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MANIFOLDS OF SMOOTH MAPS

by P. MICHOR

Let X, Y be smooth finite-dimensional manifolds, and let $C^\infty(X, Y)$ be the space of all smooth maps from X to Y. We introduce the $\mathfrak{D}^\infty$-topology on $C^\infty(X,Y)$ in Sections 1, 2, and use it then to show that $C^\infty(X, Y)$ is a smooth manifold modelled on spaces $\mathfrak{D}(F)$ of smooth sections with compact support of vector bundles F over X with the nuclear (LF)-topology of L. Schwartz. The notion of differentiation which we use is the concept C^∞_π of Keller [7], a rather strong one. We obtain a weak inversion theorem which can be applied to some notions of stability. We give a manifold structure to the tangent bundle of $C^\infty(X, Y)$, and we show that the tangent bundle behaves in a nice functional way.

We have considered only finite dimensional manifolds X, Y, since the topology $\mathfrak{D}^\infty$ depends heavily on it. It may be possible to extend the results for some infinite dimensional manifolds Y.

We considered only smooth maps, but it is clear how to adapt the theory to the case C^r, $r > 0$. If X is supposed to be compact, then our theory coincides with existing theories, e. g. Leslie [8].

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1. THE $\mathfrak{D}$-TOPOLOGY ON $C^\infty(X, Y)$.

1.1. NOTATION. Let X, Y be smooth finite dimensional manifolds, and let $C^\infty(X, Y)$ be the set of smooth mappings from X to Y ; for $n \in \mathbb{N}$, let $J^n(X, Y)$ be the fibre bundle of n-jets of maps from X to Y, equipped with the canonical manifold structure which makes $j^n f: X \to J^n(X, Y)$ into a smooth section for each $f \in C^\infty(X, Y)$, where $j^n f(x)$ is the n-jet of f at $x \in X$. Let

$$\alpha : J^n(X, Y) \to X, \quad \beta : J^n(X, Y) \to Y$$

be the components of the fibre bundle projection, i.e.

$$\alpha(\sigma) = x \quad \text{and} \quad \beta(\sigma) = y$$

if σ is a n-jet at $x \in X$ of a function

$$f \in C^\infty(X, Y) \quad \text{with} \quad f(x) = y.$$

See [5].

1.2. DEFINITION. Let $K = (K_n)$, $n = 0, 1, \ldots$ be a fixed sequence of compact subsets of X such that

$$K_0 = \emptyset, \quad K_{n-1} \subset K_n{}^\circ \text{ for each } n \quad \text{and} \quad X = \bigcup_n K_n.$$

Then consider sequences $m = (m_n)$, $U = (U_n)$ for $n = 0, 1, \ldots$ such that m_n is a nonnegative integer and U_n is open in $J^{m_n}(X, Y)$. For each such pair (m, U) of sequences define a set $M(m, U) \subset C^\infty(X, Y)$ by:

$$M(m, U) = \{ f \in C^\infty(X, Y) \mid j^{m_n} f(X \backslash K_n{}^\circ) \subset U_n \text{ for all } n \in \mathbb{N} \}.$$

The $\mathfrak{D}$-topology on $C^\infty(X, Y)$ is given by taking all sets $M(m, U)$ as a basis for its open sets. It is easy to check that this is actually a base for a topology.

1.3. Let d_n be a metric on $J^n(X, Y)$, $n = 0, 1, 2, \ldots$, which is compatible with the topology of the manifold. Consider families $\phi = (\phi_n)$, $n = 0, 1, \ldots$ of continuous nonnegative functions on X such that the family of the sets $(\operatorname{supp} \phi_n)$ is locally finite.

LEMMA. *Let $f \in C^\infty(X, Y)$. Then each family $\phi = (\phi_n)$ as described above defines an open neighborhood $V_\phi(f)$ of f by*

$$V_\phi(f) = \{ g \in C^\infty(X,Y) \mid \phi_n(x)\, d_n(j^n f(x), j^n g(x)) < 1 \text{ for all } x \in X,\ n = 0,1,\dots \}.$$

PROOF. We claim that each $V_\phi(f)$ is open. Determine (m_n), $n = 0,1,\dots$ inductively in such a way that

$$m_{-1} = 0, \quad m_n \nearrow \infty \quad \text{and} \quad K_n \cap \operatorname{supp} \phi_l = \emptyset \text{ for all } l \leqslant m_{n-1}.$$

Then consider the continuous maps $\Delta_l : J^l(X,Y) \to \mathrm{R}$ defined by

$$\Delta_l(\sigma) = \phi_l(\alpha(\sigma))\, d_l(j^l f(\alpha(\sigma)), \sigma), \quad \sigma \in J^l(X,Y),$$

and let W_l be the open set $\Delta_l^{-1}((-1,1))$. Set

$$U_n = \bigcap_{i=m_{n-1}+1}^{n} (\pi_{i,m_n})^{-1}(W_i), \quad n = 0,1,\dots,$$

where $\pi_{l,k} : J^k(X,Y) \to J^l(X,Y)$ denotes the canonical projection for $k \geqslant l$. Then it is easily checked up by writing out the definitions that

$$M(m,U) = V_\phi(f), \quad \text{where } m = (m_n),\ U = (U_n).$$

1.4. Consider sequences $L = (L_n)$, $n = 0,1,\dots$ of compact subsets L_n of X such that $(X \setminus L_n^\circ)$ is locally finite on X, and sequences $U = (U_n)$ of open subsets : $U_n \subset J^n(X,Y)$.

LEMMA. *Each pair (L,U) of sequences as above defines a set*

$$M'(L,U) = \{ f \in C^\infty(X,Y) \mid j^n f(X \setminus L_n^\circ) \subset U_n \text{ for all } n \geqslant 0 \}.$$

The family $\{M'(L,U)\}$, where L and U are as above, is a basis for the $\mathfrak{D}$-topology.

PROOF. Let $K = (K_n)$ be the fixed sequence of compact subsets of X of 1.2 ; then $(X \setminus K_n^\circ)$ is clearly locally finite. Let $M(m,U)$ be a basic open set as in 1.2 ; define $L = (L_n)$ by

$$L_j = K_n \quad \text{for } m_n \leqslant j \leqslant m_{n+1} - 1, \quad n = 0,1,\dots ;$$

define

$$U'_j = U_n \text{ for } j = m_n \quad \text{and} \quad U'_j = J^j(X,Y) \text{ for } j \notin \{m_n\}.$$

Then clearly

$$M'(L,U') = M(m,U), \quad \text{where } U' = (U'_n).$$

So the system $\{M'(L,U)\}$ contains a basis for the $\mathfrak{D}$-topology and it is itself a basis if we can show that each $M'(L,U)$ is open.

Given $M'(L,U)$ and $f \in M'(L,U)$ we will construct a neighborhood $V_\phi(f)$ of f with $V_\phi(f) \subset M'(L,U)$. For each n, $W_n = (j^n f)^{-1}(U_n)$ is open in X and $X \backslash L_n{}^\circ \subset W_n$ since $j^n f(X\backslash L_n{}^\circ) \subset U_n$. For $x \in W_n$ define

$$a_n(x) = \inf\{ d_n(j^n f(x), \sigma) \mid \sigma \in \alpha^{-1}(x) \cap (J^n(X,Y) \backslash U_n)\} .$$

We claim that a_n is bounded below by a positive constant on each compact subset C of W_n. This is seen as follows: $j^n f(C)$ is compact in $J^n(X,Y)$ and contained in the open set U_n. So U_n contains a set

$$\{ \tau \in J^n(X,Y) \mid d_n(j^n f(C), \tau) < \epsilon \} \quad \text{for some } \epsilon > 0$$

and so $a_n(x) \geqslant \epsilon$ for all $x \in C$. Now we construct a continuous function:

$$b_n : W_n \to \mathrm{R}, \quad 0 < b_n(x) \leqslant a_n(x) \quad \text{for all} \quad x \in W_n :$$

For $x \in W_n$ choose a continuous function $\delta_x : W_n \to \mathrm{R}$ such that $\delta_x(x) > 0$, and $0 \leqslant \delta_x \leqslant a_n$; this is possible since a_n is bounded below on a compact neighborhood of x in W_n. Then use a partition of unity on X subordinate to the cover

$$\{ z \in W_n \mid \delta_x(z) > 0 \}_{x \in W_n}$$

and obtain b_n.

Now $(X\backslash L_n{}^\circ)$ is locally finite, each $X\backslash L_n{}^\circ$ is closed. So there is a family W_n' of open sets such that $X\backslash L_n{}^\circ \subset \overline{W}_n' \subset W_n$ and $\overline{W}_n'$ is again locally finite. Choose continuous functions

$$u_n : X \to [0,1] \quad \text{such that} \quad u_n = 1 \text{ on } X\backslash L_n{}^\circ ,\ u_n = 0 \text{ off } W_n'$$

and let

$$\phi_n(x) = u_n(x)/b_n(x) \text{ for } x \in W_n' \quad \text{and} \quad \phi_n(x) = 0 \text{ for } x \notin W_n'.$$

Then $\phi_n : X \to [0,\infty)$ is continuous and $\phi = (\phi_n)$ is a family such that: $(\operatorname{supp}\phi_n)$ '$\subset$' (W_n') is locally finite, so by 1.3 $V_\phi(f)$ is $\mathfrak{D}$-open.

Given $g \in V_\phi(f)$, then

$$\phi_n(x)\, d_n(j^n f(x), j^n g(x)) < 1$$

for all $n > 0$ and for all $x \in X$, i.e.

$$u_n(x)\, d_n(j^n f(x), j^n g(x)) < b_n(x) \quad \text{for all} \quad x \in W'_n,$$

so
$$d_n(j^n f(x), j^n g(x)) < b_n(x) \leqslant a_n(x) \quad \text{for} \quad x \in X \backslash L_n{}^\circ.$$

Since

$$j^n g(x) \in a^{-1}(x) \subset J^n(X, Y),$$

this implies $j^n g(x) \in U_n$ for all $x \in X \backslash L_n{}^\circ$ by the definition of a_n, so we get $g \in M'(L, U)$. We have $f \in V_\phi(f) \subset M'(L, U)$.

