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SHEAVES AND CONCEPTS : A MODEL-THEORETIC INTERPRETATION OF GROTHENDIECK TOPOI

by Gonzalo E. REYES

INTRODUCTION.

The aim of this expository paper is to give a logical (more precisely: model-theoretic) interpretation of Grothendieck topoi [SGA 4].

We shall see that the theory of these topoi may be considered as an algebraic version of many-sorted logic, finitary as well as infinitary. The main theorems of the general theory will surely ring a bell when viewed in this logical perspective. In particular, a logician recognizes versions of the Lowenheim-Skolem theorem, the Tarski theorem on unions of chains, the method of diagrams and Gödel completeness theorem.

At this point, let us add that this algebraic version of logic is not the usual one in terms of polyadic or cylindric algebras and their homomorphisms, but rather in terms of categories and functors. This may be the reason that topos theory was not recognized as algebraic logic, until the work of Lawvere [L] and its development and extensions by Lawvere himself, Volger [V], Joyal-Reyes [R] and others brought to light the following far reaching analogy (cf. Appendix for relevant definitions):

Topos theory	$\longleftrightarrow$	Model theory
Site	$\longleftrightarrow$	Theory
Fiber (of the site)	$\longleftrightarrow$	Model (of the theory)
Sheaf	$\longleftrightarrow$	Concept (represented by a formula).

In particular, model-theoretic methods to construct models, for instance, may be applied in topos theory to construct fibers to obtain old and new results in a unified manner. This program has been carried out by Michael Makkai and the author [MR] and the details will appear in book form. This paper gives an outline of some aspects of their work.

We shall briefly describe the contents of this paper. In the first chapter we study two concrete examples of our analogy and we show how to associate a (possibly infinitary) theory with a site (Section 2). Intuitively, this theory «expresses elementarily» the structure of the site. This is made precise by defining a model of a theory in a topos (Section 3). An important point to notice is that the theories thus obtained have a simple syntactical structure which leads to the notion of a coherent theory. The second chapter is devoted to soundness and completeness for a formal system of coherent logic. The completeness theorem states the existence of Boolean-valued models for coherent theories and is a variant of Mansfield's theorem [M]. The applications to the theory of Grothendieck's topoi appear in the third chapter. They include several embedding theorems for topoi (Barr's [Ba], Deligne's [SGA 4], ...) as well as a categorical formulation of «conceptual» or Beth-like completeness for certain sites (pretopoi). This formulation is one instance of logical insight gained via categories. Furthermore, we construct a «universal model» of a coherent theory (the so-called classifying topos) as a «completion» of a Lindenbaum-Tarski category of certain formulas.

The last chapter is not directly concerned with the main theme of this paper, the coherent logic in both its model-theoretic and categorical aspects. Rather it deals with intuitionistic infinitary (!) logic in a Grothendieck topos. It is included, however, for the following reasons: the intuitionistic operations are «implicitely» definable in terms of the coherent ones and the use of this «implicit» intuitionistic logic does not increase the coherent consequences of a coherent theory. As an example of this phenomenon, we study Kock's principle that, as far as coherent consequences go, «local rings may be assumed to be fields, provided intuitionistic logic is used». Finally, some embedding theorems for topoi of Chapter II are given here sharper formulations. To help the reader, we added an Appendix with the main definitions and theorems of the theory of Grothendieck's topoi.

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CHAPTER I. SHEAVES AND CONCEPTS

The main theme of this chapter is a mathematical formulation of our basic analogy

$$\text{Topos theory} \longleftrightarrow \text{Model theory.}$$

The theory of Grothendieck topos is mainly concerned with set-valued functors over a small category (cf. Appendix), actually, sheaves over a small site (cf. Appendix). We may think of a functor of this type as a generalization of the notion of «family of sets». Whereas an ordinary family of sets may be viewed as «a variable set parametrized by a constant set», a set-valued functor may be thought of as «a variable set parametrized by a category». In a heraclitean vein, not only functors, but sets themselves, can be viewed as «processes of deformations», since sets may be considered as limiting cases of «variable sets» whose variations «tend to zero». «Everything flows and nothing abides; everything gives way and nothing stays fixed». These processes are usually subordinated to the deformations of a generating structure, which shall be referred to as «the basic domain of deformations». This idea is made precise in the notion of a site. In dialectical contradiction with this interpretation, we may view functors as «logical invariants» of the deformations of the basic domain; namely, our set-valued functors will get more or less identified with formulas. I. e., we may view *functors as «concepts»* of the theory of the basic domain, that is of the laws according to which the deformations of the «generating structure» take place. It is this parmenidean view of functors which is developed here. We hope that their clash will produce light.

1. Algebraic sets (cf. [GD]).

The problem is to study curves, surfaces, ..., given as zero-sets of sets of polynomials with real coefficients, say. We want to be able to distinguish the intersection T of the line $L : y - 1 = 0$ with the circle

$$C : x^2 + y^2 - 1 = 0$$

from the intersection S of L with the line $L' : x = 0$. Classical geometers already considered T as a point with an infinitesimal linear neighbourhood whereas S was seen as just a point. The trouble is that, set-theoretically,

$$S = T = \{(0,1)\} \subset |\mathbf{R}|^2 .$$

The functorial point of view considers T, S and $|\mathbf{R}|$, the underlying set of reals, as «processes of deformations» subordinated to the basic domain of deformations of $|\mathbf{R}|$, defined to be the category $FP(\mathbf{R}\text{-}alg)$ of finitely presented $\mathbf{R}$-algebras. This choice is not unique or even canonical and it has been suggested by mathematical practice.

We recall that an object of the category $\mathbf{R}\text{-}alg$ of $\mathbf{R}$-algebras is a commutative ring A with 1 together with a ring homomorphism $\mathbf{R} \xrightarrow{\phi} A$.

A *morphism* (or $\mathbf{R}$-morphism) is a commutative triangle of ring homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {}^{\phi}\nwarrow & & \nearrow^{\psi} \\ & \mathbf{R} & \end{array}$$

Intuitively we may think of f as a «basic deformation» of A into B and we say that *A is deformed into B via f*. By «abus de langage», we shall denote an $\mathbf{R}$-algebra $\mathbf{R} \xrightarrow{\phi} A$ by A (when the context is clear).

The category $FP(\mathbf{R}\text{-}alg)$ is the full subcategory of $\mathbf{R}\text{-}alg$ whose objects are of the form $\mathbf{R} \to \mathbf{R}[X_1, \ldots, X_n] / I$, where I is a «finitely generated» ideal and the homomorphism is the canonical one. Curves, surfaces, ..., can now be identified with (covariant) set-valued functors on $FP(\mathbf{R}\text{-}alg)$. Thus, T is identified with the functor which sends

$$\mathbf{R} \xrightarrow{\phi} A \quad \text{into the set} \quad \{(a, b) \mid a^2 = 0 \wedge b = 1\} \subset |A|^2 .$$

Notice that functoriality implies that whenever A is deformed into B via

f, $T(A)$ is deformed into $T(B)$ via f again and, in this sense, these deformations are subordinated to the basic ones (i.e. the $\mathbf{R}$-morphisms).

Similarly, S as well as $|\mathbf{R}|$ (the set of reals) become functors. The last one is identified with the «underlying set» functor

$$\mathcal{U}(\mathbf{R} \xrightarrow{\phi} A) = |A| .$$

Notice that $S \neq T$, as desired, since they differ e.g. for $\mathbf{D} = \mathbf{R}[X]/(X^2)$.

As a first example, for a conceptual (logical or Parmenidian) point of view, we consider our basic domain of variation or rather its dual

$$\mathcal{C} = FP(\mathbf{R}\text{-}alg)^{opp}$$

as a site by introducing the *discrete localization* consisting of identities only, i.e.,

$$Loc(A) = \{(A \xrightarrow{A} A)\}.$$

One can show (cf. [GU]) that the category $Fib_{SET}(\mathcal{C})$ (cf. Appendix) is equivalent to $\mathbf{R}$-alg, whereas a sheaf is just a functor $\mathcal{C}^{opp} \to SET$.

