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by Harvey WOLFF

0. Introduction.

Fractional categories as special cases of localizations have played an important role in many aspects of category theory and its applications. In [9] Gabriel initiated the use of such techniques in algebra and since then there have been a great number of papers dealing with them. See for example [10], [17], [18] and [19]. In topology localization has played a role in homotopy theory. For example in [2], [11], [15] and [16]. In algebraic geometry fractional categories have appeared in the notion of derived category [13] and in Grothendieck topologies [21]. Recent works of Lawvere and Tierney on Topoi [14] have made extensive use of fractional categories. It often happens that if the category we begin with has *Hom* sets which are objects in a category $\underline{V}$, then the localization also has its *Hom* sets objects in $\underline{V}$ (see for example [10], [11], [13], [15], or [16]). In the light of such examples, it seems reasonable to want to extend the concept of fractional categories to more general contexts so as to provide a single theory. One vehicle for doing this is the use of $\underline{V}$-categories, i.e. categories which are defined over a fixed symmetric monoidal closed category $\underline{V}$. In this paper we plan to provide such a theory. Our main result is an existence theorem for $\underline{V}$-fractional categories.

By a $\underline{V}$-localization we mean the following. Given a $\underline{V}$-category $\underline{A}$ and a class of morphisms Σ of the underlying *Set*-based category $\underline{A}_0$, then the $\underline{V}$-localization of $\underline{A}$ with respect to Σ consists of a $\underline{V}$-category $\underline{A}[\Sigma^{-1}]$ and a $\underline{V}$-functor $\Phi: \underline{A} \rightarrow \underline{A}[\Sigma^{-1}]$ such that $\Phi(\sigma)$ is an isomorphism for all $\sigma \in \Sigma$ and Φ is universal with regard to this. In [25], we showed that if $\underline{A}$ is small and $\underline{V}$ is cocomplete, then the $\underline{V}$-localization always exists. A $\underline{V}$-fractional category is a $\underline{V}$-localization in which the *Hom* object $\underline{A}[\Sigma^{-1}](A, B)$ is a canonical direct limit of the

Hom objects of $\underline{A}$. In this paper our basic problem is the following: given $\underline{A}$, Σ and $\underline{V}$ what conditions guarantee that $\underline{A}[\Sigma^{-1}]$ is fractional. In the case of $\underline{V} = Sets$ or abelian groups, these conditions are well known ([1] or [11]). In the general $\underline{V}$-case, since the conditions for $\underline{V} = Sets$ involve elements in the *Hom* sets, we would need a more categorical approach, but one which when applied to $\underline{V} = Sets$ yields the well known conditions. In this paper we provide such an approach. Our central observation is that the conditions for $\underline{V} = Sets$ are equivalent (see below 1.15) to the fact that each of the *Hom* sets $\underline{A}(A, B)$ can be written as a certain canonical filtered direct limit and this becomes the core of our proof for the $\underline{V}$-case.

After making the appropriate definitions we prove a sequence of results aimed at exposing some of the structure of $\underline{V}$-fractional categories. We then present our main result. The result first appeared in the author's doctoral dissertation [22] under the direction of Professor J.W. Gray. The proof we present here is far different than the proof in [22]. We end with an application to $\underline{V}$-topologies and $\underline{V}$-sheaf theory.

We use the following notation: if $\underline{A}$ is a category, $\underline{A}^{o}$ denotes the opposite category. If $\underline{A}$ and $\underline{B}$ are categories, $[\underline{A}, \underline{B}]$ denotes the functor category and $[F, G]$ denotes the natural transformations between two functors.

1. V-fractional Categories.

Throughout we assume that $\underline{V}$ is a fixed symmetric, monoidal closed category (see [5]). We assume that $\underline{A}$ is a $\underline{V}$-category and $\Sigma \subset \underline{A}_0$ is a subcategory of the underlying *Set*-based category $\underline{A}_0$ with the same objects as $\underline{A}$. By a $\underline{V}$-localization of $\underline{A}$ with respect to Σ we mean a $\underline{V}$-category $\underline{A}[\Sigma^{-1}]$ together with a $\underline{V}$-functor $\Phi : \underline{A} \to \underline{A}[\Sigma^{-1}]$ such that $\Phi(\sigma)$ is an isomorphism for all $\sigma \in \Sigma$ and every $\underline{V}$-functor $F : \underline{A} \to \underline{B}$ such that $F(\sigma)$ is an isomorphism for all $\sigma \in \Sigma$ factors uniquely through Φ. If such a localization exists then we say that Σ is $\underline{V}$-localizable. If Σ is $\underline{V}$-localizable we may assume that the objects. A of $\underline{A}[\Sigma^{-1}]$ are the same as the objects of $\underline{A}$ and that Φ is the identity on objects. We will always make this assumption.

To describe when a $\underline{V}$-localization is a $\underline{V}$-fractional category, we first of all recall that, if A is an object of $\underline{A}$, then Σ/A is the category whose objects are the maps $E \xrightarrow{s} A$, $s \in \Sigma$ and whose morphisms from $E_1 \xrightarrow{s_1} A$ to $E_2 \xrightarrow{s_2} A$ are maps $f: E_1 \to E_2$ in $\underline{A}_0$ such that $s_2 f = s_1$. Denote by $Q_A : \Sigma/A \to \underline{A}_0$ the obvious projection.

The category A/Σ is defined dually with $Q^A : A/\Sigma \to \underline{A}_0$ the projection.