Q.E.D.

1.5. COROLLARY. *We have the following equivalent descriptions of the* $\mathfrak{D}$*-topology on* $C^\infty(X, Y)$:

a) Fix a sequence $K = (K_n)$ *of compact sets in* X *such that*

$$K_0 = \emptyset, \quad K_{n-1} \subset K_n{}^\circ, \quad X = \bigcup_n K_n.$$

Then the systems of sets of the form

$$M(m, U) = \{ f \in C^\infty(X, Y) \mid j^{m_n} f(X \backslash L_n{}^\circ) \subset U_n, \; n = 0, 1, \dots \}$$

is a base for the $\mathfrak{D}$*-topology on* $C^\infty(X, Y)$, *where* $m = (m_n)$ *runs through all sequences of integers and* $U = (U_n)$, U_n *open in* $J^{m_n}(X, Y)$. *The* $\mathfrak{D}$*-topology is independent of the choice of the sequence* (K_n).

b) Fix a sequence (d_n) *of metrics* d_n *on* $J^n(X, Y)$, *compatible with the manifold topologies. Then the system of sets of the form*

$$V_\phi(f) = \{ g \in C^\infty(X, Y) \mid \phi_n(x)\, d_n(j^n f(x), j^n g(x)) < 1 \text{ for all } x \in X \text{ and } n = 0, 1, \dots \}$$

is a neighborhood base for $f \in C^\infty(X, Y)$ *in the* $\mathfrak{D}$*-topology, consisting of open sets, where* $\phi = (\phi_n)$ *runs through all sequences of continuous maps* $\phi : X \to [0, \infty]$ *such that* $(\operatorname{supp} \phi_n)$ *is locally finite. The* $\mathfrak{D}$*-topology is independent of the choice of the metrics* d_n.

c) The system of sets of the form

$$M'(L, U) = \{ f \in C^\infty(X, Y) \mid j^n f(X \backslash L_n{}^\circ) \subset U_n, \; n = 0, 1, \dots \}$$

is a base of open sets for the $\mathfrak{D}$*-topology on* $C^\infty(X, Y)$, *where* $L = (L_n)$ *runs through all sequences of compact sets* $L_n \subset X$ *such that* $(X \backslash L_n{}^\circ)$ *is locally finite and* U_n *open in* $J^n(X, Y)$.

1.6. REMARKS.

a) The $\mathfrak{D}$-topology is finer than the Whitney C^∞-topology, the topology W^∞ in [9], as e.g. described very explicitly in [5], II-3. It is the topology $\mathfrak{C}^\infty$ of Morlet [3].

b) If $X = \mathrm{R}^n$ and $Y = \mathrm{R}$, then the $\mathfrak{D}$-topology on $C^\infty(\mathrm{R}^n, \mathrm{R})$ is the one produced by applying the seminorms of $\mathfrak{D}$ (see e.g. [6]) to $C^\infty(\mathrm{R}^n, \mathrm{R})$:

$$q(f-g) = \infty \quad \text{if } f-g \text{ does not have compact support.}$$

Hence the name. The set $\mathfrak{D}$ (of all functions with compact support in the space $C^\infty(\mathrm{R}^n, \mathrm{R})$) is the maximal linear subspace of $C^\infty(\mathrm{R}^n, \mathrm{R})$ which is a topological linear space with the topology induced from the $\mathfrak{D}$-topology (and from the Whitney C^∞-topology too, but $\mathfrak{D}$ with the Whitney topology has no merits from the point of view of functional analysis and this is our reason for introducing the $\mathfrak{D}$-topology).

c) $C^\infty(X, Y)$ with the $\mathfrak{D}$-topology is a Baire space. This is proved by Morlet [3]. A quite straightforward proof is possible using description 1.5 b, which can also be used to define pseudo-metrics for a uniformity, and $C^\infty(X, Y)$ is complete in this uniformity if the metrics d_n on $J^n(X, Y)$ are chosen to be complete.

1.7. LEMMA. *A sequence (f_n) in $C^\infty(X, Y)$ converges in the $\mathfrak{D}$-topology to f iff there exists a compact set $K \subset X$ such that all but finitely many of the f_n's equal f off K and $j^l f_n \to j^l f$ «uniformly» on K, for all $l \in \mathrm{N}$.*

PROOF. Sufficiency is obvious by looking at the neighborhood base 1.5 b. Necessity follows since $f_n \to f$ in the $\mathfrak{D}$-topology implies $f_n \to f$ in the Whitney topology and there it is well known ([5], II-3, page 43).

1.8. LEMMA. *For each $k \geqslant 0$ the map*

$$j^k : C^\infty(X, Y) \to C^\infty(X, J^k(X, Y))$$

is continuous in the $\mathfrak{D}$-topology.

PROOF. For each l the mapping

$$\alpha_{k,l} : J^{k+l}(X, Y) \to J^l(X, J^k(X, Y))$$

is defined as follows: if $\sigma \in J^{k+l}(X, Y)$, $\alpha(\sigma) = x$ and $f \in C^\infty(X, Y)$ represents σ, then

$$\alpha_{k,l}(\sigma) = j^l(j^k f)(x).$$

It is a routine task to show that $\alpha_{k,l}(\sigma)$ is well defined, i.e. does not depend on the special choice of the representative f, and that it is smooth (in fact, an embedding). By definition we have

$$\alpha_{k,l} \circ j^{k+l} f = j^l(j^k f): X \to J^l(X, J^k(X, Y))$$

for each $f \in C^\infty(X, Y)$. Now we use the basis 1.5 c. Let $M'(L, U)$ be a basic open set in $C^\infty(X, J^k(X, Y))$, i.e.

$$U = (U_n),\ U_n \text{ open in } J^n(X, J^k(X, Y)), \text{ and } L = (L_n).$$

Now set

$$L'_n = \emptyset,\ n = 0, \dots, k\text{-}1,\ L'_{k+l} = L_l,\ l = 0, 1, \dots, \text{ and } L' = (L'_n).$$

Set

$$U'_n = \emptyset,\ n = 0, \dots, k\text{-}1,\ U'_{k+l} = (\alpha_{k,l})^{-1}(U_l),\ l = 0, 1, \dots,$$

which is open in $J^{k+l}(X, Y)$, and denote $U' = (U'_n)$. Then $M'(L', U')$ is a basic open set in $C^\infty(X, Y)$, and

$$(j^k)^{-1}(M'(L, U)) = M'(L', U'),$$

which can be seen by writing out the definitions.

Q.E.D.

1.9. If A, B, P are topological Hausdorff spaces and

$$\pi_A : A \to P,\quad \pi_B : B \to P$$

are continuous maps, we consider the set $A \underset{P}{\times} B$ defined by:

$$A \underset{P}{\times} B = \{ (a, b) \in A \times B \mid \pi_A(a) = \pi_B(b)\},$$

with the topology induced from $A \times B$. Then $A \underset{P}{\times} B$ is the pullback of the mappings π_A, π_B in the category of Hausdorff topological spaces.

A continuous map is called proper if the inverse image of a compact subset is compact. The subset $C^\infty prop(X, Y)$ of proper maps of $C^\infty(X, Y)$ is open in the $\mathfrak{D}$-topology, since it is open in the coarser Whitney topology (see [9], 2, Proposition 4).

LEMMA ([9], 2, *Lemma 1*). *Let A, B and P be Hausdorff topological spaces. Suppose P is locally compact and paracompact. Let $\pi_A : A \to P$ and $\pi_B : B \to P$ be continuous mappings. Let $K \subset A$ and $L \subset B$ be such that π_A / K and π_B / L are proper. Let U be an open neighborhood of $K \underset{P}{\times} L$ in $A \underset{P}{\times} B$. Then there exist open neighborhoods V of K in A and W of L in B such that $K \underset{P}{\times} L \subset V \underset{P}{\times} W \subset U$.*

1.10. PROPOSITION. *If X, Y, Z are smooth manifolds, then composition:*

$$C^\infty(Y, Z) \times C^\infty prop(X, Y) \to C^\infty(X, Z),$$

given by $(g, f) \to g \circ f$, is continuous in the $\mathfrak{D}$-topology.

PROOF. Let $(g, f) \in C^\infty(Y, Z) \times C^\infty prop(X, Y)$ and let $M'(L, K)$ be a basic $\mathfrak{D}$-open neighborhood of $g \circ f$ in $C^\infty(X, Z)$ (cf. 1.5 c), i.e. $L = (L_n)$ and $U = (U_n)$, where each L_n is compact in X with $(X \backslash L_n{}^\circ)$ locally finite, U_n open in $J^n(X, Z)$ and

$$j^n(g \circ f)(X \backslash L_n{}^\circ) \subset U_n \quad \text{for all } n \geq 0.$$

Now consider the topological pullback for each $n \geq 0$, as in 1.9:

$$\begin{array}{ccc} J^n(Y, Z) \underset{Y}{\times} J^n(X, Y) & \xrightarrow{\ \pi_2\ } & J^n(X, Y) \\ \pi_1 \Big\downarrow & & \Big\downarrow \beta \\ J^n(Y, Z) & \xrightarrow{\ \alpha\ } & Y. \end{array}$$

