CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

ROBERT J. PERRY

An extended comparison functor for triples

Cahiers de topologie et géométrie différentielle catégoriques, tome 18, n° 1 (1977), p. 61-65

<http://www.numdam.org/item?id=CTGDC_1977__18_1_61_0>

CAHIERS DE TOPOLOGIE
ET GEOMETRIE DIFFERENTIELLE

Vol. XVIII-1 (1977)

AN EXTENDED COMPARISON FUNCTOR FOR TRIPLES

by Robert J. PERRY

ABSTRACT

$(\mathcal{B}, \mathcal{C})$ generates a triple on $\mathcal{A}$ when $\mathcal{A}$ is a coreflective subcategory of $\mathcal{C}$ and $\mathcal{B}$ is a reflective subcategory of $\mathcal{C}$. For a given triple $\mathbf{T}$ on *Ens*, a category $A(\mathbf{T})$ is constructed with $(Ens^{\mathbf{T}}, A(\mathbf{T}))$ generating $\mathbf{T}$. The Eilenberg-Moore comparison functor $\phi : \mathcal{B} \to Ens^{\mathbf{T}}$ is then extended to a functor $\psi : \mathcal{C} \to A(\mathbf{T})$ satisfying similar uniqueness conditions.

If $F : \mathcal{C} \to \mathcal{B}$ and $U : \mathcal{C} \to \mathcal{A}$ are reflector and coreflector functors, respectively, with

$$r(C) : C \to F(C) \quad \text{and} \quad \epsilon(C) : U(C) \to C$$

the corresponding reflection and coreflection maps, set

$$\eta(A) = Ur(A), \quad \mu(A) = UF\epsilon F(A) \quad \text{and} \quad T = U|\mathcal{B}\, F|\mathcal{A} ;$$

then

PROPOSITION 1. *(1) $F|\mathcal{A}$ is the left adjoint of $U|\mathcal{B}$;*

(2) $\mathbf{T} = (T, \eta, \mu)$ is a triple on $\mathcal{A}$.

PROOF. It follows easily that η is a natural transformation. If $f : A \to U(B)$ then $t = F(\epsilon(B) f)$ is the unique map satisfying $U(t)\eta(A) = f$. //

We restrict our attention to the case $\mathcal{A} = Ens$.

DEFINITION 1. We say that the pair $(\mathcal{B}, \mathcal{C})$ *generates the triple* $\mathbf{T}$ *on Ens* and we call $(\mathcal{B}, \mathcal{C})$ a *generating pair.* //

The observation that *(Comp Haus, Top)* generates the ultrafilter triple B motivated our results. In [1] Barr showed $R(\mathrm{B}) = Top$, where $R(\mathbf{T})$ is the category of $\mathbf{T}$-relational algebras; Manes [3] proved

$$Ens^{\mathbf{B}} = Comp\,Haus\,.$$

It follows easily from a theorem of Manes [4] that $(Ens^{\mathbf{T}}, R(\mathbf{T}))$ generates $\mathbf{T}$ for any triple $\mathbf{T}$ on Ens.

An additional example of a generating pair is

$$([\mathfrak{D}, Ens]_{Inf}, [\mathfrak{D}, Ens])$$

(see Kennison [2]); Ens is coreflective in $[\mathfrak{D}, Ens]$ provided $\mathfrak{D}$ is connected.

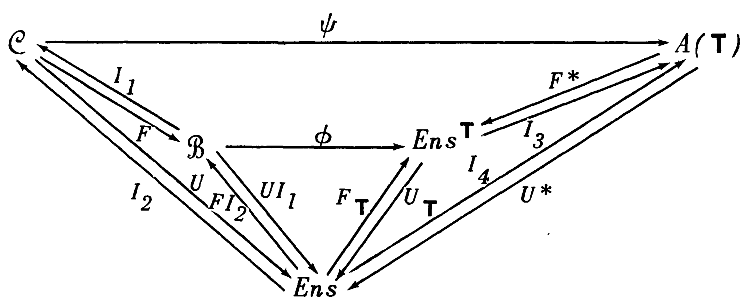
DEFINITION 2. Let $(A, \rho) \in Ens^{\mathbf{T}}$ and let $f: K \to A$. Set

$$f^{-1}(\rho) = \{(x, y) \mid \rho\, T(f)(x) = f(y)\}. \quad //$$

DEFINITION 3. If $\bar{U}: R(\mathbf{T}) \to Ens$, let $A(\mathbf{T})$ be the full subcategory of the comma category $(\bar{U}, \bar{U} \mid Ens^{\mathbf{T}})$ with objects

$$\{((K, f^{-1}(\rho)), f, (A, \rho)) \mid (A, \rho) \in Ens^{\mathbf{T}},\ f: K \to A\}. \quad //$$

