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ET GEOMETRIE DIFFERENTIELLE

ON CATEGORICAL SHAPE THEORY

by Armin FREI

0. Introduction.

The notion of *shape* was first introduced by Borsuk [2] in the study of homotopy properties of compacta. Several others, e.g. [7], [8], [14], [15], used the same principle in topological contexts. Roughly speaking the principle consists in the following: Given a homotopy category $\mathcal{T}$ of spaces and a full subcategory $\mathcal{P}$ one forms a category $\mathcal{S}$ having the same objects as $\mathcal{T}$ but larger *hom*-sets; a morphism in $\mathcal{S}$ is a system of morphisms in $\mathcal{T}$. However, for pairs of objects (X, A) in $\mathcal{T}$ with A in $\mathcal{P}$, $\mathcal{S}(X, A)$ is equivalent to $\mathcal{T}(X, A)$.

Mardesič [14] gave the general definition of shape for the homotopy category of shapes and the full subcategory of spaces having the homotopy type of CW-complexes, and showed that this definition agrees with many of the previous ones. In [11] Le Van introduced the notion of shape for full embeddings of abstract categories.

In the present paper we introduce the notion of the shape category for a general functor $K: \mathcal{P} \to \mathcal{T}$. The shape category $\mathcal{S}_K$ of K has the same objects as $\mathcal{T}$ and

$$\mathcal{S}_K(X, Y) = Nat[\mathcal{T}(Y, K-), \mathcal{T}(X, K-)],$$

the composition being composition of natural transformations. A *canonical functor* $D: \mathcal{T} \to \mathcal{S}_K$ is defined as being the identity on objects and

$$Df = \mathcal{T}(f, K-) \text{ on morphisms.}$$

Two morphisms in $\mathcal{T}$ are said to be *K-shape equivalent* or to have the *same K-shape* if they are equivalent in $\mathcal{S}_K$. Any two objects which are equivalent in $\mathcal{T}$ have, of course, the same K-shape, but two objects may have the same K-shape even if there is no morphism in $\mathcal{T}$ between them.

By imposing on the functor K the condition that the function

$$f \to \mathcal{T}(f, K-): \mathcal{T}(X, KA) \to \mathcal{S}_K(X, KA)$$

is a bijection for all pairs (X, KA) in $\mathcal{T}$ with A in $\mathcal{P}$, we call it condition C, we preserve one of the main features of «classical» shape theory and obtain many results, some of which generalise facts known for full embeddings in Topology. It appears that C is most suitable to make shape theory work nicely.

Throughout the paper we make frequent use of right Kan extensions. In Section 1 we collect, for further reference, some facts about Kan extensions along functors having an adjoint and along composite functors.

In Section 2 we introduce the notion of the shape category $\mathcal{S}_K$ of a functor $K: \mathcal{P} \to \mathcal{T}$ and the canonical functor $D: \mathcal{T} \to \mathcal{S}_K$, giving three equivalent descriptions. We show that every *pointwise* right Kan extension along K factors through D, i. e. is *shape invariant*. We recall from [13] the notion of *codense functor* and show that K is codense if and only if D is an isomorphism.

In Section 3 we introduce our condition C for K and give two conditions which are equivalent to C and are better suited for several proofs. We also give some consequences of C which lie outside the actual framework of shape theory.

Section 4 is concerned with consequences of condition C. Thus, if K satisfies condition C, then $L = DK$ is codense; the pointwise right Kan extension of a functor exists along K if and only if it exists along L; the factorization of a pointwise Kan extension over D is unique and is itself a pointwise Kan extension over D and over L; D is the pointwise Kan extension of L along K and D is codense. We also show that if K satisfies C, then $\mathcal{S}_K$ and D are characterised by a universal property, i. e. D is terminal among the functors $V: \mathcal{T} \to \mathcal{C}$ where $\mathcal{C}$ has the same objects $\mathcal{T}$ and V is fully faithful on pairs (X, KA).

Section 5 is devoted to the case where K has a left adjoint F. It turns out that $\mathcal{S}_K$ is isomorphic to the Kleisli category of the triple induced by the adjunction $F \dashv K$; in particular D then has a right adjoint H and the adjunction $D \dashv H$ induces the same triple as $F \dashv K$. Furthermore the functors F, D, KF and the left adjoint F_T in the Kleisli situation render invertible the same saturated family of morphisms in $\mathcal{T}$.

In Section 6 we investigate the implications of condition C in the case where K has a left adjoint F. We show that for any adjunction $F \dashv K$ the induced triple is idempotent if and only if the induced cotriple is idempotent. If the functor $K: \mathcal{P} \to \mathcal{T}$ has a left adjoint F, then K satisfies condition C if and only if the induced triple and cotriple are idempotent, if and only if the category $\mathcal{S}_K$ is isomorphic to the category of fractions $\mathcal{T}[S^{-1}]$ with respect to the family S of morphisms in $\mathcal{T}$ rendered invertible by F, or which is the same, by D. Also K satisfies C if and only if every morphism $f: KA \to KB$ in $\mathcal{T}$ is of the form $f = Kg \circ (Kr)^{-1}$ where r is a morphism in $\mathcal{P}$ rendered invertible by K.

NOTATION. We write *lim* for inverse or left limits. If $K: \mathcal{P} \to \mathcal{T}$ is any functor and X any object in $\mathcal{T}$, we denote $(X \downarrow K)$ the obvious comma category and $Q_X: (X \downarrow K) \to \mathcal{P}$ the projection functor. An object in $(X \downarrow K)$ will be denoted

$$\langle f: X \to KA, A \rangle \quad \text{or simply} \quad \langle f, A \rangle .$$

If G is any functor $\mathcal{P} \to \mathcal{C}$,

$$(G', \phi: G'K \to G) = Ran_K G$$

stands for:

G' is the right Kan extension of G along K with universal transformation ϕ.

We sometimes write $Ran_K G$ for the functor G'. $\dot{R}an_K G$ indicates that the Kan extension is *pointwise*, i. e. that

$$G'X = lim\,(\,(X \downarrow K) \xrightarrow{Q_X} \mathcal{P} \xrightarrow{G} \mathcal{C}\,)$$

or equivalently, that the Kan extension is preserved by all representable functors (see Theorem 3 on page 240 of [13]). $\bar{R}an_K G$ indicates that the Kan extension is *absolute*, i. e. is preserved by any functor. Where the term natural transformation appears very often we abbreviate by n.t.

The author is very grateful to Peter Hilton for much valuable information, especially for the statement of Theorem 2.1 asserting that the pointwise right Kan extension along K of any functor is shape invariant. While writing the present paper, the author learned that Aristide Deleanu and Peter Hilton are writing a paper on the same subject [6]. The two papers, although of nonempty intersection, are quite different. Both introduce the same notion of the shape category of K, and the statement mentioned above is contained in both, though with different arguments (cf. Theorem 1.5 of [6]), as well as the assertion that the functor D is an isomorphism if and only if K is codense (Proposition 2.2, cf. Proposition 2.7 of [6]). In order to have the theory working nicely, Deleanu and Hilton impose on K the condition of being *rich*, which turns out to be stronger than condition C. In the Appendix to the present paper we compare the two conditions.

There is of course a dual to shape theory, *coshape theory*, with dual properties. However, in the present paper we do not formulate any duals, except in Section 6 where an equivalence of some conditions with their own duals appear.

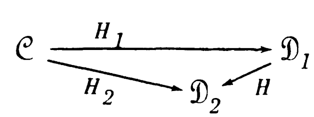
1. On Kan extensions.

Proposition 3 on page 245 of [13] asserts that if a functor $F: \mathcal{C} \to \mathfrak{D}$ is left adjoint to $G: \mathfrak{D} \to \mathcal{C}$ with counit $\epsilon: FG \to 1$, then $\bar{R}an_G 1_{\mathfrak{D}}$ exists and is equal to F with universal transformation ϵ. From this we infer immediately:

PROPOSITION 1.1. *If the functor* $F: \mathcal{C} \to \mathfrak{D}$ *is left adjoint to* $G: \mathfrak{D} \to \mathcal{C}$ *with counit* $\epsilon: FG \to 1$ *of the adjunction, then for any functor* H *with domain* $\mathfrak{D}$, $\bar{R}an_G H$ *exists, and is equal to* HF *with counit* $H\epsilon: HFG \to H$.

As in the sequel we shall have to consider Kan extensions along composite functors, we state and prove

LEMMA 1.2. *Let*



be a commutative diagram of functors and $F: \mathcal{C} \to \mathcal{E}$ *an arbitrary functor.*

(i) If $(\tilde{F};\ \tilde{F}H_1 \xrightarrow{\epsilon_1} F) = Ran_{H_1} F$ *and* $(\tilde{\tilde{F}};\ \tilde{\tilde{F}}H \xrightarrow{\epsilon} \tilde{F}) = Ran_H \tilde{F}$, *then*

$$(\tilde{\tilde{F}};\ \tilde{\tilde{F}}H_2 \xrightarrow{\epsilon H_1} \tilde{F}H_1 \xrightarrow{\epsilon_1} F) = Ran_{H_2} F.$$

(ii) If $(\tilde{F};\ \tilde{F}H_1 \xrightarrow{\epsilon_1} F) = Ran_{H_1} F$ *and* $(\tilde{\tilde{F}};\ \tilde{\tilde{F}}H_2 \xrightarrow{\epsilon_2} F) = Ran_H F$, *then there is a unique* $\epsilon: \tilde{\tilde{F}}H \to \tilde{F}$ *such that* $\epsilon_1 \circ \epsilon H_1 = \epsilon_2$; *furthermore*

$$(\tilde{\tilde{F}};\ \tilde{\tilde{F}}H \xrightarrow{\epsilon} \tilde{F}) = Ran_H \tilde{F}.$$

(iii) The statements (i) and (ii) remain valid if Ran is replaced by $\dot{Ran}$ *or by* $\bar{Ran}$.

