:

$$\bar{f}(r_i, \omega_i) = (r_i, \omega_i'), \quad i = 1, 2,$$

while the remaining 3-morphisms, representing the three squares in (11), are obvious.

We are trying to fill (11) with a suitable 4-morphism μ which finally exhibits as a 3-morphism in $\bar{F}(\nu, \xi) = (\nu, \mu)$. This task is accomplished in complete analogy with the foregoing case by translating ξ into a mapping

$$(R, \Omega): g_1 \times 1_I \to g_2 \rho,$$

where $\rho: Y \times I \to Y$ is the projection and R, resp. Ω, correspond to the homotopy ν, resp. the homotopy between homotopies ξ. For the sake of simplicity we assume all homotopies to be elementary. We perform the construction of $\bar{F}(R, \Omega)$ for this case (i.e., for X, Y replaced by $X \times I$, $Y \times I$), by applying the ideas leading to Lemma 1.4 and Proposition 1.3. This provides us with a filling of (11). Details are now easy and left to the reader. This completes the proof of Theorem 2.4.

We denote by $\bar{\mathrm{K}}_{\mathrm{h}}$ the homotopy category of $\bar{\mathrm{K}}$ and by $\bar{\mathrm{P}}_{\mathrm{h}} \subset \bar{\mathrm{K}}_{\mathrm{h}}$ the full subcategory of $\bar{\mathrm{K}}_{\mathrm{h}}$ which is determined by P.

2.5. COROLLARY. *The functor h induces an isomorphism between* P_{h} *and* $\bar{\mathrm{P}}_{\mathrm{h}}$.

REMARK. The homotopy $hh' \approx 1$ whose existence is assured by 2.4 is natural whenever this makes sense (i.e., whenever the relevant compositions are defined).

3. A UNIVERSAL PROPERTY OF THE TRIPLE $(\bar{\mathrm{K}}, h, h')$.

We can use some of the properties which we developed in Sections 1, 2 for a characterization of the shape category.

To this end let $(\mathrm{L}, \hat{h}, \hat{h}')$ be a triple consisting of the items:

1° L is a category with topological spaces as objects.

2° $\hat{h}: \mathrm{K} \to \mathrm{L}$ is a 3-functor which is the identity on the objects.

3° $\hat{h}'$ is an assignment $\hat{h}': \mathrm{L}(X, Y) \to \mathrm{K}(X, Y)$ which is defined for any $Y \in \mathrm{P}$. We assume that $\hat{h}'$ is a 3-functor whenever this makes sense (i.e., in the sense of 2.2).

These three items are subject to the following conditions:

a) One has $\hat{h}'\hat{h}(f) = f$ for any $f \in \mathrm{K}(X, Y)$, $Y \in \mathrm{P}$ correspondingly for f replaced by 2- or 3-morphisms in K.

b) Let $l \in \mathrm{L}(X, Y)$, $Y \in \mathrm{P}$, $l_1 \in \mathrm{L}(X', X)$ be any morphisms in L, then one has

$$\hat{h}'(\hat{h}\hat{h}'(l)l_1) = \hat{h}'(ll_1),$$

correspondingly for l replaced by 2- or 3-morphisms in L.

Under these circumstances we can prove the following assertion:

3.1. THEOREM. *There exists a unique 2-functor* $\phi : \mathrm{L} \to \bar{\mathrm{K}}$ *with the following properties:*

1º ϕ *is the identity on the objects.*

2º $\phi\hat{h} = h$.

3º $h'\phi(l) = \hat{h}'(l)$, $l \in \mathrm{L}(X, Y)$, $Y \in \mathrm{P}$.

PROOF. Take $l \in \mathrm{L}(X, Y)$, $g \in \mathrm{P}_Y$, then define

$$\phi(l)(g) = \hat{h}'(\hat{h}(g)l).$$

For $(r, \omega): g_1 \to g_2$ in P_Y we establish a

$$\phi(l)(r, \omega): \phi(l)(g_1) \to \phi(l)(g_2)$$

in the following way: We have

$$r\phi(l)(g_1) = r\hat{h}'(\hat{h}(g_1)l) = \hat{h}'\hat{h}(r)\,\hat{h}'(h(g_1)l) =$$
$$= \hat{h}'(\hat{h}(rg_1)l) \underset{\omega'}{\approx} \hat{h}'(\hat{h}(g_2)l) = \phi(l)(g_2),$$

where $\omega' = \hat{h}'(\hat{h}(\omega)l)$. Hence we set

$$\phi(l)(r, \omega) = (r, \omega').$$

This definition is clearly functorial and makes $\phi(l): \mathrm{P}_Y \to \mathrm{P}_X$ into a functor. For 2-morphisms $(\nu, \xi): (r_1, \omega_1) \approx (r_2, \omega_2)$ we proceed analogously:

$$\omega_2 \circ \nu\phi(l)(g_1) = \omega_2 \circ \nu\hat{h}'(\hat{h}(g_1)l) = \hat{h}'(\hat{h}(\omega_2 \circ \nu g_1)l)$$
$$\underset{\xi'}{\approx} \hat{h}'(\hat{h}(\omega_1 l)) = \omega_1,$$

where we set $\xi' = \hat{h}'(\hat{h}(\xi)l)$. This gives a 2-functor $\phi(l): \mathrm{P}_Y \to \mathrm{P}_X$ and consequently a morphism $\phi(l) \in \bar{\mathrm{K}}(X, Y)$.