DEFINITION 1.1. *A $\underline{V}$-right fractional category of $\underline{A}$ with respect to Σ* is a $\underline{V}$-localization $\Phi : \underline{A} \to \underline{A}[\Sigma^{-1}]$ such that

$$\text{for every } A,\ B \in Ob(\underline{A}[\Sigma^{-1}]),\ \underline{A}[\Sigma^{-1}](A,B) = \varinjlim_{(\Sigma/A)^o} \underline{A}(Q_A^o(\cdot), B)$$

with the universal natural transformation ψ^{AB} given by the equation

$$\psi_s^{AB} = \underline{A}[\Sigma^{-1}](\Phi(s)^{-1}, \Phi(B)).\Phi_{E,B} \quad \text{where } E \xrightarrow{s} A \in (\Sigma/A)^o .$$

(It is easily checked that it is natural.)

A $\underline{V}$-left fractional category $\underline{A}$ with respect to Σ is a $\underline{V}$-localization $\Phi : \underline{A} \to \underline{A}[\Sigma^{-1}]$ such that for every pair A, B of objects of $\underline{A}[\Sigma^{-1}]$:

$$\underline{A}[\Sigma^{-1}](A, B) = \varinjlim_{B/\Sigma} \underline{A}(A, Q^B(\cdot))$$

with the universal natural transformation θ^{AB} given by

$$\theta_t^{AB} = \underline{A}[\Sigma^{-1}](\Phi(A), \Phi(t)^{-1}).\Phi_{A,E} \quad \text{where } B \xrightarrow{t} E \in B/\Sigma.$$

(Again it is easily checked that this is natural).

Let $\underline{A}$ be a $\underline{V}$-category and $\Sigma \subset \underline{A}_0$ a subcategory containing the identities. If the $\underline{V}$-right ($\underline{V}$-left) fractional category of $\underline{A}$ with respect to Σ exists then we say that Σ admits a $\underline{V}$-calculus of right (left) fractions.

PROPOSITION 1.2. *Let $\underline{A}$ be a $\underline{V}$-category. If $\Sigma \subset \underline{A}_0$ admits a $\underline{V}$-calculus of right fractions, then $\Sigma^o \subset \underline{A}_0^o$ admits a $\underline{V}$-calculus of left fractions.*

PROOF. Clear.

In the following we will deal mainly with $\underline{V}$-calculus of right fractions. The results for $\underline{V}$-calculus of left fractions will then be clear by duality using the above proposition.

Recall that, if $\underline{V}$ has pullbacks and $P:\underline{A}\to\underline{B}$ is a $\underline{V}$-functor, then $P^{-1}(B)$ is the category such that the following diagram is a pullback (see [12])..

$$\begin{array}{ccc} P^{-1}(B) & \xrightarrow{J_B} & \underline{A} \\ \downarrow & & \downarrow P \\ I & \xrightarrow{\ulcorner B \urcorner} & \underline{B} \end{array}$$

DEFINITION 1.3. If $\underline{V}$ has pullbacks and $P:\underline{A}\to\underline{B}$ is a $\underline{V}$-functor, we say that P *left covers* $\underline{B}$ if for all $A, B \in \underline{A}$ and every $E \in P^{-1}(PB)$

$$\underline{B}(PA, PB) = \varinjlim_{P^{-1}(PA)^{\circ}} \underline{A}(J_{PA}\,\cdot, E)$$

with the universal natural transformation given by $P_{\cdot E}$.

There are many examples of left covering functors. So, every functor with a cleavage is left covering.

Since $\underline{V}$ has pullbacks we can form the $\underline{V}$-category $\underline{A}^2$ for any $\underline{V}$-category $\underline{A}$. This is the category with objects being the morphisms of $\underline{A}$ and such that, if $f:A\to B$, $g:C\to D$, then $\underline{A}^2(f,g)$ is such that the following is a pullback diagram in $\underline{V}$:

$$\begin{array}{ccc} \underline{A}^2(f,g) & \xrightarrow{\bar{D}} & \underline{A}(A,C) \\ R\downarrow & & \downarrow \underline{A}(A,g) \\ \underline{A}(B,D) & \xrightarrow[\underline{A}(f,D)]{} & \underline{A}(A,D) \end{array}$$

There are then two $\underline{V}$-functors $\bar{D}, R:\underline{A}^2\to\underline{A}$. We define Σ^2 to be the full subcategory of $\underline{A}^2$ whose objects are in Σ. Then $\bar{D}$ and R restrict to $\underline{V}$-functors from Σ^2 into $\underline{A}$. Our object is to use the category Σ^2 and the functors $\bar{D}$ and R to construct $\underline{V}$-fractional categories. Before we do this, however, we will look at some relationships between Σ^2 and fractional categories. We begin by looking at composition in $\underline{A}[\Sigma^{-1}]$.

PROPOSITION 1.4. *Let $\underline{A}(\underline{A}, M, j)$ be a $\underline{V}$-category and suppose that $\Sigma \subset \underline{A}_0$ admits a $\underline{V}$-calculus of right fractions where*

$$\underline{A}[\Sigma^{-1}] = (\underline{A}[\Sigma^{-1}], \overline{M}, \overline{j}).$$

If A, B, C are objects of $\underline{A}$, $s: E \to B$, $u: L \to A$, $t: D \to L$ are all in Σ then the following diagram commutes

$$\begin{array}{ccc}
\Sigma^2(t,s) \otimes \underline{A}(E,C) & \xrightarrow{\overline{D} \otimes id} & \underline{A}(D,E) \otimes \underline{A}(E,C) \\
\downarrow{\scriptstyle R \otimes id} & & \downarrow{\scriptstyle M} \\
\underline{A}(L,B) \otimes \underline{A}(E,C) & & \underline{A}(D,C) \\
\downarrow{\scriptstyle \psi(u) \otimes \psi(s)} & & \downarrow{\scriptstyle \psi(ut)} \\
\underline{A}[\Sigma^{-1}](A,B) \otimes \underline{A}[\Sigma^{-1}](B,C) & \xrightarrow{\overline{M}} & \underline{A}[\Sigma^{-1}](A,C).
\end{array}$$

PROOF. Consider diagram 1.5 where, writing $\underline{A}[\Sigma^{-1}] = \overline{\underline{A}}$: 1 commutes since it is $\underline{A}(E, C)$ tensored with a commutative diagram; 2 and 5 commute since Φ is a $\underline{V}$-functor; 3 commutes since $\underline{A}[\Sigma^{-1}](\cdot, \cdot)$ is a functor and 4 commutes by 8.2 of [6].