From a logical point of view, we look for the laws according to which the deformations of the «generating structure» $|\mathbf{R}|$ take place. We introduce a one-sorted language whose unique sort «denotes» $|\mathbf{R}|$, or rather $\mathcal{U}$, after our identification. The theory of the basic domain is «obviously» the theory of $\mathbf{R}$-algebras, i.e. the atomic diagram of $(\mathbf{R}, +, ., 0, 1)$, plus the axioms for commutative rings with 1. Actually, according to the general way of associating a theory to a site described below, this is what we get in this case. (The operation symbols of our language are $+$, $.$ and one constant for each real.)

A la Frege, T may be now identified with the «concept» expressed by the formula $x^2 + y^2 = 1 \wedge y = 1$ of our language, whereas S becomes the formula $x = 0 \wedge y = 1$. As already mentioned, $\mathcal{U}$ is identified with the «only» sort or, what amounts to the same, with $x = x$. This solves again our problem of distinguishing T from S : as concepts they differ, although their extensions (in $\mathbf{R}^2$) are the same.

As a second example, we make $\mathcal{C} = FP(\mathbf{R}\text{-}alg)^{opp}$ into a site by introducing the *Zariski localization* generated by the following «co-localizations» in $FP(\mathbf{R}\text{-}alg)$:

$$\begin{array}{ccl} & \nearrow & \mathbf{R}[X,Y]/(XY-1) \\ \mathbf{R}[X] & & \\ & \searrow & \mathbf{R}[X,Y]/((1-X)Y-1) \end{array}$$

and the empty family as a «co-localization» of the null ring $\mathbf{0}$.

Going over, via Yoneda, to the category of representable functors (which is equivalent to $\mathcal{C}$), we obtain the following localization corresponding to the first one

$$\begin{array}{ccl} & \swarrow & U \\ \mathfrak{U} & & \\ & \nwarrow & U' \end{array}$$

where $U, U' : FP(\mathbf{R}\text{-}alg) \to SET$ are defined by

$$U(\mathbf{R} \xrightarrow{\phi} A) = \{ (a,b) \mid ab = 1 \} \subset |A|^2 ,$$

$$U'(\mathbf{R} \xrightarrow{\phi} A) = \{ (a,b) \mid (1-a)b = 1 \} \subset |A|^2 .$$

Indeed, letting h_B be the functor represented by B,

$$h_{\mathbf{R}[X]}(A) = \mathcal{C}(A, \mathbf{R}[X]) = \mathbf{R}\text{-}alg(\mathbf{R}[X], A) \simeq$$

$$\simeq |A| = \mathfrak{U}(A),$$

since an $\mathbf{R}$-morphism from $\mathbf{R}[X]$ into A is determined by its value at X, i.e. by an element of A. Hence $h_{\mathbf{R}[X]} \simeq \mathfrak{U}$. Similarly, we can check that

$$U \simeq h_{\mathbf{R}[X,Y]/(XY-1)} \quad \text{and} \quad U' \simeq h_{\mathbf{R}[X,Y]/((1-X)Y-1)} .$$

Again, it is easy to see using [GU] that the category of fibers of this site is equivalent to the category of $\mathbf{R}$-algebras which are local rings (but with usual $\mathbf{R}$-morphisms).

To describe the logical point of view, we consider the localizations as new laws satisfied by the «generating structure» $\mathfrak{U}$. To impose

$$\begin{array}{ccc} & & U \\ & \swarrow & \\ \mathfrak{U} & & \\ & \nwarrow & \\ & & U' \end{array}$$

as a localization or covering should mean, intuitively, that the union of the projections of both functors into $\mathfrak{U}$ is the whole of $\mathfrak{U}$, i. e.

$$x = x \implies \exists y (x . y = 1) \vee \exists y ((1 - x) . y = 1)$$

by identifying, as before, $\mathfrak{U}$ with $x = x$, U with $x . y = 1$ and U' with $(1 - x) . y = 1$. Similarly, the empty localization gives $0 = 1 \implies$.

All said, we end up with the theory of local **R**-algebras as the analogue of the site $\mathcal{C}$ (with the Zariski localization). Its models are precisely the fibers ot his site!

The reader has doubtless observed our use and abuse of the word «concept».

It can be made precise once that the notion of a classifying topos of a (coherent) theory is available (cf. III Section 10). The *concept expressed by* a (coherent) formula of the language of a theory T is the canonical interpretation of the formula in the classifying topos $\mathcal{E}(T)$.

As example of sheaf for $\mathcal{C}$ with the Zariski localization, let us mention the functor Nil defined by

$$Nil(A) = \{ a \in A \mid a \text{ is nilpotent} \}.$$

Notice that, as a concept, Nil becomes the *infinitary* formula

$$\bigvee \{ x^n = 0 \mid n \in \mathbf{N} - \{ 0 \} \}.$$

On the other hand, the sheaf $Proj$ for $\mathcal{C}$ with the discrete localization defined by

$$Proj(A) = A \times U(A) \cup U(A) \times A / \sim ,$$

where $U(A)$ is the set of invertible elements of A and

$$(a, b) \sim (a', b') \iff \exists \lambda \in U(A)\ (a, b) = (\lambda a', \lambda b'),$$

cannot be written in our language, since we do not have syntactical operations to express quotients by equivalence relations (see III, Section 10, for a discussion of this problem).

As a final observation, notice that we can change throughout this example **R** by any commutative ring with 1, for instance **Z**, the ring of integers. In this case the category of sheaves for the site of the dual of the category of finitely presented commutative rings with the Zariski topology is called the *Zariski topos.* As before, we can see that the theory of this site is just the theory of non-trivial local rings.

2. Theory associated to a site.

In Section 1, we saw how to associate a theory with a site in two concrete examples so that the fibers of the site are exactly the models of the theory. We attack now the general case.

We shall be concerned with sublanguages of $L_{\infty\omega}$ (see e.g. [Bw]), but in a many-sorted version (as e.g. [F]). Such a language has a non-empty set of sorts and, for each sort, a countable set of variables of that sort, an arbitrary set of finitary sorted predicate symbols (i.e. each place is assigned a definite sort) and an arbitrary set of finitary sorted operation symbols. We shall consider only formulas with finitely many free variables.

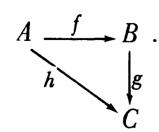
With each small site $\mathcal{C}$ we associate a language $L_{\mathcal{C}}$ of this type as follows: its sorts are the objects of $\mathcal{C}$; $L_{\mathcal{C}}$ has many sorted operation symbols: the morphisms of $\mathcal{C}$. (Besides =, there are no predicate symbols). The theory $T_{\mathcal{C}}$ associated to $\mathcal{C}$ has the following (Gentzen) sequents as axioms (which intuitively express the site structure of $\mathcal{C}$):

I. Axioms for category: a) identity

$\Rightarrow\ Id_A\, a = a$, for every diagram $A \xrightarrow{1_A} A$,

b) composition

$\Rightarrow\ g f a = h a$, for every diagram



II. Axioms for finite $\varprojlim$: a) products

$\Rightarrow\ \exists c\, (\pi_A c = a \wedge \pi_A c = b)$,

$$\pi_A c = \pi_A c' \wedge \pi_B c = \pi_B c' \Rightarrow c = c'$$

for every product diagram $A \xleftarrow{\pi_A} C \xrightarrow{\pi_B} B$;

b) equalizers

$$fa = ga \Rightarrow \exists e \; \epsilon e = a \Rightarrow f\epsilon e = g\epsilon e,$$
$$\epsilon e = \epsilon e' \Rightarrow e = e'$$

for every equalizer diagram $E \xrightarrow{\epsilon} A \overset{f}{\underset{g}{\rightrightarrows}} B$;

c) terminal object

$\Rightarrow t = t' \Rightarrow \exists t (t = t)$ for every terminal object t.