Now we have to check the functoriality of ϕ : Let

$$l_1 \in \mathrm{L}(X, Y), \quad l_2 \in \mathrm{L}(Y, Z),$$

then we deduce

$$\phi(l_2 l_1)(g) = \hat{h}'(\hat{h}(g)l_2 l_1) = \hat{h}'(\hat{h}\hat{h}'(\hat{h}(g)l_2)l_1) =$$
$$= \hat{h}'(\hat{h}\,\phi(l_2)(g)l_1) = \phi(l_1)(\phi(l_2)(g)) = (\phi(l_2)\phi(l_1))(g).$$

The corresponding result holds for 1-, resp. 2-, morphisms in P_Y . Thus, we have

$$\phi(l_2 l_1) = \phi(l_2)\phi(l_1).$$

If $l = 1: X \to X$ is the identity, then we get

$$\phi(1)(g) = \hat{h}'(\hat{h}(g)1) = g.$$

Hence ϕ is a functor.

However ϕ is also a 2-functor: Let $\lambda: l_0 \approx l_1$ be a 2-morphism in L , then we define for $g \in \mathrm{P}_Y$:

$$\phi(\lambda)(g): \phi(l_0)(g) \approx \phi(l_1)(g) \quad \text{by} \quad \phi(\lambda)(g) = \hat{h}'(\hat{h}(g)\lambda).$$

We leave the straightforward details to the reader.

We come to the uniqueness of ϕ : Let $\psi : \mathrm{L} \to \mathrm{K}$ be a second 2-functor which fulfills 1 - 3 in Theorem 3.1, then we have:

$$\psi(l)(g) = h'(h(g)\psi(l)) = h'(\psi\hat{h}(g)\psi(l)) =$$
$$= h'\psi(\hat{h}(g)l) = \hat{h}'(\hat{h}(g)l) = \phi(l)(g).$$

We can repeat this for 1- and 2-morphisms in P_Y yielding the proof of the fact that $\phi = \psi$ as 1-functors. Finally by using the 2-morphisms in L and $\bar{\mathrm{K}}$ we also get that $\phi = \psi$ as 2-functors. This completes the proof of Theorem 3.1.

4. EXPLICIT CONSTRUCTION OF MORPHISMS $\bar{f} \in \bar{\mathrm{K}}$.

The construction of a $\bar{f} \in \bar{\mathrm{K}}(X, Y)$ involves all maps $g: Y \to P$, for all $P \in \mathrm{P}$. This is from the computational point of view a rather hopeless situation which we are now trying to overcome. In Mardešić's shape category one is using systems

$$\mathbf{Y} = \{ P_\alpha , q^\alpha_\beta \}, \quad P \in \mathrm{P} \quad \text{with} \quad \varprojlim \mathbf{Y} = Y$$

and in consequence only the projections $q_\alpha : Y \to P$ instead of all $g: Y \to P$. It is not trivial that every shape mapping needs only be defined on these q_α. This holds for compact metric spaces, while for arbitrary spaces one is running into severe trouble. K. Morita [7] changed the whole viewpoint by proposing the concept of an «associated system» for a topological space.

In case of our shape theory we are confronted with even more trouble. It turns out that (even for compact metric spaces) inverse systems are not sufficient. What we actually need are special subcategories of P_X :

Let $\mathrm{P}^1_X \subset \mathrm{P}_X$ be the subcategory with the same objects as P_X but with mappings $(r, 1) \in \mathrm{P}_X(g_1, g_2)$ (i.e., all homotopies are identities).

4.1. DEFINITION. A subcategory $\mathrm{P}'_X \subset \mathrm{P}^1_X \subset \mathrm{P}_X$ is called a *st-category* (= strong tree category) if the following three conditions hold:

1° There exists a functor $\Phi : \mathrm{P}^1_X \to \mathrm{P}'_X$ as well as natural transformations $\phi : i\Phi \to 1$ (= identity of P^1_X). Here $i : \mathrm{P}'_X \subset \mathrm{P}^1_X$ is the inclusion, which will be omitted from our notation in most cases.

2° One has

$$\Phi i = 1 : \mathrm{P}'_X \to \mathrm{P}'_X \quad \text{and} \quad (\phi i : i\Phi i \to i) = \text{identity}.$$

Let $\mathrm{P}'_X \times I$ be the category with objects

$$g \times 1_I : X \times I \to P \times I, \quad g \in \mathrm{P}'_X ,$$

resp. for the morphisms. We will assume that

$$P \in \mathrm{P} \quad \text{implies} \quad P \times I \in \mathrm{P} .$$

Then we have an inclusion $i : \mathrm{P}'_X \times I \subset \mathrm{P}^1_{X \times I}$.

3° There exists a functor $\Psi : \mathrm{P}^1_{X\times I} \to \mathrm{P}'_X \times I$ and a natural transformation $\psi : i\Psi \to 1$ such that 1 , 2 mutatis mutandis hold.

Furthermore we assume commutativity in the following diagram where the horizontal arrows indicate the functor $g \mapsto g \times 1_I$.

$$\begin{array}{ccc} P_X^1 & \longrightarrow & P_{X\times I}^1 \\ \Phi \downarrow & & \downarrow \Psi \\ P_X' & \longrightarrow & P_X' \times I \end{array}$$

A category P_X which contains such a st-category is called a *t-category* (= tree-category).