LEMMA 1.6. *Let $\underline{A}$ be a $\underline{V}$-category and $\Sigma \subset \underline{A}_0$ admit a $\underline{V}$-calculus of right fractions. For every $f: B \to C$ in $\underline{A}_0$,*

$$\underline{A}[\Sigma^{-1}](A, \Phi(f)) = \varinjlim_{(\Sigma/A)^{o}} \underline{A}(Q^{o}_{A}(\cdot), f).$$

PROOF. It is easy to show that $\underline{A}[\Sigma^{-1}](A, \Phi(f))$ satisfies the same universal property as $\varinjlim_{(\Sigma/A)^{o}} \underline{A}(Q^{o}_{A}(\cdot), f)$.

PROPOSITION 1.7. *Let $\Sigma \subset \underline{A}_0$ be a subcategory which contains the identities. Let $T: \underline{A} \to \underline{B}$ be a $\underline{V}$-functor such that:*

(1) $T_0(s)$ is an isomorphism for each $s \in \Sigma$,

(2) T is the identity on objects,

(3) for every $A, B \in Ob(\underline{B})$, $\underline{B}(A,B) = \varinjlim_{(\Sigma/A)^{o}} \underline{A}(Q^{o}_{A}(\cdot), B)$

with universal natural transformation given by

$$\psi_s^{AB} = \underline{B}(T(s)^{-1}, B) . T_{E,B} \quad \textit{where} \quad E \xrightarrow{s} A \in (\Sigma/A)^{o}.$$

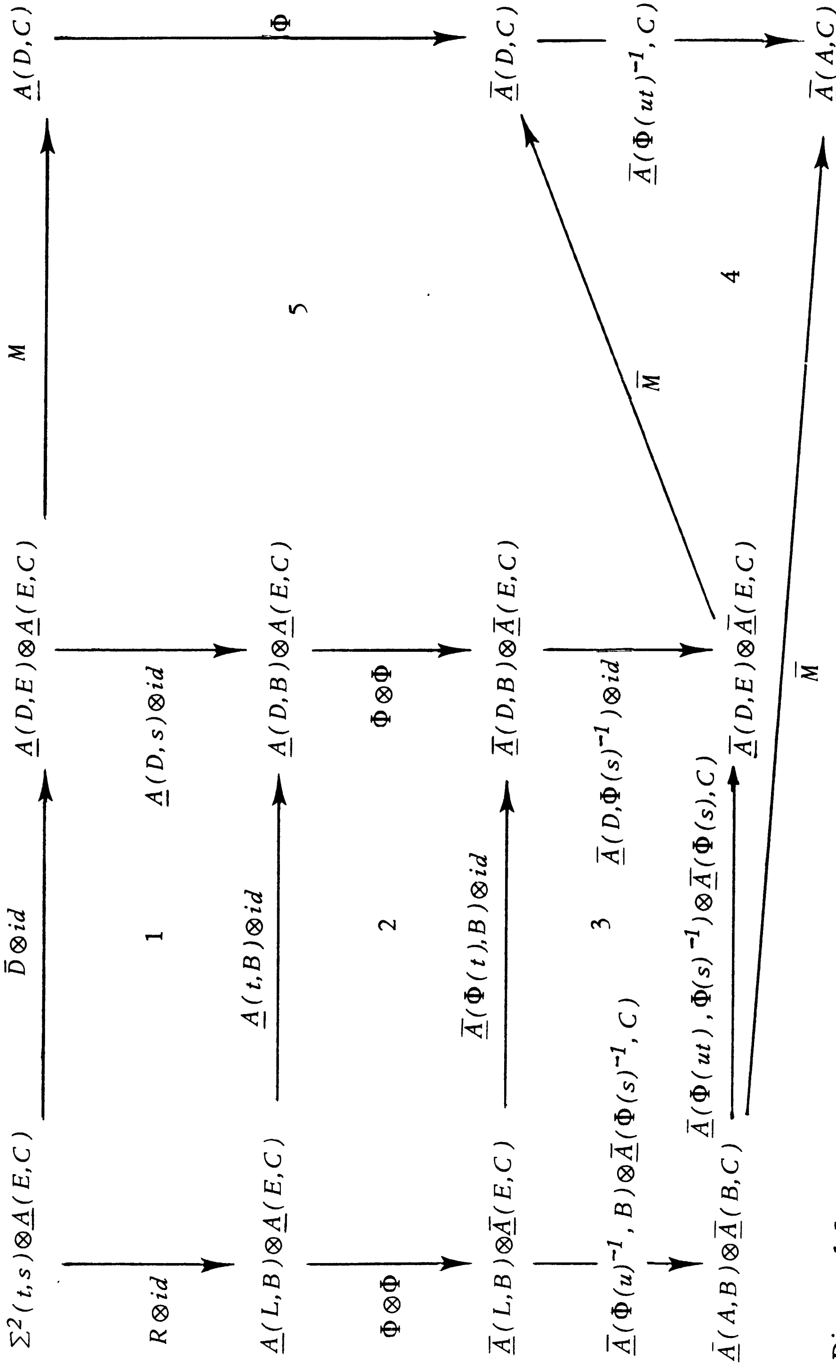

A(D,C)
Φ
Ā(D,C)
Ā(Φ(ut)^-1, C)
Ā(A,C)
M
5
M̄
4
A(D,E)⊗A(E,C)
A(D,s)⊗id
A(D,B)⊗A(E,C)
Φ⊗Φ
Ā(D,B)⊗Ā(E,C)
Ā(D,Φ(s)^-1)⊗id
Ā(D,E)⊗Ā(E,C)
M̄
D̄⊗id
1
A(t,B)⊗id
2
Ā(Φ(t),B)⊗id
3
Ā(Φ(ut), Φ(s)^-1)⊗Ā(Φ(s),C)
Σ^2(t,s)⊗A(E,C)
R⊗id
A(L,B)⊗A(E,C)
Φ⊗Φ
Ā(L,B)⊗Ā(E,C)
Ā(Φ(u)^-1, B)⊗Ā(Φ(s)^-1, C)
Ā(A,B)⊗Ā(B,C)

Diagram 1.5

If $R:\Sigma^2\to\underline{A}$ *is left covering, then* $\underline{B}\cong\underline{A}[\Sigma^{-1}]$ *with* $T=\Phi$.