The maps

$$\gamma_n : J^n(Y, Z) \underset{Y}{\times} J^n(X, Y) \to J^n(X, Z), \quad \gamma_n(\sigma, \tau) = \sigma \circ \tau,$$

are well defined, since

$$\alpha(\sigma) = \beta(\tau) \quad \text{for} \quad (\sigma, \tau) \in J^n(Y, Z) \underset{Y}{\times} J^n(X, Y),$$

and they are smooth, since they are locally just composition of polynomials without constant term, followed by truncation to order n. We have

$$\begin{aligned} &\gamma_n(j^n g(Y) \underset{Y}{\times} j^n f(X \backslash L_n{}^\circ)) = \\ &= \gamma_n(\{(\sigma, \tau) \in j^n g(Y) \times j^n f(X \backslash L_n{}^\circ) \mid \alpha(\sigma) = \beta(\tau)\}) = \\ &= \{\sigma \circ \tau \mid \sigma \in j^n g(Y),\ \tau \in j^n f(X \backslash L_n{}^\circ),\ \alpha(\sigma) = \beta(\tau)\} = \end{aligned}$$

$$= j^n(g \circ f)(X \setminus L_n^{\circ}).$$

So

$$j^n g(Y) \underset{Y}{\times} j^n f(X \setminus L_n^{\circ}) \subset \gamma_n^{-1}(U_n) \quad \text{for all} \quad n \geq 0.$$

$\alpha / j^n g(Y)$ is proper, since it is a homeomorphism, inverse to

$$j^n g : Y \to j^n g(Y).$$

$\beta / j^n f(X \setminus L_n^{\circ})$ is proper : If C is compact in Y, then

$$(\beta / j^n f(X \setminus L_n^{\circ}))^{-1}(C) = j^n f((f/(X \setminus L_n^{\circ}))^{-1}(C)) =$$

$$= j^n f((X \setminus L_n^{\circ}) \cap f^{-1}(C)) =$$

$$= j^n f(\text{compact set}) = \text{compact set},$$

since f is proper. So all the hypotheses of Lemma 1.10 are fulfilled, so there exist neighborhoods

V_n of $j^n g(Y)$ in $J^n(Y, Z)$ and W'_n of $j^n f(X \setminus L_n^{\circ})$ in $J^n(X, Y)$

for each $n \geq 0$ such that

$$j^n g(Y) \underset{Y}{\times} j^n f(X \setminus L_n^{\circ}) \subset V_n \underset{Y}{\times} W'_n \subset \gamma_n^{-1}(U_n).$$

Since f is proper, Y is locally compact and $X \setminus L_n^{\circ}$ is locally finite, the family $(f(X \setminus L_n^{\circ}))$ is locally finite too. So there exists a sequence of compact sets K_n in Y such that $f(X \setminus L_n^{\circ}) \subset Y \setminus K_n$ and $(Y \setminus K_n^{\circ})$ is locally finite. Then we have

$$\beta(j^n f(X \setminus L_n^{\circ})) = f(X \setminus L_n^{\circ}) \subset Y \setminus K_n$$

and $Y \setminus K_n$ is open. So $\beta^{-1}(Y \setminus K_n)$ is open in $J^n(X, Y)$, and if we set $W_n = W'_n \cap \beta^{-1}(Y \setminus K_n)$, then we have $j^n f(X \setminus L_n^{\circ}) \subset W_n$ and W_n is open in $J^n(X, Y)$. Moreover

$$j^n g(Y) \underset{Y}{\times} j^n f(X \setminus L_n^{\circ}) \subset V_n \underset{Y}{\times} W_n \subset V_n \underset{Y}{\times} W'_n \subset j_n^{-1}(U_n).$$

Now let $K = (K_n)$, $V = (V_n)$, $W = (W_n)$. Then $g \in M'(K, V)$ since

$$j^n g(Y \setminus K_n^{\circ}) \subset j^n g(Y) \subset V_n \quad \text{for all} \quad n \geq 0,$$

and $f \in M'(L, W)$ since $j^n f(X \setminus L_n^{\circ}) \subset W_n$ for all $n \geq 0$. Now if

$$g' \in M'(K, V) \quad \text{and} \quad f' \in M'(L, W),$$

then for all $n \geq 0$ and $x \in X \setminus L_n^{\circ}$ we have

$$f'(x) = \beta j^n f'(x) \epsilon \beta j^n f'(X \backslash L_n{}^\circ) \subset \beta(W_n) \subset Y \backslash K_n \subset Y \backslash K_n{}^\circ,$$

so

$$(j^n g'(f'(x)), j^n f'(x)) \epsilon V_n \underset{Y}{\times} W_n \subset \gamma_n{}^{-1}(U_n),$$

so

$$j^n(g' \circ f')(x) = \gamma_n(j^n g'(f'(x)), j^n f'(x)) \epsilon U_n,$$

hence $g' \circ f' \epsilon M'(L, U)$.

Q.E.D.

Note that we used Y locally compact in the proof.

2. THE $\mathfrak{D}^\infty$-TOPOLOGY ON $C^\infty(X, Y)$.

2.1. DEFINITION. Let X, Y be smooth finite dimensional manifolds. If $f, g \epsilon C^\infty(X, Y)$ and the set

$$\{ x \epsilon X \mid f(x) \neq g(x) \}$$

is relatively compact in X, we call f *equivalent to* g ($f \sim g$).

This is clearly an equivalence relation. The $\mathfrak{D}^\infty$-topology on the set $C^\infty(X, Y)$ is now the weakest among all topologies on $C^\infty(X, Y)$ which are finer than the $\mathfrak{D}$-topology and for which all equivalence classes of the above relation are open.

2.2. REMARK. The $\mathfrak{D}^\infty$-topology on $C^\infty(X, Y)$ is given by the following process: take all equivalence classes with the topology induced from the $\mathfrak{D}$-topology and take their disjoint union. It is clear how to translate the different descriptions of the $\mathfrak{D}$-topology given in 1.5: In 1.5 a and c, just take all intersections of basic $\mathfrak{D}$-open sets with equivalence classes. In 1.5 b, add $f \sim g$ to the definition of $V_\phi(f)$.

2.3. COROLLARY. *A sequence (f_n) in $C^\infty(X, Y)$ converges in the $\mathfrak{D}^\infty$-topology iff there exists a compact set $K \subset X$ such that all but a finite number of the f_n's equal f off K and $j^l f_n \to j^l f$ «uniformly on K» for all l.*

2.4. COROLLARY. *For each $k \geq 0$ the map*

$$j^k : C^\infty(X, Y) \to C^\infty(X, J^k(X, Y))$$

is continuous in the $\mathfrak{D}^{\infty}$*-topology.*

PROOF. j^k respects the equivalence relation.

2.5. COROLLARY. *Let* X, Y, Z *be smooth manifolds. Then the subset of proper maps* $C^{\infty}prop(X,Y)$ *is* $\mathfrak{D}^{\infty}$*-open in* $C^{\infty}(X,Y)$ *and composition*

$$C^{\infty}(Y,Z)\times C^{\infty}prop(X,Y)\to C^{\infty}(X,Z)$$

is continuous in the $\mathfrak{D}^{\infty}$*-topology.*

PROOF. If

$$g_1 \sim g_2 \quad \text{and} \quad f_1 \sim f_2 ,$$

where f_1, f_2 are proper, then $g_1 \circ f_1 \sim g_2 \circ f_2$.

2.6. REMARKS.

a) By 2.3 and 1.7 convergence of sequences in $C^{\infty}(X,Y)$ is equivalent for the Whitney C^{∞}-topology, the $\mathfrak{D}$-topology and the $\mathfrak{D}^{\infty}$-topology. Therefore the Thom Transversality-Theorem and the Multijet-Transversality-Theorem (cf. [5], II, Theorems 4.9 and 4.13) hold for the $\mathfrak{D}$-topology and the $\mathfrak{D}^{\infty}$-topology too, since in the proofs of these, convergent sequences are constructed.

b) $C^{\infty}(X,Y)$ with the $\mathfrak{D}^{\infty}$-topology is in general no Baire space. The reason for this will become clear in Section 3. However, it is paracompact and normal (cf. 3.9).

2.7. PROPOSITION. *Let* $\pi : E \to X$ *be a smooth finite dimensional vector bundle. Let* $\mathfrak{D}(E)$ *denote the space of all smooth sections with compact support of this bundle. Then* $\mathfrak{D}(E)$, *with the topology induced from the* $\mathfrak{D}^{\infty}$*-topology on* $C^{\infty}(X,E)$, *is a locally convex topological linear space, in fact a dually nuclear (LF)-space. Furthermore it is a Lindelöf space, hence paracompact and normal.*

PROOF. $C^{\infty}(E)$, the space of all sections, is clearly $\mathfrak{D}^{\infty}$-closed in the space $C^{\infty}(X,E)$, and $\mathfrak{D}(E)$ is closed and open in $C^{\infty}(E)$. There exists a vector bundle $\pi_1 : F \to X$ such that the Whitney sum $E \oplus F \to X$ is trivial, thus a space $X \times \mathrm{R}^n$. Then $C^{\infty}(E \oplus F) = C^{\infty}(X, \mathrm{R}^n)$ and the $\mathfrak{D}^{\infty}$-topology on it is exactly the topology induced from the $\mathfrak{D}^{\infty}$-topology on $C^{\infty}(X, E \oplus F)$.

Then $\mathfrak{D}(E)$ is a topological subspace, in fact, a direct summand of

$$\mathfrak{D}(E \oplus F) = \mathfrak{D}(X, \mathrm{R}^n) = \mathfrak{D}(X \times \mathrm{R})^n$$

and the topology is exactly the topology of L. Schwartz, as is seen by comparing 1.5 b with one of the well known systems of seminorms for $\mathfrak{D}(X \times \mathrm{R})^n$ (compare [6] page 170). Therefore it is a (LF)-space, nuclear and dually nuclear, and locally convex inductive limit of countably separable Frechet spaces (which can be identified with

$$\{ f \in \mathfrak{D}(X \times \mathrm{R})^n \mid supp\, f \subset K_l \}$$

where $K = (K_l)$ is a sequence of compact sets of X as in 1.2. Each of of these is a Lindelöf space (separable and metrizable), so $\mathfrak{D}(X \times \mathrm{R})^n$ and its closed subspace $\mathfrak{D}(E)$ is a Lindelöf space, and clearly completely regular, hence paracompact and normal.