In the sequel we are proceeding into two directions:

a) We prove that every functor $T: P_Y' \to P_X$ under certain very mild restrictions allows an extension to a $\bar{f} \in \bar{K}(X, Y)$. Here P_Y is of course supposed to be a t-category.

b) We assure that, for compact metric X, P_X is a t-category provided P is the category of spaces having the homotopy type of a CW-space.

There are numerous applications of these results (cf. [1, 2]). In this paper the proof of Theorem 6.1 can be viewed as an additional application.

4.2. THEOREM. *Let* P_Y *be a t-category and* $T: P_Y' \to P_X$, $X, Y \in K$ *be any functor with the following properties:*

T1) $g \in P_Y'$, $g: Y \to P \implies T(g): X \to P$.

T2) $r = (r, 1): g_1 \to g_2$ *in* $P_Y' \implies T(r, 1) = (r, \omega)$ *for suitable* ω.

Then there exists a $\bar{f} \in \bar{K}(X, Y)$ *such that* $T = \bar{f}|P_Y'$.

PROOF. At a first step we construct a functor $T^1: P_Y^1 \to P_X$ which extends T: We define

$$T^1(g) = \phi_g T\,\Phi(g), \quad g \in P_Y^1,$$

and for $r = (r, 1) \in P_Y^1(g, rg)$ we need a homotopy μ in

$$(r, \mu) = T^1(r, 1): \quad \begin{array}{ccc} T^1(g) & \longrightarrow & T^1(rg) \\ \| & & \| \\ r\phi_g T\Phi(g) & & \phi_{rg} T(\Phi(rg)). \end{array}$$

Since ϕ is natural we have $r\phi_g = \phi_{rg}\Phi(r)$. Moreover there exists a homotopy

$$\nu: T\,\Phi(rg) \approx \Phi(r)\, T\Phi(g),$$

due to $T2$. Here we have by an abuse of notation denoted the first (and by assumption only relevant) component of $\phi(r,1)$ by $\phi(r)$. Thus we have achieved a

$$T^1(r,1)=(r,\mu),\quad \mu: r\phi_g T\Phi(g)\approx \phi_{rg} T\Phi(rg).$$

This definition is clearly functorial: For

$$g \xrightarrow[(r,1)]{} rg \xrightarrow[(s,1)]{} srg$$

in P_Y^1 we have

$$T^1(s,1)\,T^1(r,1)=T^1(sr,1).$$

Our next aim is the extension of T^1 to a $\bar{f}: P_Y \to P_X$, the ultimate step in our construction (according to Proposition 1.3). On the objects of P_Y we set evidently $\bar{f}=T^1$. In the same way we proceed for the morphisms $(r,1)\in P_Y$. Let $\omega: g_0\approx g_1$ be any homotopy between $g_0, g_1 \in P_Y$. Let us assume that ω is an elementary homotopy (see Section 7). Then ω is represented by a map $F: Y\times I\to P$. Using Definition 5.1.3 we obtain:

$$\Psi F=(g\times 1_I): Y\times I\to P'\times I \quad \text{and} \quad \psi_F: P'\times I\to P$$

such that $F=\psi_F\circ\Phi F$, for suitable $g\in P_Y'$. Let

$$r_t=\psi_F\, i_t \quad (i_t: P'\to P'\times I \text{ the inclusion, } t\in I),$$

then we have

$$g_j=r_j g,\quad j=0,1,$$

and a chain of homotopies:

$$\bar{f}(g_0)=T^1(g_0)\approx r_0T^1(g)\approx r_1T^1(g)\approx T^1(g_1)=\bar{f}(g_1).$$

This will be our definition of $\bar{f}(1,\omega): \bar{f}(g_0)\approx \bar{f}(g_1)$ for elementary homotopies. Let $\omega=\omega_1\ldots\omega_n$ be any homotopy, represented as a product of elementary homotopies. Then we set:

$$\bar{f}(1,\omega)=\prod_{i=1}^{n}\bar{f}(1,\omega_i).$$

For any $(r,\omega): g_1\to g_2$ in P_Y we have the canonical decomposition

$$(r,\omega)=(1,\omega)(r,1).$$

Therefore we can set

$$\bar{f}(r,\omega)=\bar{f}(1,\omega)\bar{f}(r,1)=\bar{f}(1,\omega)\,T^1(r,1).$$

and in particular

$$\bar{f}(r,1) = T^1(r,1).$$

We are still obliged to prove that this assignment gives a functor $\bar{f}: \mathrm{P}_Y \to \mathrm{P}_X$. This follows immediately from the following two assertions:

1° Let $\omega : g_0 \to g_1$ be an elementary homotopy and ω^{-1} its inverse then we have

$$\bar{f}((1,\omega)(1,\omega^{-1})) = \bar{f}(1,\omega)\,\bar{f}(1,\omega^{-1}) = (1,1).$$

2° One has $\bar{f}(1,1) = (1,1)$.

The proof of 1, 2 is straightforward and omitted.

We can make $\bar{f}$ into a 2-functor by using Proposition 1.3. This completes the proof of Theorem 4.2.

We are now entitled to give an existence proof for a P'_X. Here X is supposed to be a compact metric space and P is the category of spaces having the homotopy type of CW-spaces.