PROOF. We just need to show that, if $F:\underline{A}\to\underline{C}$, $\underline{C}=(\underline{C},O,k)$, is a $\underline{V}$-functor such that $F_0(s)$ is an isomorphism for each $s\in\Sigma$, then there exists a unique $\underline{V}$-functor $\tilde{F}:\underline{B}\to\underline{C}$ such that $\tilde{F}.T=F$.

Define $\tilde{F}(B)=F(B)$ for each $B\in\underline{B}$. If $A,B\in\underline{B}$, to define $\tilde{F}_{A,B}$ we first of all define a natural transformation

$$w:\underline{A}(Q^o_A-,B)\to\underline{C}(FA,FB)$$

as follows. If $s:D\to A\in(\Sigma/A)^o$, then

$$w(s)=\underline{C}(F(s)^{-1},FB).F_{D,B}.$$

This is clearly natural and thus by the universal property of direct limits there exists a unique $\tilde{F}_{A,B}:\underline{B}(A,B)\to\underline{C}(FA,FB)$ such that

$$\tilde{F}_{A,B}\psi(s)=w(s) \text{ for every } s:D\to A \text{ in } (\Sigma/A)^o.$$

To show that $\tilde{F}$ is a $\underline{V}$-functor we note that, since $\otimes$ commutes with colimits and since

$$\varinjlim_{(\Sigma/D)^o}\Sigma^2(-,u)=\underline{A}(D,u)$$

for any $u:E\to B$ in Σ/B, it suffices to show that, for $s:D\to A$, $s':D'\to B$ and $t:E\to D$ all in Σ:

$$O_{FA,FB,FC}\cdot\tilde{F}_{A,B}\otimes\tilde{F}_{B,C}\cdot\psi(s)\otimes\psi(s').R(t,s')\otimes\underline{A}(D',C)$$
$$=\tilde{F}_{A,C}\cdot M_{A,B,C}\cdot\psi(s)\otimes\psi(s').R(t,s')\otimes\underline{A}(D',C).$$

Consider diagram 1.8: 1 commutes by 1.4 (note that the proof did not use the induced functor property); 2 and 3 commute by the definition of $\tilde{F}$; 4 commutes since F is a $\underline{V}$-functor; 5 commutes by 8.2 of [6]. Since the outer diagram is clearly commutative we have

$$\tilde{F}\cdot M=O.\tilde{F}\otimes\tilde{F}.$$

Now

$$\tilde{F}_{A,A}\cdot\tilde{j}_A=\tilde{F}.T.j=\tilde{F}.\psi(id).j=F.j=k_{FA,FA}$$

Hence $\tilde{F}$ is a $\underline{V}$-functor.