PROOF. (i) Let $G: \mathfrak{D}_2 \to \mathcal{E}$ be an arbitrary functor and $GH_2 = GHH_1 \xrightarrow{\psi} F$ a natural transformation. As $\tilde{F} = Ran_{H_1} F$, there is a unique n.t.

$$\theta: GH \to \tilde{F} \quad \text{such that } \epsilon_1 \circ \theta H_1 = \psi .$$

As $\tilde{\tilde{F}} = Ran_H \tilde{F}$, there is a unique n.t.

$$\nu: G \to \tilde{\tilde{F}} \quad \text{such that } \epsilon \circ \nu H = \theta .$$

Then

$$\epsilon_1 \circ \epsilon H_1 \circ \nu H H_1 = \epsilon_1 \circ \theta H_1 = \psi .$$

ν is the unique n.t. satisfying this last identity. Indeed if

$$\epsilon_1 \circ \epsilon H_1 \circ \nu' H H_1 = \psi ,$$

then

$$\epsilon_1 \circ (\epsilon H_1 \circ \nu H H_1) = \epsilon_1 \circ (\epsilon H_1 \circ \nu' H H_1),$$

thus, by the universal property of ϵ_1, $\epsilon \circ \nu H = \epsilon \circ \nu' H$, and by the universal property of ϵ, $\nu = \nu'$.

(ii) As $(\tilde{F},\ \tilde{F}H_1 \xrightarrow{\epsilon_1} F) = Ran_{H_1} F$, the n.t.

$$\tilde{\tilde{F}}HH_1 = \tilde{\tilde{F}}H_2 \xrightarrow{\epsilon_2} F$$

determines a n.t.

$$\epsilon: \tilde{\tilde{F}}H \to \tilde{F} \quad \text{such that } \epsilon_1 \circ \epsilon H_1 = \epsilon_2 .$$

Let now $G: \mathfrak{D}_2 \to \mathcal{E}$ be an arbitrary functor, and $\psi: GH \to \tilde{F}$ a n.t. As

$$(\overset{\approx}{F};\ \overset{\approx}{F} H_2 \xrightarrow{\epsilon_2} F) = Ran_{H_2} F ,$$

the n. t.

$$GH_2 = GHH_1 \xrightarrow{\psi H_1} \tilde{F} H_1 \xrightarrow{\epsilon_1} F$$

determines a unique n. t.

$$\eta : G \to \overset{\approx}{F} \quad \text{such that} \quad \epsilon_2 \circ \eta H_2 = \epsilon_1 \circ \psi H_1 .$$

This entails $\epsilon_1 \circ \epsilon H_1 \circ \eta H H_1 = \epsilon_1 \circ \psi H_1$ and by the universal property of ϵ_1, $\epsilon \circ \eta\ H = \psi$. One verifies that η is the unique natural transformation with this property.

(iii) Suppose that

$$\tilde{F} = \bar{R}an_{H_1} F \quad \text{and} \quad \overset{\approx}{F} = \bar{R}an_H F$$

and let $G : \mathcal{E} \to \mathcal{F}$ be any functor. Then

$$(G\tilde{F};\ G\overset{\approx}{F} H_1 \xrightarrow{G\epsilon_1} GH) = Ran_{H_1} GF$$

and

$$(G\overset{\approx}{F};\ G\overset{\approx}{F} H \xrightarrow{G\epsilon} G\tilde{F}) = Ran_H G\tilde{F} ;$$

hence, by (i),

$$(G\overset{\approx}{F};\ G\overset{\approx}{F} H_2 \xrightarrow{G\epsilon H_1} G\tilde{F} H_1 \xrightarrow{G\epsilon_1} GF) = Ran_{H_2} GF .$$

The same argument works if we replace Ran by $\bar{R}an$ and «G any functor» by «G any representable functor». By a similar argument one shows that (ii) still holds if Ran is replaced by $\bar{R}an$ or by $\dot{R}an$.

2. Shape.

We generalize the notion of shape theory for inclusion functors of [11] to general functors. Let $K : \mathcal{P} \to \mathcal{T}$ be any functor. We define the *shape category* $\mathcal{S}_K$ of K by:

$\mathcal{S}_K$ has the same objects as $\mathcal{T}$,

$\mathcal{S}_K(X, Y) = Nat[\mathcal{T}(Y, K-), \mathcal{T}(X, K-)]$,

the composition of morphisms in $\mathcal{S}_K$ is the composition of natural transformations.

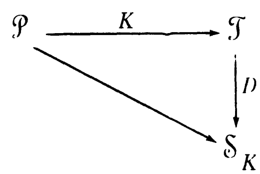
Notice that in general $\mathcal{S}_K$ does not belong to the same universe $\mathcal{U}$ as $\mathcal{T}$ but to a higher universe $\mathcal{U}'$ of which $\mathcal{U}$ is an element. In most «practical» cases however, $\mathcal{S}_K$ does belong to the same universe as $\mathcal{T}$.

We define a canonical functor $D: \mathcal{T} \to \mathcal{S}_K$ by: D is the identity on objects, and for a morphism $f: X \to Y$ in $\mathcal{T}$,

$$D(f) = \mathcal{T}(f, K-) = f^*: \mathcal{T}(Y, K-) \to \mathcal{T}(X, K-),$$

We denote by L the composite functor $L = DK$. The situation is illustrated in the diagram

(2.1)



If X, Y are objects in $\mathcal{T}$ we say that they are *K-shape equivalent* or that they have the *same K-shape* if they are isomorphic in $\mathcal{S}_K$, i.e. if $DX \approx DY$. If two objects X, Y in $\mathcal{T}$ are equivalent in $\mathcal{T}$, then they are of course K-shape equivalent, but in order that they be K-shape equivalent, it is not even necessary that there be a morphism in $\mathcal{T}$ between them as the following trivial example shows. Let $\mathcal{T}$ be any discrete category, $\mathcal{P} = \mathbf{1}$, the category consisting of one object and its identity and $K: \mathcal{P} \to \mathcal{T}$ any functor. Then any two objects of $\mathcal{T}$ are K-shape equivalent.

We say that a functor $G: \mathcal{T} \to \mathcal{A}$ is *K-shape invariant* if there is a functor $\bar{G}: \mathcal{S}_K \to \mathcal{A}$ such that $\bar{G} D = G$.

As $\mathcal{T}$ and $\mathcal{S}_K$ have the same objects and D is the identity on them we will often (but not consistently) write Y for DY where Y is an object of $\mathcal{T}$.

Before we proceed to prove any theorems we give two alternative descriptions of the shape category and of the canonical functor. This will enable us to keep some proofs rather simple and to establish connections to other concepts.

A. For any object X in $\mathcal{T}$ the functor $\mathcal{T}(X, K-)$ can be identified with the comma category $(X\downarrow K)$, and a natural transformation

$$\omega : \mathcal{T}(Y, K-) \to \mathcal{T}(X, K-)$$

can be identified with an obvious functor $\Omega : (Y\downarrow K) \to (X\downarrow K)$ for which the diagram

$$(2.2)\qquad \begin{array}{ccc} (Y\downarrow K) & \xrightarrow{\ \Omega\ } & (X\downarrow K) \\ & {\scriptstyle Q_X}\searrow \quad \swarrow{\scriptstyle Q_Y} & \\ & \mathcal{P} & \end{array}$$

commutes, and the composition of natural transformations corresponds to composition of functors. Thus $\mathcal{S}_K$ is isomorphic to the dual of the category having objects the comma categories $(X\downarrow K)$ for all objects X in $\mathcal{T}$ and morphisms the functors Ω for which (2.2) commutes. If $f: X \to Y$ is a morphism in $\mathcal{T}$, then in this interpretation, $D(f)$ simply becomes the functor $(Y\downarrow K) \to (X\downarrow K)$ induced by f.

B. Let $(\mathcal{P}, \mathcal{E}ns)^{o}$ be the dual of the category of functors $\mathcal{P} \to \mathcal{E}ns$ and $S: \mathcal{T} \to (\mathcal{P}, \mathcal{E}ns)^{o}$ the functor defined by:

$SX = \mathcal{T}(X, K-)$ on objects,

and for $f: X \to Y$, $Sf = \mathcal{T}(f, K-)$.

Thus $\mathcal{S}_K$ is the full subcategory of $(\mathcal{P}, \mathcal{E}ns)^{o}$ generated by the objects which are images under S of objects in $\mathcal{T}$. The functor D is the same as S, considered as a functor with codomain $\mathcal{S}_K$.

This point of view is closely related to the dual of the theory of categories with models of Appelgate-Tierney [1]. In a forthcoming paper we will consider the relationship between categories with models and co-shape.

We now proceed to prove a few facts about shape:

THEOREM 2.1. *In the situation* (2.1) *let* $F: \mathcal{P} \to \mathcal{A}$ *be a functor for which* $\tilde{F} = \dot{R}an_K F$ *exists. Then there is a functor* $\dot{F}: \mathcal{S}_K \to \mathcal{A}$ *with* $\tilde{F} = \dot{F} D$, *i.e. pointwise Kan extensions along* K *are shape invariant.*

PROOF. We consider the description of $\mathcal{S}_K$ given in A and define $\dot{F}$ on objects by $\bar{F}(X\downarrow K) = FX$. If $\Omega: (Y\downarrow K) \to (X\downarrow K)$ is the dual of a morphism in $\mathcal{S}_K$ we define $\bar{F}\Omega$ to be the canonical morphism:

$$\bar{F}X = \lim_{(X\downarrow K)} FQ_X \xrightarrow{\bar{F}\Omega} \lim_{(Y\downarrow K)} FQ_X\Omega = \lim_{(Y\downarrow K)} FQ_Y = \bar{F}Y\,;$$

with this $\bar{F}$ becomes a functor $\bar{F}: \mathcal{S}_K \to \mathfrak{A}$ and clearly $\bar{F}D = \bar{F}$ on objects and on morphisms.