4.3. THEOREM. *The category P_X for a compact metric space X is a t-category. Moreover the st-category $\mathrm{P}'_X \subset \mathrm{P}^1_X \subset \mathrm{P}_X$ has the following additional properties:*

The category P'_X has countably many objects $p_i : X \to P_i$, $i = 1, 2, \ldots$.

All morphisms $(r, 1)$ in P'_X are compositions of (in P'_X) indecomposable morphisms $(r_i, 1) : p_i \to p_{i-1}$, where the $r_i : P_i \subset P_{i-1}$ are inclusions. Furthermore all P_i are compact ANR-spaces, namely finite unions of ϵ-neighborhoods of points in the Hilbert cube Q.

PROOF. The proof runs as in [1] and consists in an extensive application of Dugundji's mapping theorem ([5], page 188, Theorem 6.1). We embed X in a Hilbert cube Q and construct a decreasing sequence of spaces

$$P_i \in \mathrm{P},\quad P_i \subset P_{i-1} \quad \text{such that} \quad \bigcap_1^\infty P_i = X,$$

as was accomplished in 7.3. The mapping theorem provides us with a canonical extension ϕ_g for a map $g : X \to P \in \mathrm{P}$ over some P_i. We assume the index i to be minimal with respect to this property and set

$$\phi(g) = p_i : X \subset P_i \text{, the inclusion.}$$

We have clearly $\phi_g p_i = g$. This Φ is functorial: For $(r, 1): g_1 \to g_2$ in P_X^1 we ensure that the index i_1 in $\Phi(g_1) = p_{i_1}$ is less or equal to i_2 in $p_{i_2} = \Phi(g_2)$. Therefore we can set

$$\Phi(r, 1) = (P_{i_1} \subset P_{i_2}, 1).$$

Let $i: \mathrm{P}'_X \subset \mathrm{P}_X^1$ be the inclusion functor, then we have:

$$\Phi i = 1 \quad \text{and} \quad (\phi i : i\Phi i \to i) = \text{identity}.$$

Moreover we can easily replace X by $X \times I$ and the space P_i by $P_i \times I$. An analogous argument provides us with a functor Ψ and a transformation ψ as required in Definition 5.1 3. This completes the proof of Theorem 4.3.

5. MARDESIČ'S SHAPE CATEGORY.

In [6] S. Mardesič succeeded in giving Borsuk's shape category a treatment which fits into a categorical framework. The definition is very close to our construction of $\bar{\mathrm{K}}$. We simply have to use $\mathrm{H} = \mathrm{Top}_\mathrm{h}$, the homotopy category, instead of $\mathrm{H} = \mathrm{Top}$ and in addition to forget about 2-, 3- etc... morphisms in K (resp. to require all these to be identities). Then a shape morphism $\alpha: X \to Y$ in the sense of Mardesič is an assignment which assigns to each homotopy class $g: Y \to P$, $P \in \mathrm{P}$, a homotopy class

$$\alpha([g]) : X \to P$$

such that a homotopy commutative diagram $[r][g_1] = [g_2]$ is converted into a homotopy commutative diagram $[r]\alpha[g_1] = \alpha[g_2]$.

5.1. DEFINITION. The Mardesič shape category $\bar{\mathrm{H}}$ has the same objects as H but the previously defined assignments α as morphisms.

Similar to $\bar{\mathrm{K}}$ we have a functor $h: \mathrm{Top}_\mathrm{h} \to \bar{\mathrm{H}}$ which is defined by:

$$h([f])([g]) = [gf],$$

and as the identity on the objects. In the same way we can establish an assignment $h'(\alpha)$, defined for each $\alpha \in \bar{\mathrm{H}}(X, Y)$, $Y \in \mathrm{P}$, by

$$h'(\alpha) = \alpha([1_Y]).$$

Now we have

$$h'h = 1 : \mathrm{Top_h} \to \mathrm{Top_h} \quad \text{and} \quad hh' = 1$$

whenever these relations make sense. The proof is at this time almost trivial and does not require any of the work of Section 2.

The relation between $\bar{\mathrm{H}}$ and $\bar{\mathrm{K}}$ is usually expressed by means of the functor $\eta : \bar{\mathrm{K}}_h \to \bar{\mathrm{H}}$ which is constructed in the following way:

On the objects η has to be the identity. Let $\bar{f} \in \bar{\mathrm{K}}(X, Y)$, then we set

(1) $$\eta([\bar{f}])([g]) = [\bar{f}(g)] \quad \text{for any} \quad g \in [g] ,$$

and correspondingly on the morphisms. This defines obviously a morphism in $\bar{\mathrm{H}}$ and does not depend on the choice of $\bar{f}$ or of g within their homotopy classes. Everything is clearly functorial.

5.2. PROPOSITION. *There exists a functor* $\eta : \bar{\mathrm{K}}_h \to \bar{\mathrm{H}}$ *which is the identity on the objects such that the diagram*

(2) $$\begin{array}{ccccc} & & \bar{\mathrm{K}} & \xrightarrow{\ \bar{\rho}\ } & \bar{\mathrm{K}}_h \\ & \overset{h}{\nearrow} & & & \downarrow \eta \\ \mathrm{K} & & & & \\ & \underset{\rho}{\searrow} & & & \\ & & \mathrm{K}_h = \mathrm{H} & \xrightarrow{\ h\ } & \bar{\mathrm{H}} \end{array}$$

is commutative. Here $\rho, \bar{\rho}$ *are the projections into the related homotopy categories.*

The proof of the commutativity is immediate.

