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FREE, ITERATIVELY CLOSED CATEGORIES OF COMPLETE LATTICES

by Mitchell WAND

Let CL denote the cartesian closed category of complete lattices, with morphisms continuous over directed chains. It is well-known [5] that there is a morphism $Y \in CL(M^M, M)$ whose underlying map takes each continuous function $f \in CL(M, M)$ to the least $x \in M$ such that $f(x) = x$. Let T be an algebraic theory and $A: T \to CL$ a faithful product-preserving functor. We say that A is *iteratively closed* iff for each $t \in T(n+m, m)$ there exists a (unique) $\mu(t) \in T(n, m)$ such that $A(\mu(t)) = Y.\widehat{At}$ (where «$\hat{\ }$» denotes exponentiation). The existence of iteratively closed A is also well-known, and these structures are important for formal language theory and other theoretical areas in computer science [8,6]. Our object in this Note is to construct free iteratively closed algebras.

1. Definitions and notations.

We will regard a theory T as a category whose set of objects is ω and in which the object n is the n-fold product of the object 1. Theories form a category Th when equipped with product-preserving functors as morphisms.

We denote by RS («ranked sets») the category $(Sets, \omega)$ of maps $r: \Omega \to \omega$ and rank-preserving maps $h: \Omega \to \Omega'$. If $r: \Omega \to \omega$ is a ranked set, we use Ω_n to denote $r^{-1}(n)$. There is a forgetful functor $V: Th \to RS$ sending T to $r: \Omega \to \omega$ where $\Omega_n = T(n, 1)$. As is well-known, V has a left adjoint, the free-theory functor.

Let μTh denote the category whose objects are iteratively-closed faithful product-preserving functors, and with morphisms from $A: T \to CL$ to $A': T' \to CL$ precisely those morphisms $h: T \to T'$ such that

$$h(\mu(t)) = \mu(h(t)) \quad \text{for each morphism } t \in T.$$

Let V also denote the forgetful functor $V: \mu Th \to RS$. Our main result is that V has a left adjoint.

Since A is faithful, we can enrich T over the category of posets by setting

$$t \leqslant t' \text{ in } T(n,m) \quad \text{iff } At \leqslant At' \text{ in the lattice } [An \to Am].$$

If $t \in T(k+n, n)$ then $Y.\widehat{At}$ is given as follows :

$$\text{let } t_0 = \bot \in T(k,n), \text{ let } t_{p+1} = t.(1, t_p); \text{ then } Y.\widehat{At} = \cup At_p.$$

PROPOSITION 1.1. *If A is iteration-closed, then $\mu(t) = \cup t_p$.*

PROOF. If A is iteration-closed, $\bot \in T(k,n)$, so the t_p are well-defined. $A(\mu(t)) = \cup A(t_p)$, by definition. So $A(\mu(t)) \geqslant At_p$ for each p, whence $\mu(t) \geqslant t_p$. Assume that for all p, $u \geqslant t_p$. Then

$$Au \geqslant At_p \quad \text{and} \quad Au \geqslant \cup A(t_p).$$

So

$$Au \geqslant A(\mu(t)), \quad \text{and} \quad u \geqslant \mu(t).$$

So $\mu(t)$ is a least upper bound for $\{t_p\}$. ∎

2. Lattice-theoretic preliminaries.

In constructing the free theory over Ω the most important step is finding the smallest set X such that

$$X \cong \coprod \{ X^{r(s)} \mid s \in \Omega \}.$$

In that case, the solution was the set of finite Ω-trees.

In our case, we seek a lattice L such that

$$L \cong \coprod \{ L^{r(s)} \mid s \in \Omega \}.$$

In general, we may obtain solutions to such fixed-point equations as a series [4,2], but in this case we can obtain a tractable representation as a lattice of (possibly infinite) trees. We restate the theorem here for precision and completeness. Recall that for any set S, we denote by S^* the (underlying set of the) free monoid generated by S; we denote the identity of S^* by e.