REMARK. We will see in Section 4 that $\bar{F}$ is unique provided that K satisfies condition C.

We recall from [12] that a functor $F: \mathcal{C} \to \mathfrak{D}$ is *codense* if it satisfies one of the equivalent conditions:

(i) For each object X in $\mathfrak{D}$, $\lim_{(X\downarrow F)} FQ_X = X$ with limiting cone given by $\lambda\langle f: X \to FY, Y\rangle = f$.

(ii) $(I_{\mathfrak{D}}\,;\ F \xrightarrow{I} F) = \mathrm{Ran}_F F$.

(iii) For all objects X, X' in $\mathfrak{D}$, the correspondence sending $f: X \to X'$ to $\mathfrak{D}(f, F-)$ is a bijection

$$\mathfrak{D}(X, X') \to Nat[\mathfrak{D}(X', F-),\ \mathfrak{D}(X, F-)].$$

From (iii) and the definition of shape category we have immediately:

PROPOSITION 2.2. *A functor $K: \mathcal{P} \to \mathcal{T}$ is codense if and only if the canonical functor $D: \mathcal{T} \to \mathcal{S}_K$ is an isomorphism.*

EXAMPLES. 1° In [12] Linton defines a category T_U, the *(full) clone of operations* on a functor $U: \mathfrak{X} \to \mathfrak{A}$, and T_U is the dual of the shape category of U.

2° Let S, R be rings with identity, $K: S \to R$ a ring homomorphism preserving the identity, considered as an additive functor. The shape category $\mathcal{S}_K$ is then a category with one object $1R$ and morphisms

$$\mathcal{S}_K(1R, 1R) = Nat[R(1R, K1S),\ R(1R, K1S)].$$

A set map $\eta: R \to R$ is a natural transformation if and only if

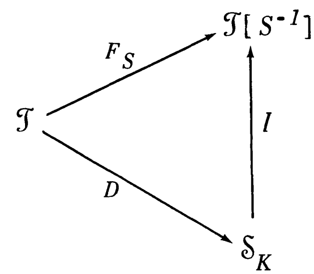
$$K(s)\circ\eta(r) = \eta(K(s)\circ r).$$

Thus $\mathcal{S}_K(1R, 1R)$ consists of those maps $R \to R$ which respect the operation of S on R via K. Under composition and addition in R, then $\mathcal{S}_K(1R, 1R)$ is a ring. The functor $D: R \to \mathcal{S}_K$ is the ring homomorphism taking r in R to $-.r$ in $\mathcal{S}_K$.

3° Let $\mathcal{T}$ be a category, S a family of morphisms in $\mathcal{T}$, $\mathcal{T}[S^{-1}]$ the category of fractions and $F_S : \mathcal{T} \to \mathcal{T}[S^{-1}]$ the canonical functor. Assume that every object Y in $\mathcal{T}$ is S-completable, i.e. that the functor

$$\mathcal{T}[S^{-1}](-, Y): \mathcal{T}^{\circ} \to \mathcal{E}ns$$

is representable for all $Y \in |\mathcal{T}|$, which is the same as to assume that F_S has right adjoint G_S. If η denotes the unit of this adjunction, then the S-complete objects are those objects X in $\mathcal{T}$ for which $\eta X: X \to G_S F_S X$ is an isomorphism. Let $K: \mathcal{T}_S \to \mathcal{T}$ be the embedding of the full subcategory generated by the S-complete objects. As pointed out in [9], there is an isomorphism I rendering commutative the diagram



thus the shape category $\mathcal{S}_K$ may be identified with the category of fractions $\mathcal{T}[S^{-1}]$.

3. Condition C.

In order to have shape theory working nicely we have to impose some condition on the functor K. Condition C below appears to be quite suitable.

DEFINITION 3.1. We say that a functor $K: \mathcal{P} \to \mathcal{T}$ *satisfies condition* C if, for any object A of $\mathcal{P}$ and any object Y of $\mathcal{T}$, the correspondence which takes $f: Y \to KA$ to $\mathcal{T}(f, K-)$ is a bijection

$$\partial: \mathcal{T}(Y, KA) \to Nat[\mathcal{T}(KA, K-), \mathcal{T}(Y, K-)].$$

In terms of (2.1) K satisfies condition C if and only if the functor D is fully faithful on pairs (Y, KA) of objects.

From Proposition 2.2 it follows immediately that K satisfies condition C whenever it is codense. It is also quite easy to see that if K is full then it satisfies condition C.

We defer studying the implications of condition C on shape theory to the next section. We first exhibit some conditions which are equivalent to C and some consequences which lie outside the strict framework of shape theory. In Section 6 we shall see the implications of C in the theory of triples.

LEMMA 3.2. *Let $K: \mathcal{P} \to \mathcal{T}$ be a functor and A an object in $\mathcal{P}$. Then, for every object Y in $\mathcal{T}$ the correspondence*

$$\begin{aligned} \mathcal{T}(Y, KA) &\longrightarrow Nat[\mathcal{T}(KA, K-), \mathcal{T}(Y, K-)] \\ f &\longmapsto \mathcal{T}(f, K-) \end{aligned}$$

is a bijection if and only if

$$KA = lim((KA \downarrow K) \xrightarrow{Q_{KA}} \mathcal{P} \xrightarrow{K} \mathcal{T})$$

with limiting cone $\lambda_{<\beta: KA \to KB\, ;\, B>} = \beta$, where $<\beta, B>$ is an object in $(KA \downarrow K)$.

PROOF. (Only if) A natural transformation $\omega: \mathcal{T}(KA, K-) \to \mathcal{T}(Y, K-)$ determines a cone $Y \xrightarrow{\omega(\beta)} KQ_{KA}$. By the universal property of $lim\, KQ_{KA}$ there is a unique morphism $w: Y \to KA$ such that

$$\omega(\beta) = \beta w \quad \text{for all } \beta: KA \to KB = KQ_{KA}<\beta, B>.$$

Thus $\omega = \mathcal{T}(w, K-)$.

(If) Any cone $\omega: Y \to KQ_{KA}$ is of the form $\omega_{<\beta, B>} = \omega(\beta)$ for a natural transformation $\omega: \mathcal{T}(KA, K-) \to \mathcal{T}(Y, K-)$. But every such natural transformation is induced by a unique morphism $w: Y \to KA$, i.e. there is a unique w such that $\omega_{<\beta, B>} = \beta w$; thus ω is a limiting cone.

«Globalising» Lemma 3.2 we obtain

THEOREM 3.3. *The functor $K: \mathcal{P} \to \mathcal{T}$ satisfies condition* C *if and only if* *for every object A in $\mathcal{P}$, $KA = \lim_{(KA\downarrow K)} KQ_{KA}$ with limiting cone as in Lemma 3.2.*

Whenever we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{H_1} & \mathcal{D}_1 \\ & \searrow^{H_2} & \downarrow H \\ & & \mathcal{D}_2 \end{array}$$

of functors, H induces, for every object X in $\mathcal{D}_1$, a functor

$$\vec{H}_X : (X\downarrow H_1) \to (HX\downarrow H_2)$$

in an obvious way and

$$Q_{HX}\vec{H}_X = Q_X : (X\downarrow H_1) \to \mathcal{C}\,.$$

Furthermore if H is fully faithful, then $\vec{H}_X$ is an isomorphism.

Let

$$\mathcal{P} \xrightarrow{K''} \mathcal{E} \xrightarrow{K'} \mathcal{T}$$

be the canonical factorization of $K : \mathcal{P} \to \mathcal{T}$: $\mathcal{E}$ is the full subcategory of $\mathcal{T}$ generated by those objects which are image under K of objects in $\mathcal{P}$, K' is the embedding and K'' does the same as K to objects and to morphisms. By the remark above, $\vec{K}'_{K''A}$ is an isomorphism, hence clearly an initial functor for any object $K''A$ in $\mathcal{E}$ and

(3.1)
$$\begin{array}{ccc} (K''A\downarrow K'') & \xrightarrow{\vec{K}'_{K''A}} & (KA\downarrow K) \\ & {}_{Q_{K''A}}\searrow \quad \swarrow_{Q_{KA}} & \\ & \mathcal{P} & \end{array}$$

commutes. We are now ready to prove

THEOREM 3.4. *$K : \mathcal{P} \to \mathcal{T}$ satisfies condition C if and only if*

$$(K' ; 1 : K'K'' \to K) = \mathrm{Ran}_{K''}K\,.$$

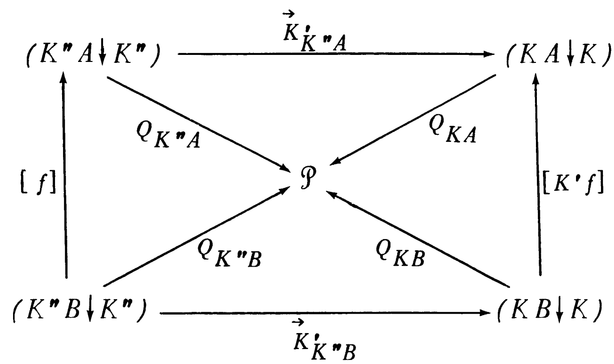
PROOF. As (3.1) commutes and $\vec{K}'_{K''A}$ is an isomorphism,

$$\lim KQ_{KA} = \lim KQ_{K''A}$$

with the same limiting cone on both sides. If K satisfies condition C, then $KA = \lim KQ_{KA}$ with limiting cone as in Lemma 3.2, hence

$$K'(K''A) = KA = \lim KQ_{K''A} = \dot{R}an_{K''}K(K''A),$$

i.e. $K' = \dot{R}an_{K''}K$ on objects. Given any morphism $f: K''A \to K''B$ in $\mathcal{E}$, the diagram



commutes, where $[f]$ and $[K'f]$ are the functors induced by f and $K'f$ respectively. This shows that $K' = \dot{R}an_{K''}K$ on morphisms. The component of the limiting cone, corresponding to the object $<1, K''A>$ in $(K''A \downarrow K'')$ is $1: KA \to KA$, thus the counit of $\dot{R}an_{K''}K$ is the identity.