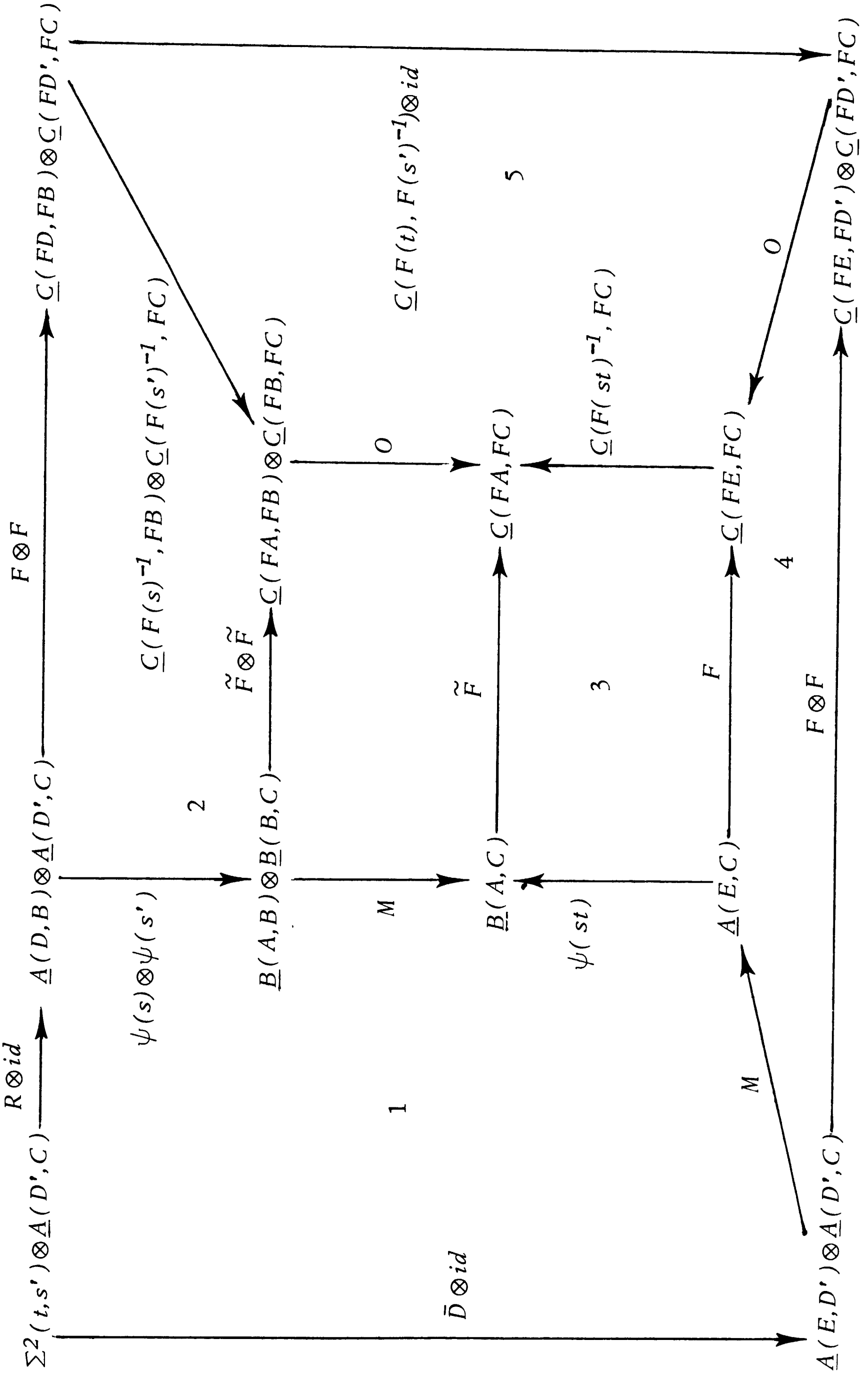

Σ2(t,s')⊗A(D',C)
R⊗id
A(D,B)⊗A(D',C)
F⊗F
C(FD,FB)⊗C(FD',FC)
ψ(s)⊗ψ(s')
2
C(F(s)-1,FB)⊗C(F(s')-1,FC)
B(A,B)⊗B(B,C)
F̃⊗F̃
C(FA,FB)⊗C(FB,FC)
1
M
O
C(F(t), F(s')-1)⊗id
5
B(A,C)
F̃
C(FA,FC)
ψ(st)
3
C(F(st)-1,FC)
A(E,C)
F
C(FE,FC)
4
M
O
D̄⊗id
A(E,D')⊗A(D',C)
F⊗F
C(FE,FD')⊗C(FD',FC)

Diagram 1.8

Since, for every $A, B \in \underline{A}$, $\psi(id) = T_{A,B}$, we have

$$(\tilde{F} . T)_{A,B} = \tilde{F}_{A,B} \cdot T_{A,B} = \tilde{F} . \psi(id) = w(id) = F_{A,B} .$$

Thus $\tilde{F} . T = F$. The uniqueness of $\tilde{F}$ is clear.

COROLLARY 1.9. *Let* $\Sigma \subset \underline{A}_0$ *and let* $T : \underline{A} \to \underline{B}$ *be a* $\underline{V}$*-functor which satisfies (1), (2) and (3) of 1.7. If* $\varinjlim_{(\Sigma/A)^o}$ *commutes with pullbacks, then* $\underline{B} \cong \underline{A}[\Sigma^{-1}]$ *and* $T = \Phi$.

PROOF. The following is a pullback of functors for $s : E \to B$

$$\begin{array}{ccc} \Sigma^2(\cdot, s) & \xrightarrow{\quad R \quad} & \underline{A}(Q^o_A(\cdot), E) \\ {\scriptstyle \bar{D}}\downarrow & & \downarrow{\scriptstyle \underline{A}(\cdot, s)} \\ \underline{A}(A, B) & \xrightarrow[\underline{A}(\cdot, B)]{} & \underline{A}(Q^o_A(\cdot), B) \end{array} .$$

Lemma 1.6 shows that

$$\varinjlim_{(\Sigma/A)^o} \underline{A}(Q^o_A \cdot, s) = \underline{B}(A, T(s)).$$

But $\underline{B}(A, T(s))$ is an isomorphism of $\underline{V}$. So if we take the $\varinjlim_{(\Sigma/A)^o}$ of the above pullback we get

$$\varinjlim_{(\Sigma/A)^o} \Sigma^2(\cdot, s) = \underline{A}(A, B).$$

Hence the result follows by 1.7.

DEFINITION 1.10 (Almkvist [1]). Let $\underline{A}$ be a $\underline{V}$-category, $\Sigma \subset \underline{A}_0$ a subcategory such that objects of Σ and $\underline{A}$ are the same. Σ is said to be *nice* if, for each $A \in \underline{A}$, $(\Sigma/A)^o$ has a small final subcategory.

COROLLARY 1.11. *Let* $\underline{V}$ *be cocomplete and have pullbacks such that filtered colimits commute with pullbacks. Let* $\underline{A}$ *be a* $\underline{V}$*-category and* $\Sigma \subset \underline{A}_0$ *be nice such that* $(\Sigma/A)^o$ *is filtered for all* A. *Then* Σ *admits a* $\underline{V}$*-calculus of right fractions iff there exists a* $\underline{V}$*-category* $\underline{B}$ *and a* $\underline{V}$*-functor* $T : \underline{A} \to \underline{B}$ *satisfying (1), (2) and (3) of 1.7.*

We now come to the main existence theorem.

THEOREM 1.12. *Let* V *be complete and cocomplete. Let* $\Sigma \subset \underline{A}_0$ *be a subcategory with the same objects as* $\underline{A}$. *If:*

(1) $R : \Sigma^2 \to \underline{A}$ *is left covering and*

(2) If s, t *are morphisms with* $s \in \Sigma$ *and with the same codomain, then there exist* s', t' *with* $s' \in \Sigma$ *such that* $s t' = t s'$,

then Σ *admits a* $\underline{V}$*-calculus of right fractions.*

PROOF. The proof proceeds by defining a $\underline{V}$-triple on the $\underline{V}$-functor category $[\underline{A}^o, \underline{V}]$ and then using some results of [7]. The results we need are the following:

1 (3.6 of [7]). If T is a triple on $[\underline{A}^o, \underline{V}]$, there is a triple T' on $[\underline{A}^o, \underline{V}]$ where T' is cocontinuous and a triple map $\varepsilon : T' \to T$ which is universal with respect to cocontinuous triples. T' is obtained by restricting T to the representables and then Kan extending. Hence T and T' agree on representables.