THEOREM 2.1 [3]. *Let $r: \Omega \to \omega$ be a ranked set, and let D_Ω denote the lattice $\coprod \{ s \mid s \in \Omega \}$. Let $L_\Omega = L$ be the set of functions $t: \omega^* \to D_\Omega$ sat-*

isfying the following conditions:

(Ranking) If $t(w) = s$ *and* $r(s) = n$, *then* $t(wj) = \bot$ *for all* $j \geqslant n$;

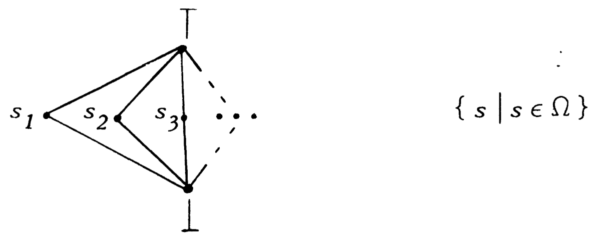
(Truncation) If $t(w) \in \{\bot, \top\}$, *then* $t(wx) = t(w)$ *for all* $x \in \omega^*$.

Let $t \leqslant t'$ *iff* $t(w) \leqslant t'(w)$ *for all* w. *Then* L *is a complete lattice and*

$$L_\Omega \cong \amalg\{ L_\Omega^{r(s)} \mid s \in \Omega \}.$$

PROOF. Figure 2.1 illustrates the lattice D_Ω. To see that L is a complete

FIGURE 2.1



lattice, let $\{ t_\alpha \mid \alpha \in I \} \subseteq L_\Omega$. Define $t(w)$ by induction:

(i) $t(e) = \cup\{ t_\alpha(e) \mid \alpha \in I \}$,

(ii)

$$t(wn) = \begin{cases} \top & \text{if } t(w) = \top & \text{(ii-a)} \\ \bot & \text{if } t(w) = \bot & \text{(ii-b)} \\ \cup\{ t_\alpha(wn) \mid \alpha \in I \} & \text{otherwise} & \text{(ii-c)} \end{cases} .$$

We claim t is the least upper bound in L of the t_α.

First of all, $t \in L$; clauses (ii-a) and (ii-b) force t to satisfy the truncation axiom, and it is easy to verify that the ranking axiom holds also: if $t(w) = s \in \Omega_n$, then it must be that $t_\alpha(w) \leqslant s$ for every α, and hence:

$$t_\alpha(wj) = \bot \text{ for } j \geqslant n.$$

For each α, $t_\alpha \leqslant t$, by an easy induction on w. To show t is the least upper bound, assume that

$$t_\alpha \leqslant u \text{ for all } \alpha .$$

Then $t(e) \leqslant u(e)$. Assume $t(w) \leqslant u(w)$. If $t(w) = \top$, then

$$u(w) = \top, \quad \text{hence} \quad \top = t(wn) \leqslant u(wn) = \top.$$

If $t(w) = \bot$, then $t(wn) = \bot \leqslant u(wn)$. Else

$$t(wn) = \cup\{ t_\alpha(wn) \mid \alpha \in I \} \leqslant u(wn).$$

So $t = \cup t_\alpha$.

Last, we must show

$$L \cong \amalg \{ L^{r(s)} \mid s \in \Omega \} .$$

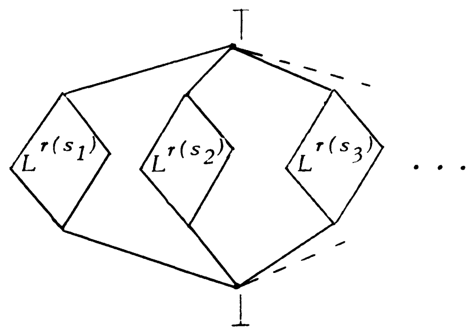
The lattice on the right-hand side consists of the following components:

(i) a bottom $\bot$,

(ii) a top $\top$,

(iii) for each $s \in \Omega$, a copy of $L^{r(s)}$. (See figure 2.2)

FIGURE 2.2



Define a map $I: L \to \amalg L^{r(s)}$ as follows:

(i) if $t(e) = \bot$, $I(t) = \bot$,

(ii) if $t(e) = \top$, $I(t) = \top$,

(iii) if $t(e) = s \in \Omega$ and $r(s) = n$, $I(t) = (t_1, \ldots, t_n)$ in the s-th component of $\amalg L^{r(s)}$, where $t_i(w) = t(iw)$.

This is clearly a bijection and I is clearly continuous over directed chains. ■

L_Ω is in fact an initial object in an appropriate category of solutions to [2]

$$D \cong \amalg \{ D^{r(s)} \mid s \in \Omega \} .$$

We view t as defining a countable labelled tree, truncated at nodes labelled $\bot$ or $\top$. We say t is finite iff $\{ w \mid t(w) \in \Omega \}$ is finite. We use $E(\Omega)$ to denote the underlying set of L_Ω.