Suppose now that

$$\dot{R}an_{K''}K = (K';\ 1: K'K'' \to K).$$

Then

$$K'K''A = KA = \lim KQ_{K''A}.$$

If $\lambda_{<f,B>}$ denotes the component of the limiting cone corresponding to the object $<f,B>$ in $(K''A \downarrow K'')$, then, by the hypothesis, $\lambda_{<1,A>} = 1_{KA}$, for all objects A in $\mathcal{P}$. By what was said at the beginning of the proof,

$$\lambda_{<f,B>} = \nu_{<K'f,B>}$$

where ν is the limiting cone of $\lim KQ_{KA}$. We have to show that

$$\nu_{<K'f,B>} = K'f.$$

For this consider $f: K''A \to K''B$ simply as a morphism in $\mathcal{E}$. As we have $K' = \dot{R}an_{K''}K$, $K'f$ is the unique morphism in $\mathcal{T}$ for which

$$\begin{array}{ccc} & KC & \\ {\scriptstyle \lambda_{<fg,\,C>}} \nearrow & & \nwarrow {\scriptstyle \lambda'_{<g,\,C>}} \\ KA = \dot{R}an_{K''}K(K''A) & \xrightarrow[K'f]{} & \dot{R}an_{K''}K(K''B) = KB \end{array}$$

commutes for all objects $<g, C>$ in $(K''B \downarrow K'')$, where λ' is the limiting cone associated with $\dot{R}an_{K''}K(K''B)$. In particular, for

$$C = B \quad \text{and} \quad g = 1_{K''B}\,,$$

we have the commutative diagram

$$\begin{array}{ccc} & KB & \\ {\scriptstyle \lambda_{<f,\,B>}} \nearrow & & \nwarrow {\scriptstyle \lambda'_{<1,\,B>} = 1_{KB}} \\ KA & \xrightarrow[K'f]{} & KB \end{array}$$

Thus $\nu_{<K'f,\,B>} = \lambda_{<f,\,B>} = K'f$.

We now give a necessary condition for K to satisfy condition C.

PROPOSITION 3.5. *If K satisfies condition* C, *then K'' is codense.*

PROOF. We show that for any $K''A$ in $\mathcal{E}$,

$$K''A = lim((K''A \downarrow K'') \xrightarrow{Q_{K''A}} \mathcal{P} \xrightarrow{K''} \mathcal{E})$$

with limiting cone

$$\lambda_{<\beta':K''A \to K''B,\; B>} = \beta'.$$

As K satisfies condition C, by Lemma 3.2 we have

$$K'K''A = lim((KA \downarrow K) \xrightarrow{Q_{KA}} \mathcal{P} \xrightarrow{K} \mathcal{E})$$

with limiting cone

$$\lambda_{<\beta:KA \to KB,\; B>} = \lambda_{<K'\beta':K'K''A \to K'K''B,\; B>} = K'\beta'.$$

As $\vec{K}'$ is an isomorphism,

$$K'K''A = lim((K''A \downarrow K'') \xrightarrow{Q_{K''A}} \mathcal{P} \xrightarrow{K''} \mathcal{E} \xrightarrow{K'} \mathcal{T})$$

with the same limiting cone as above. But K', being fully faithful, reflects limits, hence

$$K''A = lim((K''A \downarrow K) \xrightarrow{Q_{K''A}} \mathcal{P} \xrightarrow{K''} \mathcal{E})$$

with limiting cone $\lambda_{<\beta' : K''A \to K''B,\, B>} = \beta'$.

If the functor K admits a right Kan extension along itself, then it admits a *codensity triple* (cf. [13], Exercise 3, p. 246, there called *codensity monad*). For functors satisfying condition C we have:

PROPOSITION 3.6. *Let $K: \mathcal{P} \to \mathcal{T}$ be a functor which satisfies condition* C *and admits a codensity triple with $(T; \psi: TK \to K) = Ran_K K$. Let*

$$\mathcal{P} \xrightarrow{K''} \mathcal{E} \xrightarrow{K'} \mathcal{T}$$

be the canonical factorization of K. Then K' admits a codensity triple, and this codensity triple is the same as the one of K.

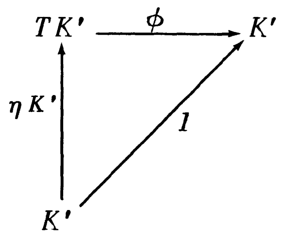
PROOF. By Theorem 3.4,

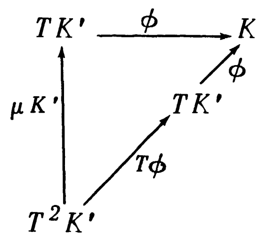
$$(K'; 1: K'K'' \to K) = Ran_{K''}K$$

and by Theorem 1.2, (ii), there is a unique natural transformation ϕ satisfying $\phi K'' = \psi$, and

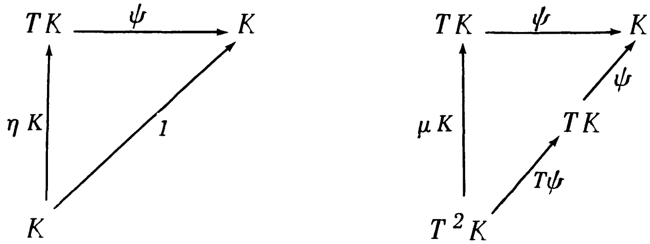
$$(T; \phi: TK' \to K') = Ran_{K'}K'.$$

The unit η and multiplication μ of the codensity triple of K' are the natural transformations for which





commute. Hence, as $\phi K'' = \psi$ and $K'K'' = K$, also



commute, showing that η and μ are also the unit and multiplication of the codensity triple of K.

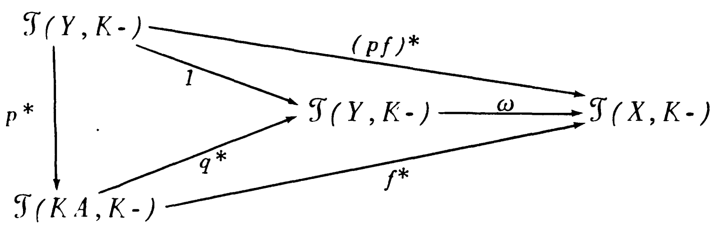
Suppose that the functor K satisfies condition C and let the object Y in $\mathcal{T}$ be dominated by an object KA with A in $\mathcal{P}$, i.e. assume that there are morphisms

$$Y \xrightarrow{q} KA \xrightarrow{p} Y \quad \text{in } \mathcal{T} \quad \text{with } pq = 1.$$

For every object X in $\mathcal{T}$ and every natural transformation

$$\omega : \mathcal{T}(Y, K\text{-}) \to \mathcal{T}(X, K\text{-})$$

we have a commutative diagram



where f^* exists as K satisfies condition C. Hence $\omega = (pf)^*$. If we have $\omega = g_1^* = g_2^*$ then

$$g_1^*(q) = g_2^*(q), \quad \text{i.e.} \quad qg_1 = qg_2, \quad \text{hence} \quad g_1 = g_2.$$

Thus, as described in [12] where K is an inclusion, we have:

PROPOSITION 3.7. *If the functor $K : \mathcal{P} \to \mathcal{T}$ satisfies condition* C, *then for every object Y in $\mathcal{T}$ which is dominated by an object KA, the correspondence which takes f to f^* is a bijection*

$$\mathcal{T}(X, Y) \longrightarrow Nat[\mathcal{T}(Y, K\text{-}), \mathcal{T}(X, K\text{-})].$$

In [4] a functor $K: \mathcal{P} \to \mathcal{T}$ is said to be *dominant* if every object in $\mathcal{T}$ is dominated by an object of the form KA. From Propositions 3.7 and 2.2 we immediately have

COROLLARY 3.8. *If the functor K in (2.1) is dominant and satisfies condition C then D is an isomorphism and K is codense.*

EXAMPLES.

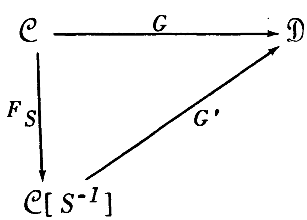
As pointed out before, any full functor, hence any full embedding, satisfies condition C.

Codense functors also satisfy condition C. In order to construct a class of codense functors, let S be a family of morphisms in a category $\mathcal{C}$, $\mathcal{C}[S^{-1}]$ the category of fractions and $F_S: \mathcal{C} \to \mathcal{C}[S^{-1}]$ the canonical functor. We then have

PROPOSITION 3.9. *Any canonical functor $F_S: \mathcal{C} \to \mathcal{C}[S^{-1}]$ is codense.*

The above Proposition is a direct consequence of the more general statement:

PROPOSITION 3.10. *If*



is a commutative diagram, then

(i) $(G'; 1: G'F_S \to G) = \dot{Ran}_{F_S} G$ *and*

(ii) $(G'; 1: G \to G'F_S) = \dot{Lan}_{F_S} G$, *where* $\dot{Lan}$ *stands for: absolute left Kan extension.*

PROOF. It is well known that F_S induces a bijection

$$Nat(M, N) \to Nat(MF_S, NF_S)$$

for any pair of functors $M, N: \mathcal{C}[S^{-1}] \to \mathcal{D}$. Thus, given any natural transformation $\psi: MF_S \to G = G'F_S$, there is a unique natural transformation

$$\psi' : M \to G' \quad \text{with} \quad 1 \circ \psi' F_S = \psi ,$$

hence $(G', 1) = Ran_{F_S} G$. This Kan extension is clearly preserved by any functor, and (i) is proved. Similarly one proves (ii).