We construct a functor $\psi: \mathcal{C} \to A(\mathbf{T})$ that extends the comparison functor of Eilenberg-Moore as indicated in the following diagram:



where

$$U^* I_3 = U_{\mathbf{T}}, \quad U^*\psi = U, \quad \phi F = F^*\psi,$$
$$F^* I_4 = F_{\mathbf{T}}, \quad \psi I_2 = I_4, \quad \psi I_1 = I_3 \phi\,.$$

Moreover, ψ is unique with respect to these conditions.

We consider an alternative view of the situation. Given a functor $G: \mathcal{C} \to \mathcal{C}'$, where $(\mathfrak{B}, \mathcal{C})$ and $(\mathfrak{B}', \mathcal{C}')$ are generating pairs for a fixed triple $\mathbf{T}$ on Ens, the following properties are essential to G's preservation of the generation process:

(1) $G|\mathcal{B}:\mathcal{B}\to\mathcal{B}'$, (2) $G|Ens = 1_{Ens}$,

(3) $G\epsilon = \epsilon' G$, (4) $Gr = r'G$.

DEFINITION 4. If G has properties (1)-(4), we call G *an admissible functor*. The category of generating pairs and admissible functors will be denoted by $G(\mathbf{T})$. //

DEFINITION 5. (1) Let $(\mathcal{B},\mathcal{C})\in G(\mathbf{T})$ with $U:\mathcal{C}\to Ens$ faithful. We let $\mathcal{C}^*$ be the full subcategory of $(U, U|\mathcal{B})$ with objects

$$\{(C, U(f), Y) \mid C\in\mathcal{C},\ Y\in\mathcal{B},\ f: C\to Y\};$$

(2) Set $I(C) = (C, Ur(C), F(C))$ and $I(g) = (g, F(g))$.

Through I we can regard $\mathcal{C}\subset\mathcal{C}^*$.

PROPOSITION 2. $(\mathcal{B},\mathcal{C}^*)\in G(\mathbf{T})$.

PROOF. $(f, 1_Y): (C, U(f), Y)\to(Y, U1_Y, Y)$ and

$$(\epsilon(C), F(\epsilon(Y)U(f))): (U(C), UrU(C), FU(C))\to(C, U(f), Y)$$

are the reflection and coreflection maps, respectively. //

Since $\bar{U}$ is faithful, $(Ens^{\mathbf{T}}, A(\mathbf{T}))\in G(\mathbf{T})$. The existence of ψ with the required properties is equivalent to showing $(Ens^{\mathbf{T}}, A(\mathbf{T}))$ is a terminal object in $G(\mathbf{T})$. Then ψ will be the unique admissible functor from $(\mathcal{B},\mathcal{C})$ to $(Ens^{\mathbf{T}}, A(\mathbf{T}))$.

DEFINITION 6. (1) Let $(\mathcal{B},\mathcal{C})\in G(\mathbf{T})$. Define $\lambda(C)\subset TU(C)\times U(C)$ by

$$(x, y)\in\lambda(C) \quad\text{iff}\quad UF\epsilon(C)(x) = Ur(C)(y).$$

(2) Set $H(\mathcal{B},\mathcal{C})(C) = (U(C), \lambda(C))$ and $H(\mathcal{B},\mathcal{C})(f) = U(f)$. //

PROPOSITION 3. $H(\mathcal{B},\mathcal{C}): \mathcal{C}\to R(\mathbf{T})$ *with* $H(\mathcal{B},\mathcal{C})|\mathcal{B} = \phi$.