L_Ω is a subposet of $D_\Omega^{\omega^*}$ but not, in general, a sublattice. However, we have:

LEMMA 2.2. *Let $u_0 \leq u_1 \leq \ldots$ be an increasing chain in L_Ω. Then*

$$(\cup u_i)(w) = \cup u_i(w) .$$

PROOF. $(\cup u_i)(e) = \cup u_i(e)$; we calculate $(\cup u_i)(wn)$ from the definition. If $(\cup u_i)(w) = \top$, then $(\cup u_i)(wn) = \top$. But

$$(\cup u_i)(w) = \cup u_i(w) = \top$$

implies that $u_j(w) = \top$ for some j. Hence

$$u_j(wn) = \top \quad \text{as well, and} \quad \cup u_i(wn) = \top.$$

Similarly, if $(\cup u_i)(w) = \bot$, then $u_i(w) = \bot$ for all i, and $u_i(wn) = \bot$ for all i. So

$$(\cup u_i)(wn) = \bot = \cup u_i(wn). \quad \blacksquare$$

COROLLARY. *Let* $\{u_i\}$ *be an increasing chain as before. If* $s \in \Omega$,

$$(\cup u_i)^{-1}(s) = \cup u_i^{-1}(s).$$

Furthermore, if $u_i^{-1}(\top) = \emptyset$, *then* $(\cup u_i)^{-1}(\top) = \emptyset$. $\blacksquare$

If $r: \Omega \to \omega$ is a ranked set, let $\Omega + X$ denote the ranked set

$$r' = \langle r, X \to \{0\} \hookrightarrow \omega \rangle : \Omega \amalg X \to \omega,$$

that is, the ranked set consisting of Ω with the members of X adjoined as new 0-ary elements ($r'(X) = 0$). We will write the new elements of $\Omega + k$ ($k = \{0, 1, \dots, k-1\}$) as $x_1, \dots, x_k$.

Consider the theory TE_Ω constructed as follows:

$$TE_\Omega(k, 1) = E(\Omega + k), \quad TE_\Omega(k, n) = [E(\Omega + k)]^n,$$

with the following composition rule:

If $t \in TE_\Omega(n, 1)$ and $\langle u_1, \dots, u_n \rangle \in TE_\Omega(k, n)$, then

$$t.\langle u_1, \dots, u_n \rangle(w) = \begin{cases} u_i(v) & \text{if } w = zv \text{ and } t(z) = x_i \\ t(w) & \text{otherwise.} \end{cases}$$

Note that since $r(x_i) = 0$, $t(zw') = \bot$ for all $w' \neq e$, and hence z is unique if it exists.

LEMMA 2.3. *Let* $v = t.\langle u_1, \dots, u_n \rangle$ *in* $TE_\Omega(k, 1)$. *Then for* $s \in \Omega$

$$v^{-1}(s) = t^{-1}(s) \cup \bigcup_{i=1}^{n} t^{-1}(x_i) u_i^{-1}(s),$$

$$v^{-1}(x_j) = \bigcup_{i=1}^{n} t^{-1}(x_i) u_i^{-1}(x_j). \quad \blacksquare$$

Let $AE_\Omega : TE_\Omega \to CL$ be the product-preserving functor given by :

$AE_\Omega(1) = L_\Omega$ and $AE_\Omega(t) : (u_1, \dots, u_n) \mapsto t.<u_1, \dots, u_n>$.

All of our theories will be subtheories of TE_Ω and our algebras will be constructed as composites

$$T \longrightarrow TE_\Omega \xrightarrow{AE_\Omega} CL .$$

We will usually delete the AE_Ω and write $A : T \to TE_\Omega \to CL$.

Now, $TE_\Omega(k, n)$ has a lattice structure imposed upon it by $L^n_{\Omega+k}$. Thus AE_Ω , acting on $TE_\Omega(k, n)$, is a function whose domain and codomain are both underlying sets of lattices. In fact, AE_Ω is (the underlying map of) a morphism of lattices :

LEMMA 2.4. *For any k, n,*

$$AE_\Omega : L^n_{\Omega+k} \longrightarrow [L^k_\Omega \longrightarrow L^n_\Omega]$$

is a continuous morphism of lattices.