In [2] we have settled the problem whether η is an equivalence of categories or not. It turned out that even for compact metric spaces there exist mappings $\bar{f}, \bar{g} \in \bar{\mathrm{K}}(X, Y)$ such that

$$\bar{f} \not\simeq \bar{g} \ \text{in}\ \bar{\mathrm{K}} \quad \text{but} \quad \eta([\bar{f}]) = \eta([\bar{g}]) .$$

This counterexample is essentially due to N. Steenrod.

There remains the question whether the classification of objects X, Y in both categories is presumably the same. This is answered in the affirmative in Section 6 for compact metric spaces.

As an application of the results of Section 4 we prove [2] :

5.3. THEOREM. *Let* K *be the category of compact metric spaces,* $K_h = H$ *the corresponding homotopy category,* P *the subcategory of spaces having the homotopy type of a CW-space, then the functor* $\eta : \bar{K}_h \to \bar{H}$ *(between the corresponding shape categories) is surjective (i. e., every* $a \in \bar{H}(X, Y)$ *has a counterimage* $\bar{f} \in \bar{K}_h(X, Y)$ *under* η *).*

PROOF. Due to Theorem 4.3, P_Y is a t-category with a st-category P'_Y. Make for any $q_i : Y \to P_i$ in P'_Y a choice $\bar{f}(q_i) \epsilon a([q_i])$ arbitrarily. Furthermore for any $r_i : P_i \subset P_{i-1}$ in P'_Y (see Theorem 4.3) we take any homotopy

$$\omega_i : r_i \bar{f}(q_i) \approx \bar{f}(q_{i-1}).$$

According to Theorem 4.2 this is sufficient for determining a $\bar{f} \epsilon \bar{K}(X, Y)$. Moreover one has $\eta([\bar{f}]) = a$. This completes the proof of Theorem 5.3.

6. COMPARISON OF $\bar{K}$ AND $\bar{H}$.

The expositions of the two preceding sections serve as a preparation for a proof of the following theorem:

6.1. THEOREM. *For compact metric spaces* X *and* Y *(based or unbased) the following two conditions are equivalent:*

a) X *and* Y *are equivalent in* $\bar{H}$.

b) X *and* Y *have the same homotopy type in* $\bar{K}$.

Recall that we use for P the category of spaces having the homotopy type of CW-spaces.

This theorem assures that the classification of compact metric spaces in Mardesič's category coincides with the classification in the category $\bar{K}$.

We have the functor $\eta : \bar{K}_h \to \bar{H}$ which is an identity on the objects. Hence a homotopy equivalence $\bar{f} : X \to Y$ in $\bar{K}$ is transformed into an equivalence in $\bar{H}$. This proves the assertion «b $\Longrightarrow$ a».

The verification of the reverse implication is more delicate and it needs some additional preparation. Before entering into the details we will outline the main idea of the proof:

According to 5.3 we can find for any

$$\alpha \in \bar{H}(Y,X), \quad \beta \in \bar{H}(X,Y) \quad \text{with} \quad \alpha\beta = 1, \quad \beta\alpha = 1$$

in $\bar{H}$ morphisms

$$\bar{a} \in \bar{K}(Y,X), \quad \bar{b} \in \bar{K}(X,Y) \quad \text{with} \quad \eta(\bar{a}) = \alpha, \quad \eta(\bar{b}) = \beta .$$

It cannot be expected that any such pair $\bar{a}$, $\bar{b}$ has the property

$$\bar{a}\bar{b} \approx 1_Y, \quad \bar{b}\bar{a} \approx 1_X \quad \text{in } \bar{K}_h .$$

However we can prove the following assertion:

6.2. LEMMA. *Let* $\alpha, \beta \in \bar{H}$ *be as above and* $\bar{b} \in \bar{K}(X,Y)$ *any morphism with* $\eta(\bar{b}) = \beta$. *Then there exist a* $\bar{a} \in \bar{K}(Y,X)$ *such that* $\eta(\bar{a}) = \alpha$ *and a homotopy* $\bar{a}\bar{b} \approx 1_X$ *in* $\bar{K}$.

We deduce Theorem 6.1 from Lemma 6.2:

Let α, β be as above, then we find for prescribed

$$\bar{b} \in \bar{K}(X,Y) \quad \text{with} \quad \eta(\bar{b}) = \beta$$

an $\bar{a} \in \bar{K}(Y,X)$ such that $\bar{a}\bar{b} \approx 1_X$. Analogously we obtain for this $\bar{a}$ a

$$\bar{b}_1 \in \bar{K}(X,Y) \quad \text{such that} \quad \bar{b}_1\bar{a} \approx 1_Y .$$

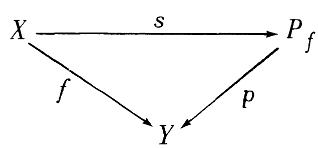
Now by an elementary computation we get:

$$\bar{b}_1 \approx \bar{b}_1\bar{a}\bar{b} \approx \bar{b} .$$

Thus we also have $\bar{b}\bar{a} \approx 1_Y$ and the theorem is proved.

The proof of Lemma 6.2 depends on two well-known facts. The first one is a lemma on fibrations ([3], page 165).

