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## DUALITY OF BANACH SPACES

by Michael BARR

### INTRODUCTION

In a sense, the genesis of this paper was an offhand remark by Eilenberg many years ago that it was clear that a morphism of Banach spaces is not a continuous linear map but a norm reducing one. Only then does an isomorphism preserve norm. Only then does the category allow such important constructions as products and sums.

An almost inevitable corollary of this observation is that the real underlying set of a Banach space is not the set of all the elements but rather those of its unit ball. Only then does the underlying functor have an adjoint. Only then does it take the obvious internal *hom* to the external one.

This raises the possibility of treating the unit ball as the object of study. There is no particular difficulty in doing this. The unit ball  $B$  of a Banach space is closed under absolutely convex combinations, and is separated and complete in a norm defined by

$$\|b\| = \inf \{ \lambda \mid b \in \lambda B \}.$$

Now an interesting thing happens. A morphism of unit balls need only preserve the convex structure. The fact that it reduces norm (and is, in particular, continuous) is automatic. This suggests, at least, that from a certain point of view the unit ball of a Banach space is a discrete object in a certain category.

It is easy to see that finite dimensional Banach spaces are reflexive. It is more or less standard that to extend a duality from finite things to infinite ones you must topologize. The most suggestive example is the extension of the duality theory of finite groups to a duality between compact and discrete groups and then to a duality of locally compact groups.

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This led me to construct a category whose objects are unit balls of Banach spaces but topologized, not necessarily by the norm. Maps are continuous and preserve the absolutely convex structure. Among the objects, the ones which are topologized by the norm play the role of discrete ones. A duality theory, in fact two, can be constructed and it turns out that for one of them the discrete objects and compact ones are dual.

To some extent this has been done by Alexiewicz, Semadeni and Wieweger (see [4] and many further references found there). They studied the notion of a mixed-topology Banach space which is essentially a Banach space equipped with a second topology. They prove many results and deduce (a result equivalent in their setting to) the above mentioned duality between discrete and compact balls. However the entire direction of their work is different. For one thing, they do not consider the possibility of endowing the dual with a mixed-topology Banach space structure.

The paper [1] on vector spaces over a discrete field was a paradigm theory of this type. Many of the arguments used here appeared there, usually in a much simpler form.

Some remarks on notation and terminology are needed. All topological vector spaces considered here are locally convex. I use, generally,  $A, B, C, \dots$  to denote balls, elements of which are  $a, b, c, \dots$ ;  $M, N, P$  denote subsets of balls, usually neighborhoods;  $U, V, W, \dots$  are Banach spaces, usually with a mixed topology;  $p, q, r, \dots$  denote seminorms;  $\phi, \psi, \dots$  denote linear functionals on  $A, \dots$  (but on  $A^*$  I have used, *faute de mieux*,  $f, g, \dots$ ). The letter  $\Omega$  is an index set whose elements are  $\omega$ .  $\mathbf{R}$  denotes the field of scalars  $\lambda, \mu, \nu, \dots$  either real or complex,  $I$  its unit disc, while  $J$  is the closed interval  $[0, 1]$  with elements  $\varepsilon, \delta, \dots$ . If  $\varepsilon \in J$  and  $A$  is a ball,  $\varepsilon A$  naturally denotes all things in  $A$  of norm  $\leq \varepsilon$ . If  $\lambda$  and  $\mu$  are real numbers, we write  $\lambda \vee \mu$  for the larger and  $\lambda \wedge \mu$  for the smaller. If  $p, q$  are real-valued functions, we write  $p \vee q$  for their pointwise *sup* and  $p \wedge q$  for their pointwise *inf*.

A subset of a (real or complex) vector space is called *absolutely convex* (AC) provided it is convex and closed under scalar multiplication by scalars of absolute value  $\leq 1$ .

**1. Preliminaries.**

By a *ball*  $B$  we mean a subset of a separated locally convex topological vector space  $V$  which is bounded, closed and absolutely convex (circled convex) and complete in its self-induced norm (see below). This has the following structure.

- i) If  $b_1, b_2 \in B, \lambda_1, \lambda_2 \in \mathbf{R}, |\lambda_1| + |\lambda_2| \leq 1$ , then  $\lambda_1 b_1 + \lambda_2 b_2 \in B$  ;
- ii) It is a topological space ;
- iii) It is a complete metric space.

The third requires an explanation. Define

$$\|b\| = \inf \{ |\lambda| \mid b \in \lambda B \},$$

Using the fact that  $B$  is bounded and  $V$  is separated, one can easily see that  $\|b\| = 0$  iff  $b = 0$ . Then the distance between  $b_1$  and  $b_2$  is

$$2 \|1/2 b_1 - 1/2 b_2\|.$$

The topology is not that of the metric. The norm and the metric are lower semi-continuous but not in general continuous. We do suppose, however, that  $B$  is complete in the norm.