The assertion of Proposition 3.10 still holds, of course, if we replace F_S by any functor F which induces a bijection $Nat(M, N) \to Nat(MF, NF)$ and these functors are precisely the ones for which the assertion of Proposition 3.10 holds.

Some other more specific examples of functors satisfying condition C will follow in Section 6, where we study functors having left adjoints.

4. The implications of condition C.

In this Section we refer again to the situation (2.1).

PROPOSITION 4.1. *If* K *satisfies condition* C, *then* $\mathcal{S}_K$ *is isomorphic to* $\mathcal{S}_L$ *and* L *is codense.*

PROOF. If K satisfies C, then for every object A in $\mathcal{P}$,

$$\mathcal{T}(Y, KA) \xrightarrow{\ \xi\ } \mathcal{S}_K(Y, KA)$$

is a natural bijection. But

$$\mathcal{S}_K(Y, KA) = \mathcal{S}_K(Y, DKA), \text{ as } D \text{ is the identity on objects,}$$
$$= \mathcal{S}_K(Y, LA).$$

Hence $\mathcal{T}(Y, K-)$ and $\mathcal{S}_K(Y, L-)$ are equivalent. Thus

$$Nat[\mathcal{T}(Y, K-), \mathcal{T}(X, K-)]$$

is equivalent to

$$Nat[\mathcal{S}_K(Y, L-), \mathcal{S}_K(X, L-)],$$

i.e. $\mathcal{S}_K(X, Y)$ is equivalent to $\mathcal{S}_L(X, Y)$, and as $\mathcal{S}_K$ and $\mathcal{S}_L$ have the same objects, they are isomorphic. L is then codense by Proposition 2.2.

For the sequel we shall need the technical

LEMMA 4.2. K *satisfies condition* C *if and only if for any object* Y *in* $\mathcal{T}$, *the functor* $\vec{D}_Y : (Y \downarrow K) \to (Y \downarrow L)$ *is an isomorphism.*

PROOF. It suffices to recall that D is the identity on objects and K satisfies condition C means that D is fully faithful on pairs of objects of the type (Y, KA).

Lemma 4.2 entails immediately that if K satisfies C, then $\vec{D}_Y$ is initial for any object Y in $\mathcal{T}$. This in turn entails same nice properties about pointwise Kan extensions, in particular the uniqueness of the factorization described in Theorem 2.1.

THEOREM 4.3. *If in the situation* (2.1) *the functor* $\vec{D}_Y$ *is initial for any object* Y *in* $\mathcal{T}$, *in particular if* K *satisfies condition* C, *then:*

(i) For any functor $F : \mathcal{P} \to \mathcal{A}$, $\dot{R}an_K F$ *exists if and only if* $\dot{R}an_L F$ *exists; if they exist, then* $\dot{R}an_K F$ *is canonically isomorphic to* $(\dot{R}an_L F) D$, *and* $\dot{R}an_D (\dot{R}an_K F)$ *is canonically isomorphic to* $\dot{R}an_L F$.

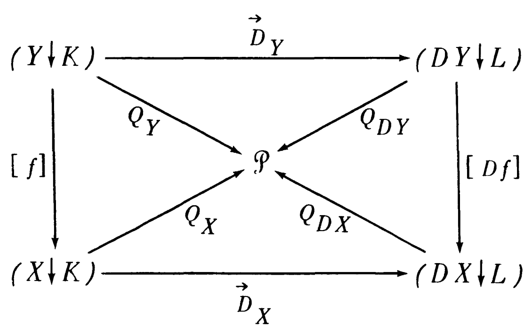
(ii) If $F : \mathcal{P} \to \mathcal{A}$ *is a functor and*

$$(\tilde{F}; \epsilon_1 : \tilde{F} K \to F) = \dot{R}an_K F,$$

then there is a unique functor $\dot{F}$ *with* $\dot{F} D = \tilde{F}$; *furthermore*

$$(\dot{F}; 1 : \dot{F} D \to \tilde{F}) = \dot{R}an_D \tilde{F} \quad \text{and} \quad (\dot{F}; \epsilon_1 : \dot{F} L \to F) = \dot{R}an_L F.$$

PROOF. (i) Let $f : X \to Y$ be any morphism in $\mathcal{T}$. f induces the commutative diagram



where $[f]$ and $[Df]$ are induced by f and Df. Composing with $F : \mathcal{P} \to \mathcal{A}$ and taking limits we obtain the following commutative diagram in $\mathcal{A}$, where θ_X, θ_Y, induced by initial functors, are isomorphisms, and the limits on the left hand side exist if and only if the ones on the right hand side do.

$$\begin{array}{ccc}
(\dot{Ran}_K F)(Y) = \lim F Q_Y = \lim F Q_X[f] & \xleftarrow{\ \theta_Y\ } & \lim F Q_{DY} = (\dot{Ran}_L F)(DY) = \lim F Q_{DX}[Df] \\
{\scriptstyle (\dot{Ran}_K F)(f)} \Big\uparrow & & \Big\uparrow {\scriptstyle (\dot{Ran}_L F)(Df)} \\
(\dot{Ran}_K F)(X) = \lim F Q_X & \xleftarrow[\ \theta_X\ ]{} & \lim F Q_{DX} = (\dot{Ran}_L F)(DX)
\end{array}$$

Hence $\dot{Ran}_K F$ and $\dot{Ran}_L F$ exist simultaneously and

$$\dot{Ran}_K F \xleftarrow{\ \theta\ } (\dot{Ran}_L F) D$$

is a canonical isomorphism. The last assertion follows directly from Lemma 1.2, (ii).

(ii) As $\dot{Ran}_K F = (\tilde{F};\ \epsilon_1 : \tilde{F} K \to F)$ exists, so does

$$\dot{Ran}_L F = (\tilde{\tilde{F}};\ \epsilon_2 : \tilde{\tilde{F}} L \to F)$$

by (i). By the very nature of θ, $\epsilon_1 \circ \theta K = \epsilon_2$, and by Lemma 1.2 (ii), (iii),

$$(\tilde{\tilde{F}};\ \theta : \tilde{\tilde{F}} D \to \tilde{F}) = \dot{Ran}_D \tilde{F} .$$

Let $\bar{F}$ be such that $\bar{F} D = \tilde{F}$. Then $1 : \bar{F} D \to \tilde{F}$ induces a unique natural transformation $\psi : \bar{F} \to \tilde{\tilde{F}}$ with $\theta \circ \psi D = 1$. As θ is an equivalence and D is onto objects, ψ is an equivalence, hence

$$(\bar{F};\ 1 : \bar{F} D \to \tilde{F}) = \dot{Ran}_D \tilde{F} .$$

If G is a functor with $GD = \tilde{F}$, then $1 : GD \to \tilde{F}$ induces a unique natural transformation $\tau : G \to \bar{F}$ such that $\tau D = 1$, hence $\tau = 1$, again as D is onto objects, hence $\bar{F}$ is the unique functor with $\bar{F} D = \tilde{F}$. Finally

$$(\bar{F};\ \epsilon_1 : \bar{F} L \to F) = \dot{Ran}_L F$$

follows from

$$(\tilde{F};\ \epsilon_1 : \tilde{F} K \to F) = \dot{Ran}_K F \quad \text{and} \quad (\bar{F};\ 1 : \bar{F} D \to \tilde{F}) = \dot{Ran}_D \tilde{F}$$

by Lemma 1.2 (i), (iii).

COROLLARY 4.4. *If K in (2.1) satisfies condition* C, *then*

$$(D;\ 1 : DK \to L) = \dot{Ran}_K L .$$

PROOF. If K satisfies C, then, by Proposition 4.1, L is codense, i.e.

$$(4.1) \qquad (1\mathcal{S}_K;\ 1: L \to L) = \dot{R}an_L L,$$

and by Theorem 4.3 (i),

$$(4.2) \qquad (\tilde{L};\ \epsilon_1 : \tilde{L}K \to L) = \dot{R}an_K L$$

exists; furthermore by (ii) of the same Theorem, there is a unique functor $\bar{L}: \mathcal{S}_K \to \mathcal{S}_K$ with $\bar{L}D = \tilde{L}$, and $\bar{L}$ also satisfies

$$(4.3) \qquad (\bar{L};\ 1: \bar{L}D \to \tilde{L}) = \dot{R}an_D \tilde{L}.$$

From (4.2), (4.1), we have, by Lemma 1.2, (ii), (iii) that there is a unique n.t. $\epsilon: 1\mathcal{S}_K D \to \tilde{L}$ with $\epsilon_1 \circ \epsilon K = 1$ and

$$(4.4) \qquad (1\mathcal{S}_K;\ \epsilon: D \to \tilde{L}) = \dot{R}an_D \tilde{L}.$$

From (4.3) and (4.4) we infer that $\epsilon: D \to \tilde{L}$ is an equivalence, but this entails that

$$(D;\ \epsilon_1 \circ \epsilon K = 1: DK \to L) = \dot{R}an_K L.$$

From $(D;\ 1: DK \to L) = \dot{R}an_K L$ and (4.1) we have by Lemma 1.2, (ii), (iii), that $(1\mathcal{S}_K;\ 1: D \to D) = \dot{R}an_D D$, i.e.

COROLLARY 4.5. *If K in (2.1) satisfies condition C, then D is codense.*

We conclude this Section by exhibiting the fact that when K satisfies condition C then the functor D is terminal in the family of functors $V: \mathcal{T} \to \mathcal{C}$ characterized by

(a) $\mathcal{C}$ has the same objects as $\mathcal{T}$,

(b) V is fully faithful on pairs (X, KA) with A in $\mathcal{P}$.