2 (3.12 of [7]). There is an equivalence of categories between cocontinuous triples on $[\underline{A}^o, \underline{V}]$ and the category of pairs $(x, \underline{A}')$ where $\underline{A}'$ is a $\underline{V}$-category and $x : \underline{A} \to \underline{A}'$ is a surjection on objects. Given T, $\underline{A}'$ is defined by

$$ob\,\underline{A}' = ob\,\underline{A} \quad \text{and} \quad \underline{A}'(A, B) = T\underline{A}(\cdot, B)(A).$$

Furthermore x corresponds to the unit η of T.

To define the $\underline{V}$-triple $\underline{T} = (T, \eta, \mu)$ on $[\underline{A}^o, \underline{V}]$ (which turns out to be idempotent) we first define T. Let $F \in [\underline{A}^o, \underline{V}]$ and define

$$T(F)(A) = \varinjlim_{(\Sigma/A)^o} F \cdot Q^o_A$$

with universal transformation ε. To give a $\underline{V}$-functor structure to $T(F)$ we note that

$$\underline{V}(TF(A), TFB) = \varinjlim_{\Sigma/A} \underline{V}(F \cdot Q^o_A \cdot, TFB).$$

Hence to define $T(F) : \underline{A}^o(A, B) \to \underline{V}(TFA, TFB)$ it suffices to define a natural family

$$\Gamma : \underline{A}^o(A, B) = \underline{A}(B, A) \to \underline{V}(F Q^o_A \cdot, TF(B)).$$

Fix $s:E \to A$ in Σ/A. By hypothesis $\underline{A}(B,A) = \varinjlim_{(\Sigma/A)^o} \Sigma^2(\cdot, s)$. So to define $\Gamma(s)$ it suffices to define a natural family

$$\Sigma^2(\cdot, s) \to \underline{V}(FE, TFB).$$

Let $t:C \to B$ in $(\Sigma/B)^o$ and set

$$\Gamma(s).R(t,s) = \underline{V}(FE, \varepsilon(t)).F.\bar{D}(t,s).$$

This is easily checked to be natural in t and consequently there exists a unique $\Gamma(s)$ such that

$$\Gamma(s).R(t,s) = \underline{V}(FE, \varepsilon(t)).F.\bar{D}(t,s).$$

It is then easy to check that Γ is natural. Hence there exists a unique

$$T(F):\underline{A}^o(A,B) \to \underline{V}(TFA, TFB)$$

such that

$$\underline{V}(\varepsilon(s), TFB).T(F) = \Gamma(s).$$

A moderate size diagram which we omit shows that with the above definitions $TF:\underline{A}^o \to \underline{V}$ is a $\underline{V}$-functor.

We claim now that we can give T the structure of a $\underline{V}$-functor $[\underline{A}^o, \underline{V}] \to [\underline{A}^o, \underline{V}]$. For notational convenience let us denote $\hat{\underline{A}} = [\underline{A}^o, \underline{V}]$. Recall (see [5]) that $\hat{\underline{A}}[F,G] = \int_A \underline{V}(FA, GA)$. To define a $\underline{V}$-functor structure on T we need $T:\hat{\underline{A}}[F,G] \to \hat{\underline{A}}[TF, TG]$. So to define T we need a $\underline{V}$-natural family

$$\delta_A : \hat{\underline{A}}[F,G] \to \underline{V}(TFA, TGA).$$

To define δ_A we need a natural family $\hat{\underline{A}}[F,G] \to \underline{V}(FQ^o_A\cdot, TGA)$. Let $s:E \to A$ be in Σ, define

$$\underline{V}(\varepsilon(s), TGA).\delta_A = \underline{V}(FE, \varepsilon(s)).\psi(E),$$

where ψ is the $\underline{V}$-natural family $\int_A \underline{V}(FA, GA) \to V(F\cdot, G\cdot)$. A short check shows that this is natural and so δ_A is well defined. We claim now that $\{\delta_A\}$ is a $\underline{V}$-natural family. Before we do this, however, we need to note two things. First, let us define $\eta F: F \to TF$ by $\eta FA = \varepsilon(id)$. Then using the definition of TF it is easy to see that ηF is

$\underline{V}$-natural. Secondly, we note that if $s:E\to B$ is in Σ then $TF(s): TFB\to TFE$ is an isomorphism. To see this, define $m:TFE\to TFB$ by $m.\varepsilon(t)=\varepsilon(st)$. Now by the definition of TF and hypothesis (2) we have $TF(s).\varepsilon(l)=\varepsilon(n).F(d)$ where $l:L\to B$, $n:M\to E$ are in Σ and $s.n=l.d$. Then

$$m.TF(s).\varepsilon(l)=m.\varepsilon(n).F(d)=\varepsilon(sn).F(d)=\varepsilon(l)$$

and

$$TF(s).m.\varepsilon(t)=TF(s).\varepsilon(st)=\varepsilon(t).F(id)=\varepsilon(t).$$

Now to show $\underline{V}$-naturality we need to show $\sigma_0\,\delta(B)=\sigma_0\,\delta(A)$. Consider diagram 1.13: 1 commutes by definition, 2 (resp. 3) by $\underline{V}$-naturality of ψ (resp. $\varepsilon(id)$), 4 since $m.TF(s)$ is the identity; 5 commutes by naturality properties of σ_0 and the fact that $m'.TF(t)$ is the identity; 6 and 8 commute by naturality of σ_0 ; 7 and 9 commute by definition; 10 commutes by functoriality of $\underline{V}(\cdot,\cdot)$; 11 commutes by definition of Σ^2 ; and finally 12 commutes by definition of TF. Hence $\{\delta(A)\}$ is a $\underline{V}$-natural family and consequently there exists a unique morphism

$$T:\hat{A}[F,G]\to\hat{A}[TF,TG] \quad \text{with} \quad \psi(A).T=\delta(A).$$

It is easily checked that T is a $\underline{V}$-functor and that the map $\eta:1\to T$ defined by $\eta FA=\varepsilon(id)$ is a $\underline{V}$-natural transformation.