There is a partial operation of addition and this is continuous. If  $M_1$  and  $M_2 \subset B$ , let  $M_1 + M_2$  denote the set of sums, so far as they are defined (i. e.  $(M_1 + M_2) \subset B$ ). Then for any  $0$ -neighborhood  $M$ , there exists a  $0$ -neighborhood  $N$  with  $N + N \subset M$ . If  $M$  is a neighborhood of  $b$ , there is a  $0$ -neighborhood  $N$  with  $b + N \subset M$ .

In the sequel we will describe the structure in i) as the absolutely convex (AC) structure.

If  $B$  is a ball, we may retopologize  $V$  by the coarsest topology such that every linear map which is continuous on  $B$  is continuous. This space is called  $vB$ . It is clear that the topology on  $vB$  is finer than that of  $V$  and induces the original topology on  $B$ . It is in fact the finest such linear topology.

It is possible to begin with a set equipped with the structure defined above and find conditions that it be a ball. Aside from some obvious ones

it requires that for all  $0$ -neighborhoods  $M$ , there is a sequence  $N_1, N_2, \dots$  of  $0$ -neighborhoods such that for all  $n$ ,

$$2^{-n} M \supset 2^{-n} B \cup (2^{-n} N_1 + 2^{-n+1} N_2 + \dots + 1/2 N_{n-1} + N_n).$$

See [5], p. 49, for a fuller treatment.

The space  $vB$  with its topology and the norm induced by that of  $B$  is an example of mixed topology space (see [4] or [5]). That is, it is a topological space vector which is also a normed linear space in such a way that the norm is lower semi-continuous and the norm topology finer than the given one. When it has the topology induced by the unit ball (as  $vB$  does), it is said to have the mixed topology (of [4] or [5]).

A morphism of balls is a continuous function which preserves the AC structure.

If  $B$  has the norm topology, we call it *discrete*. This is equivalent to  $vB$  being a Banach space. We call  $B$  a *compact* ball if it is compact in the topology.

If  $B$  is a ball, a semi-norm on  $B$  is the restriction to  $B$  of a continuous semi-norm on  $vB$  which is bounded by the norm. Explicitly, it is a continuous map  $p: B \rightarrow J$  such that

$$p(\lambda b) = |\lambda| p(b) \quad \text{and} \quad p(b_1 + b_2) \leq p(b_1) + p(b_2).$$

If  $V$  is a locally convex topological vector space and  $M$  is an AC  $0$ -neighborhood, then the function  $p$  defined by

$$p(v) = \sup \{ |\lambda| \mid v \in \lambda M \}.$$

It is a continuous semi-norm called the *gauge* of  $M$ . Every semi-norm  $p$  on  $V$  is the gauge of some set, namely  $p^{-1}(J)$ . Not every semi-norm on  $vB$  is less than the norm but every one is continuous in the norm topology, hence bounded, and so some scalar multiple of it is a semi-norm on  $B$ . Thus the topology on  $B$  is the weak topology for the semi-norms. We let  $sB$  denote the set of semi-norms on  $B$ .

PROPOSITION 1.1. *Let  $B$  be a ball and  $b \in B$ . Then*

$$\|b\| = \sup \{ p(b) \mid p \in sB \}.$$

PROOF. Let  $\|b\| = \lambda$ . If  $\mu < \lambda$ ,  $b \notin \mu B$  and that is closed in  $vB$ . For  $B$  is closed in  $V$ , hence in  $vB$ , and then so is  $\mu B$ . Thus there is an AC 0-neighborhood  $M$  in  $vB$  with  $(b+M) \cap \mu B = \emptyset$ , and then  $(1/2M+b)$  is disjoint from  $(1/2M+\mu B)$ . The latter is still an AC 0-neighborhood and

$$p = \mu, \text{ gauge of } (1/2M + \mu B)$$

is a semi-norm on  $B$  with  $p(b) > \mu$ .

Let  $p \in sB$ . We will suppose  $p: vB \rightarrow \mathbb{R}$ . We let  $\ker p$  be its null set. It is clearly a closed subspace. The quotient space  $vB / \ker p$  admits  $p$ . For if  $p(b) = 0$ ,  $b' \in B$ ,

$$p(b'+b) \leq p(b') = p(b'+b-b) \leq p(b'+b).$$

Its completion is a Banach space whose unit ball we call  $B_p$ . There is a natural map  $\pi_p: B \rightarrow B_p$ , and  $p$  is just the composite

$$B \xrightarrow{\pi_p} B_p \xrightarrow{\|\cdot\|} J.$$

If  $p \leq q$  are semi-norms, then  $\ker q \subset \ker p$ , and there is induced

$$vB / \ker q \rightarrow vB / \ker p$$

and finally a map  $\pi_{pq}: B_q \rightarrow B_p$ . These satisfy

$$\pi_{pp} = \text{identity}, \quad \pi_{pq} = \pi_q, \quad \pi_{pq}\pi_{qr} = \pi_{pr} \text{ for } r \leq q \leq p.$$