This generalizes Theorem 3.1 of [14].

THEOREM 4.6. *If K in (2.1) satisfies condition C and $V: \mathcal{T} \to \mathcal{C}$ satisfies (a), (b) above, then there is a unique functor $W: \mathcal{C} \to \mathcal{S}_K$ such that $WV = D$.*

PROOF. We interpret $\mathcal{S}_K$ and D as in A of Section 2. We define W on objects by $W(X) = (X \downarrow K)$ which amounts to the identity in our «usual» description of $\mathcal{S}_K$ and is the only way to define W on objects so that $WV = D$.

Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. f induces a functor

$$(Y\!\downarrow\! VK) \xrightarrow{\quad [f] \quad} (X\!\downarrow\! VK)$$

By (b) the functor $\vec{V}_X\colon (X\!\downarrow\! K) \to (X\!\downarrow\! VK)$ is an isomorphism for every object X in $\mathcal{T}$. Hence $[f]$ gives raise to a functor $\bar{f}\colon (Y\!\downarrow\! K) \to (X\!\downarrow\! K)$ which commutes with Q_X and Q_Y, i.e. to a morphism in $\mathcal{S}_K(X, Y)$. The correspondence $f \to \bar{f}$ clearly preserves identities and composition, hence is a functor.

Let $g\colon X \to Y$ be a morphism in $\mathcal{T}$. The diagram

$$\begin{array}{ccc} (Y\!\downarrow\! K) & \underset{\overline{Vg}}{\overset{[g]}{\rightrightarrows}} & (X\!\downarrow\! K) \\ \big\downarrow \vec{V}_Y & & \big\downarrow \vec{V}_X \\ (Y\!\downarrow\! VK) & \xrightarrow{\quad [Vg] \quad} & (X\!\downarrow\! VK) \end{array}$$

clearly commutes with both top arrows, hence $\mathbb{W}Vg = (\overline{Vg}) = Dg$. This shows that $\mathbb{W}V = D$ on morphisms.

Let now $\mathbb{W}'$ be a functor with $\mathbb{W}'V = D$. $\mathbb{W}$ and $\mathbb{W}'$ are clearly identical on objects, and, as K and V satisfy (b), they are identical on morphisms in $\mathcal{C}$ which are of the type $X \to VKA$, i.e. of the type Vn for an n in $\mathcal{T}(X, KA)$. Let now $f\colon X \to Y$ be a morphism in $\mathcal{C}$ and let

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow^{Vn} & \big\downarrow Vm \\ & & VKA \end{array}$$

be commutative in $\mathcal{C}$. (For any morphism $Y \to VKA$ in $\mathcal{C}$ we can write such a diagram by (b).) From this we obtain a diagram

$$\begin{array}{ccc} \mathbb{W}X = \mathbb{W}'X & \underset{\mathbb{W}'f}{\overset{\mathbb{W}f}{\rightrightarrows}} & \mathbb{W}Y = \mathbb{W}'Y \\ & {}_{Dn}\searrow \qquad \swarrow_{Dm} & \\ & DKA & \end{array}$$

which commutes, with both horizontal arrows, for all m in $\mathcal{T}(Y, KA)$ and all A in $\mathcal{P}$. By the Lemma below this entails that $W(f) = W'(f)$.

LEMMA 4.7. *Let $K: \mathcal{P} \to \mathcal{T}$ be any functor and $G, G': (Y \downarrow K) \to (X \downarrow K)$ two functors which commute with the projections Q_X, Q_Y. If $G[i] = G'[i]$ for all functors $[i]: (KA \downarrow K) \to (Y \downarrow K)$ induced by morphisms $i: Y \to KA$ in $\mathcal{T}$, then $G = G'$. In other words: if $\omega, \omega': \mathcal{T}(Y, K\text{-}) \to \mathcal{T}(X, K\text{-})$ satisfy $\omega \circ i^* = \omega \circ i'^*$ for any $i: Y \to KA$, then $\omega = \omega'$.*

PROOF. Let $k: Y \to KA$ be an object in $(Y \downarrow K)$. Then

$$Gk = G[k](1KA) \quad \text{and} \quad G'k = G'[k](1KA),$$

hence G and G' are equal on objects, and as they commute with Q_X, Q_Y, they are equal on morphisms.

REMARK. In the proof of Theorem 4.6 we have tacitly assumed that there are morphisms $Vm: Y \to VKA$, i.e. that $\mathcal{C}(Y, VKA)$ is not empty for all A. If it is empty, then the category $(Y \downarrow K) = W(Y)$ is also empty, and $W(f)$ is the embedding of the empty category $(Y \downarrow K)$ into $(X \downarrow K)$.

5. The adjoint situation.

The purpose of this Section is to collect some information about the case where the functor K of (2.1) has a left adjoint. We will actually consider the more general case where $Ran_K K$ exists and adapt some results of [12] to our situation. In order to visualize to which category objects and morphisms belong, we denote by $X, Y, \dots$ the objects in $\mathcal{T}$ and by DX, $DY, \dots$ the ones in $\mathcal{S}_K$.

As already observed in the proof of Lemma 3.2, given a functor $KQ_X: (X \downarrow K) \to \mathcal{T}$, every cone $Y \to KQ_X$ can be identified with a natural transformation $\omega: \mathcal{T}(X, K\text{-}) \to \mathcal{T}(Y, K\text{-})$, i.e. with a morphism $\omega: DY \to DX$ in $\mathcal{S}_K$, and every such morphism represents a cone; then a limiting cone $Z \to KQ_X$ is identified with a morphism $\lambda: DZ \to DX$ having the universal property: for every morphism $\omega: DY \to DX$, there is a unique morphism $f: Y \to Z$ in $\mathcal{T}$ such that $\lambda \circ D(f) = \omega$. We call such a morphism *universal.*

Suppose given, for every object X in $\mathcal{T}$, a universal morphism

$$\lambda X : DTX \to DX.$$

The family $\{\lambda X\}$ determines a functor $T : \mathcal{T} \to \mathcal{T}$; for a morphism $f: X \to Y$ in $\mathcal{T}$, $Tf: TX \to TY$ is defined to be the unique morphism with

$$\lambda Y \circ D(Tf) = Df \circ \lambda X. \tag{5.1}$$

With this λ becomes a natural transformation $DT \to D$ and

$$(T;\ \lambda K\text{-}(1K\text{-}): TK \to K) \dot{=} Ran_K K.$$

Thus the family $\{\lambda X\}$ determines a specific (within its equivalence class) codensity triple $\mathbf{T} = (T, \eta, \mu)$, where ηX and μX are defined to be the unique morphisms satisfying respectively

$$\lambda X \circ D(\eta X) = 1DX \quad \text{and} \quad \lambda X \circ D(\mu X) = \lambda X \circ \lambda TX. \tag{5.2}$$

Let $\mathcal{T}_T$ denote the Kleisli category of $\mathbf{T}$ and

$$F_T : \mathcal{T} \to \mathcal{T}_T, \quad U_T : \mathcal{T}_T \to \mathcal{T}$$

the canonical functors. We define a functor $I : \mathcal{T}_T \to \mathcal{S}_K$ by

$IX = DX$ on objects,

$I(f: Y \to TX) = \lambda X \circ Df : DY \to DTX \to DX$ on morphisms.

Using (5.1) and (5.2) one verifies easily that I is a functor and that the functor $IF_T = DI$ is fully faithful by the universal property of the morphisms in $\{\lambda X\}$. As it is the identity on objects, I is an isomorphism. $H = U_T I^{-1}$ is then right adjoint to D with unit η, counit ϵ given by $\epsilon DX = \lambda X$ and satisfies $HD = T$. The right adjoint H to D with $HD = T$ and counit $\epsilon DX = \lambda X$ is uniquely determined by $\{\lambda X\}$. As I is an isomorphism, the triple induced by $\eta, \epsilon : D \dashv H$ is $\mathbf{T}$. $I : \mathcal{T}_T \to D$ is clearly the comparison functor.

On the other hand, if H is a given right adjoint to D with counit ϵ, then for every X in $\mathcal{T}$,

$$\{\lambda X = \epsilon DX : DHDX \to DX\}$$

is a family of universal morphisms, and with this family HD satisfies (5.1). Thus we have a one to one correspondence:

$\{$ families $\{\lambda X\}$ of universal morphisms$\} \Longleftrightarrow \{$ adjunctions $D \dashv H\}$.

Summarizing we have:

THEOREM 5.1. *The functor D of (2.1) has a right adjoint if and only if $\mathrm{Ran}_K K$ exists and in this case $\mathcal{S}_K$ is isomorphic to the Kleisli category of the codensity triple of K. More precisely, there is a one to one correspondence between families of universal morphisms $\{\lambda X\}$ and adjunctions $D \dashv H$ given by*

$\{\lambda X\} \Longleftrightarrow H: \mathcal{S}_K \to \mathcal{T}$ *such that* $D \dashv H$ *with counit* $\epsilon DX = \lambda X$.

The triple induced by $D \dashv H$ is the codensity triple $\mathbf{T}$ of K determined by the corresponding $\{\lambda X\}$ via (5.1) and (5.2). The comparison functor $\mathcal{T}_T \to \mathcal{S}_K$ ($\mathcal{T}_T$ the Kleisli category of $\mathbf{T}$) is an isomorphism.

Suppose that K has a left adjoint F with unit η and counit ϵ. This adjunction determines a family $\{\lambda X\}$ of morphisms $\lambda X: DKFX \to DX$ given, for f in $\mathcal{T}(X, KA)$, by

$$\lambda X(f) = K\epsilon A \circ KFf \quad \text{in} \quad \mathcal{T}(KFX, KA).$$

One verifies that every λX is universal and that the codensity triple of K determined by $\{\lambda X\}$ via (5.1) and (5.2) is identical to the one induced by the adjunction $\eta, \epsilon: F \dashv K$.