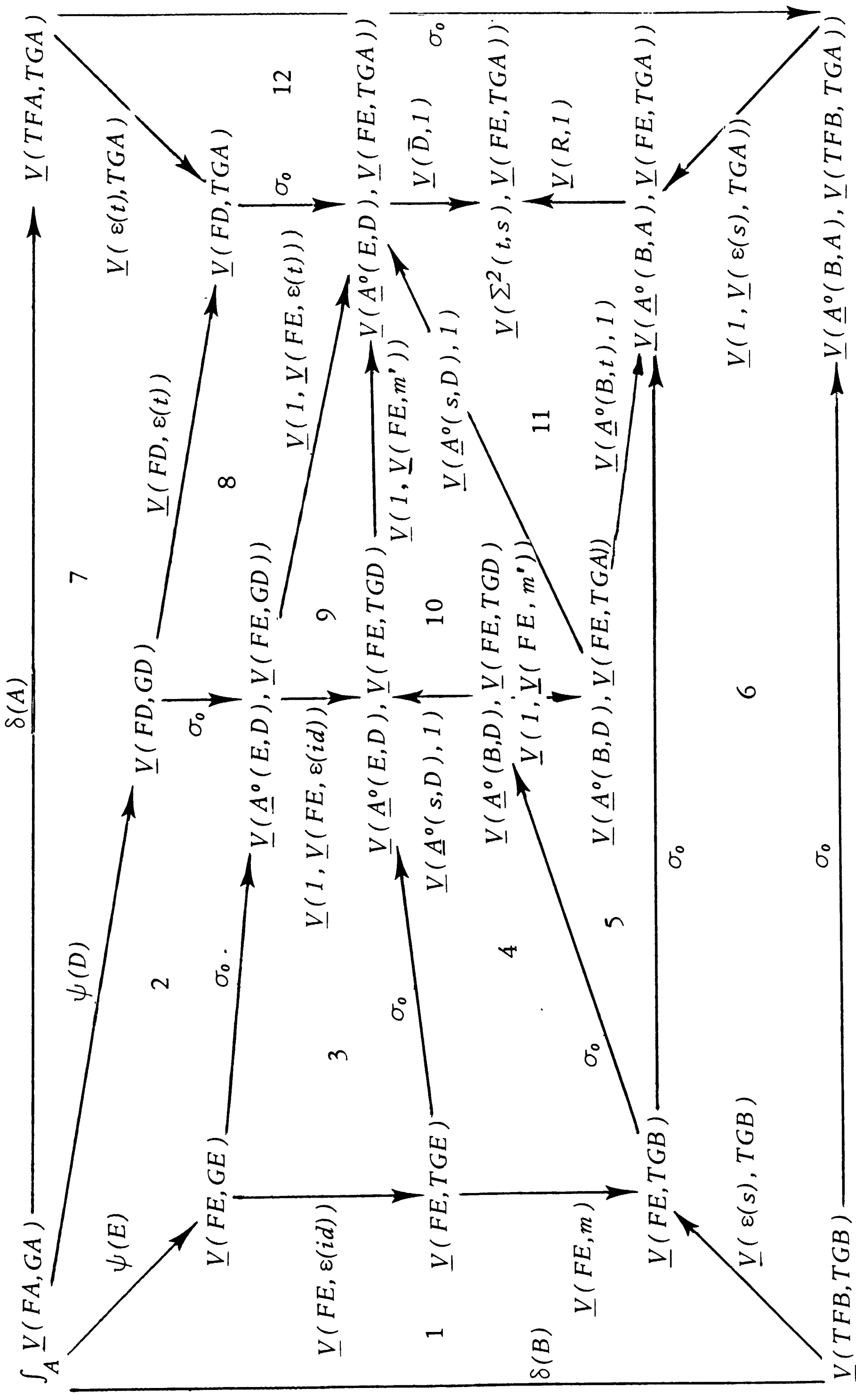
Now consider $\eta T:T\to T^2$. We claim that ηT is an isomorphism. To see this we define for each $F\in\hat{\underline{A}}$ and each $A\in\underline{A}^o$ an inverse μ_A to $\eta TFA=\varepsilon(id)$. Now

$$T^2FA=\varinjlim_{(\Sigma/A)^o} TFQ^o_A.$$

So to define μ_A we need a natural transformation $\mu'_A:TFQ^o_A\to TFA$. Let $s:E\to A$ be in Σ. Define $\mu'_A(s):TFE\to TFA$ by $\mu'_A(s)=TF(s)^{-1}$. That this is natural is clear. Hence there exists a unique $\mu_A:T^2FA\to TFA$ such that $\mu_A.\varepsilon(s)=TF(s)^{-1}$. Now

$$\mu_A.\eta TFA.\varepsilon(t)=\mu_A.\varepsilon(id).\varepsilon(t)=TF(id).\varepsilon(t)=\varepsilon(t).$$