A subset  $t \subset sB$  is called a base provided the norm is the *sup* of all semi-norms in  $t$  and if for  $p \in sB$ , there exist  $q \in t$  and a  $\lambda > 0$  with  $\lambda p \leq q$ . It is called a subbase if the set of finite *sup*s of semi-norms in  $t$  is a base. It is readily checked that if  $t$  is a base (resp. a subbase) for  $sB$ , then the set of all sets

$$\{b \mid p(b) < \varepsilon\} \text{ for all } \varepsilon > 0, \text{ all } p \in t,$$

is a base (resp. a subbase) for the neighborhood system of 0. Of course a ball is discrete iff there is a semi-norm base with one element.

Note that semi-norms are required to be continuous. The norm is not, in general, and thus is not a semi-norm. It is, as we have seen, the *sup* of all the semi-norms.

PROPOSITION 1.2. *Let  $A$  and  $B$  be balls. A function  $f: A \rightarrow B$  which preserves the AC structure is continuous iff it transforms  $sB$  to  $sA$ .*

PROOF. Trivial.

If  $B$  is a ball, a subset  $A$  of  $B$  closed under the AC operations and complete in its own norm is called a *subball*. It is called *pure* if the inclusion preserves norm. This is the same as requiring  $\lambda b \in A \implies b \in A$ .

If  $B$  is a ball, we let  $|B|$  be the same point set with the same absolutely convex combinations, topologized by its norm. In other words  $|B|$  is the unit ball of a Banach space.

If  $f: A \rightarrow B$  is a monomorphism of balls, we naturally say that  $f$  preserves norm if  $\|f(a)\| = \|a\|$ . When  $f$  is not a monomorphism, this is impossible. Nevertheless there is still a useful notion of being norm preserving. We say that  $f$  *preserves norm* if

$$\|f(a)\| = \inf \{ \|a'\| \mid f(a') = f(a) \}.$$

This means that if  $\|f(a)\| = \lambda$ , then for all  $\varepsilon > 0$  there is an  $a'$  such that:

$$f(a) = f(a') \quad \text{and} \quad \|a'\| < \lambda + \varepsilon.$$

The same definition applies as well to maps of normed linear spaces.

## 2. Categorical constructions.

Let  $\{B_\omega\}$ ,  $\omega \in \Omega$  be a family of balls. Their cartesian product is a ball when equipped with the *sup* norm and the product topology. It is obvious that that is the product in the category. A subbase for the semi-norms are those of the form

$$\prod B_\omega \rightarrow B_\omega \rightarrow J$$

where the first map is a product projection and the second a semi-norm on  $B_\omega$ .

If  $A \rightrightarrows B$  are two morphisms of balls, the subset  $E \subset A$  consisting of the set theoretic equalizer is easily seen to be a pure subball. It is then norm complete and is the equalizer. A base for its semi-norms consists of the restrictions of those of  $A$ . It is seemingly not invariably the case, however, that these are all the semi-norms on  $E$ .

If  $\{B_\omega\}$  is a collection of balls, let  $\Sigma B_\omega$  denote the subset of  $\prod B_\omega$  consisting of all sequences  $(b_\omega)$  for which  $\Sigma \|b_\omega\| \leq 1$ . If

$$p = (p_\omega) \in \prod s B_\omega \text{ and } \Sigma \|b_\omega\| \leq 1,$$

then  $\Sigma p_\omega b_\omega \leq 1$  and defines a semi-norm on  $\Sigma B_\omega$ . These constitute a base for a topology and with that topology and  $\Sigma \|b_\omega\|$  as norm,  $\Sigma B_\omega$  is a ball. It is readily seen to be the categorical sum.

Before dealing with coequalizers, we observe that 1.1 may be rephrased as follows:

PROPOSITION 2.1. *Let  $B$  be a ball. The natural map  $B \rightarrow \prod B_p$ ,  $p \in sB$ , embeds  $B$  as a pure subball.*

PROOF. It is a subball because the semi-norms determine a base of 0-neighborhoods. It is pure because 1.1 says that the map preserves the norm.