Let $\mathbf{T} = (T, \eta, \mu)$ be the triple induced by $\eta, \epsilon: F \dashv K$, $\mathcal{T}_T$ its Kleisli category and

$$F_T: \mathcal{T} \to \mathcal{T}_T, \quad U_T: \mathcal{T}_T \to \mathcal{T}$$

the canonical functors. We then have:

COROLLARY 5.2. *If K has a left adjoint F with unit η and counit ϵ, then D has a unique right adjoint H with $HD = FK$, unit η and counit ϵ' given by $\epsilon' DX(f) = K\epsilon A \circ KFf$ for f in $\mathcal{T}(X, KA)$. The triples induced by $F \dashv K$ and $D \dashv H$ are identical and the comparison functor $I: \mathcal{T}_T \to \mathcal{S}_K$ is an isomorphism.*

REMARK. If $\mathrm{Ran}_K K$ exists, in particular if K has a left adjoint, $\mathcal{S}_K$ belongs to the same universe as $\mathcal{T}$. Indeed $\mathcal{S}_K$ is isomorphic to $\mathcal{T}_T$ which belongs to the same universe as $\mathcal{T}$.

COROLLARY 5.3. *If K has a left adjoint F, then the functors F, KF, F_T and D render invertible the same family S of morphisms of $\mathcal{T}$.*

PROOF. The comparison functors

$$I: \mathcal{T}_T \to \mathcal{S} \quad \text{and} \quad C: \mathcal{T}_T \to \mathcal{P}$$

are fully faithful, hence $F = CF_T$ and $D = IF_T$ render invertible the same morphisms as F_T. The canonical functor $U_T: \mathcal{T}_T \to \mathcal{T}$ satisfies $U_T = U^T C'$ where U^T is the forgetful functor from the Eilenberg-Moore category $\mathcal{T}^T$ of $\mathbf{T}$ to $\mathcal{T}$ and $C': \mathcal{T}_T \to \mathcal{T}^T$ the comparison functor. C' is fully faithful and U^T reflects isomorphisms, thus U_T reflects isomorphisms. Hence we get: $KF = U_T F_T$ and F_T render invertible the same morphisms.

6. Condition C in the adjoint situation.

In this Section we study the implications of condition C in the case where K of (2.1) has a left adjoint F. As in the last section, we denote by η, ϵ the unit and counit respectively of the adjunction $F \dashv K$ and by $\mathbf{T} = (T, \eta, \mu)$ the triple induced on $\mathcal{T}$. We denote

$$\mathbf{C} = (C, \epsilon, \delta) = (FK, \epsilon, F\eta K)$$

the cotriple induced on $\mathcal{P}$. We recall that the triple $\mathbf{T}$ is said to be *idempotent* if μ is an equivalence; similarly we say that the cotriple is idempotent if δ is an equivalence.

THEOREM 6.1. *Suppose that the functor K of (2.1) has a left adjoint F. Then K satisfies condition C if and only if the cotriple $\mathbf{C}$ is idempotent.*

PROOF. By definition K satisfies condition C if and only if D is fully faithful on pairs (X, KA) of objects in $\mathcal{T}$, hence, by Corollary 5.2 if and only if F_T is fully faithful on the same pairs of objects. By the definition of the morphisms in $\mathcal{T}_T$ and of F_T the diagram of natural transformations

$$\begin{array}{ccc} \mathcal{T}(Y,KA) & \xrightarrow{(F_T)} & \mathcal{T}_T(Y,KA) \\ & \searrow^{\mathcal{T}(Y,\eta KA)} & \| \updownarrow \\ & & \mathcal{T}(Y,TKA) \end{array}$$

is commutative. (F_T) is a natural bijection if and only if ηK is an equivalence. But if ηK is an equivalence, so is $\delta = F\eta K$. If, on the other hand, $F\eta K$ is an equivalence, we have from $K\epsilon \circ \eta K = 1K$ that

$$F(\eta K \circ K\epsilon) = 1FK.$$

As F is fully faithful on the image of K, $\eta K \circ K\epsilon = 1K$, thus ηK is an equivalence.

The theory of shape of a functor K which has a left adjoint and condition C turns out to be very closely related to idempotent triples and cotriples. We therefore prove a few statements about the latter. We consider our adjunction $F \dashv K$ with its induced triple $\mathbf{T}$ and cotriple $\mathbf{C}$ and restate Proposition 2.1 of [4] together with its dual.

PROPOSITION 6.2. *For the triple* $\mathbf{T}$ *the following are equivalent:*

(i) $\mathbf{T}$ *is idempotent,*

(ii) ϵF *is an equivalence,*

(iii) $F\eta$ *is an equivalence,*

(iv) $F\eta \circ \epsilon F = 1$,

and for the cotriple $\mathbf{C}$ *the following are equivalent:*

(i)^{o} $\mathbf{C}$ *is idempotent,*

(ii)^{o} ηK *is an equivalence,*

(iii)^{o} $K\epsilon$ *is an equivalence,*

(iv)^{o} $\eta K \circ K\epsilon = 1$.

The eight conditions just enumerated are all equivalent; i.e., we have:

PROPOSITION 6.3. *The triple* $\mathbf{T}$ *induced by an adjunction* $F \dashv K$ *is idempotent if and only if the cotriple induced by the same adjunction is idempotent.*

PROOF. Suppose that $\mathbf{T}$ is idempotent. Then, by Proposition 6.1, (ii), ϵF is an equivalence, thus $\epsilon FK = \epsilon C$ is an equivalence. As $\mathbf{C}$ is a cotriple, $\epsilon C \circ \delta = 1C$, thus δ is an equivalence. Dually, $\mathbf{C}$ idempotent entails $\mathbf{T}$ idempotent.

Let now R and S be the families of morphisms rendered invertible by K and F respectively, let $\mathcal{T}[S^{-1}]$ be the category of fractions with respect to S and $F_S : \mathcal{T} \to \mathcal{T}[S^{-1}]$ the canonical functor. By Proposition 1.1 of [3], S and R are saturated.

THEOREM 6.4. *For the adjunction $F \dashv K$ the following are equivalent:*

(i) K satisfies condition C.

(ii) The triple $\mathbf{T}$ *is idempotent.*

(iii) For every morphism $f: KA \to KB$ there is a pair of morphisms

$$A \xleftarrow{\;r\;} A' \xrightarrow{\;g\;} B$$

in $\mathcal{P}$ with r in R and $f = Kg \circ (Kr)^{-1}$.

(iv) There is an isomorphism (unique)

$$I' : \mathcal{T}[S^{-1}] \to \mathcal{T}_T \quad \text{with} \quad I'F_S = F_T.$$

(v) There is an isomorphism (unique)

$$I'' : \mathcal{T}[S^{-1}] \to \mathcal{S}_K \quad \text{with} \quad I''F_S = D.$$

PROOF. (i) $\Longleftrightarrow$ (ii) by Theorem 6.1 and Proposition 6.3.

(ii) $\Longrightarrow$ (iii). Let $f: KA \to KB$ be a morphism, and consider the morphisms

$$A \xleftarrow{\;\epsilon A\;} FKA \xrightarrow{\;Ff\;} FKB \xrightarrow{\;\epsilon B\;} B$$

Then ϵA is in R by Proposition 6.2, (iii)° and

$$\begin{array}{ccc} KA & \xrightarrow{\quad f \quad} & KB \\ {\scriptstyle K\epsilon A}\uparrow & & \uparrow{\scriptstyle K\epsilon B} \\ KFKA & \xrightarrow[\quad KFf \quad]{} & KFKB \end{array}$$

commutes. Hence $f = (K\epsilon B \circ KFf) \circ (K\epsilon A)^{-1}$.

(iii) $\Longrightarrow$ (i). The map

$$\delta : \mathcal{T}(Y, KA) \to Nat[\mathcal{T}(KA, K\text{-}), \mathcal{T}(Y, K\text{-})]$$

has, in general, a left inverse κ, taking $\omega : \mathcal{T}(KA, K\text{-}) \to \mathcal{T}(Y, K\text{-})$ to

$\omega A(1KA)$. Let now $f: KA \to KB$ be a morphism with $f = Kg \circ (Kr)^{-1}$ and r in R and let ω be a natural transformation as above. Then

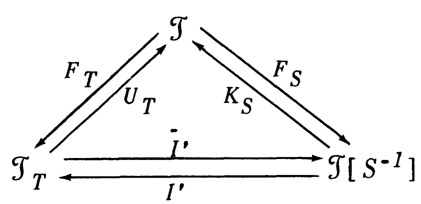
$$\begin{array}{ccc}
\mathcal{T}(KA,KB) & \xrightarrow{\omega B} & \mathcal{T}(Y,KB) \\
{\scriptstyle \mathcal{T}(KA,Kg)}\uparrow & & \uparrow{\scriptstyle \mathcal{T}(Y,Kg)} \\
\mathcal{T}(KA,KA') & \xrightarrow{\omega A'} & \mathcal{T}(Y,KA') \\
{\scriptstyle \mathcal{T}(KA,Kr)}\downarrow & & \downarrow{\scriptstyle \mathcal{T}(Y,Kr)} \\
\mathcal{T}(KA,KA) & \xrightarrow{\omega A} & \mathcal{T}(Y,KA)
\end{array}$$

commutes. Thus, taking $1KA$ from the lower left to the upper right corner along the perimeter, we have $\omega B(f) = f \circ \omega A(1KA)$, hence

$$\omega = (\omega A(1KA))^* = \delta_K(\omega)$$

Thus δ is a bijection, i.e. K satisfies condition C.