6.3. LEMMA. *Let* $f \in \mathrm{Top}(X,Y)$ *be any map, then there exists a commutative triangle*



where s *is a homotopy equivalence and* p *a (Hurewicz-)fibration. Further-*

more there exists a map $r: P_f \to X$ *such that* $rs = 1$ *and* $sr \approx 1$.

Lemma 6.3 confirms the well-known fact that «every map can be converted into a fibration».

The second statement deals with the homotopy extension property (HEP) and follows from the fact that every inclusion of metric compacta has the (HEP) for ANR-spaces [5]:

6.4. LEMMA. *Let* X *be compact metric,* P *an ANR,* $p: X \subset P$ *an inclusion, then* p *has the HEP for ANR-spaces* Q. *More precisely: Let* $f: P \to Q$ *be any map,* $F: fp \approx g$ *be any homotopy, then there exists a homotopy* $G: f \approx f'$ *such that* $G(p \times 1) = F$ *and* $G i_0 = f$.

We have to use 6.4 for spaces Q which are compact subsets of the Hilbert cube (see 7.3). By an ANR-space we mean a space which has this property with respect to the class of metric spaces.

Now we are providing a proof of Lemma 6.2:

Let X, Y be compact metric spaces as in Lemma 6.2, then we have a st-category P'_X for X which is ordered: The objects are mappings

$$p_i: X \to P_i, \quad i = 1, 2, \ldots,$$

P_i = ANR-spaces, and morphisms

$$(r_i, 1): p_i \to p_{i-1}, \quad \text{i.e.,} \quad r_i p_i = p_{i-1}.$$

The objects of P'_Y are denoted by q_i, $i = 1, 2, \ldots$. The $(r_i, 1)$ are of course indecomposable in P'_X. Let $\bar{b} \in \bar{\mathrm{K}}(X, Y)$ be a given morphism. We are going to construct a suitable

$$\bar{a} \in \bar{\mathrm{K}}(Y, X) \quad \text{such that} \quad \eta(\bar{a}) = \alpha.$$

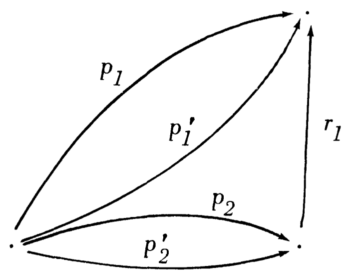
Let $\bar{a}(p_1)$ be any map in $\alpha([p_1])$ and set $(\bar{a}\bar{b})(p_1) = p'_1$. Since by assumption $[p'_1] = [p_1]$, we can choose a homotopy $\bar{F}(p_1): p_1 \approx p'_1$ (where we identify $\bar{F}(p_1)$ with the corresponding map $\bar{F}(p_1): X \times I \to P_1$ describing this homotopy; furthermore $\bar{F}(p_2)$ is assumed to be elementary, which is permitted without loss of generality because of Lemma 7.2). Now we try to detect a

$$\bar{F}(p_2): p_2 \approx p'_2 = (\bar{a}\bar{b})(p_2)$$

for a suitable $\bar{a}(p_2)$ as well as a homotopy $\omega_1 : r_1 \bar{a}(p_2) \approx \bar{a}(p_1)$ and a homotopy between homotopies

$$\xi_1 : \gamma \circ r_1 \bar{F}(p_2) \approx \bar{F}(p_1), \text{ where } (\bar{a}\bar{b})(r_1, 1) = (r_1, \gamma).$$

The situation is probably clarified by drawing the following 3-dimensional diagram (Figure 1):



The base is interpreted as $\bar{F}(p_2)$, the top as $\bar{F}(p_1)$ while the two remaining faces represent γ, resp. the identical homotopy (recall that $r_1 p_2 = p_1$). The 3-morphism ξ_1 represents the «interior» of Figure 1. •

We need two simple observations which are more or less immediate:

6.5. LEMMA. *a) Replacing r_1 by a fibration $\hat{r}_1$ (in the sense of 6.3) alters Figure 1 into a new one which is related to the first by a homotopy between homotopies.*

b) Replacing r_1 by a map $\hat{r}_1 \approx r_1$ alters Figure 1 into a new one which is related to the first by a homotopy between homotopies.

Let $\bar{a}(p_2) \in \alpha([p_2])$ be any map and $\nu : r_1 \bar{a}(p_2) \approx \bar{a}(p_1)$ be any homotopy, which exists since

$$[r_1]\alpha([p_2]) = \alpha([p_1]).$$

We take $\gamma : r_1 p_2' \approx p_1'$ from

$$(r_1, \gamma) = (\bar{a}\bar{b})(r_1, 1) = \bar{b}(r_1, \nu)$$

(observe that $\bar{b}$ is supposed to be already defined). Now we replace r_1 by a fibration (which for simplicity, by an abuse of notation, is again denoted by r_1) and lift $\bar{F}(p_1)^{-1} \circ \gamma$ to a homotopy $\Phi : p_2 \approx \hat{p}_2$ for a suitable $\hat{p}_2$. Hence we have $r_1 \Phi = \bar{F}(p_1)^{-1} \circ \gamma$. Since

$$\hat{p}_2 \epsilon(\beta\alpha)([p_2]) = [p_2],$$

we have a homotopy $\Psi: \hat{p}_2 \approx p_2$. Because the mapping p_2 fulfills all requirements of Lemma 6.4, we obtain a homotopy

$$\mu: \hat{r}_1 \approx r_1 \quad \text{such that} \quad r_1 \Psi = \mu p_2.$$

Now we set $\bar{F}(p_2) = \Phi^{-1} \circ \Psi^{-1}$ and find

$$\gamma \circ r_1 \bar{F}(p_2) = \gamma \circ \gamma^{-1} \bar{F}(p_1) \mu^{-1} p_2 \approx \bar{F}(p_1),$$

according to Lemma 6.5 b.