Let  $f, g: A \rightarrow B$  be a pair of morphisms. Consider

$$c(f, g) = \{p \in sB \mid \pi_p f = \pi_p g\}.$$

Let  $C$  denote the pure subobject generated by the image of  $B$  in

$$\prod \{B_p \mid p \in c(f, g)\}.$$

Then

PROPOSITION 2.2. *The ball  $C$  with obvious map  $M \rightarrow C$  is the coequalizer of  $f$  and  $g$ .*

PROOF. Suppose  $h: B \rightarrow D$  is a morphism with  $hf = hg$ . For a semi-norm  $q$  on  $D$ ,  $p = qb \in c(f, g)$ . Thus there is induced  $B_p \rightarrow D_q$  and

$$\prod \{B_p \mid p \in c(f, g)\}$$

maps to  $\prod D_q$ . The elements of the image of  $B$  are mapped to  $D$  and since  $D$  is pure (and the map norm reducing) this map extends to the pure subobject generated by  $B$ . The uniqueness is clear since  $C$  is generated by the image of  $B$ .

### 3. Duality.

In this section we define the dualities and prove the basic theorem that every functional on the dual is an evaluation.



If  $A$  and  $B$  are balls,  $\text{hom}(A, B)$  denotes the set of continuous maps  $A \rightarrow B$  which preserve the AC structure. It is clear that  $\text{hom}(A, B)$  itself has an AC structure inherited from

$$v \text{hom}(A, B) \subset \text{hom}(vA, vB).$$

There are two reasonable ways of topologizing  $\text{hom}(A, B)$ . We let  $(A, B)$  denote the ball topologized by pointwise convergence. A subbase of  $0$ -neighborhoods is given by

$$\{f \mid f(a) \in M\} \text{ where } M \text{ is a } 0\text{-neighborhood in } B.$$

This is called the *weak hom*. The second one, called the *strong hom*, is the topology of uniform convergence on compact subballs. A base of  $0$ -neighborhoods is given by

$$\{f \mid f(A_0) \subset M\}$$

where  $A_0$  is a compact subball of  $A$  and  $M$  is a  $0$ -neighborhood in  $B$ . We denote this ball by  $[A, B]$ .

It is easy to see that both are balls. In the first case, there is a subbase of semi-norms given by

$$f \mapsto p f(a) \text{ for a fixed } a \in A, p \in s B.$$

In the second, take

$$f \mapsto \sup \{p f(a) \mid a \in A_0\} \text{ where } A_0 \text{ is a compact subball of } A.$$

Both result in global *sup* norm as the norm of the *hom* set.

We let

$$B^* = (B, I) \text{ and } \hat{B} = [B, I].$$

There are known as the weak and strong duals, respectively.

PROPOSITION 3.1. *Let*

$$U \xrightarrow{f} V \xrightarrow{g} W$$

*be maps of normed linear spaces. If the composite preserves norm, so does  $f$ .*

PROOF. Let  $v \in V$ ,  $\|v\| = \lambda$ . Every preimage of  $v$  has norm at least  $\lambda$ . If

$$\|g(v)\| = \mu < \lambda,$$

the preimages in  $U$  of  $g(v)$  are the same as those of  $v$ , since  $g$  is 1-1. But we know that some preimage has norm  $< \lambda$  in that case, and that is impossible.

For the purposes of the next proposition, we define, for a mixed topological space  $V$ , the dual  $V^*$  to be the set of maps  $V \rightarrow \mathbf{R}$  which are continuous in the topology and bounded in the norm. The norm of a map is defined as usual, by the *sup* over the unit ball of  $V$ . We are not concerned with any topology on  $V^*$ .

PROPOSITION 3.2. *Suppose  $A$  is a subball of  $B$ . If  $B^* \rightarrow A^*$  preserves norm, its image includes all elements of  $A^*$  of norm  $< 1$ .*

PROOF. Consider  $vA \rightarrow V \rightarrow vB$  where  $V$  is the image of  $vA$  in  $vB$ , given the subspace topology. Considered as mixed topological spaces,  $vA$  and  $vB$  have the mixed topology. That is not in general the case for  $V$ , but since  $A$  is a subspace of  $B$ ,  $v vA \cong vV$ , so that  $vA$  is the mixed topology associated to  $V$ . Then by [4], 13.5,  $(vA)^*$  is the norm closure of  $V^*$  in  $|V|^*$ . Now if  $B^* \rightarrow A^*$  preserves norm, so does  $(vB)^* \rightarrow (vA)^*$ . Now we have

$$(vB)^* \rightarrow V^* \rightarrow (vA)^* .$$

The first map is onto by the Hahn-Banach theorem. The composite preserves norm and hence the first map does. In particular every element of  $V^*$  of norm  $< 1$  is the image of an element of  $vB$  of norm  $< 1$ . We know that

$$(vB)^* \cong v(B^*)$$

is complete as a normed space and hence so is the quotient  $V^*$  ([2], I.6.3) in its own norm, which is that of  $(vA)^*$ . But we know that  $V^*$  is norm dense in  $(vA)^*$  and thus they are equal.

We remark that this is not really a theorem about  $B^*$  and  $A^*$  but rather their associated discrete spaces. Thus the same statement is true, for example, of  $B^\wedge \rightarrow A^\wedge$ .