Q. E. D.

3. THE MANIFOLD STRUCTURE ON $C^\infty(X, Y)$ EQUIPPED WITH THE $\mathfrak{D}^\infty$-TOPOLOGY.

3.1. DEFINITION. Let X be a submanifold of the smooth manifold Y. A tubular neighborhood of X in Y is an open subset Z of Y together with a submersion $\pi : Z \to X$ such that:

a) $\pi : Z \to X$ is a vector bundle;

b) the embedding $X \to Z$ is the zero section of this bundle.

3.2. PROPOSITION. *Let X, Y be smooth finite dimensional manifolds, let $f \in C^\infty(X, Y)$ and denote the graph*

$$\{ (x, f(x)) \mid x \in X \} \subset X \times Y \quad of\ f$$

by X_f. Then there exists a tubular neighborhood Z_f of X_f in $X \times Y$ with vertical projection, i. e. the submersion $\pi : Z_f \to X_f$ is just the restriction to Z_f of the mapping $(x, y) \to (x, f(x))$ from $X \times Y$ onto X_f.

PROOF. We have $X_f \subset X \times Y$, so $T_{X_f}(X_f)$ is a subbundle of $T_{X_f}(X \times Y)$. We claim that for each $(x, f(x)) \in X_f$ the space $T_{(x, f(x))}(X_f)$ is transversal to

$$T_{(x,f(x))}(\{x\}\times Y) = \{0_x\}\times T_{f(x)}(Y)$$

in $T_{(x,f(x))}(X\times Y)$: any vector $v \in T_{(x,f(x))}(X_f)$ has the form $(dc)_0(\frac{\partial}{\partial t})$, where $c: \mathrm{R}\to X_f$ is a smooth path with $c(0) = (x, f(x))$ and $\frac{\partial}{\partial t}$ is the unit vector in $T_0(\mathrm{R})$. Such a path is of the form

$$c(t) = (c_1(t), f(c_1(t)))$$

for a smooth path $c_1: \mathrm{R}\to X$ with $c_1(0) = x$.

$$v \neq 0 \quad \text{iff} \quad (dc)_0 \neq 0 \quad \text{iff} \quad (dc_1)_0 \neq 0,$$

so if $v \neq 0$ then

$$v = ((dc_1)_0(\frac{\partial}{\partial t}), (df)_x(dc_1)_0(\frac{\partial}{\partial t}))$$

has non-zero first coordinate and

$$v \notin T_{(x,f(x))}(\{x\}\times Y) = \{0_x\}\times T_{f(x)}(Y).$$

Thus

$$T_{(x,f(x))}(X_f)\cap T_{(x,f(x))}(\{x\}\times Y) = \{0\}$$

and transversality follows since $\dim X_f = \dim X$. So

$$E = \bigcup_{X_f} T_{(x,f(x))}(\{x\}\times Y)$$

is a vector bundle over X_f since it is just the pullback of the vector bundle $\bigcup_{X\times Y} T_{(x,y)}(\{x\}\times Y)$ over $X\times Y$ via the embedding $X_f\to X\times Y$, and it is a realization of the normal bundle to $T_{X_f}(X_f)$ in $T_{X_f}(X\times Y)$. Now let:

$$exp^X : U\subset T(X)\to X, \quad exp\, Y: V\subset T(Y)\to Y$$

be exponential maps, defined on neighborhoods U and V of the zero sections respectively. Then

$$(exp^X, exp^Y): U\times V\subset T(X)\times T(Y)\to X\times Y$$

is an exponential for $X\times Y$, given explicitly by

$$(exp^X, exp^Y)_{(x,y)}(v_x, w_y) = (exp^X_x(v_x), exp^Y_y(w_y)).$$

$(exp^X, exp^Y)/(U\times V)\cap E$ gives a diffeomorphism of

$$\bigcup_x(\{0_x\}\times V)\subset \bigcup_x(\{0_x\}\times T_{f(x)}(Y)) = E$$

onto our open neighborhood Z_f of X_f in $X \times Y$, given by

$$(0, v_x) \in T_{(x,f(x))}(\{x\} \times Y) = E_{(x,f(x))} \to (x, \exp^Y_{f(x)}(v_x)),$$

if $v_x \in V$. We denote it by exp.

Clearly there is a diffeomorphism ϕ from the neighborhood $W = \bigcup_x (\{0_x\} \times V)$ of the zero section of E onto the whole of E, which is fibre preserving (cf. [5], II, Lemma 4.7), i.e. such that $\pi \circ \phi = \pi / W$, where $\pi : E \to X_f$ is the projection. It is easily checked that the diagram

$$\begin{array}{ccccc} E & \xrightarrow{\phi^{-1}} & W & \xrightarrow{exp} & Z \\ & \searrow^{\pi} & & \swarrow_{\rho} & \\ & & X_f & & \end{array}$$

commutes, where $\rho : Z \to X_f$ is the restriction to Z of the map

$$X \times Y \to X_f, \quad (x, y) \to (x, f(x)).$$

So we have a fibre preserving diffeomorphism $\tau = exp \circ \phi^{-1} : E \to Z$, thus making $\rho : Z_f \to X_f$ into a vector bundle.

Q. E. D.

3.3. For further reference we repeat the situation in detail: For each f in $C^\infty(X, Y)$, we have a vector bundle $\pi_f : E(f) \to X_f$ given by

$$E(f) = \bigcup_X T_{(x,f(x))}(\{x\} \times Y) \subset T_{X_f}(X \times Y),$$

an open neighborhood Z_f of X_f in $X \times Y$ together with the vertical projection $\rho_f : Z_f \to X_f$,

$$\rho_f(x, y) = (x, f(x)) \quad \text{for} \quad (x, y) \in Z_f,$$

and a fibre preserving diffeomorphism $\tau_f : E(f) \to Z_f$, i.e. the diagram

$$\begin{array}{ccc} E(f) & \xrightarrow{\tau_f} & Z_f \\ & \searrow^{\pi_f} \quad \swarrow_{\rho_f} & \\ & X_f & \end{array}$$

commutes. For $f, g \in C^\infty(X, Y)$ we have the diffeomorphisms

$$\sigma^f_g : X_f \to X_g, \quad \sigma^f_g(x, f(x)) = (x, g(x))$$

and

$$\sigma^f : X_f \to X, \quad \sigma^f(x, f(x)) = x.$$

They satisfy

$$\sigma_g^f \circ \sigma_f^h = \sigma_g^h, \quad \sigma^f \circ \sigma_f^g = \sigma^g, \quad \sigma_f^f = id_{X_f}.$$

3.4. Since the main idea of the construction of the manifold will be blurred up by technicalities later on, we give an outline of it now, disregarding topologies, continuity and differentiability.

For $f \in C^\infty(X, Y)$ we have the setting of 3.3. Let

$$U_f = \{ h \in C^\infty(X, Y) \mid X_h \subset Z_f \},$$

and denote by $C^\infty(Z_f)$ the space of smooth sections of the vector bundle Z_f. Then define $\phi_f : U_f \to C^\infty(Z_f)$ by

$$\phi_f(g)(x, f(x)) = (x, g(x)) \quad \text{for} \quad g \in U_f, \ (x, f(x)) \in X_f$$

and $\psi_f : C^\infty(Z_f) \to U_f$ by

$$\psi_f(s)(x) = \pi_Y s(x, f(x)) \quad \text{for} \quad s \in C^\infty(Z_f), \ x \in X,$$

where $\pi_Y : X \times Y \to Y$ is the canonical projection. Then $\psi_f = \phi_f^{-1}$ since

$$\phi_f \circ \psi_f(s)(x, f(x)) = (x, \psi_f(s)(x)) = (x, \pi_Y s(x, f(x)) =$$
$$= (\pi_X s(x, f(x)), \pi_Y s(x, f(x)) = s(x, f(x)).$$

($\pi_X s(x, f(x)) = x$ uses the vertical projection of Z_f), and

$$\psi_f \circ \phi_f(g)(x) = \pi_Y \phi_f(g)(x, f(x)) = \pi_Y(x, g(x)) = g(x).$$

We now declare that U_f is a chart for f, and that ϕ_f is the coordinate mapping. We will check the coordinate change now. Let $g \in C^\infty(X, Y)$ be a second map such that $U_f \cap U_g \neq \emptyset$. That means that there is $h \in C^\infty(X, Y)$ with $X_h \subset Z_f \cap Z_g$. Let us check the map $\phi_f \circ \psi_g / \phi_g(U_f \cap U_g)$. If

$$s \in \phi_g(U_f \cap U_g) \subset C^\infty(Z_g),$$

then

$$\phi_f \circ \psi_g(s)(x, f(x)) = (x, \psi_g(s)(x)) =$$
$$= (\pi_X s(x, g(x)), \pi_Y s(x, g(x)) = s(x, g(x)).$$

So $\phi_f \circ \psi_g(s) = s \circ \sigma_g^f$ in the notation of 3.3. But, of course, the linear structure changes.

3.5. REMARK. If U_f in 3.4 should be a chart for f, then we have to introduce a topology on $C^\infty(X,Y)$ such that U_f is open. From the form of U_f it is clear that such a topology must be finer than the Whitney C^0-topology. But with this topology (and the Whitney C^∞-topology too) the space $C^\infty(Z_f)$ is not a topological vector space, only a topological module over the topological ring $C^\infty(X_f, \mathbb{R})$. One could choose this solution and use differential calculus on such modules, as sketched by F. Berquier [1]. The other possible solution, presented here, is to look at the maximal linear subspace of $C^\infty(Z_f)$ which is a topological linear space with the Whitney C^0-topology; this is the space $\mathfrak{D}(Z_f)$ of smooth sections with compact support. But the Whitney C^∞-topology on it has no merits from the point of view of functional Analysis (cf. the topological conclusions of 2.7), so we choose to introduce the $\mathfrak{D}^\infty$-topology. The equivalence relation 2.1 is necessary, if we want to model the manifold on topological vector spaces. It is clear how to modify the construction to obtain one of the other models just mentioned, and most of the proofs which we will give remain valid.