PROOF. It will clearly suffice to show the case $n = 1$. So let $u_0 \leqslant u_1 \leqslant \dots$ be a chain in $L_{\Omega+k}$. We want to show that for any $t_1, \dots, t_k \in L_\Omega$,

$$(\cup u_i).<t_1, \dots, t_k> = (\cup u_i.<t_1, \dots, t_k>) .$$

We refer to these two objects as F and G , and we must show that, for all $w \in \omega^*$, $F(w) = G(w)$. We can calculate $F(w)$ from the definition as follows:

$$(\cup u_i).<t_1, \dots, t_k>(w) = \begin{cases} t_j(v) & \text{if } (\exists z)((\cup u_i)(z) = x_j \ \& \ w = zv) \\ u_i(w) & \text{otherwise.} \end{cases}$$

If for some z and j , $(\cup u_i)(z) = x_j$, then for every i , $u_i(z) = \bot$ or x_j . If $u_i(z) = x_j$, then $u_i.<t_1, \dots, t_k> = t_j(v)$. If $u_i(z) = \bot$ and if α is a proper initial segment of z , then $u_i(\alpha) \leqslant (\cup u_i)(\alpha) \in \Omega$. Hence for no α and no j is $u_i(\alpha) = x_j$. So $u_i.<t_1, \dots, t_k> = u_i(w)$, which is $\bot$ by the truncation rule. So $G(w) = t_j(v) = F(w)$.

Now assume that for no j is there a z such that $(\cup u_i)(z) = x_j$. Then

$$G(w) = \cup\{ u_i(w) \mid \sim(\exists z)(\exists j)\ [u_i(z) = x_j \ \& \ w = zv] \} \\ \cup\{ t_j(v) \mid (\exists z)\ [w = zv \ \& \ u_i(z) = x_j] \} .$$

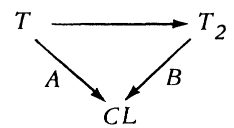
Since the u_i form a nondecreasing chain and for every j and initial segment z of w, $\cup u_i(z) \neq x_j$, if the second set is nonempty there must be some i such that $u_i(z) > x_j$. Hence $u_i(z) = \top$. So $u_i(w) = \top$. Hence

$$G(w) = \cup u_i(w) = F(w). \quad \blacksquare$$

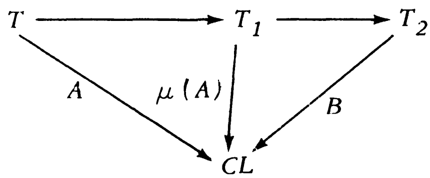
3. Main Theorems.

The first theorem asserts that iteration-closures exist.

THEOREM 3.1. *If $A : T \to CL$ is a product-preserving functor, there exists an object $\mu(A) : T_1 \to CL$ of μTh and a Th-morphism $T \to T_1$ such that, if $B : T_2 \to CL$ is any other object of μTh satisfying*



then T_1 is a subtheory of T_2 and



Furthermore, if A is faithful, so is $T \to T_1$.

PROOF. $T_1(n, k)$ will be a subset of $CL(An, Ak)$, given inductively as follows:

$$T_{(0)}(n,k) = AT(n,k),$$

$$T_{(i+1)}(n,k) = AT(n,k) \cup \{ tt' \mid t \in T_{(i)}(m,k),\ t' \in T_{(i)}(n,m) \}$$
$$\cup \{ \langle t_1, \ldots, t_k \rangle \mid t_j \in T_{(i)}(m,1) \}$$
$$\cup \{ Y.\hat{t} \mid t \in T_{(i)}(n+k,k) \}.$$

Then $T_1(n,k) = \cup T_{(i)}(n,k)$ clearly has the required properties.

LEMMA. *For any $A : T \to CL$ the constant function $\perp$ is a morphism of $\mu(A)$.*

PROOF. $\perp = \mu(id_1)$.

Let T_Ω be the free theory generated by Ω. There is an obvious inclusion $T_\Omega \to TF_\Omega$, since T_Ω consists precisely of finite trees in which

$\bot$ and $\top$ do not appear, except for $\bot$'s introduced by truncation. This observation enables us to reach our main theorem.