PROPOSITION 3.3. *Let  $\{B_\omega\}$ ,  $\omega \in \Omega$ , be a collection of balls. Of the following natural maps, the first is 1-1 and onto and the second is an isomorphism:*

- i)  $\Sigma B_\omega^* \rightarrow (\prod B_\omega)^*$  .  
 ii)  $(\Sigma B_\omega)^* \rightarrow \prod B_\omega^*$  .

PROOF. The category is pointed, so there are natural maps

$$B_\omega \rightarrow \prod B_\omega \quad \text{and} \quad \Sigma B_\omega \rightarrow B_\omega.$$

These are where the indicated homomorphisms all come from. Also, it is trivial to see that the maps are 1-1. For example, if a formal sum  $\Sigma b_\omega$  induces the 0 map on the product, then by applying it to  $B_\omega \subset \prod B_\omega$ , we see  $b_\omega = 0$ . Next the fact that ii) is 1-1 and onto follows from the categorical properties of  $\Sigma$  together with the fact that the underlying set functor preserves products. We now consider the cases separately.

i) If  $\phi: \prod B_\omega \rightarrow I$ , let  $\phi_\omega = \phi|_{B_\omega}$ . If  $\Sigma \|\phi_\omega\| > 1$  (e.g., if more than countably many were non-zero), then there would already be a finite subset

$$\{\omega_1, \dots, \omega_n\} \subset \Omega \quad \text{such that} \quad \Sigma \|\phi_\omega\| > 1,$$

summed over  $\omega_1, \dots, \omega_n$ . Supposing the sum is  $1 + \varepsilon$ , we can find  $b_\omega \in B_\omega$  such that

$$\phi_\omega(b_\omega) > \|\phi_\omega\| - \frac{\varepsilon}{2n}$$

and then defining  $b$  to be  $b_\omega$  in these coordinates and 0 in the others, we have

$$\phi(b) > 1 + \varepsilon/2,$$

which is impossible. Hence  $\Sigma \|\phi_\omega\| < 1$  and the sequence

$$\{\phi_\omega\} = \psi \in \Sigma B_\omega^*.$$

Now it is clear that  $\phi$  and  $\psi$  agree on the  $B_\omega$ , and hence on the subspace of the product they generate. But that subspace is dense.

ii) A subbasic 0-neighborhood on the left is

$$\{\phi \mid |\phi(b)| < \varepsilon\}, \quad \text{where } b \in \Sigma B_\omega.$$

Let  $b = \Sigma b_\omega$ . Choose

$$\omega_1, \dots, \omega_n \quad \text{such that} \quad \Sigma \|b_\omega\| < \varepsilon/2,$$

summed on the complementary set of indices.

Now if  $(\phi_\omega) \in \prod B_\omega^*$  has  $\|\phi_\omega\| < \varepsilon/2n$ ,  $\omega = \omega_1, \dots, \omega_n$ , it is clear that

$$(\phi_\omega)(\sum b_\omega) < \varepsilon.$$

Those restrictions define a 0-neighborhood on the right.

COROLLARY 3.4. *The natural maps*

$$\sum B_\omega^\wedge \rightarrow (\prod B_\omega)^\wedge, \quad (\sum B_\omega)^\wedge \rightarrow \prod B_\omega^\wedge$$

are 1-1 and onto.

We will see later that the first one is an isomorphism (6.5).

There is an obvious map from the underlying set of  $B$  to that of its second dual,  $b \mapsto$  evaluation at  $b$ . This does not always underlie a morphism in the category. Nonetheless we will abuse notation and write

$$B \rightarrow B^{**}, \quad B \rightarrow B^{\wedge\wedge}, \text{ etc.}$$

In fact, the first of these is continuous while the second is open. What we wish to do here is show that they are each 1-1 and onto. We know that for any  $b \neq 0$  in  $B$  there is a functional  $\phi$  on  $vB$  with  $\phi b \neq 0$ . Since  $\phi$  is continuous on the Banach space  $|B|$ , it is bounded, and there is a  $\lambda > 0$  such that  $\lambda\phi : B \rightarrow I$ . From this we see that the natural map is 1-1.

PROPOSITION 3.5. *The natural maps*

$$B \rightarrow B^{**} \text{ and } B \rightarrow B^{\wedge\wedge}$$

*preserve norm.*

PROOF. This is just a restatement of 1.1.

COROLLARY 3.6. *Let  $B$  be the unit ball of a finite dimensional space. Then  $B \rightarrow B^{**}$  and  $B \rightarrow B^{\wedge\wedge}$  are isomorphisms.*

PROOF. There is only one possible topology on  $vB$ , that of  $\mathbf{R}^n$ . Thus

$$(vB)^* = v(B^*)$$

is the unit ball of  $\mathbf{R}^n$  in some norm and so is  $vB^{**}$ . Thus  $vB \rightarrow vB^{**}$  is an isomorphism and since they have the same norm, so is  $B \rightarrow B^{**}$ . In the finite dimensional case there is no difference between  $*$  and  $\wedge$ .