III. Axioms for coverings :

$$a = a \Rightarrow \bigvee \{ \exists a_i (f_i a_i = a) \mid i \in I \}$$

for every localization $(A_i \xrightarrow{f_i} A)_{i \in I}$ of $\mathcal{C}$.

We can ask whether the models of this theory are precisely the fibers of the site $\mathcal{C}$ and the answer is yes, of course. However, this is not interesting since the theory, being infinitary (because of III) may fail to have models.

The right thing to do is to consider models in arbitrary Grothendieck topos by extending the notion of a set-valued interpretation of a language. Then, for any topos $\mathcal{E}$, there is an equivalence between the category of $\mathcal{E}$-models of $T_{\mathcal{C}}$ and the $\mathcal{E}$-fibers of $\mathcal{C}$ (cf. Section 4).

3. Interpreting languages in a topos.

In this Section, we shall define an $\mathfrak{R}$-interpretation of a language L (of the type described in Section 2) in a topos $\mathfrak{R}$, or more generally, in a category $\mathfrak{R}$ with finite $\varprojlim$ in such a way that it reduces to the usual set-theoretical interpretation when $\mathfrak{R} = SET$. For simplicity, languages with unary function symbols will be discussed only.

An $\mathfrak{R}$-interpretation M of L is a map assigning to every:
sort s, an object $M(s) \in |\mathfrak{R}|$,
function symbol f (with sorts s_1, s), a morphism $M(f) : M(s_1) \rightarrow M(s)$.

Notice that in case $\mathfrak{R} = SET$, M is just an ordinary many-sorted structure of similarity type L, possibly with empty partial domains $M(s)$.

For a sequence $\vec{x} = (x_1, \dots, x_n)$ of variables of sorts $s_1, \dots, s_n$, we define

$$M(\vec{x}) = M(s_1) \times \dots \times M(s_n).$$

For a term t and a formula ϕ of L, whose free variables are among $\vec{x}$, $M_{\vec{x}}(t)$ will be a morphism $M(\vec{x}) \to M(s)$, where s is the sort of the values of t, and $M_{\vec{x}}(\phi)$ will be a subobject of $M(\vec{x})$ or will remain undefined if not all the operations called for by ϕ can be performed in $\mathfrak{R}$.

If $t = x$, the i'th variable in x, $M_{\vec{x}}(t)$ is defined to be the canonical projection $\pi_i : M(\vec{x}) \to M(x)$.

If $t = ft_1$ where the values of t, t_1 are of sorts s, s_1 respectively, $M_{\vec{x}}(t)$ is the composite

$$M(\vec{x}) \xrightarrow{M_{\vec{x}}(t_1)} M(s_1) \xrightarrow{M(f)} M(s).$$

The only atomic formulas are of the form $t_1 = t_2$ and they are interpreted by the equalizer diagram

$$M_{\vec{x}}(t_1 = t_2) \rightarrowtail M(\vec{x}) \underset{M_{\vec{x}}(t_2)}{\overset{M_{\vec{x}}(t_1)}{\rightrightarrows}} M(s)$$

(s is the common sort of the values of t_1, t_2).

Propositional connectives give no trouble. Thus, e.g.,

$$M_{\vec{x}}(\vee \Theta) = \vee \{ M_{\vec{x}}(\theta) \mid \theta \in \Theta \}.$$

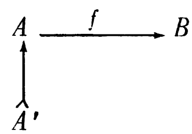
Naturally, this is defined only if $M_{\vec{x}}(\theta)$ is defined for every $\theta \in \Theta$ (as a subobject of $M(\vec{x})$) and if the lattice-theoretic operation of taking the *sup* of these subobjects of $M(\vec{x})$ can be performed in $\mathfrak{R}$. (This is always the case if $\mathfrak{R}$ is a topos.)

Finally, we arrive to quantifiers. If the interpretation of $\phi(\vec{x}, y)$ is given, i.e., $M_{\vec{x}y}(\phi) \to M(\vec{x}, y)$, we define $M_{\vec{x}}(\exists y\, \phi(x, y))$ to be the image of $M_{\vec{x}y}(\phi)$ under the canonical projection $\pi: M(\vec{x}, y) \to M(\vec{x})$. In symbols :

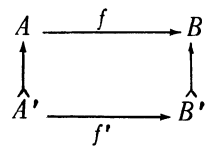
$$M_{\vec{x}}(\exists y\, \phi(\vec{x}, y)) = \exists_\pi M_{\vec{x}y}(\phi).$$

Again, this need not to exist ; however, in a topos every morphism has an image.

Recall that given a diagram



the *image of* A' *under* f, $\exists_f A'$, is the smallest (if it exists) subobject $B' \rightarrowtail B$ such that for some $A' \xrightarrow{f'} B'$, the following diagram commutes :



The rest of the quantifiers and connectives ($\forall, \to, \neg, \ldots$) will be dealt with in the last chapter.

If $\sigma = \phi \Rightarrow \psi$ is a (Gentzen) sequent with finite sets ϕ, ψ of formulas, we say that M *satisfies* σ, and write $M \models \sigma$, if

$$M_{\vec{x}}(\wedge\, \phi) \leqslant M_{\vec{x}}(\vee\, \psi),$$

including the condition that the required interpretation exist, for $\vec{x}$ the sequence of free variables in $\phi \cup \psi$, $\leqslant$ the ordering of subobjects of $M(\vec{x})$.

Of course, we say that M is a *model of a theory* T (i.e. a set of sequents) if $M \models \sigma$ for every $\sigma \in T$.

4. Fibers and models.

If $\mathfrak{R}$ is any category with finite $\varprojlim$, we can make the $\mathfrak{R}$-models of a coherent theory T into a category $Mod_{\mathfrak{R}}(T)$ in a natural way which, in case $\mathfrak{R} = SET$, amounts to this : its objects are set-theoretical models and its morphisms are «algebraic», i.e., functions $f: \mathfrak{A} \to \mathfrak{B}$ which preserve the basic relations and operation symbols in the sense that

$$(a_1, \dots, a_n) \in \mathfrak{R}^{\mathfrak{A}} \Rightarrow (f(a_1), \dots, f(a_n)) \in \mathfrak{R}^{\mathfrak{B}}$$

(with a corresponding clause for the operation symbols).

We are now able to state the following mathematical formulation of our analogy.

THEOREM. *The canonical (or identical) interpretation of $L_{\mathcal{C}}$ into $\mathcal{C}$ induces an equivalence between the categories $Fib_{\mathcal{E}}(\mathcal{C})$ and $Mod_{\mathcal{E}}(T_{\mathcal{C}})$, for any topos $\mathcal{E}$.*

The *proof* is obvious from the notion of a model.

The last part of our analogy, i.e. sheaves vs. concepts (or formulas) will be dealt with in Chapter III, Section 10.

CHAPTER II. COHERENT LOGIC

5. Coherent theories and their models.

We shall notice the following syntactical feature of the sets of axioms $T_{\mathcal{C}}$ in I, Section 2. Let us say that a formula of an $L_{\infty\omega}$ language L is *coherent* if it is obtained from the atomic formulas by using arbitrary $\bigvee$, finite $\bigwedge$ and $\exists$ only. A *coherent* theory will be a set of (Gentzen) sequents $\phi \Rightarrow \psi$, where ϕ, ψ are finite (possibly empty) sets of coherent formulas. With this terminology, we see that the theory $T_{\mathcal{C}}$ is coherent.

Finitary coherent formulas and theories have nice model-theoretical caracterizations.