THEOREM 3.2. *The forgetful functor* $V: \mu Th \to RS$ *has a left adjoint.*

PROOF. The object function is calculated as follows: Given $\Omega \in RS$, construct $T_\Omega \to TE_\Omega \to CL$. Then the desired object is the closure of this functor. Call this object $B_\Omega: TB_\Omega \to CL$. We must then show that for any $A \in \mu Th$ any RS-morphism $h: \Omega \to VA$ extends uniquely to a μTh-morphism

$$h^*: B_\Omega \to A \quad \text{such that} \quad \begin{array}{ccc} \Omega & \longrightarrow & VB_\Omega \\ & {\scriptstyle h}\searrow & \downarrow {\scriptstyle Vh^*} \\ & & VA \end{array}$$

Let $A: T \to CL$ be any object of μTh and $h: \Omega \to VA = VT$ be an RS-morphism. Hence h extends uniquely to a Th-morphism h' satisfying:

$$\begin{array}{ccc} \Omega & \longrightarrow & VT_\Omega \\ & {\scriptstyle h}\searrow & \downarrow {\scriptstyle Vh'} \\ & & VT \end{array}$$

Now for $t \in TB_\Omega(n,k)$ let

$$h^*(t) = \cup\{ h' t_i(\bot) \mid t_i \in T_\Omega(p,k) \;\&\; (\exists u \in B_\Omega(n,p)) [t = t_i \cdot u] \};$$

h^* is easily shown to be a Th-morphism, and continuous on the (enriched) morphism sets. To show h^* preserves iteration-closure, let $t \in TB_\Omega(k,n)$. Recalling Proposition 1.1, set

$$t_0 = \bot, \quad t_{p+1} = t.(1, t_p)$$

and then $\mu(t) = \cup t_p$. However, we then have

$$h^*(t_0) = \bot, \quad h^*(t_{p+1}) = h^*(t).(1, h^*(t_p))$$

and

$$h^*(\mu(t)) = h^*(\cup t_p) = \cup h^*(t_p) = \mu(h^*(t)).$$

Furthermore, by the proof of Theorem 3.1, h^* is clearly unique. ■

4. Characterization of TB_Ω.

Our last result is a characterization of TB_Ω. Again, let S^* denote the free monoid generated by S. We say $G \subseteq S^*$ *is recognizable* iff there exists a finite monoid M, a monoid homomorphism $Q: S^* \to M$ and a subset $F \subseteq M$ such that $G = Q^{-1}(F)$. We say $t \in L_\Omega$ *is rational* iff:

(i) $\{ s \in \Omega \mid t^{-1}(s) \neq \emptyset \}$ is finite,

(ii) $t^{-1}(\mid) = \emptyset$,

(iii) $(\forall s \in \Omega)(t^{-1}(s)$ is a recognizable subset of $\omega^*)$.

TB_Ω may now be characterized as follows:

THEOREM 4.1. *TB_Ω is the subtheory of TE_Ω consisting of the rational trees.*

The proof is a tedious but comparatively straightforward exercise in automata theory, relying heavily on Lemma 2.3. Since quite similar results have appeared elsewhere [8,7,1], we will forego reproducing the proof here.

While this characterization of the iteration-closure of

$$T_\Omega \longrightarrow TE_\Omega \longrightarrow CL$$

was known, its freeness was not. One may then apply the triangular identities to find identities in μTh. These, in turn, yield a number of interesting results, primarily in the area of formal languages [8,7] and the semantics of programming languages.

References.

1. BEKIC H., Definable operations in general algebras and the Theory of Automata and Flowcharts, *I.B.M. Vienna*, 1969.

2. REYNOLDS J.C., Notes on a lattice-theoretical approach to the Theory of Computation, *Dept. of Systems & Info. Sc.*, Syracuse University, 1972.

3. SCOTT D., The lattice of flow diagrams, *Oxford U. Comp. Lab.*, Rep. PRG-3, 1970.

4. SCOTT D., Data types as lattices, *Lecture Notes, Amsterdam*, 1972.

5. TARSKI A., A lattice-theoretical fixpoint theorem and its applications, *Pacific J. of Math.* 5 (1955), 285-309.

6. WAGNER E.G., An algebraic theory of recursive definitions and recursive languages, *Proc. 3d ACM Symp. Th. Comp.* (1971), 12-23.

7. WAND M., A concrete approach to abstract recursive definitions, *Automata, Languages & Programming* (M. Nivat ed.), North-Holland, 1973, 331-341.

8. WAND M., Mathematical foundations of Formal Language Theory, *Project MAC TR-108*, M.I.T., Cambridge, Mass. 1973.

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