PROPOSITION 3.7. *Let  $B$  be discrete. Then  $B \rightarrow B^{**}$  is onto.*

PROOF. Let  $\{B_\omega\}$  range over the finite dimensional subballs of  $B$ . Then the natural map  $f: \Sigma B_\omega \rightarrow B$  is onto and evidently norm preserving. The map  $g: B^* \rightarrow \prod B_\omega^*$  embeds  $B^*$  as a pure subspace of the product. This is one description of the pointwise convergence topology and *sup* norm. Taking duals, we get a commutative diagram

$$\begin{array}{ccc}
 \Sigma B_\omega & \xrightarrow{b} & (\prod B_\omega^*)^* \\
 f \downarrow & & \downarrow g^* \\
 B & \xrightarrow{k} & B^{**}
 \end{array}$$

where  $b$  is the natural map (when  $B_\omega$  and  $B_\omega^{**}$  are identified). Now  $b$  is a norm isomorphism and  $f$  and  $k$  preserve norm, hence  $g^*$  does and thus (by 2.2) its image includes all elements of norm  $< 1$ . Since  $b$  is onto, the image of  $k$  includes all elements of norm  $< 1$ , but since  $k$  is 1-1 and  $B$  is norm complete, that implies it is onto.

PROPOSITION 3.8. *Let  $B$  be discrete. Then  $B^\wedge = B^*$ .*

PROOF. Let  $C$  be a compact subball of  $B$ . Then  $C$  is totally bounded ([3] p. 25) as a subspace of  $B$  with its natural uniformity inherited from  $\nu B$ . Hence for any  $\varepsilon > 0$  there is a finite set  $c_1, \dots, c_n$  of points of  $C$  such that  $C \subset \cup (c_i + \varepsilon B)$ . A basic open set in  $B^\wedge$  is

$$\{ \phi \mid |\phi(C)| < \varepsilon \}.$$

Now if  $c_1, \dots, c_n$  are chosen so that  $C \subset \cup (c_i + \varepsilon/2 B)$  and if

$$|\phi(c_1)|, \dots, |\phi(c_n)| \text{ are all } < \varepsilon/2,$$

it is clear that  $|\phi(C)| < \varepsilon$ .

COROLLARY 3.9. *When  $B$  is discrete, the natural map  $B \rightarrow B^{\wedge\wedge}$  is onto.*

PROOF. For  $B^{\wedge\wedge}$  has the same elements as  $B^{**} = B^{**}$ .

PROPOSITION 3.10. *For any ball  $B$ , the natural maps*

$$B \rightarrow B^{**} \quad \text{and} \quad B \rightarrow B^{\wedge\wedge}$$

*are continuous.*

PROOF. A subbasic open set in either second dual is

$$\{ f \mid |\phi(f)| < \varepsilon \}$$

and its inverse image in  $B$  is

$$\{ b \mid |\phi(b)| < \varepsilon \},$$

which is open because  $\phi$  is continuous.

**COROLLARY 3.11.** *When  $C$  is compact, the natural map  $C \rightarrow C^{**}$  is continuous.*

**PROOF.** For  $C^*$  is topologized by the uniform, i.e. norm, topology and is thus discrete. Then  $C^{**} = C^{*\ast}$ .

**PROPOSITION 3.12.** *For any compact ball  $C$ , the natural map  $C \rightarrow C^{**}$  is an isomorphism.*

**PROOF.** Since  $C$  is compact, we need only show that the image is dense. The topology on  $C^{**}$  is that of pointwise convergence, so it is sufficient to show that for any  $f: C^* \rightarrow I$  and any  $\phi_1, \dots, \phi_n \in C^*$  there is a

$$c \in C \quad \text{with} \quad \phi_i(c) = f(\phi_i), \quad i = 1, \dots, n.$$

The set  $(\phi_i)$  defines a map  $\phi: C \rightarrow I^n$  whose image is a subball  $D \subset I^n$ . Then  $D^*$  is that subball of  $C^*$  generated by  $\phi_1, \dots, \phi_n$ . When  $f$  is restricted to  $D^*$ , it follows from (3.6) that for some  $d \in D$ ,  $f$  is evaluation at  $d$ . Choose any  $c$  which maps to  $d$  and then

$$\phi_i(c) = f(\phi_i), \quad i = 1, \dots, n.$$

**THEOREM 3.13.** *For any ball  $B$ , the natural maps*

$$B \rightarrow B^{*\ast} \quad \text{and} \quad B \rightarrow B^{**}$$

*are onto.*

**PROOF.** The second implies the first. Let  $\{B_\omega\}$  range over the family of compact subballs of  $B$ . Then by definition  $B^*$  is topologized as a subball of  $\prod B_\omega^*$ . Now exactly the same arguments as in the proof of 3.7 apply.