Letting $Mod(T) = Mod_{SET}(T)$ be the category of models mentioned in I, Section 4. Every coherent (infinitary) formula $\phi(x_1, \dots, x_n)$ gives rise to a functor $\phi^{(\ )}: Mod(T) \to SET$ defined by

$$\phi^{(\mathfrak{A})} = \{ (a_1, \dots, a_n) \in |\mathfrak{A}|^n : \mathfrak{A} \models \phi [a_1, \dots, a_n] \}.$$

Indeed, these formulas are preserved by the morphisms of $Mod(T)$. For finitary formulas, however, these are the only ones [CK]. This can be stated as follows :

PROPOSITION [*CK*]. *Let* $\phi(x_1, \dots, x_n)$ *be a finitary formula of* L. *Then* $\phi^{(\ )}$ *is a subfunctor of* $|\ |^n$ *iff* ϕ *is T-equivalent to a coherent formula.*

On the other hand, from [K] and [CK] we obtain :

THEOREM. *Let* T *be a finitary theory. Then* T *has a coherent set of axioms iff* $Mod(T)$ *admits filtered colimits.*

The connection of coherent theories with sites is established by the following

THEOREM. *Let* T *be a finitary theory. Then* T *has a coherent set of axioms iff* $Mod(T)$ *is equivalent to* $Fib_{SET}(\mathcal{C})$, *for some algebraic site* $\mathcal{C}$.

6. A formal system for coherent logic.

Let L be an $L_{\infty\omega}$ many-sorted language and let F be a fragment (i.e., a class of formulas closed under subformulas and substitution) consisting of coherent formulas.

Capital Greek letters will denote finite sets of formulas of F and lower case Greek letters will denote single formulas (in F).

AXIOMS. $\Theta \Rightarrow \theta$, if $\theta \in \Theta$.

RULES OF INFERENCE.

$$(\mathrm{R}\wedge_1) \qquad \frac{\Phi, \wedge\Theta, \theta \Rightarrow \psi}{\Phi, \wedge\Theta \Rightarrow \psi} \qquad \text{if } \theta \in \Theta.$$

$$(\mathrm{R}\wedge_2) \qquad \frac{\Phi, \wedge\Theta \Rightarrow \psi}{\Phi \Rightarrow \psi} \qquad \text{if } \Theta \subset \Phi.$$

$$(\mathrm{R}\vee_1) \qquad \frac{\Phi, \theta, \vee\Theta \Rightarrow \psi}{\Phi, \theta \Rightarrow \psi}$$

provided that $\theta \in \Theta$ and all the free variables in $\vee\Theta$ occur free in the conclusion.

$$(\mathrm{R}\vee_2) \qquad \frac{\{\Phi, \vee\Theta, \theta \Rightarrow \psi : \theta \in \Theta\}}{\Phi, \vee\Theta \Rightarrow \psi}.$$

$$(\mathrm{R}\,\exists_1)\qquad \frac{\Phi,\ \theta(t/y),\ \exists x\,\theta(x/y) \Rightarrow \psi}{\Phi,\ \theta(t/y) \Rightarrow \psi}\ ;$$

here, of course, y and t are of the same sort of x.

$$(\mathrm{R}\,\exists_2)\qquad \frac{\Phi,\ \exists x\,\theta(x/y),\ \theta \Rightarrow \psi}{\Phi,\ \exists x\,\theta(x/y) \Rightarrow \psi}$$

provided that y does not occur free in the conclusion.

$$(\mathrm{RT})\qquad \frac{\Phi, \Theta(t_1, \dots, t_n),\ \varphi(t_1, \dots, t_n) \Rightarrow \psi}{\Phi,\ \Theta(t_1, \dots, t_n) \Rightarrow \psi}$$

provided that for some $\Theta(x_1, \dots, x_n) \Rightarrow \varphi(x_1, \dots, x_n)$ belonging to T and for some terms $t_1, \dots, t_n$, $\Theta(t_1, \dots, t_n)$ is the set of all substitution instances $\theta(t_1, \dots, t_n)$ of all $\theta \in \Theta$, all free variables in Θ or φ are among $x_1, \dots, x_n$; $\varphi(t_1, \dots, t_n)$ is the result of substituting t_i for x_i in φ and, finally, all free variables that occur in the premise occur in the conclusion. (Notice that this rule depends on a fixed set T of sequents.)

$$(\mathrm{R} =_1)\qquad \frac{\Phi,\ t = t \Rightarrow \psi}{\Phi \Rightarrow \psi}$$

provided that every free variable in t occurs free in the conclusion.

$$(\mathrm{R} =_2)\qquad \frac{\Phi,\ t_1 = t_2,\ t_2 = t_1 \Rightarrow \psi}{\Phi,\ t_1 = t_2 \Rightarrow \psi}\ .$$

$$(\mathrm{R} =_3)\qquad \frac{\Phi,\ t_1 = t_2,\ \varphi(t_1),\ \varphi(t_2) \Rightarrow \psi}{\Phi,\ t_1 = t_2,\ \varphi(t_1) \Rightarrow \psi}$$

provided that $\varphi(t_i)$ is obtained from φ by substituting t_i for x.

The notion of *formal consequence of* T is defined in an obvious way. We write $T \models \sigma$ for « σ is a formal consequence of T ».

We notice a few features of this system. First, no rule changes the right hand side of a sequent. Second, every rule makes the left hand side shorter. Finally, in every rule, except $(\mathrm{R}\,\exists_2)$, the conclusion contains no less free variables than (each of) the premise(s).

7. Soundness and completeness.

Let L be a many-sorted $L_{\infty\omega}$ language and let F be a fragment of coherent formulas. We shall consider interpretations in topos and pretopos.

SOUNDNESS THEOREM. *If* $\mathcal{E}$ *is a topos and* M *an* $\mathcal{E}$*-interpretation of* L *, then all axioms in* F *are valid and all rules of inference in* F *are sound. The same is true if* $\mathcal{E}$ *is a pretopos, provided that* F *consists of finitary formulas.*

(Needless to say, a rule is sound if from the validity of the premise(s) we can conclude the validity of the conclusion.)

Just as in the set-theoretical case, the *proof* requires the *substitution* lemma which says, roughly, that the interpretation of $\varphi(t/x)$ is the pull-back of the interpretation of φ along the interpretation of t. This lemma is proved by induction on φ and uses the stability of coproducts and of equivalence relations (for the $\vee$, $\exists$ clauses). A version of this lemma can be found in Benabou [Be].

To state the completeness theorem, let $\mathbf{H}$ be a complete Heyting algebra. By defining

$$(a_i)_{i\in I} \text{ is a localization of } a \text{ iff } a = \bigvee_{i\in I} a_i ,$$

we make $\mathbf{H}$ into a site. We let $Sh(\mathbf{H})$ be the topos of sheaves over $\mathbf{H}$.

A *Heyting-valued model* of a coherent theory T is an interpretation of T in some $Sh(\mathbf{H})$ such that all axioms are valid, i.e., a $Sh(\mathbf{H})$-model of T. Naturally, if $\mathbf{H}$ is a boolean algebra, we speak of a *boolean-valued model.*

COMPLETENESS THEOREM. *a) For a theory* T *in* F *and a sequent* σ *in* F, $T \vdash \sigma$ *iff* σ *is valid in every boolean-valued model of* T.

b) If F *is countable or if* F *consists of finitary formulas only,*

$$T \vdash \sigma \quad iff \quad T \models \sigma$$

(i.e., σ *is valid in ordinary set-theoretical models of* T *).*

The *proof* follows the lines of Mansfield [M] with technical modifications due to the fact that some of the interpretations of our sorts may be empty. In particular, the usual notion of boolean-valued model of Mostowski (cf. Rasiowa-Sikorski [RS]) that Mansfield uses has to be modified accordingly. Finally, using the work of Higgs [Hi], one shows that such a modified boolean-valued model is exactly a boolean-valued model in the categorical sense defined here.