3.6. THEOREM. *Let X, Y be smooth manifolds. Then $C^\infty(X,Y)$ with the $\mathfrak{D}^\infty$-topology is a smooth manifold.*

PROOF. We postpone the discussion of differentiability to 3.7, and we use the notation set up in 3.3. For $f \in C^\infty(X,Y)$ we define the chart U_f by:

$$U_f = \{ g \in C^\infty(X,Y) \mid g \sim f,\ X_g \subset Z_f \}$$
$$= \{ g \in C^\infty(X,Y) \mid j^0 g(X) \subset Z_f \} \cap \{ g \mid g \sim f \}.$$

So U_f is $\mathfrak{D}^\infty$-open. Let $\mathfrak{D}(E(f))$ be the space of smooth sections with compact support of $\pi_f : E(f) \to X$. We define $\phi_f : U_f \to \mathfrak{D}(E(f))$ by:

$$\phi_f(g)(x, f(x)) = \tau_f^{-1}(x, g(x)) \quad \text{for } g \in U_f,\ (x, f(x)) \in X_f,$$

and $\psi_f : \mathfrak{D}(E(f)) \to U_f$ by

$$\psi_f(s)(x) = \pi_Y \circ \tau_f \circ s(x, f(x)) \quad \text{for } s \in \mathfrak{D}(E(f)).$$

Then

$$\phi_f = C^\infty(X_f, \tau_f^{-1}) \circ C^\infty((\sigma^f)^{-1}, Z_f) \circ j^0 / U_f$$

and

$$\psi_f = C^\infty(X, \pi_Y) \circ C^\infty(\sigma^f, E(f))/\mathfrak{D}(E(f)),$$

so both maps are $\mathfrak{D}^\infty$-continuous by 2.5. We claim that ϕ_f and ψ_f are inverse to each other. Note that $\pi_Y \circ j^0 g = g$. Let $s \in \mathfrak{D}(E(f))$. Then

$$\begin{aligned}
&\phi_f \circ \psi_f(s)(x, f(x)) = \tau_f^{-1}(x, \psi_f(s)(x)) = \\
&= \tau_f^{-1}(x, \pi_Y \circ \tau_f \circ s(x, f(x))) = \\
&= \tau_f^{-1}(\pi_X \circ \tau_f \circ s(x, f(x)), \pi_Y \circ \tau_f \circ s(x, f(x)) = \\
&= \tau_f^{-1} \circ \tau_f \circ s(x, f(x)) = s(x, f(x)).
\end{aligned}$$

If on the other hand $g \in U_f$, then

$$\begin{aligned}
\psi_f \circ \phi_f(g)(x) &= \pi_Y \circ \tau_f \circ \phi_f(g)(x, f(x)) = \\
&= \pi_Y \circ \tau_f \circ \tau_f^{-1}(x, g(x)) = g(x).
\end{aligned}$$

Now we study the coordinate change. Let $g \in C^\infty(X, Y)$ such that $U_f \cap U_g$ be non-void. So there is $h \in C^\infty(X, Y)$ with

$$f \sim h \sim g \quad \text{and} \quad X_h \subset Z_f \cap Z_g.$$

We have to check the map $\phi_f \circ \psi_g / \phi_g(U_g \cap U_f)$. Clearly $\phi_g(U_g \cap U_f)$ is open in $\mathfrak{D}(E(f))$. Let $s \in \phi_g(U_g \cap U_f) \subset \mathfrak{D}(E(g))$.

$$\begin{aligned}
&\phi_f \circ \psi_g(s)(x, f(x)) = \tau_f^{-1}(x, \psi_g(s)(x)) = \\
&= \tau_f^{-1}(x, \pi_Y \circ \tau_g \circ s(x, g(x))) = \\
&= \tau_f^{-1}(\pi_X \circ \tau_g \circ s(x, g(x)), \pi_Y \circ \tau_g \circ s(x, g(x))) \\
&= \tau_f^{-1} \circ \tau_g \circ s(x, g(x)).
\end{aligned}$$

So

$$\phi_f \circ \psi_g(s) = \tau_f^{-1} \circ \tau_g \circ s \circ \sigma_g^f$$

or

$$\phi_f \circ \psi_g = C^\infty(X_g, \tau_f^{-1} \circ \tau_g) \circ C^\infty(\sigma_g^f, E(g))/\phi_g(U_g \cap U_f),$$

$C^\infty(\sigma_g^f, E(g)): s \to s \circ \sigma_g^f$ is just carrying over sections of $E(g)$ to the pullback $(\sigma_g^f)^* E(g)$, a vector bundle over X_f, and so is linear:

$$\mathfrak{D}(E(g)) \to \mathfrak{D}((\sigma_g^f)^* E(g)).$$

So we have just to check the differentiability of the map

$$\mathfrak{D}(\tau_f^{-1} \circ \tau_g): \mathfrak{D}((\sigma_g^f)^* E(g)) \to \mathfrak{D}(E(f)),$$

given by $s \to r_f^{-1} \circ r_g \circ s$, and this is assured by Lemma 3.8.

Q.E.D.

3.7. We now have to fix the notion of differentiability we want to use. It is a rather strong concept, the notion C_π^∞ of Keller [7], valid for arbitrary locally convex spaces and strong enough for the chain rule and the Taylor expansion to hold.

Let E and F be locally convex linear spaces, let $f: E \to F$ be a continuous mapping. Then f is of class C_c^1, if for all $x, y \in E$ and $\lambda \in \mathbb{R}$, we have

$$\lim_{|\lambda| \to 0} \frac{1}{\lambda}(f(x+\lambda y) - f(x)) = Df(x)y \quad \text{in } F,$$

where $Df(x)$ is a linear mapping $E \to F$ for each $x \in E$, the derivative of f at x, and the mapping $(x,y) \to Df(x)y$ is continuous as a mapping from $E \times E$ to F. f is of class C_c^2 if $(x,y) \to Df(x)y$ is of class C_c^1, and so on. Keller [7] has shown that $C_c^\infty = C_\pi^\infty$.

If f is of class C_π^1, then it is actually differentiable in an apparently stronger sense: the remainder

$$R(x,y) = f(x+y) - f(x) - Df(x)y$$

fulfills the following condition (see Keller [7], 1.2.8):

(HL') For each seminorm p on F there is a seminorm q on E such that

$$\lim_{y \to 0} \frac{p(R(x,y))}{q(y)} = 0 \quad \text{for all } x \in E.$$

If f is of class C_π^∞, then even the following stronger condition holds:

(HL) For each seminorm p on F there is a seminorm q on E with

$$\lim_{q(y) \to 0} \frac{p(R(x,y))}{q(y)} = 0 \quad \text{for all } x \in E$$

Similar conditions hold for the remainders in Taylor series.

3.8. LEMMA. *Let X be a smooth manifold, let $\pi: E \to X$ and $\rho: F \to X$ be vector bundles over X. Let U be an open neighborhood of the image of a smooth section with compact support s of E and let $\alpha: U \to F$ be a smooth* *fibre preserving map. Then the map*

$$\mathfrak{D}(a): V \to \mathfrak{D}(F), \quad s \to a \circ s,$$

is of class C_π^∞, *where*

$$V = \{ s \in \mathfrak{D}(E) \mid s(X) \subset U \}$$

is open in $\mathfrak{D}(E)$. *Furthermore,* $D\mathfrak{D}(a) = \mathfrak{D}(d_f a)$, *where* $d_f a$ *is the fibre derivative of* a,

$$d_f a(e_x) = d(a/E_x)(e_x) \quad for \quad e_x \in E_x, \quad x \in X.$$

PROOF. If we can show that

$$D\mathfrak{D}(a)(s)\hat{s} = \mathfrak{D}(d_f a)(s, \hat{s}) = d_f a \circ (s, \hat{s})$$

holds in the $\mathfrak{D}^\infty$-topology for $s \in V$, $\hat{s} \in \mathfrak{D}(E)$, then $\mathfrak{D}(a)$ is of class C_π^∞, since the continuity condition for $\mathfrak{D}(d_f a)$ is automatically fulfilled by 2.5, so $\mathfrak{D}(a)$ is of class C_π^1, and $D\mathfrak{D}(a) = \mathfrak{D}(d_f a)$ is of the same form as $\mathfrak{D}(a)$, so it is of class C_π^1 again, and so on.

So we have to show that, for $\lambda \in \mathbb{R}\backslash\{0\}$:

$$(1) \quad \lim_{\lambda \to 0} \frac{1}{\lambda}(\mathfrak{D}(a)(s + \lambda\hat{s}) - \mathfrak{D}(a)(s)) = \mathfrak{D}(d_f a)(s, \hat{s})$$

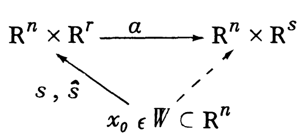
holds in the $\mathfrak{D}^\infty$-topology; we will make use of Corollary 2.3. For $x \in X$,

$$(2) \quad \frac{1}{\lambda}(a \circ (s + \lambda\hat{s}) - a \circ s)(x) = \frac{1}{\lambda}(a(s(x) + \lambda\hat{s}(x)) - a(\hat{s}(x)))$$

converges by the definition of $d_f a$ to

$$(3) \quad d_f a(s(x))\,\hat{s}(x).$$

Outside of the compact support of $\hat{s}$ both expressions give zero. So we only have to show that on the compact support of the section $\hat{s}$ all «partial derivatives» of (2) with respect to x converge uniformly to those of (3). And for this it suffices to show that for each $x_0 \in supp\,\hat{s}$ all «partial derivatives» converge uniformly on some neighborhood of x_0. So we can take a locally trivializing chart W about x_0 and have now the situation



In this set up we have

$$s(x) = (x, t(x)), \quad \hat{s}(x) = (x, \hat{t}(x)) \quad \text{and} \quad a(x,y) = (x, a_x(y))$$

for $x \in W$, where $t, \hat{t}: W \to \mathbb{R}^r$ are smooth and $a_x : \mathbb{R}^r \to \mathbb{R}^s$.