This construction gives rise to an inductive argument: Given

$$p_k', \; \bar{F}(p_k), \; \bar{a}(p_k), \; \xi_{k-1} \quad \text{for} \quad k < n-1,$$

we construct a

$$p_n', \; \bar{F}(p_n), \; \bar{a}(p_n), \; \xi_{n-1}.$$

This provides us with a

$$\bar{a} \in \bar{\mathrm{K}}(X, Y) \quad \text{such that} \quad \eta(\bar{a}) = \alpha$$

and with a homotopy $\bar{F} \in \bar{\mathrm{K}}(X \times I, X)$ between $\bar{a}\bar{b}$ and the identity.

The proof of Lemma 6.2 is thereby complete.

7. APPENDIX.

Although we are not using anything particular from the meanwhile widely developed theory of n-categories which cannot be immediately deduced for our special case (where all higher morphisms are some kinds of homotopies between homotopies) we are obliged to mention that our concept of a 2-category is considerably weaker than the one which is ordinarily used in the literature:

We take simply advantage of the fact that $\mathrm{K}(X, Y)$ for fixed objects $X, Y \in \mathrm{K}$ carries again the structure of a category with the following properties: Any $f \in \mathrm{K}(X', X)$, resp. $g \in \mathrm{K}(X, Y')$ induces a functor

$$f^*: \mathrm{K}(X, Y) \to \mathrm{K}(X', Y), \quad \text{resp.} \quad g_*: \mathrm{K}(X, Y) \to \mathrm{K}(X, Y')$$

which operates by composition on the objects (e.g.,

$$a \in \mathrm{K}(X, Y) \mapsto f^*(a) = fa \;)$$

with the following properties:

a) For $f_1 \in \mathrm{K}(X'', X')$, $f_2 \in \mathrm{K}(X', X)$ (resp.

$$g_1 \in \mathrm{K}(Y, Y'),\quad g_2 \in \mathrm{K}(Y', Y''))$$

one has

$$(f_2 f_1)^* = f_1^* f_2^*, \quad (g_2 g_1)_* = g_{2*} g_{1*}.$$

b) The identities 1_X, resp. 1_Y, induce the identities as functors.

Let K, L be two 2-categories, then a 2-functor $F: \mathrm{K} \to \mathrm{L}$ is:

1° an ordinary functor between the two categories K, L, and

2° for fixed $X, Y \in \mathrm{K}$ the assignment $\mathrm{K}(X, Y) \to \mathrm{L}(F(X), F(Y))$ induces a functor

$$F: \mathrm{K}(X, Y) \to \mathrm{L}(F(X), F(Y))$$

such that for f, g as above one has commutativity in the squares:

$$\begin{array}{ccc} \mathrm{K}(X,Y) & \longrightarrow & \mathrm{L}(F(X),F(Y)) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle F(f)^*} \\ \mathrm{K}(X',Y) & \longrightarrow & \mathrm{L}(F(X'),F(Y)) \end{array} \qquad \begin{array}{ccc} \mathrm{K}(X,Y) & \longrightarrow & \mathrm{L}(F(X),F(Y)) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle F(g)_*} \\ \mathrm{K}(X,Y') & \longrightarrow & \mathrm{L}(F(X),F(Y')) \end{array}$$

A n-category is defined by induction: Assume that every $\mathrm{K}(X, Y)$ carries the structure of a $(n-1)$-category and that f, g induce $(n-1)$-functors such that a and b hold.

We collect some facts from general topology which are necessary for an understanding of the preceding sections although they are not new.

There is first of all the question of turning K = Top or Top_0 or any other suitable category of topological spaces into a 2-category, where the 2-morphisms are supposed to be homotopies. Let

$$\omega: f_0 \approx f_1, \quad \nu: f_1 \approx f_2$$

be two homotopies, then we need a composition $\nu \circ \omega: f_0 \approx f_2$ of these two homotopies. If we agree to define ω and ν by mappings

$$G, F: X \times I \to Y \quad (f_i: X \to Y), \text{ where } I = [0, 1],$$

and $\nu \circ \omega$ by the mapping $H: X \times I \to Y$:

$$(1) \qquad H(x,t) = \begin{cases} F(x, 2t)\ldots & 0 \leqslant t \leqslant 1/2 \\ G(x, 2t-1)\ldots & 1/2 \leqslant t \leqslant 1, \end{cases}$$

then this does not make $\mathrm{K}(X, Y)$ into a category. The composition (1) is neither associative nor is there any identity available. Hence we have to proceed differently.

Let $I_n = [0, 1]$ for $0 \leqslant n \in \mathrm{N}$ be an interval, then a homotopy is a mapping $F: X \times I_n \to Y$. Let $G: X \times I_m \to Y$ be a second homotopy such that $F i_n = G i_0$ (i.e., both homotopies fit together, $i_t(x) = (x, t)$), then we define $G \circ F = H$ as the mapping $H: X \times I_{n+m} \to Y$ with

$$H \mid X \times I_n = F, \quad H \mid X \times [n, n+m] = G.$$

In other words, we paste both homotopies together without contracting the interval. We abbreviate I_1 simply by I.