Notice that b follows from a by taking appropriate ultrafilters (using the Rasiowa-Sikorski lemma).

An important feature of the proof is that the Boolean-valued model constructed is *conservative* in the following sense: we can find a complete boolean algebra **B** such that

$$T \vdash \sigma \quad \text{iff} \quad Sh(\mathbf{B}) \models \sigma .$$

We just mention that **B** is constructed as the regular open algebra of the topological space defined by the order $\mathcal{P} = \langle \{ \sigma : T \vdash \sigma \}, \leq \rangle$, where

$$\sigma = \phi \Rightarrow \psi \leq \phi' \Rightarrow \psi' \quad \text{iff} \quad \phi \supseteq \phi' \text{ and } \psi = \psi'.$$

CHAPTER III. APPLICATIONS TO THE THEORY OF TOPOI

8. Embedding of topoi.

Our completeness theorem has consequences for embedding «coherently» an arbitrary topos in a topos of a simple type. Indeed, a new proof can be given of

THEOREM *(Barr [Ba]). Every Grothendieck topos $\mathcal{E}$ has a boolean point which is «surjective», i.e., there is a complete boolean algebra* **B** *and a geometric morphism $Sh(\mathbf{B}) \xrightarrow{p} \mathcal{E}$ for which $p^*: \mathcal{E} \to Sh(\mathbf{B})$ is faithful.*

PROOF (in sketch). Let $\mathcal{C}$ be a small site such that $\mathcal{E} \simeq Sh(\mathcal{C})$. Without loss of generality, we may assume that $\mathcal{C}$ is closed under images and subobjects of $\mathcal{E}$. Let $T_{\mathcal{C}}$ be its associated coherent theory. By the completeness theorem, we can find a conservative boolean-valued model, i.e., a $Sh(\mathbf{B})$-model of $T_{\mathcal{C}}$, for some complete boolean algebra **B**. By the equivalence between fibers and models (Section 4), this gives a $Sh(\mathbf{B})$-fiber and, by general topos theory (cf. Appendix), a geometric morphism

$$Sh(\mathbf{B}) \xrightarrow[p]{} \mathcal{E} .$$

One concludes that p^* is faithful from the fact that the original model of $T_{\mathcal{C}}$ is conservative (and the closure conditions of $\mathcal{C}$).

In some cases, we can strengthen the conclusion of this theorem and replace **B** by a set algebra 2^Y of subsets of Y. Indeed, this is the case whenever $T_{\mathcal{C}}$ generates a countable fragment since our completeness theorem for ordinary set-theoretical interpretations apply. In this way, we obtain for the first case

THEOREM *(Deligne [SGA 4]). Every coherent topos* $\mathcal{E}$ *has a surjective boolean point* $Sh(2^Y) \xrightarrow{p} \mathcal{E}$, *for some set* Y.

We notice that $Sh(2^Y) \sim SET^Y$.

Let us say that a Grothendieck topos is *separable* if it is equivalent to some $Sh(\mathcal{C})$ for a small site $\mathcal{C}$ such that:

$Ob(\mathcal{C})$ is countable,

$Hom_{\mathcal{C}}(A, B)$ is countable for every $A, B \in |\mathcal{C}|$,

and the localization of $\mathcal{C}$ is generated by a countable family of coverings. In this case, $T_{\mathcal{C}}$ generates a countable fragment and we may conclude:

THEOREM. *Every separable Grothendieck topos* $\mathcal{E}$ *has a surjective boolean point* $Sh(2^Y) \to \mathcal{E}$, *for some set* Y.

9. Is a topos determined by its points ?

There are several ways of making this question precise. Since a topos may fail to have points, as the example of $Sh(\mathbf{B})$ for a complete atomless boolean algebra **B** shows [Ba], we must reformulate the question either for coherent topos and set-valued points or for arbitrary topos and boolean points.

In the first case, we may ask:

Does $Point(\mathcal{E}) \simeq Point(\mathcal{E}')$ imply $\mathcal{E} \simeq \mathcal{E}'$ for coherent topoi $\mathcal{E}, \mathcal{E}'$?

The answer is obviously no. Indeed, pick any two complete, non-isomorphic boolean algebras $\mathbf{B}_1, \mathbf{B}_2$ such that their Stone spaces X_1, X_2 have the same cardinality. Make $\mathbf{B}_1, \mathbf{B}_2$ into algebraic sites by defining: a *finite* family $(a_i)_{i \in I}$ is a localization of A iff $A = \bigvee_{i \in I} a_i$. Let $\tilde{\mathbf{B}}_1, \tilde{\mathbf{B}}_2$ the category of sheaves. Then, for $i = 1, 2$,

$$Points(\tilde{\mathbf{B}}_i) \simeq card(X_i) \text{ and hence } Points(\tilde{\mathbf{B}}_1) \simeq Points(\tilde{\mathbf{B}}_2).$$

On the other hand, $\tilde{\mathbf{B}}_1 \not\simeq \tilde{\mathbf{B}}_2$, since

$$\mathbf{B}_i \simeq \text{subobjects of } 1 \text{ in } \tilde{\mathbf{B}}_i, \text{ for } i = 1, 2.$$

We reformulate the question as follows: let $\mathcal{E} \xrightarrow{i} \mathcal{E}'$ be a coherent geometric morphism between coherent topos which induces, via composition, an equivalence

$$Points(\mathcal{E}) \simeq Points(\mathcal{E}').$$

Is it true that i is an equivalence ?

This time the answer is yes and it turns out to have applications in Algebraic Geometry (see [MR]) since it allows, in some cases, to find the theory of which a given coherent topos is its «universal model» or classifying topos.

Via the duality pretopos - coherent topos (cf. [SGA 4]) we can reformulate the answer as follows.

THEOREM. *Let $I : \mathcal{P} \to \mathcal{P}'$ be a coherent functor between pretopoi (i. e., which preserves finite* $\varprojlim$, *coproducts and quotients by equivalence relations). Assume that I induces, via composition, an equivalence*

$$Fib(\mathcal{P}') \xrightarrow[I^*]{\sim} Fib(\mathcal{P}).$$

Then I is an equivalence.

We shall say some words about the *proof*, since it is closely related to Preservations theorems in Model theory.

Indeed, the theorem follows from two results which have independent interest.

THEOREM A. *If I^* is full, then I is full with respect to subobjects (i. e. if $S \rightarrowtail I(P) \in \mathcal{P}'$, then*

$$S \simeq I(P_0) \quad \textit{for some } P_0 \in |\mathcal{P}|\,).$$

If, in addition, I^* is surjective on objects, then I is full.

A rough idea of the *proof*: by considering pretopoi as sites with the precanonical localization (cf. Appendix), we are reduced to study the interpretation of theories $T_{\mathcal{P}} \xrightarrow{I} T_{\mathcal{P}'}$. Using a general form of Beth's theorem, our hypotheses imply that S (as a sort of $L_{\mathcal{P}'}$) is definable by $I(\phi)$, where ϕ is a formula in the (full) language $L_{\mathcal{P}}$. This formula, however, is shown to be preserved by monomorphisms of models of $T_{\mathcal{P}}$ and hence (Section 5) it is $T_{\mathcal{P}}$-equivalent to a coherent formula in $L_{\mathcal{P}}$ whose interpretation in $\mathcal{P}$ gives the required $P_0 \in |\mathcal{P}|$.

To state the other result needed, let us say that an object $P' \in |\mathcal{P}'|$ is *finitely covered via I* if there are finitely many objects $P_1, \dots, P_n \in |\mathcal{P}|$, there are corresponding subobjects $P_i' \rightarrowtail I(P_i)$ in $\mathcal{P}'$ and morphisms:

$$P_i' \xrightarrow{f_i} P' \text{ in } \mathcal{P}' \text{ such that } P' = \bigvee_{i=1}^{n} \exists_{f_i}(P_i').$$

THEOREM B. *Assume that I^* is faithful. Then every P' is finitely covered by $\mathcal{P}$ via I.*

Proceeding (as before) to reduce the problem to one of the interpretations of theories, the *proof* uses the compactness theorem as well as the method of diagrams.