#### 4. The weak bom.

**PROPOSITION 4.1.** *For a fixed ball  $A$ , the functor  $(A, -)$  preserves projective limits and has a left adjoint  $- \otimes A$ .*

PROOF. There is no question about the underlying point set. The only question is the topology. If  $B = \prod B_\omega$ , a basic 0-neighborhood in  $(A, B)$  is

$$\{f \mid f(a) \in M\},$$

where  $M$  is a 0-neighborhood in  $B$ . We may restrict  $M$  to be in a base of 0-neighborhoods and thus suppose  $M = \prod M_\omega$ , where  $M_\omega$  is a 0-neighborhood in  $B_\omega$  and  $M_\omega = B_\omega$  except for finitely many indices. Then

$$\{f \mid f(a) \in M\} = \prod \{f \mid f(a) \in M_\omega\}$$

and that is a 0-neighborhood in the product. An analogous argument shows that  $(A, \cdot)$  preserves subspaces and thus equalizers. The disc  $I$  is a cogenerator and since, as is easily shown, monomorphisms are 1-1, it follows that the category is well-powered. Thus the special adjoint functor theorem applies to give the adjoint.

This means there is a 1-1 correspondence between maps  $A \otimes B \rightarrow C$  and maps  $A \rightarrow (B, C)$ .

PROPOSITION 4.2. *Maps  $A \otimes B \rightarrow C$  are characterized as the bilinear maps  $A \times B \rightarrow C$  which are, for each  $a \in A$ , continuous on  $B$  and for each  $b \in B$ , continuous on  $A$ .*

PROOF. In order that we have a function  $F: A \rightarrow (B, C)$ , continuous or not, the first condition is necessary and sufficient. It is continuous iff for all  $b \in B$ , 0-neighborhood  $M$  of  $C$ ,

$$F^{-1} \{f \mid f(b) \in M\}$$

is open in  $C$ . But this is precisely the condition that  $F(\cdot, b)$  be continuous on  $A$ .

COROLLARY 4.3. *There is a natural isomorphism  $A \otimes B \cong B \otimes A$  which interchanges the arguments in bilinear maps.*

PROOF. Trivial on account of the symmetry of the above characterization.

PROPOSITION 4.4. *Composition of functions gives a continuous map*

$$(B, C) \otimes (A, B) \rightarrow (A, C).$$

PROOF. Composition is bilinear, so we have only to verify the separate

continuity. If  $f: B \rightarrow C$  is a fixed continuous function,  $a \in A$ ,  $M$  a 0-neighborhood in  $C$ , then

$$\{g: A \rightarrow B \mid fg(a) \in M\} = \{g: A \rightarrow B \mid g(a) \in f^{-1}M\}$$

and the latter is a 0-neighborhood, since  $f^{-1}M$  is one in  $A$ . If  $g: A \rightarrow B$  is continuous,

$$\{f: B \rightarrow C \mid fg(a) \in M\}$$

is a 0-neighborhood in  $(B, C)$  by definition.

**THEOREM 4.5.** *The category  $B$  is, equipped with the weak hom, a symmetric monoidal closed category.*

**PROOF.** See [2], II.3 and II.4.

### 5. Weak reflexive spaces.

Recall that  $B$  is weak reflexive if the natural map  $B \rightarrow B^{**}$  is an isomorphism.

**PROPOSITION 5.1.** *The natural map  $B \rightarrow B^{**}$  is continuous.*

**PROOF.** See [1], 3.3.

**COROLLARY 5.2.** *For any  $B$ ,  $B^*$  is reflexive.*

**PROOF.** See [1], 2.6.

**PROPOSITION 5.3.** *If  $A$  is a ball and  $B$  a weak reflexive ball,  $(A, B)$  is weak reflexive.*

**PROOF.** See [1], 3.3.

**THEOREM 5.4.** *The full subcategory of weak reflexive balls is a reflexive subcategory of  $B$  which is itself a symmetric closed monoidal category.*

**PROOF.** The reflector is given by  $V \mapsto V^{**}$ , easily seen to be left adjoint to the inclusion. The hom is the same and  $(A \otimes B)^{**}$  is used for the tensor. The same composition is used and so again the result follows from [2], II.3.

Thus this category of weak reflexive balls is seen to be very well-



behaved. The trouble is, no infinite dimensional Banach spaces are included, since a weak reflexive  $B$  is a subball of  $l^{B^*}$  and is hence totally bounded while it is well known that that is never true of the unit ball of an infinite dimensional Banach space.