Step 1. We will show that for $\lambda \to 0$ the expression

$$(4) \qquad \frac{1}{\lambda}(a_x(t(x) + \lambda \hat{t}(x)) - a_x(t(x)))$$

converges uniformly on some neighborhood of x_0 to

$$(5) \qquad da_x(t(x))\hat{t}(x).$$

For that we use the mean value theorem in the form of Dieudonné [4] (8. 5.4):

Let E and F be two Banach spaces, let f be a continuous mapping into F of a neighborhood of a segment S joining two points a, b of E. If f is differentiable on S, then

$$\|f(b) - f(a)\| \leqslant \sup_{c \in S} \|f'(c)\| \,.\, \|b - a\|.$$

We set $f - f'(d)$ for f and get

$$\|f(b) - f(a) - f'(d)(b-a)\| \leqslant \sup_{c \in S} \|f'(c) - f'(d)\| \,.\, \|b-a\|.$$

In our case this looks like

$$(6) \quad \left\|\frac{1}{\lambda}(a_x(t(x)+\lambda\hat{t}(x)) - a_x(t(x))) - da_x(t(x))\hat{t}(x)\right\|$$

$$\leqslant |\lambda| \, \|\hat{t}(x)\| \,.\, \sup_{r \in [0,1]} \frac{1}{|\lambda|} \|da_x(t(x)+r\lambda\hat{t}(x)) - da_x(t(x))\|$$

$$\leqslant \sup_{x \in W} \|\hat{t}(x)\| \,.\, \epsilon < K \,.\, \epsilon,$$

if $|\lambda| < \omega(\epsilon)$ and $\|x - x_0\| < \omega_0$ by the uniform continuity of the mapping

$$(\lambda, x) \to da_x(t(x)+\lambda\hat{t}(x)) - da_x(t(x)).$$

Step 2. We now show that the differential of (4) with respect to x converges uniformly on some neighborhood of x_0 to the differential of (5). We do this by reducing to Step 1. So we compute first the differentials of (4) and (5). But first some preliminaries: Let $p: \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^s$ be the canonical projection, and for $x \in X$ let $j_x : \mathrm{R}^r \to \mathrm{R}^{n+r}$ be the injection such that $j_x(y) = (x, y)$. Then we have $a_x = p \circ a \circ j_x$, so

$$(7) \quad da_x(y) = d(p \circ a \circ j_x)(y) = dp(\dots) \circ d(a \circ j_x)(y) = = p \circ d_2 a(x, y),$$

where d_2 designs the second partial derivative. $d_2 a$ is a mapping:

$$\mathrm{R}^n \times \mathrm{R}^r \to L(\mathrm{R}^r, \mathrm{R}^n \times \mathrm{R}^s);$$

to it corresponds a mapping

$$\overline{d_2 a} : \mathrm{R}^n \times \mathrm{R}^r \times \mathrm{R}^r \to \mathrm{R}^n \times \mathrm{R}^s,$$

given by $\overline{d_2 a}(x, y, z) = d_2 a(x, y) z$. If

$$e : L(\mathrm{R}^r, \mathrm{R}^n \times \mathrm{R}^s) \times \mathrm{R}^r \to \mathrm{R}^n \times \mathrm{R}^s$$

designs evaluation, i.e. $e(f, y) = f(y)$, then we have

$$\overline{d_2 a} = e \circ (d_2 a \times Id),$$

since

$$e \circ (d_2 a \times Id)((x, y), z) = e(d_2 a(x, y), z) = d_2 a(x, y) z.$$

So we can compute the derivative of $\overline{d_2 a}$:

$$\begin{aligned}(8) \quad & d(\overline{d_2 a})(x, y, z)(h, k, l) = \\ & = \{ de(d_2 a(x, y), z) \circ [d(d_2 a)(x, y) \times dId(z)] \}(h, k, l) \\ & = de(d_2 a(x, y), z)(d(d_2 a)(x, y)(h, k), l) \\ & = d_2 a(x, y) l + [d(d_2 a)(x, y)(h, k)] z,\end{aligned}$$

since e is bilinear, so

$$de(f, z)(\hat{f}, \hat{z}) = e(f, \hat{z}) + e(\hat{f}, z).$$

Now we are ready to compute the derivative of (5):

$$\begin{aligned} & da_x(t(x))\hat{t}(x) = p[d_2 a(x, t(x))\, \hat{t}(x)] \\ & = p[\overline{d_2 a}(x, t(x), \hat{t}(x))] = [p \circ \overline{d_2 a} \circ (Id, t, \hat{t})](x). \\ & d(p \circ d_2 a \circ (Id, t, \hat{t}))(x)(y) = \\ & = [dp(\dots) \circ d(\overline{d_2 a})(x, t(x), \hat{t}(x)) \circ d(Id, t, \hat{t})(x)](y) \\ & = p[d(\overline{d_2 a})(x, t(x), \hat{t}(x))(y, dt(x)y, d\hat{t}(x)y)] \end{aligned}$$

$$= p\{ d_2 a(x,t(x))d\hat{t}(x)y + [d(d_2 a)(x,t(x))(y,dt(x)y)]\hat{t}(x)\}$$
$$= da_x(t(x))d\hat{t}(x)y + p\{[\dots]\hat{t}(x)\}.$$

Now we set

$$p* = L(Id,p): L(\mathrm{R}^r, \mathrm{R}^n \times \mathrm{R}^s) \to L(\mathrm{R}^r, \mathrm{R}^s),$$

then $p*$ is again linear and so for any function f into $L(\mathrm{R}^r, \mathrm{R}^n \times \mathrm{R}^s)$, we have

$$(9) \qquad d(p* \circ f)(\xi) = dp*(\dots) \circ df(\xi) = p* \circ df(\xi).$$

So:

$$p\{[d(d_2 a)(x,t(x))(y,dt(x)y)]\hat{t}(x)\} =$$
$$= \{[p* \circ d(d_2 a)(x,t(x))](y,dt(x)y)\}\hat{t}(x) =$$
$$= [d(p* \circ d_2 a)(x,t(x))(y,dt(x)y)]\hat{t}(x) =$$
$$= [d(d_2 a)(x,t(x))(y,dt(x)y)]\hat{t}(x),$$

since

$$(p* \circ d_2 a)(x,y) = p*(d_2 a(x,y)) =$$
$$= p \circ d_2 a(x,y) = da_x(y) = d_f a(x,y)$$

by (7). So the derivative of (5) is the following expression

$$(10) \qquad d(p \circ \overline{da} \circ (Id,t,\hat{t}))(x)(y) =$$
$$= d_f a(x,t(x))d\hat{t}(x)y + [d(d_f a)(x,t(x))(y,dt(x)y)]\hat{t}(x).$$

Now we compute the derivative with respect to x of (4):

$$\frac{1}{\lambda}[a_x(t(x)+\lambda\hat{t}(x)) - a_x(t(x))] =$$
$$= \frac{1}{\lambda}[(p \circ a)(x,t(x)+\lambda\hat{t}(x)) - (p \circ a)(x,t(x))] =$$
$$= \frac{1}{\lambda}[p \circ a \circ (Id, t+\lambda\hat{t}) - p \circ a \circ (Id,t)](x).$$

$$(11) \qquad d[\frac{1}{\lambda}[p \circ a \circ (Id,t+\lambda\hat{t}) - p \circ a \circ (Id,t)]](x)(y) =$$
$$= \frac{1}{\lambda}[dp(\dots) \circ da(x,t(x)+\lambda\hat{t}(x)) \circ d(Id, t+\lambda\hat{t})(x)$$
$$- dp(\dots) \circ da(x,t(x)) \circ d(Id,t)(x)](y) =$$

$$= \frac{1}{\lambda} p[da(x, t(x) + \lambda \hat{t}(x))(y, dt(x)y + \lambda d\hat{t}(x)y)$$

$$- da(x, t(x))(y, dt(x)y)]$$

$$= \frac{1}{\lambda} p[d_1 a(x, t(x) + \lambda \hat{t}(x))y$$

$$+ d_2 a(x, t(x) + \lambda \hat{t}(x))(dt(x)y + \lambda d\hat{t}(x)y)$$

$$- d_1 a(x, t(x))y - d_2 a(x, t(x))dt(x)y]$$

$$= \frac{1}{\lambda}[p\, d_1 a(x, t(x) + \lambda \hat{t}(x))y + d_f a(x, t(x) + \lambda \hat{t}(x))dt(x)y$$

$$+ \lambda d_f a(x, t(x) + \lambda \hat{t}(x))d\hat{t}(x)y - p\, d_1 a(x, t(x))y$$

$$- d_f a(x, t(x))dt(x)y].$$

Now we put together pieces from (11). First we consider

$$(12) \quad p([\tfrac{1}{\lambda}(d_1 a(x, t(x) + \lambda \hat{t}(x)) - d_1 a(x, t(x)))](y));$$

the expression in the square bracket is of the same form as (2), so by Step 1 we can conclude that this converges uniformly on a neighborhood of x_0 for $\lambda \to 0$ to

$$(13) \quad p([d_2 d_1 a(x, t(x))(\hat{t}(x))](y))$$

$$= p[d_2 d_1 a(x, t(x))(\hat{t}(x), y)]$$

$$= p[d_1 d_2 a(x, t(x))(y, \hat{t}(x))]$$

$$= [p_* \circ d_1(d_2 a)(x, t(x))](y)(\hat{t}(x))$$

$$= [d_1(p_* \circ d_2 a)(x, t(x))](y)(\hat{t}(x))$$

$$= [d_1(d_f a)(x, t(x))(y)]\hat{t}(x).$$

Then we consider

$$(14) \quad \tfrac{1}{\lambda}[(d_f a(x, t(x) + \lambda \hat{t}(x)) - d_f a(x, t(x))]\, dt(x)y.$$

This is again of the same form as (2), so by Step 1 this again converges uniformly on a neighborhood of x_0 for $\lambda \to 0$ to

$$(15) \quad [d_2(d_f a)(x, t(x))(\hat{t}(x))]\, dt(x)y$$

$$= [d_2(p_* \circ d_2 a)(x, t(x))(\hat{t}(x))]\, dt(x)y$$

$$= p[d_2 d_2 a(x, t(x))(\hat{t}(x), dt(x)y)]$$

$$= p[\, d_2 d_2 a(x, t(x))(dt(x)y, \hat{t}(x))]$$
$$= [\, d_2(d_f\, a)(x, t(x))(dt(x)y)]\, \hat{t}(x).$$

Then

$$(13) + (15) = [\, d(d_f a)(x, t(x))(y, dt(x)y)]\, \hat{t}(x)$$

is one half of (10). The remaining member of (11),

$$d_f a(x, t(x) + \lambda\, \hat{t}(x))d\hat{t}(x)y$$

converges uniformly on a neighborhood of x_0 by uniform continuity of $d_f a$ for $\lambda \to 0$ to $d_f a(x, t(x))d\hat{t}(x)y$, the remaining member of (10).