The analogous question for arbitrary topos and boolean points (i.e. let $\mathcal{E} \xrightarrow{i} \mathcal{E}'$ be a geometric morphism which induces, via composition, an equivalence

$$Top(Sh(\mathbf{B}), \mathcal{E}) \xrightarrow{\sim} Top(Sh(\mathbf{B}), \mathcal{E}'),$$

for every complete boolean algebra $\mathbf{B}$; is it true that i is an equivalence?) remains open. Only partial results, using the notion of *admissible* pretopos, are known (cf. [MR]).

10. Classifying topos of a coherent theory.

Let T be a coherent theory in a $L_{\infty\omega}$-language L. We construct a «universal model» or «classifying topos» of T, $\mathcal{E}(T)$, in three steps: we first construct a category $\mathcal{F}(T)$ of formulas (or concepts) which is formally similar to the Lindenbaum-Tarski algebra of T. Then we add, in a formal way, disjoint coproducts to obtain $\mathcal{F}'(T)$ and finally, again formally, quotients by equivalence relations to obtain $\mathcal{E}(T)$. From the construction, we shall obtain a canonical interpretation of L into $\mathcal{E}(T)$ which will allow us to state the sense in which $\mathcal{E}(T)$ is a «universal model»:

THEOREM. *The canonical interpretation of* L *into* $\mathcal{E}(T)$ *is a model of* T. *Furthermore it induces an equivalence between*

$$Mod_{\mathcal{E}}(T) \text{ and } Top(\mathcal{E}, \mathcal{E}(T)),$$

for any topos $\mathcal{E}$.

From this theorem and our completeness theorem for coherent logic we obtain

COROLLARY. *Let* σ *be a coherent sequent in the language of a coherent theory* T. *Then the following are equivalent:*

1° $T \vdash \sigma$.

2° $T \models^{b} \sigma$, *i. e.* σ *is valid in every boolean-valued model.*

3° $\mathcal{E}(T) \models \sigma$.

(The obvious version for set-valued models instead of boolean-valued ones is true for T, σ finitary.)

The construction of $\mathcal{E}(T)$ has been described in detail in [D] (first step) and [A] (the other two steps) and will appear in the book mentioned in the Introduction. Here we merely sketch it.

The *objects* of $\mathcal{F}(T)$ are the coherent formulas of L. The *morphisms* of $\mathcal{F}(T)$ are «definable provably functional relations» between $\phi(\vec{x})$ and $\psi(\vec{y})$, i.e., equivalence classes (under provability in T) of coherent formulas $\alpha(\vec{x};\vec{y})$ such that

$$T \vdash \text{« } \alpha \text{ is a function from } \phi \text{ to } \psi \text{ ».}$$

The *composition* is the obvious relative product of relations.

To obtain $\mathcal{F}'(T)$, we consider as *objects* functions from *Sets* into $|\mathcal{F}(T)|$, which we write $\sum_{i\in I} A_i$. Since we want these to be coproducts, a *morphism* $f: \sum_{i\in I} A_i \to \sum_{j\in J} B_j$ may be defined as a family

$$\left(A_i \xrightarrow{f_i} \sum_{j\in J} B_j \right)_{i\in I},$$

and we are reduced to define a morphism $A \xrightarrow{f} \sum_{j \in J} B_j$. Since our coproducts should be disjoint and stable (under pullbacks), such an f should determine a partition of A. Hence, we define f to be a family $(f_i)_{i \in I}$ of morphisms of $\mathfrak{F}'(T)$ such that $(dom\, f_i)_{i \in I}$ is a partition in $\mathfrak{F}(T)$ of A and

$$f_i : dom\, f_i \to B_j \in \mathfrak{F}(T) \text{ for some } j \in J .$$

Families giving the same «glueing» in A are identified.

Finally, the *objects* of $\mathcal{E}(T)$ are couples (X, R), where R is an equivalence relation on X (i.e. it satisfies the axioms of equivalence relation). Intuitively (X, R) should represent X/R. With this in mind, a morphism $(X, R) \xrightarrow{V} (Y, S)$ is a subobject of $X \times Y$ satisfying

$$x = x \implies \exists y\ V(x, y),$$
$$V(x, y) \wedge V(x, y') \implies S(y, y'),$$
$$V(x, y) \iff \exists x' (R(x, x') \wedge V(x', y)),$$
$$V(x, y) \iff \exists y' (S(y, y') \wedge V(x, y')).$$

The *composition* of two morphisms

$$(X, R) \xrightarrow{V} (Y, S) \xrightarrow{W} (Z, Q)$$

is the relative product, i.e. the subobject of $X \times Z$ defined by the formula

$$\exists y (V(x, y) \wedge W(y, z)).$$

We notice that there are obvious versions of this construction to obtain a κ-pretopos from a $L_{\kappa\omega}$ coherent theory. (The cardinal κ is assumed to be regular.) In particular, starting from a finitary coherent theory T, we first construct $\mathfrak{F}_\omega(T)$ by restricting the formulas used in constructing $\mathfrak{F}_\omega(T)$ to be finitary. Further, we construct $\mathfrak{F}_\omega(T)$ as functions

$$\sum_{i \in I} A_i \text{ with finite } I .$$

The last step, i.e. adjoining quotients by equivalence relations, is left untouched. In this way, we obtain a pretopos $\mathcal{P}(T)$ having the following universal property:

THEOREM. *The canonical interpretation* $L \to \mathcal{P}(T)$ *is a model of* T. *Furthermore, it induces an equivalence between the categories* $Mod_{\mathcal{P}}(T)$ *and* $Fib_{\mathcal{P}}(\mathcal{P}(T))$, *for every pretopos* $\mathcal{P}$.

Notice that although $\mathcal{E}(T)$ has more logical operations than $\mathcal{F}(T)$, namely coproducts and quotients by equivalence relations, the canonical functor $\mathcal{F}(T) \to \mathcal{E}(T)$ induces, via composition, an equivalence between their $\mathcal{E}$-models, for a topos $\mathcal{E}$. (Clearly e.g. (X, R) is uniquely interpreted up to isomorphism in $\mathcal{E}$ as the quotient of the interpretation of X by that of R). This means that these new operations are «implicitly» definable in $\mathcal{F}(T)$, although they are not «explicitly» definable. Because the ordinary logical language fails to contain logical operations for coproducts and quotients, we cannot identify sheaves and formulas.

The same considerations hold for the pretopos case replacing $\mathcal{F}(T)$, $\mathcal{E}(T)$ and the topos $\mathcal{E}$ by $\mathcal{F}_{\omega}(T)$, $\mathcal{P}(T)$ and a pretopos $\mathcal{P}$. We find here a *defect* in the usual formulations of many-sorted languages. Notice that we had already encountered this problem at the end of Section 1.

From this point of view, we may consider the main theorem of Section 9 as stating a «conceptual completeness» for a pretopos: every «implicitly» defined logical operation is «explicitly» defined in a pretopos.

For the sake of completeness we finish with two results.

A. Every topos is the classifying topos of a coherent theory.

B. Every coherent topos is the classifying topos of a finitary coherent theory.

For the *proof*, let $\mathcal{E}$ be a topos. By general topos theory (cf. Appendix), $\mathcal{E} \simeq Sh(\mathcal{C})$, for some small site $\mathcal{C}$. Then $Sh(\mathcal{C}) \simeq \mathcal{E}(T_{\mathcal{C}})$, since they both satisfy the same universal property.

Other constructions of the classifying topos of a finitary coherent theory may be found in [Co_2] and [R].