**THEOREM 5.5.** *A ball is reflexive iff it is totally bounded.*

**PROOF.** Let  $B$  be totally bounded. Then it is a subball of a compact ball  $C$ . We have a commutative diagram

$$\begin{array}{ccc}
 B & \xrightarrow{\quad} & C \\
 \uparrow & & \uparrow \\
 B^{**} & \xrightarrow{\quad} & C^{**}
 \end{array}$$

and thus  $B^{**} \rightarrow B \rightarrow C$  is continuous and so  $B^{**} \rightarrow B$  is. For the converse, it is sufficient to observe that every dual space is totally bounded, which is clear, as  $B^* \rightarrow l^B$ .

### 6. The strong dual.

We say that a ball  $B$  is *strong reflexive* - or simply reflexive, if the context is clear - if the natural map  $B \rightarrow B^{**}$  is an isomorphism.

**THEOREM 6.1.** *Discrete balls and compact balls are reflexive and the two subcategories are dual to each other by the strong dual.*

**PROOF.** We already know that compact balls are reflexive (3.12) and it is evident that the dual of a compact ball is discrete. The dual of a discrete ball  $D$  is compact. This is classically known and described by saying that the dual of a Banach space is weakly compact. It can also be seen to be closed in  $l^D$  using 3.8. But then  $D \rightarrow D^{**}$  is 1-1 and onto, hence norm preserving, and both are discrete, so it is an isomorphism.

**PROPOSITION 6.2.** *For any  $B$ , the natural map  $B \rightarrow B^{**}$  is open.*

**PROOF.** A basic  $0$ -neighborhood is the inverse image of  $\varepsilon D$  under a map  $B \rightarrow D$  with  $D$  discrete. After double dualization this goes to the same set under  $B^{**} \rightarrow D^{**} \cong D$  and is hence open.

COROLLARY 6.3. For any ball  $B$ ,  $B^\wedge$  is reflexive.

PROPOSITION 6.4. Any semi-norm on  $B^\wedge$  is sup on a compact subball in  $B$ .

PROOF. A semi-norm  $p$  is of the form

$$B^\wedge \longrightarrow D \xrightarrow{\|\cdot\|} J$$

with  $D$  discrete. We may suppose that the image of  $B^\wedge$ , purified, is dense in  $D$ . Dualizing gives us a 1-1 map  $D^\wedge \rightarrow B$  with  $D^\wedge$  compact. It is now a matter of tracing through identifications to show that  $p$  is sup on  $D^\wedge$ .

COROLLARY 6.5. Let  $\{B_\omega\}$  be a collection of balls. Then the natural map  $\sum B_\omega^\wedge \rightarrow (\prod B_\omega)^\wedge$  is an isomorphism.

PROOF. We already know it is 1-1 and onto (3.4). A semi-norm  $p = (p_\omega)$  on the sum is defined by  $p(\sum b_\omega) = \sum p_\omega b_\omega$  where, for each  $\omega$ ,  $p_\omega$  is a semi-norm on  $B_\omega$ . If  $p_\omega$  is the sup on the compact subball  $C_\omega \subset B_\omega$ , then  $p$  is sup on  $\prod C_\omega$  and hence is a semi-norm on  $(\prod B_\omega)^\wedge$ .

REMARK. This argument shows why the corresponding statement for weak duals fails. For at the analogous stage you get a product of finite sets and that, while compact, is not finite. As an equational variety, compact sets are generated by products of finite sets (in fact powers of 2), so that from a certain point of view this suggests the inevitability of the compact-open topology.

PROPOSITION 6.6. An inductive limit of reflexive spaces is reflexive.

PROOF. Let  $\Omega$  be an index category and  $B_\cdot: \Omega \rightarrow B$  be a functor,  $\omega \mapsto B_\omega$ . Then we have  $B_\omega \rightarrow \text{indlim } B_\omega$  gives

$$B_\omega \cong B_\omega^{\wedge\wedge} \rightarrow (\text{indlim } B_\omega)^{\wedge\wedge}$$

and then

$$\text{indlim } B_\omega \rightarrow (\text{indlim } B_\omega)^{\wedge\wedge}.$$

That is, the natural map is continuous and hence an isomorphism.

### 7. Alternative approaches.

There are several alternative approaches to this theory. One is to

deal with mixed topology spaces as in [4], [5] and further references found there. A morphism of such spaces must reduce the norm as well as be continuous in the topology.

Very similar is to consider the category of pairs  $(V, B)$  where  $V$  is a topological space and  $B$  is a closed, bounded absolutely convex set. It is necessary to suppose that  $V$  is complete in the norm defined by the gauge of  $B$  and that  $V \cong \nu B$ .

A third possibility is to consider an object as a vector space  $V$  equipped with a family  $s$  of semi-norms which are pointwise bounded. It is necessary to suppose that the space is separated and complete in the norm defined by their *sup*. The topology is defined on the unit ball by the semi-norms and then extended.

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