(ii) $\Longrightarrow$ (iv). By Corollary 5.3, the family of morphisms rendered invertible by T and by F_T is S, and, by Theorem 2.4 of [4], F_S has a right adjoint K_S such that the adjoint pair generates **T**. Thus in



I' exists (uniquely) with $I'F_S = F_T$ by the universal property of F_S and $\bar{I}'$ exists (uniquely) with

$$\bar{I}'F_T = F_S \quad \text{and} \quad K_S\bar{I}' = U_T$$

by the universal property of the Kleisli situation. Thus $\bar{I}'I'F_S = F_S$, hence $\bar{I}'I' = 1$; moreover

$$U_T I'\bar{I}' = K_S\bar{I}' = U_T, \quad \text{and} \quad I'\bar{I}'F_T = F_T,$$

hence $I'\bar{I}' = 1$, i.e. $\bar{I}' = I'^{-1}$.

(iv) $\Longrightarrow$ (ii). If the isomorphism I' exists, $K_S = U_T I'$ is right adjoint to F_S and the adjoint pair induces **T**, and, again by Theorem 2.4 of [4], **T** is idempotent.

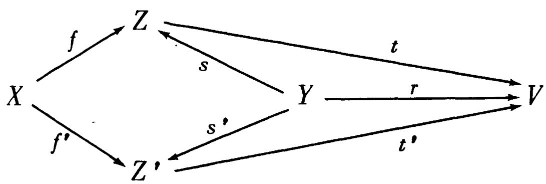
(iv) $\Longrightarrow$ (v) is clear in the light of Corollary 5.2.

Each one of the conditions of Theorem 6.4 has, of course, a dual and, by Proposition 6.3, these duals are equivalent to the original conditions of Theorem 6.4.

I' and I'^{-1} can be given explicitly: on objects, both are clearly the identity. By Theorem 2.8, (ii), (vii) of [3], S has a calculus of left fractions, hence every morphism $X \to Y$ in $\mathfrak{T}[S^{-1}]$ is an equivalence class f/s of pairs

$$X \xrightarrow{\ f\ } Z \xleftarrow{\ s\ } Y$$

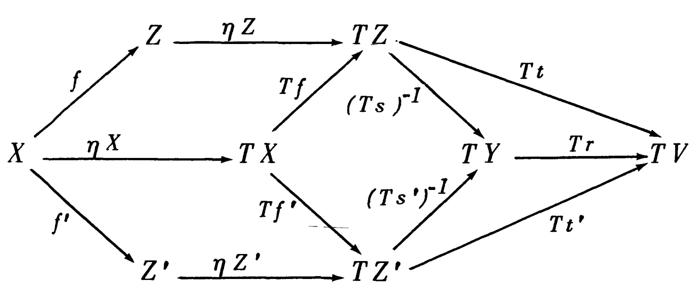
of morphisms in $\mathfrak{T}$ with s in S, two pairs (f, s), (f', s') being in the same class if there is a commutative diagram



in $\mathfrak{T}$ with s, s', t, t' and r in S. We define

$$I'(f/s) = (\ X \xrightarrow{\ f\ } Z \xrightarrow{\ \eta Z\ } TZ \xrightarrow{\ (Ts)^{-1}\ } TY\ )$$

I' is well defined, for, if (f, s) and (f', s') are in the same class, the diagram



commutes and Tr is an isomorphism. For $g: X \to TY$ in $\mathfrak{T}_T(X, Y)$ we define

$$I'^{-1} g = (g/\eta Y) = (\ X \xrightarrow{\ g\ } TY \xleftarrow{\ \eta Y\ } Y\ ).$$

By Proposition 6.2, ηY is in S. A straightforward verification shows that I' and I'^{-1} just defined are inverse to each other. Both plainly respect identities and I'^{-1} respects composition. Indeed for

$$f: X \to TY \quad \text{and} \quad g: Y \to TZ$$

in $\mathcal{T}_T$, $gf = \mu Z \circ Tg \circ f$ while

$$I'^{-1}(gf) = (\mu Z \circ Tg \circ f)/\eta Z,$$

which, by the commutativity of the diagram

$$\begin{array}{ccccccccc}
X & \xrightarrow{f} & TY & \xrightarrow{Tg} & T^2 Z & \xrightarrow{\mu Z} & TZ & \xleftarrow{\eta Z} & Z \\
 & & \uparrow{\scriptstyle \eta Y} & & \uparrow{\scriptstyle \eta TZ} & & & & \\
 & & Y & \xrightarrow[g]{} & TZ & & & &
\end{array}$$

is equal to $I'^{-1}(g) \circ I'^{-1}(f)$. As I'^{-1} respects composition, so does I' and they are both functors. It remains to show that $I'F_S = F_T$ on morphisms: for $h: X \to Y$ in $\mathcal{T}$,

$$I'F_S\, h = I'(h/1) = \eta Y \circ h = F_T(h).$$

Clearly $K_S = U_T I'$, thus we obtain an explicit description of K_S as:

$K_S(X) = TX$ on objects,

and for (f/s) in $\mathcal{T}[S^{-1}](X, Y)$,

$$K_S(f/s) = U_T((Ts)^{-1} \circ \eta Z \circ f) = \mu Y \circ (T^2 s)^{-1} \circ T\eta Z \circ Tf = (Ts)^{-1} \circ Tf.$$

EXAMPLES.

We first bring two examples of functors satisfying condition C, as announced in Section 3.

1° Let X be a topological space, $\mathfrak{X}$ the category of open sets in X and inclusions. Let $\mathcal{T}\!op$ denote the category of topological spaces and continuous maps. Let

$$s: (\mathcal{T}\!op \downarrow X) \longrightarrow (\mathfrak{X}^{\circ}, \mathcal{S}ets)$$

be the section functor. s takes an object $p: E \to X$ of $(\mathcal{T}\!op \downarrow X)$ to the functor sp defined by

$sp(u) = \{\phi : u \to E \mid p\phi = \text{inclusion of } u \text{ in } X\}$ on objects,

$sp(j)(\phi) = \phi \circ j$ on morphisms of $\mathfrak{X}^\circ$.

In [1] it is shown that s has a left adjoint r, the etale space functor of Godement, and that the cotriple induced by the adjunction is idempotent. Hence, by Proposition 6.3, the triple induced is idempotent, and by Theorem 6.4, s satisfies condition C.

2° Let $\mathcal{C}$ be the category of compact Hausdorff spaces and continuous maps, $I : \mathcal{C} \to \mathcal{T}op$ the inclusion. Let $s : \mathcal{T}op \to (\mathcal{C}^\circ, \mathcal{S}ets)$ be the canonical functor sending X in $\mathcal{C}$ to $\mathcal{T}op(I-, X)$. In [1] it is shown that s has a left adjoint and that the induced cotriple is idempotent. Thus, s satisfies condition C.

3° We recall Example 3 of Section 2. By Theorem 1.1 of [9], K has a left adjoint F and the adjunctions $F \dashv K$ and $F_S \dashv G_S$ generate the same idempotent triple $\mathbf{T}$. In [9] Corollary 2.3 it was shown that the shape category $\mathcal{S}_K$, the category of fractions $\mathcal{T}[S^{-1}]$, the Kleisli category $\mathcal{T}_T$ of $\mathbf{T}$ are all canonically isomorphic. Furthermore, by Theorem 3.4, 1.c, two objects in $\mathcal{T}$ have isomorphic S-completions if and only if they have the same K-shape.

Two instances of this situation are:

a) See [10]. $\mathcal{T} = \mathfrak{N}$, the category of nilpotent groups, S the family of *P-isomorphisms*, where P is a family of primes. $\mathfrak{N}_S$ is then the full subcategory of $\mathfrak{N}$ consisting of *P-local* nilpotent groups.

b) See [10]. $\mathcal{T} = \mathfrak{N}\mathcal{H}$, the homotopy category of nilpotent CW-complexes, S the family of those morphisms ϕ in $\mathfrak{N}\mathcal{H}$ for which $\pi_n(\phi)$ is a P-isomorphism for all $n \geqslant 1$. $(\mathfrak{N}\mathcal{H})_S$ then consists of the *P-local* CW-complexes.

7. Appendix. Comparison between rich functors and functors satisfying condition C.

In [6] Deleanu and Hilton introduced the notion of *rich* functors: A functor $V : \mathcal{C} \to \mathfrak{D}$ is rich if for any morphism $g : VC \to VC'$ in $\mathfrak{D}$, C, C' in $\mathcal{C}$, there is a diagram

$$C \xrightarrow{f_1} A_1 \xleftarrow{f_2} \cdots \xrightarrow{f_{2k-1}} A_{2k-1} \xleftarrow{f_{2k}} C'$$

in $\mathcal{C}$ such that Vf_{2i} is invertible for $1 \leqslant i \leqslant k$ and

$$g = (Vf_{2k})^{-1} Vf_{2k-1} \cdots (Vf_2)^{-1} Vf_1 .$$

Theorem 2.10 of [6] says that a rich functor satisfies condition C. A functor which satisfies condition C is not necessarily rich as the following example shows. (See Example 2.9, (ii) of [6].) Let $\mathcal{G}$ be the category of groups and $\mathcal{P}$ the subcategory consisting of $\mathbf{Z}$ and $\mathbf{Z}*\mathbf{Z}$ (free group on two generators, with morphisms the injections $g_1, g_2 : \mathbf{Z} \to \mathbf{Z}*\mathbf{Z}$, the comultiplication $\mu : \mathbf{Z} \to \mathbf{Z}*\mathbf{Z}$ and the identities. Let $K : \mathcal{P} \to \mathcal{G}$ be the inclusion. Then K is codense, hence satisfies C. K is not rich as for the trivial homomorphism $\mathbf{Z} \to \mathbf{Z}$ in $\mathcal{G}$ there is no corresponding diagram in $\mathcal{P}$.

If the functor K has a left adjoint, then K satisfies condition C if and only if it is rich, as Theorem 6.4, (i), (iii), shows.

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