PROOF. (1) If $f: C_1\to C_2$ with $(x,y)\in\lambda(C_1)$, then

$$UF\epsilon(C_2)UFU(f)(x) = UF(f)UF\epsilon(C_1)(x) =$$
$$= UF(f)Ur(C_1)(y) = Ur(C_2)U(f)(y),$$

implying $(TU(f)(x), U(f)(y))\in\lambda(C_2)$ and

$$U(f): (U(C_1), \lambda(C_1)) \to (U(C_2), \lambda(C_2)).$$

(2) If $B \in \mathfrak{B}$, $\lambda(B) = UF\epsilon(B)$; it follows that $H(\mathfrak{B}, \mathcal{C}) | \mathfrak{B} = \phi$.

(3) $UF\epsilon(C)\eta U(C) = Ur(C)$, since they are coequalized by $\epsilon F(C)$. Thus $(\eta U(C)(y), y) \in \lambda(C)$ for all $y \in U(C)$ and $(U(C), \lambda(C))$ is reflexive.

(4) $UF\epsilon(C)\mu U(C) T(p_1) = UF\epsilon(C) T(p_2)$, since they are **T**-homomorphisms equalized by $\eta\lambda(C)$. Let $(x, y) \in \lambda(C)\hat{T}(\lambda(C))$, then there exists $z \in TU(C)$ with

$$(x, z) \in \hat{T}(\lambda(C)) \text{ and } (z, y) \in \lambda(C) ;$$

also there exists $w \in \hat{T}(\lambda(C))$ with

$$T(p_1)(w) = x \text{ and } T(p_2)(w) = z .$$

Thus

$$UF\epsilon(C)\mu U(C)(x) = UF\epsilon(C)\mu U(C) T(p_1)(w) =$$
$$= UF\epsilon(C) T(p_2)(w) = UF\epsilon(C)(z) = Ur(C)(y).$$

Whence $(\mu U(C)(x), y) \in \lambda(C)$, so $(U(C), \lambda(C))$ is transitive, and

$$H(\mathfrak{B}, \mathcal{C})(C) \in R(\mathbf{T}). \quad //$$

DEFINITION 7. For $(\mathfrak{B}, \mathcal{C}) \in G(\mathbf{T})$, define

$$H^*(\mathfrak{B}, \mathcal{C})(C) = (H(\mathfrak{B}, \mathcal{C})(C),\ Ur(C),\ H(\mathfrak{B}, \mathcal{C})(F(C)))$$

and $H^*(\mathfrak{B}, \mathcal{C})(f) = (U(f), UF(f))$. //

PROPOSITION 4. $\{H^*(\mathfrak{B}, \mathcal{C})\} = Hom_{G(\mathbf{T})}((\mathfrak{B}, \mathcal{C}), (Ens^{\mathbf{T}}, A(\mathbf{T})))$.

PROOF. Range $H^*(\mathfrak{B}, \mathcal{C}) \subset A(\mathbf{T})$ follows by observing that

$$UF\epsilon F(C) TUr(C) = UF\epsilon(C).$$

The remaining details are straightforward given the observation that

$$H(\mathfrak{B}', \mathcal{C}')G = H(\mathfrak{B}, \mathcal{C}) \text{ when } G: (\mathfrak{B}, \mathcal{C}) \to (\mathfrak{B}', \mathcal{C}'). \quad //$$

EXAMPLES. By Lemma 2 of [5], it follows that $(K, f^{-1}(\rho))$ is a completely regular relational algebra. Thus $f^{-1}(\rho)$ is the largest completely regular relation on K making f admissible.

(1) If **T** is the identity triple, then $(A, \rho) = A$ and $f^{-1}(\rho)$ is the

equivalence relation on K induced by f.

(2) In $A(\mathrm{B})$, (A, ρ) is a compact Hausdorff space, while $(K, f^{-1}(\rho))$ has the smallest completely regular topology on K making f continuous.

Descriptions of the completely regular relational algebras of additional triples are given in [5].

REFERENCES

1. BARR, M., Relational algebras, *Lecture Notes in Math.* 137, Springer (1970), 39 - 55.

2. KENNISON, J. F., On limit-preserving Functors, *Ill. J. Math.* 12 (1968), 616-619.

3. MANES, E., A triple theoretic construction of compact algebras, *Lecture Notes in Math.* 80, Springer (1969), 91 - 118.

4. MANES, E., *Relational Models I*, preprint, 1970.

5. PERRY, R. J., Completely regular relational algebras, *Cahiers Topo. et Géo. Dif.* XVII - 2 (1976), 125 - 133.

Department of Mathematics
Worcester State College
WORCESTER, Mass. 01602
U. S. A.