Step 3: The higher derivatives. In Step 2 we have shown that the derivative with respect to x of the convergence situation (4) → (5) is the sum of two convergence situations (4) → (5) and something which converges clearly uniformly in all derivatives. So for the second derivative we just apply Step 2 to the two parts, and we continue in that way for the higher derivatives.

Q. E. D.

3.9. REMARKS.

(a) In the proof of Lemma 3.8, we used heavily that all sections we considered have compact support; so introducing the equivalence relation 2.1 brought advantage.

(b) By 2.7 each chart $\phi_f : U_f \to \mathfrak{D}(Z_f)$ of $C^\infty(X, Y)$ is paracompact, so $C^\infty(X, Y)$ with the $\mathfrak{D}^\infty$-topology is locally paracompact, thus paracompact. But $\mathfrak{D}(Z_f)$ is not a Baire space, if X is not compact, so $C^\infty(X, Y)$ is no longer a Baire space. One would hope that $C^\infty(X, Y)$ turns out to be an absolute neighborhood retract, but the theory is not very much developped for non-metrizable spaces.

4. MISCELLANY.

4.1. LEMMA. *Let $\pi : E \to X$ be a vector bundle and $a : E \to E$ be a fibre preserving smooth map. If the derivative $D\mathfrak{D}(a)(s_0) : \mathfrak{D}(E) \to \mathfrak{D}(E)$ of $\mathfrak{D}(a)$ at $s_0 \in \mathfrak{D}(E)$ is surjective, then there is an open neighborhood U of the* *image of s_0 and an open neighborhood V of the image of $a \circ s_0$, such that $a: U \to V$ is a diffeomorphism.*

PROOF.

$$D\mathfrak{D}(a)(s_0)s = d_f a(.,s_0(.))s(.)$$

by 3.8. Fix $x \in X$. Let $e_x \in E_x$ and let $\hat{s} \in \mathfrak{D}(E)$ such that $\hat{s}(x) = e_x$ Then by hypothesis there is a $s \in \mathfrak{D}(E)$ such that

$$D\mathfrak{D}(a(s_0))s = d_f a(.,s_0(.))s(.) = \hat{s}(.),$$

i.e.

$$d_f a(x, s_0(x))s(x) = \hat{s}(x) = e_x .$$

So the linear map $d_f a(x, s_0(x)): E_x \to E_x$ is surjective, thus invertible. The Jacobi differential matrix of a (in a local trivialization of E) looks like

$$da = \begin{pmatrix} Id & 0 \\ * & d_f a \end{pmatrix}$$

so it is invertible too on the image of s_0, and by the classical inversion Theorem there is an open neighborhood U_x of $s_0(x)$ in E and an open neighborhood V_x of $a \circ s_0(x)$ in E such that $a: U_x \to V_x$ is a diffeomorphism. If we take

$$U = \bigcup_x U_x, \quad V = \bigcup_x V_x,$$

then $a: U \to V$ is a local diffeomorphism and is trivially surjective. So we have to force injectivity.

Let r be a metric on the bundle E, i.e. a section $r: X \to S^2(E^*)$ such that $r(x)$ is a positive definite, symmetric bilinear form for all $x \in X$. We consider sets of the following form:

$$M(W,\epsilon) = \{ p \in E \mid \pi(p) \in W \text{ and } r(\pi(p))(p - s_0\pi(p), p - s_0\pi(p)) < \epsilon \},$$

where W is open in X and $\epsilon > 0$. We assert that each $s_0(x)$ has a basis of neighborhoods consisting of sets of the form $M(W,\epsilon)$. To prove that, we choose a locally trivializing chart S about x_0, so $E/S = S \times \mathbb{R}^n$ and we equip each fibre $E_x = \mathbb{R}^n$ with the inner product $r(x)$. Let $k(x)$ be the linear automorphism which carries $r(x)$ into the standard inner product of R^n.

$$k: S\times R^n \to S\times R^n, \quad k(x,y) = (x, k(x)y) - s_0(x)$$

is a homeomorphism (in fact a diffeomorphism). The map

$$E/S \longrightarrow S\times R^n \xrightarrow{k} S\times R^n$$

is then a homeomorphism which carries each set of the form $M(W,\epsilon)$ with $W\subset S$ exactly on an open set

$$W\times(\text{ball of radius } \epsilon \text{ about } 0),$$

these are obviously a base of neighborhoods.
So we assume without loss of generality that each U_x is of the form $M(W,\epsilon)$.
We now assert that $a: U\to V$ is injective. If

$$p\neq \hat{p} \text{ in } U \text{ and } \pi(p)\neq\pi(\hat{p}),$$

then

$$\pi a(p)\neq \pi a(\hat{p}), \quad \text{so } a(p)\neq a(\hat{p}).$$

If $p\neq\hat{p}$ in U and $\pi(p)=\pi(\hat{p})$, then

$$p\in M(W,\epsilon) \text{ and } \hat{p}\in M(\hat{W},\hat{\epsilon}).$$

Then $\pi(p)=\pi(\hat{p})\in W\cap\hat{W}$ and if e.g. $\epsilon>\hat{\epsilon}$, then

$$r(\pi(\hat{p}))(\hat{p}-s_0\pi(\hat{p}), \hat{p}-s_0\pi(\hat{p}))<\hat{\epsilon}<\epsilon,$$

so

$$p, \hat{p}\in M(W,\epsilon) \text{ and } a(p)\neq a(\hat{p}),$$

since $a/M(W,\epsilon)$ is a diffeomorphism.

Q.E.D.

4.2. COROLLARY *(Inversion Theorem for special mappings). Let $\pi: E\to X$ be a vector bundle and $a: E\to E$ be a fibre preserving smooth map. If the derivative $D\mathfrak{D}(a)(s_0): \mathfrak{D}(E)\to\mathfrak{D}(E)$ is surjective, then there are open neighborhoods U of s_0 in $\mathfrak{D}(E)$ and V of $a\circ s_0$ in $\mathfrak{D}(E)$ such that $\mathfrak{D}(a): U\to V$ is a diffeomorphism.*

PROOF. By 4.1, there are open neighborhoods U_0 of $s_0(X)$ and V_0 of $a s_0(X)$ in E such that $a: U_0\to V_0$ is a diffeomorphism. If

$$U = \{ s \in \mathfrak{D}(E) \mid s(E) \subset U_0 \} \text{ and } V = \{ s \in \mathfrak{D}(E) \mid s(X) \subset V_0 \},$$

then $\mathfrak{D}(a): U \to V$ has the smooth inverse $\mathfrak{D}((a/U_0)^{-1}): V \to U$.

Q. E. D.

4.3. If $f \in C^\infty(X, Y)$, let us denote by $\mathfrak{D}_f(X, TY)$ the space of «vector-fields along f» with compact support, i. e. the space of smooth maps

$$\phi: X \to TY \text{ such that } \pi_Y \circ \phi = f \text{ and } \{ x \mid \phi(x) \neq 0 \} \text{ is relatively compact in } X.$$

We have that $\mathfrak{D}_f(X, TY) = \mathfrak{D}(f^*TY)$, the space of smooth sections with compact support of the pullback f^*TY. Comparing with 3.3 we see that $\mathfrak{D}(f^*TY) = \mathfrak{D}(E(f))$ via the linear map $s \to s \circ \sigma^f$. Thus we have:

PROPOSITION. *a) $T_f C^\infty(X, Y) = \mathfrak{D}_f(X, TY)$ for all $f \in C^\infty(X, Y)$.*

b) $TC^\infty(X, Y) = \mathfrak{D}(X, TY)$, the space of all smooth mappings a such that $a: X \to TY$ and $\{ x \in X \mid a(x) \neq 0 \}$ is relatively compact in X. If $\pi_Y: TY \to Y$ is the projection, then

$$\mathfrak{D}(X, \pi_Y) = \pi_{C^\infty(X,Y)}: \mathfrak{D}(X, TY) \to C^\infty(X, Y), \quad a \to \pi_Y \circ a,$$

is the projection of $TC^\infty(X, Y)$ and $TC^\infty(X, Y)$ becomes a vector bundle in the obvious sense.

c) If $f \in C^\infty(Y, Y')$ and $g \in C^\infty \text{prop}(X', X)$, then

$$TC^\infty(X, f) = \mathfrak{D}(X, Tf): \mathfrak{D}(X, TY) \to \mathfrak{D}(X, TY'),$$

given by $a \to (Tf) \circ a$, and

$$TC^\infty(g, Y) = \mathfrak{D}(g, TY): \mathfrak{D}(X', TY) \to \mathfrak{D}(X, TY),$$

given by $a \to a \circ g$.