IV. COHERENT VS. INTUITIONISTIC LOGIC IN TOPOI

11. The charm of coherent logic.

If the reader looks at the formal system for coherent logic in Section 2, he will notice that all axioms as well as rules of inference are intuitionistically valid (we assume F a fragment consisting of finitary formulas), as well as classically valid. The distinction between the intuitionistic and classical interpretations of logical operations become irrelevant for this «absolute» logic.

Furthermore, this logic besides being a part, may be considered as a generalization of classical logic. Indeed, any classical theory may be rendered coherent by extending the language [A] . The idea is trivial and may be seen from the following example: take

$$\exists x \; \exists y \; (x \neq y)$$

as our theory. Adding a new binary relation symbol D (to be thought of as $\neq$), the desired coherent theory is

$$\begin{aligned} &\Longrightarrow \exists x \; \exists y \; D(x,y) \\ &\Longrightarrow x = y \vee D(x,y) \\ x = y \wedge D(x,y) &\Longrightarrow \end{aligned}$$

The point is: the models which respect the coherent logic of the new theory are the usual classical models (respecting the full logic) of the old one.

From the point of view of model theory, apart from nice characterizations, completeness as well as conceptual (or Beth-like) completeness are true. The word-by-word analogues of Beth, Craig and Robinson theorems, however, all fail.

From the topos-theoretic point of view, the significance of the coherent logic lies in the obvious fact that «it is preserved by inverse image of geometric morphism of topoi». This means that, whenever M is a model of a coherent theory T in a topos $\mathcal{E}$ and $\mathcal{E}' \xrightarrow{p} \mathcal{E}$ is a geometric morphism, then $p^* \circ M$ is a model of T in $\mathcal{E}'$. In particular, for a coherent (or separable) topos $\mathcal{E}$, our embedding theorems give:

M is an $\mathcal{E}$-model of T iff for every point p of $\mathcal{E}$, $p^* \circ M$ is a set-valued model of T.
This principle which reduces questions of topos to questions of sets is constantly used by the Grothendieck school of Algebraic Geometry who obviously succombed to the charm and mood of this logic.

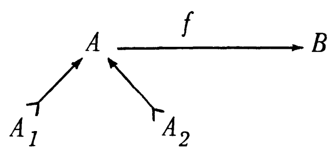
12. Intuitionistic logic.

It was the merit of Lawvere and Tierney to bring «incoherence» in topos theory, by pointing out that

1° first-order, as well as higher-order *intuitionistic* logical operations are interpretable in a topos,

2° the main constructions «inside» a topos may be carried out from this «intuitionistic core» *without* using infinitary operations.

Limiting ourselves to (infinitary) first-order logic, we can make 1 precise as follows : given a diagram



in a topos $\mathcal{E}$, the dual image of $A_1 \to A_2$ under f, written

$$\forall_f(A_1 \to A_2),$$

is the largest subobject $B' \to B$ such that $f^*(B') \wedge A_1 \leq A_2$ (in the ordering $\leq$ of subobjects of A). Here $f^*(B')$ denotes the pull-back of $B' \rightarrowtail B$ along f.

If M is an $\mathcal{E}$-interpretation of an $L_{\infty\omega}$ language L, then *all* formulas of L (and not only the coherent ones) are interpretable in $\mathcal{E}$ via M ; indeed, it is enough to interpret formulas of the form

$$\forall \vec{y}\,(\phi(\vec{x},\vec{y}) \to \psi(\vec{x},\vec{y})).$$

The obvious choice is to interpret them as dual images

$$M_{\vec{x}\,\vec{y}}(\phi) \to M_{\vec{x}\,\vec{y}}(\psi)$$

under the canonical projection $M(\vec{x},\vec{y}) \to M(\vec{x})$. Now, we can set up a formal system for infinitary intuitionistic many-sorted logic and prove a soundness theorem (as before). We omit details (cf. the book already mentioned).

Let us state the following results on embedding of topoi which preserve this intuitionistic operation, besides the coherent ones.

THEOREM. *For every Grothendieck topos* $\mathfrak{D}$ *there is a complete Heyting algebra* $\mathbf{H}$ *and a surjective* $\mathbf{H}$*-valued point*

$$Sh(\mathbf{H}) \xrightarrow{\quad p \quad} \mathcal{E}$$

such that $p^*: \mathcal{E} \to Sh(\mathbf{H})$ *preserves all infs (of subobjects of* $\mathcal{E}$*) as well as dual images in* $\mathcal{E}$.

The *proof* proceeds by extracting a Heyting algebra $\mathbf{H}$ from the boolean algebra $\mathbf{B}$ in the proof of Barr's theorem.

In some cases, this result may be strengthened.

THEOREM. *If the topos* $\mathcal{E}$ *has a surjective boolean point*

$$Sh(2^Y) \xrightarrow{\quad p \quad} \mathcal{E}$$

for some set Y, *then the topos* $Sh(\mathbf{H})$ *in the previous theorem may be chosen to be the category of sheaves over a topological space. In particular this is true for coherent and separable topoi.*

Other results of this type can be found in [Ro] .

As far as 2 is concerned, this «intuitionistic core» was formalized in the notion of an *elementary topos* (in the sense of Lawvere-Tierney).

We notice that a precise logical interpretation of elementary topoi, similar to the one given here for Grothendieck topos, is possible. In fact, Fourman [Fo] has shown that an elementary topos may be viewed as the algebraic version of an intuitionistic higher order theory. His formalization of higher order intuitionistic logic, however, seems rather unusual. «Usual» formalizations have been given by Coste [Co_1] and Boileau [Bo] (in collaboration with Joyal).

Several theorems of elementary topos may be naturally interpreted and proved at the logical level and then reinterpreted at the categorical level. The logical methods used, however, are wholly syntactical. The deduction Theorem e.g. appears as a tool to «internalize» some proofs. Details may be found in Boileau [Bo] .

13. Kock's principle.

In view of the completeness theorem for coherent logic, it is clear that classical logic is a «conservative» extension of coherent logic. In other words, the coherent consequences of a coherent theory are not increased if the *full* classical logic is used in proof. A fortiori, this is true of intuitionistic logic, instead of the classical one.

In the course of «lifting» projective Geometry over fields to «universal projective Geometry» over local rings, A. Kock [Ko] found and effectively used a striking formulation of this fact.

THEOREM *(A. Kock [Ko]). A coherent sequent which is an intuitionistic consequence of the theory T of non-trivial local rings together with the axioms :*

$$* \ \neg \bigwedge \{ x_i = 0 : 1 \le i \le n \} \Longrightarrow \bigvee \{ x_i \text{ is invertible} : 1 \le i \le n \}$$
$$\text{for } n = 1, 2, \ldots$$

is already a consequence of T.

The *proof* proceeds by showing that all these axioms are true in the coherent classifying topos (or universal model) $\mathcal{E}(T)$ of T, i.e. the Zariski topos (Hakim's theorem [Ha]). By the soundness theorem for intuitionistic logic already mentioned in Section 12, any coherent sequent σ intuitionistically provable from T together with * is true in $\mathcal{E}(T)$. By the corollary of Section 10, σ is a consequence of T.

This theorem suggests that theories such as T plus * (which may be called «an intuitionistic theory of fields») have mathematical interest even for «classical» mathematics as «ideal» conservative extensions of coherent theories, local rings in this case. From this point of view, we cannot prejudge «a priori» the existence of an «apartness» relation. Indeed, the only candidate for such a relation is

$$x \# y \underset{dy}{\Longleftrightarrow} \exists z \ (x - y) . z = 1 ,$$

which is equivalent, via * to $\neg x = y$. However, this relation is seen to satisfy only the first two axioms

$$x \# y \implies y \# x,$$

$$x \# y \implies x \# z \vee y \# z$$

for an apartness relation. The third one

$$\neg x \# y \implies x = y$$

is false in $\mathcal{E}(T)$.