It is customary to introduce two relations:

1° If $F: X \times I_n \to Y$ is any homotopy and $G: X \times I_1 \to Y$ a homotopy such that $G \circ F$ is defined but $G(x, t)$ is independent of t, then we set $G \circ F = F$. We denote G by 1_f, where $F(x, n) = f(x)$.

2° Let $F: X \times I_1 \to Y$ be a homotopy and $F^{-1}: X \times I_1 \to Y$ be defined by $F^{-1}(x, t) = F(x, 1-t)$, then we require that

$$F^{-1} \circ F = 1_f, \quad \text{where} \quad f = F i_0 \quad (\text{i.e., } f(x) = F(x, 0)).$$

Now we agree to define compositions of homotopies up to these relations.

We call homotopies of the form $F: X \times I_1 \to Y$ *elementary* and have thereby accomplished a proof of the following assertion:

7.1. LEMMA. *Every homotopy $\omega \in \mathrm{K}(f_0, f_1)$, $f_i \in \mathrm{K}(X, Y)$, allows a unique reduced decomposition $\omega = \epsilon_1 \ldots \epsilon_k$ of elementary homotopies, where «reduced» means (analogously as in group theory) that no $\epsilon_i = 1$ appears and that we never have $\epsilon_i = \epsilon_{i-1}^{-1}$.*

Homotopies between homotopies are defined in the same way: Let us start with an elementary homotopy between homotopies:

$$\xi : \omega_0 \approx \omega_1 \quad \text{where} \quad \omega_0, \omega_1 : X \times I_1 \to Y$$

are elementary homotopies between maps $f_0, f_1 : X \to Y$. We define

$$\xi : X \times I_1 \times I_1 \to Y$$

to be a mapping such that:

$$\xi(x, t, i) = \omega_i(x, t), \quad \xi(x, i, s) = f_i(x), \quad i = 0, 1.$$

Let ω_0, ω_1 be such that

$$\omega_0 : X \times I_n \to Y, \quad \omega_1 : X \times I_m \to Y,$$

then we can always assume that $n = m$ (by simply inserting sufficiently many constant homotopies). Furthermore an easy observation on deformations of squares $I_n \times I_m$ assures us that we have:

7.2. LEMMA. *Let $F : X \times I_n \to Y$ be any homotopy and $F' : X \times I_1 \to Y$ be the homotopy which results from F by contracting I_n to I_1 by a linear homeomorphism, then F and F' are related by a homotopy between homotopies (i. e., a 3-morphism $\xi : F \approx F'$ in* K *).*

REMARK. In [1] we have defined compositions of homotopies slightly differently, however the final 2-categories are isomorphic.

This turns K into a 2-category; higher 3-, 4-, etc..., category structures are defined analogously.

In Section 5 we need for a compact metric space X, which we can embed in a Hilbert cube, a sequence of spaces P_i, $i = 1, 2, \ldots$, such that:

1° $P_i \subset P_{i-1}$,

2° $P_i \in$ P = category of spaces having the homotopy type of a CW-space,

3° $\bigcap_{i=1}^{\infty} P_i = X$.

7.3. PROPOSITION. *There exists a system of P_i's fulfilling 1-3.*

PROOF. Let $\{U_i\}$ be a decreasing sequence of open sets in Q, the Hilbert cube,

$$U_i \supset X, \quad \cap U_i = X.$$

We can find a $1/m$-net for Q, which we denote by N_m, $m = 1, 2, \ldots$. This is a discrete set of points such that the union of all $1/m$-balls $K(P; 1/m)$,

$P \in N_m$, covers Q. We call

$$\mathcal{S}_i = \{ K(P; 1/i) \mid P \in N_i \} \quad \text{and} \quad \mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n .$$

Because X is compact, we get finitely many $K \in \mathcal{S}$ which cover X such that $P_i = \cup K \subset U_i$. This so defined P_i has the required properties: One has:

$$X \subset P_i \quad \text{and} \quad X \subset \bigcap_{i=1}^{\infty} P_i \subset \bigcap_{i=1}^{\infty} U_i = X,$$

hence $X = \bigcap_{i=1}^{\infty} P_i$. This completes the proof of Proposition 7.3.

REFERENCES.

1. F. W. BAUER, A shape theory with singular homology, *Pacific J. of Math.* 62, n°1 (1976).

2. F. W. BAUER, A characterization of movable compacta, *J. f. die reine u. angew. Math.* 293/294 (1977), 394-417.

3. F. W. BAUER, *Homotopietheorie*, B. I.-Hochschultaschenbücher Bd. 475 a,b, 1971.

4. T. A. CHAPMAN, On some applications of infinite dimensional manifolds to the theory of shape, *Fund. Math.* LXXVI (1972), 181-193.

5. J. DUGUNDJI, *Topology*, Allyn & Bacon, Boston, 1960.

6. S. MARDEŠIĆ, Shapes for topological spaces, *Gen. Topology and Applications*, 3 (1973), 265-282.

7. K. MORITA, On shapes of topological spaces, *Fund. Math.* LXXXVI (1975), 251.

8. T. PORTER, Coherent prohomotopy theory, *Cahiers Topo. et Géo. Diff.* XIX-1 (1978), 3-46.

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