REMARK. b and c show that the tangent bundle $TC^\infty(X, Y)$ has nice functorial properties with the only exception that the contravariant partial functor may only be applied to proper mappings.

PROOF. a is clear from the discussion preceding the proposition. We only note that the tangent space to $s_0 \in \mathfrak{D}(E)$ is again $\mathfrak{D}(E)$ where E is a vector bundle over X. We will show in 4.4 that the definition using smooth paths is equivalent.

b) By the usual definition of the tangent bundle

$$T C^\infty(X, Y) = \bigcup_{f \in C^\infty(X,Y)} T_f C^\infty(X, Y) = \bigcup_{f \in C^\infty(X,Y)} \mathfrak{D}_f(X, TY) =$$

$$= \mathfrak{D}(X, TY).$$

We give $\mathfrak{D}(X, TY)$ the differentiable structure which it inherits by being an open subset of $C^\infty(X, TY)$. Clearly $\mathfrak{D}(X, \pi_Y)$ is the canonical projection, and that it is smooth follows from c. Now we prove that

$$\mathfrak{D}(X, \pi_Y) : \mathfrak{D}(X, TY) \to C^\infty(X, Y)$$

is a vector bundle. Let $f \in C^\infty(X, Y)$ and let O_Y be the zero vectorfield on Y. We again operate with the data from 3.3. It is easily seen that

$$(\sigma^f)_* E(f) = f^* TY,$$

and if Z_f is chosen to be so small that for each $x \in X$ the set

$$\rho_f^{-1}(X) = Z_{f(x, f(x))} \subset Y$$

is a trivializing chart for $f(x) \in Y$, then

$$\bigcup_{x \in X} Z_{f(x, f(x))} \times T_{f(x)} Y = Z_{O_Y \circ f}$$

is a tubular neighborhood with vertical projection of $X_{O_Y \circ f} \subset X \times TY$ and for $E(O_Y \circ f)$ we can choose the Whitney sum

$$(\sigma^{O_Y \circ f})^* f^* TY \oplus (\sigma^{O_Y \circ f})^* f^* TY = (\sigma^{O_Y \circ f})^* f^*(TY \oplus TY),$$

and then

$$\phi_{O_Y \circ f} : U_{O_Y \circ f} \to \mathfrak{D}((\sigma^{O_Y \circ f})^* f^*(TY \oplus TY))$$

and the latter space is linearly homeomorphic to

$$\mathfrak{D}(f^* TY \oplus f^* TY) = \mathfrak{D}(f^* TY) \times \mathfrak{D}(f^* TY).$$

$\mathfrak{D}(X, \pi_Y)$ becomes the projection

$$\mathfrak{D}(f^* TY) \times \mathfrak{D}(f^* TY) \to \mathfrak{D}(f^* TY)$$

under these identifications. If we choose another trivializing chart U_g with $f \in U_g$ (i.e. Z_f small enough), then

$$f^* TY = (\sigma^f)_* E(f) = (\sigma^g)_* (\sigma_g^f)_* E(f) = (\sigma^g)_* E(g) = g^* TY,$$

so $\mathfrak{D}(f^*TY)$ is linearly homeomorphic to $\mathfrak{D}(g^*TY)$.

c) Let $h \in C^\infty(X,Y)$ and consider again the data from 3.3: We have to check that

$$\phi_{f\circ h} \circ C^\infty(X,f) \circ \psi_h : \mathfrak{D}(E(h)) \to \mathfrak{D}(E(f\circ h))$$

is smooth. For $s \in \mathfrak{D}(E(h))$ we have

$$\begin{aligned}
&\{[\phi_{f\circ h} \circ C^\infty(X,f)\circ\psi_h](s)\}(x, fh(x)) = \\
&= (\tau_{f\circ h})^{-1}(x, [f_* \circ \psi_h(s)](x)) = \\
&= (\tau_{f\circ h})^{-1}(x, f\circ \pi_Y \circ \tau_h \circ s(x,h(x))) = \\
&= (\tau_{f\circ h})^{-1} \circ (Id_X \times f)(\pi_X \circ \tau_h \circ s(x,h(x)), \pi_Y \circ \tau_h \circ s(x,h(x))) \\
&= (\tau_{f\circ h})^{-1} \circ (Id_X \times f)\circ \tau_h \circ s(x,h(x)). \\
&[\phi_{f\circ h}\circ C^\infty(X,f)\circ\psi_h](s) = \\
&\quad = (\tau_{f\circ h})^{-1} \circ (Id_X \times f) \circ \tau_h \circ s \circ (\sigma^h)^{-1} \circ (\sigma^{f\circ h})\ ;
\end{aligned}$$

so the mapping is just composition with a fibre preserving smooth function followed by pulling back to another vector bundle, and so is smooth by 3.8. (Compare with the last argument in the proof of 3.6.) Under the identification of $\mathfrak{D}(E(h))$ with $\mathfrak{D}_h(X,TY)$ and of $\mathfrak{D}(E(f\circ h))$ with $\mathfrak{D}_{f\circ h}(X,TY')$ the mapping $TC^\infty(X,f)$ just coincides with $\mathfrak{D}(X,Tf)$ which is seen by writing out the definitions. Likewise one can check that

$$\phi_{h\circ g} \circ C^\infty(g,Y) \circ \psi_h : \mathfrak{D}(E(h)) \to \mathfrak{D}(E(h\circ g))$$

is just pulling back sections, thus linear, if $E(h\circ g)$ is chosen to be $[(\sigma^h)^{-1} \circ \sigma^{h\circ g}]^* E(h)$. So $C^\infty(g,Y)$ is smooth too and $TC^\infty(g,Y)$ is easily seen to coincide with $\mathfrak{D}(g,TY)$ under the appropriate identifications.

Q. E. D.

4.4. Given U open in a space $\mathfrak{D}(E)$, then we can define $T_{s_0} U$ for $s_0 \in U$ as the space of all equivalence classes of smooth paths $\phi : \mathbb{R} \to U$ with $\phi(0) = s_0$, where

$$\phi \sim \psi \quad \text{iff} \quad (d\phi)_0 (\frac{\partial}{\partial t}) = (d\psi)_0 (\frac{\partial}{\partial t}).$$

Via $\phi \to (d\phi)_0(\frac{\partial}{\partial t})$ and $s \to (t \to s_0 + ts)$ it is easily seen that $T_{s_0}U$ coincides with $\mathfrak{D}(E)$.

Furthermore we have :

LEMMA. *If* $\phi : \mathrm{R} \to C^\infty(X, Y)$ *is a smooth path then*

$$(T\phi)(0) = 0 : \mathrm{R} \to T_{\phi(0)}C^\infty(X, Y)$$

iff

$$T(e_x \circ \phi)(0) = 0 : \mathrm{R} \to T_{\phi(0)(x)}Y$$

for all $x \in X$, *where* $e_x : C^\infty(X, Y) \to Y$ *is evaluation at* x. *That is, two paths through* $f \in C^\infty(X, Y)$ *are equivalent iff they are pointwise equivalent at* $f(x)$ *in* Y *for all* $x \in X$.

PROOF.

$$T(e_x \circ \phi)(0) = T(e_x) \circ T(\phi)(0) = e_{(x,\phi(0)(x))}T(\phi)(0)$$

locally.

Q.E.D.

4.5. We give now a sketchy development of an application of the inversion principle 4.1 to stability. The result is probably well known to specialists.

Let us denote by $Diff(X)$ the open subset of diffeomorphisms of $C^\infty(X, X)$. It is a group and there is a right action of it on $C^\infty(X, Y)$. A mapping $f \in C^\infty(X, Y)$ is called source-$\mathfrak{D}^\infty$-stable if the orbit of f under $Diff(X)$ is $\mathfrak{D}^\infty$-open in $C^\infty(X, Y)$.

$$f_*: Diff(X) \to C^\infty(X, Y), \quad \alpha \to f \circ \alpha$$

is smooth by 4.3, and

$$Df_*(Id_X): \mathfrak{D}(TX) \to \mathfrak{D}_f(X, TY)$$

is its derivative at Id_X. f is called infinitesimally source-$\mathfrak{D}^\infty$-stable if $Df_*(Id_X)$ is surjective. Following the lines of [5], V-5-6 it is possible to show that if f is source-$\mathfrak{D}^\infty$-stable, then it is infinitesimally source-$\mathfrak{D}^\infty$-stable.

Now let $\phi_{Id_X} : U_{Id_X} \to \mathfrak{D}(Z_{Id_X})$ be a chart of Id_X in $Diff(X)$, and $\phi_f : U_f \to \mathfrak{D}(Z_f)$ be a chart of f in $C^\infty(X, Y)$. Then

$$\phi_f \circ f_* \circ \psi_{Id_X} : \mathfrak{D}(Z_{Id_X}) \to \mathfrak{D}(Z_f)$$

is given by $s \to (Id_X \times f) \circ s$, followed by a linear isomorphism (pullback) as is seen in the proof of 4.3. If f is infinitesimally source-$\mathfrak{D}^\infty$-stable, then f_* is in a neighborhood of the image of Id_X $(=X)$ given by composition with the fibre preserving map $Id_X \times f$, and the derivative is surjective. As in 4.1 we conclude that the fibre maps $(Id_X \times f)/(Z_{Id_X})_*$ are locally submersions, i. e. f is a submersion, and submersions are source-$\mathfrak{D}^\infty$-stable. So we obtain the result:

The source-$\mathfrak{D}^\infty$-stable mappings in $C^\infty(X, Y)$ are exactly the submersions.

4.6. In an analogous way one can characterize the image-$\mathfrak{D}^\infty$-stable mappings in $C^\infty prop(X, Y)$ as the proper immersions, but one has to replace the argument of 4.5 by one using the adjoint of $D(f^*)(Id_Y)$, and has to consider vector bundle valued distributions. Maybe we will tackle this problem in a later article, as we will do with the canonical Lie-group structure on $Diff(X)$.

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