Insisting in having the last axiom yields an intuitionistic theory of fields which is a conservative extension (as far as coherent sequents are concerned) of the coherent theory of local rings without not-zero nilpotent elements. It is, of course, only mathematical practice which can decide between the merits of including nilpotent elements in the theory of local rings!

It would be interesting to study intuitionistic «ideal» extensions of the coherent theory of strictly local rings (whose classifying topos is the Etale topos, as shown in [MR]) as well as other coherent theories appearing in Algebraic Geometry.

ADDED IN PROOF. For an application of Kock's principle in commutative algebra, see the author's forthcoming paper:

«Cramer's rule in the Zariski topos».

APPENDIX. GENERAL TOPOS THEORY

For convenience of the reader, we recall the main definitions and theorems of the theory of Grothendieck topoi which are used in this paper.

A *site* is a category $\mathcal{C}$ with finite $\varprojlim$ together with a notion of *localization*, i. e., for every $A \in |\mathcal{C}|$ we are given a non-empty class $Loc(A)$ of families

$$(A_i \xrightarrow{f_i} A)_{i \in I}$$

called the localizations of A, which are stable under pull-backs in the sense that for every arrow $B \to A \in \mathcal{C}$, the family

$$(A_i \underset{A}{\times} B \longrightarrow B)_{i \in I}$$

is a localization of B. Furthermore we assume that

$$(A \xrightarrow{1_A} A) \in Loc(A).$$

(Notice that any class of families of the form $(A_i \xrightarrow{f_i} A)_{i \in I}$ may be extended to a localization, by closing it under pull-backs.)

This notion of site allows us to give a precise meaning to the intuitive idea that some concepts (or functors) have a local character. We say that a functor $F: \mathcal{C}^{opp} \to SET$ is a *sheaf* over $\mathcal{C}$ (or that it is *local*) whenever the elements of $F(A)$ can be recovered from those of $F(A_i)$ for every localization $(A_i \xrightarrow{f_i} A)_{i \in I}$ in the sense that

1° if $\xi, \eta \in F(A)$ are such that

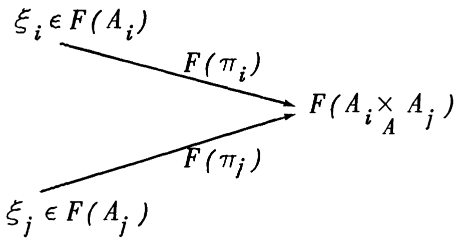
$$\xi_i = F(f_i)(\xi) = F(f_i)(\eta) = \eta_i$$

for all $i \in I$, then $\xi = \eta$,

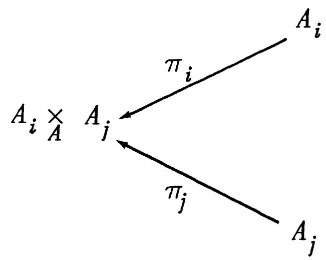
2° if $(\xi_i)_{i \in I}$ is a family such that $\xi_i \in F(A_i)$ for all $i \in I$ and is compatible (i. e., we have

$$F(\pi_i)(\xi_i) = F(\pi_j)(\xi_j))$$

in the diagram



obtained, via F, from



then there is a $\xi \in F(A)$ such that

$$\xi_i = F(f_i)\xi \quad \text{for all} \quad i \in I.$$

We now let $Sh(\mathcal{C})$ be the category of sheaves over a small site $\mathcal{C}$ with natural transformations as morphisms.

THEOREM A. *The category* $Sh(\mathcal{C})$ *has the following properties:*

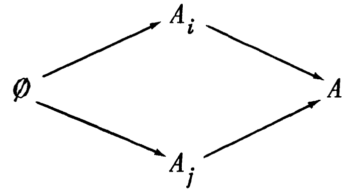
i) it has finite $\varprojlim$;

ii) it has disjoint arbitrary stable coproducts;

iii) the equivalence relations are effective and stable;

iv) it has a set of generators.

REMARKS. Ad ii) The coproduct $A = \coprod_{i \in I} A_i$ is disjoint if



is a pull-back for $i \neq j$, where $\emptyset$ is the initial object (the coproduct of the empty family). Stability (under pull-back) means that, whenever $B \to A$ is given, $B \simeq \coprod_{i \in I} B \times_A A_i$.

Ad iii) An equivalence relation $R \rightarrowtail A \times A$ is effective if there is a $A \xrightarrow{f} B$ such that

$$\begin{array}{ccccc} & & A & & \\ & \nearrow & & \overset{f}{\searrow} & \\ R & & & & B \\ & \searrow & & \nearrow_{f} & \\ & & A & & \end{array}$$

is a pull-back as well as a co-equalizer diagram. We say that R is stable if the diagram remains a co-equalizer under a pull-back along any $C \to A$.

Ad iv) $\mathcal{G}$ is a set of generators if $\mathcal{G}$ has the property that the obvious functor $Sh(\mathcal{C}) \to SET^{\mathcal{G}^{opp}}$ is conservative for monos, i.e., if the image of a monomorphism f is an isomorphism, then f is an isomorphism.

A category satisfying i-iv is called a *Grothendieck topos*. A *pretopos* is a category satisfying i, iii, iv and

ii') it has disjoint *finite* stable coproducts.

It is easily checked that any topos has a *canonical* structure of site:

$$(A_i \xrightarrow{f_i} A)_{i \in I}$$

is a localization of A iff the canonical morphism $\coprod_{i \in I} A_i \to A$ is an epimorphism. This is called the *canonical localization*. Similarly, any pretopos has a so-called *precanonical localization* (as above, but with finite I).

THEOREM B *(Giraud [SGA 4]). For any Grothendieck topos $\mathcal{E}$, there is a small site $\mathcal{C}$ such that $\mathcal{E} \simeq Sh(\mathcal{C})$.*

To state the next result, we assume that $\mathcal{C}, \mathcal{D}$ are sites. A $\mathcal{D}$-*fiber* of $\mathcal{C}$ is a functor $F: \mathcal{C} \to \mathcal{D}$ which preserves finite $\varprojlim$ and is continuous in the sense that it preserves localizations. If $\mathcal{D}$ is small, we let $Fiber_{\mathcal{D}}(\mathcal{C})$ be the full subcategory of the functor category $\mathcal{D}^{\mathcal{C}}$ consisting of $\mathcal{D}$-fibers of $\mathcal{C}$.

A *geometric morphism* $p : \mathcal{E} \to \mathcal{E}'$ between topoi is a couple $p = (p^*, p_*)$, where $p^* : \mathcal{E}' \to \mathcal{E}$ is an $\mathcal{E}$-fiber of $\mathcal{E}'$ and $p_* : \mathcal{E} \to \mathcal{E}'$ is a right adjoint of p^*.

We let $Top(\mathcal{E}, \mathcal{E}')$ be the obvious category of geometric morphisms, with natural transformations. A geometric morphism $p : SET \to \mathcal{E}$ is called a *point of* $\mathcal{E}$.

THEOREM C *(Universal Property). Let* $\mathcal{C}$ *be a small site. Then there is a canonical* $Sh(\mathcal{C})$*-fiber* F *of* $\mathcal{C}$ *which induces (via composition) an equivalence* $Fiber_{\mathcal{E}}(\mathcal{C}) \simeq Top(\mathcal{E}, Sh(\mathcal{C}))$, *for every topos* $\mathcal{E}$.

An *algebraic* site is a small site $\mathcal{C}$ such that, for every localization

$$(A_i \xrightarrow{f_i} A)_{i \in I}$$

of A there is a finite $F \subset I$ such that

$$(A_i \xrightarrow{f_i} A)_{i \in F}$$

is again a localization of A.

A topos $\mathcal{E}$ is *coherent* if $\mathcal{E} \simeq Sh(\mathcal{C})$ for some algebraic site $\mathcal{C}$.

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