Hence $\mu_A.\eta TFA=1$. Also



V(TFA, TGA)
V(ε(t), TGA)
V(FD, TGA)
V(A°(E,D), V(FE, TGA))
V(Σ²(t,s), V(FE, TGA))
V(A°(B,A), V(FE, TGA))
V(A°(B,A), V(TFB, TGA))
V(1, V(ε(s), TGA))
V(D̄,1)
V(R,1)
V(FD, ε(t))
V(1, V(FE, ε(t)))
V(1, V(FE,m'))
V(A°(s,D), 1)
V(A°(B,t), 1)
δ(A)
V(FD, GD)
V(A°(E,D), V(FE, GD))
V(A°(E,D), V(FE, TGD))
V(A°(B,D), V(FE, TGD))
V(A°(B,D), V(FE, TGA))
V(1, V(FE, m'))
V(1, V(FE, ε(id)))
V(A°(s,D), 1)
ψ(D)
ψ(E)
∫_A V(FA, GA)
V(FE, GE)
V(FE, TGE)
V(FE, TGB)
V(FE, ε(id))
V(FE, m)
V(ε(s), TGB)
δ(B)
V(TFB, TGB)
σ₀
1
2
3
4
5
6
7
8
9
10
11
12

Diagram 1.13

$$\eta TFA.\mu_A.\varepsilon(t)=\eta TFA.TF(t)^{-1}=\varepsilon(id).TF(t)^{-1}=\varepsilon(t).$$

Consequently $\eta TFA.\mu_A=1$ and ηTF is an isomorphism.

If we set $\mu=(\eta T)^{-1}$ then $\underline{T}=(T,\eta,\mu)$ is an (idempotent) monad on $[\underline{A}^o,\underline{V}]$. By 3.6 and 3.12 of [7] (see above) there exists a category $\underline{A}[\Sigma^{-1}]$ with the same objects as $\underline{A}$ with

$$\underline{A}[\Sigma^{-1}](A,B)=\varinjlim_{(\Sigma/A)^o}\underline{A}(Q^o_A\ \cdot,B)=T(\underline{A}(\cdot,B))(A)$$

and a $\underline{V}$-functor $\Phi:\underline{A}\to\underline{A}[\Sigma^{-1}]$ which is the identity on objects. Φ is defined by

$$\Phi:\underline{A}(A,B)\to T(\underline{A}(\cdot,B))(A)\quad\text{is}\quad\eta(\underline{A}(\cdot,B))(A)=\varepsilon(id).$$

Since, for all $s\in\Sigma$, $TF(s)$ is an isomorphism we have $\Phi(s)$ is an isomorphism. Note also that $\varepsilon(s)$, by the above universal natural transformation for $\varinjlim_{(\Sigma/A)^o}\underline{A}(Q^o_A\cdot,B)$, can be written as

$$\varepsilon(s)=\varepsilon(s.1)=T(\underline{A}(\cdot,B))(s)^{-1}.\varepsilon(id)=$$
$$=\underline{A}[\Sigma^{-1}](\Phi(s)^{-1},B).\Phi_{E,B}$$

if $s:E\to A$ is in $(\Sigma/A)^o$. Hence by 1.7 and the results of [7] we get the result.

REMARKS: 1. In 1.12 we assumed that $\underline{A}$ was small. There is also a similar result when $\underline{A}$ is large. In this case it is necessary to make the assumption that Σ is nice and that the pullback of the covering functor R along $\overline{D}$ is also covering. The proof is a long but direct calculation. The details appear in [22].

2. Note that condition (2) of 1.12 is equivalent to the canonical morphism $Z:\varinjlim_{(\Sigma/A)^o}V(\Sigma^2(\cdot,s))\to\underline{A}_0(A,B)$ being surjective.

COROLLARY 1.14. *Let $\underline{V}$ be cocomplete with pullbacks such that filtered colimits commute with pullbacks in $\underline{V}$. Suppose $V:\underline{V}\to Sets$ preserves filtered colimits. Let $\underline{A}$ be small and $\Sigma\subset\underline{A}_0$ such that $(\Sigma/A)^o$ is filtered for each A. Then Σ admits a $\underline{V}$-calculus of right fractions iff $R:\Sigma^2\to\underline{A}$ is left covering.*

So for example $\underline{V} = Cat$ and $\underline{V} = \hat{R}$-*modules* over a commutative ring $\hat{R}$ satisfy the conditions of 1.14.

The next proposition shows how the well known conditions for $\underline{V} = Sets$ (see [11]) can be derived directly from our conditions.

PROPOSITION 1.15. *Let* $\underline{V}$ *be Sets and* $\Sigma \subset \underline{A}$ *a subcategory. Then the following are equivalent:*

(1) $(\Sigma/A)^{o}$ *is filetered for each* $A \in \underline{A}$ *and* $\Sigma^2 \xrightarrow{R} \underline{A}$ *is left covering.*

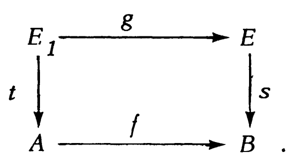
(2) *(a) for every* $f \in \underline{A}$, $s \in \Sigma$ *such that codomain* $f =$ *codomain* s *there exists* $g \in \underline{A}$, $t \in \Sigma$ *such that* $sg = ft$.

(b) If $s.f = s.g$, $s \in \Sigma$, *then there exists* $t \in \Sigma$ *such that* $ft = gt$.

PROOF. (1) implies (2): (a) Let $f: A \to B$, $s: E \to B$. Since

$$\varinjlim_{(\Sigma/A)^{o}} \Sigma^2(\cdot, s) = \underline{A}(A, B)$$

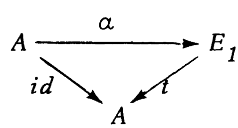
there exists $t: E_1 \to A$ in $(\Sigma/A)^{o}$ and $m \in \Sigma^2(t, s)$ such that $R(m) = f$. Hence we have the following commutative diagram



(b) Suppose $sf = sg$ where $s: B \to C$ in Σ and $f, g: A \to B$. Then $\underline{A}(A, C) = \varinjlim_{(\Sigma/A)^{o}} \Sigma^2(\cdot, s)$. Taking $id: A \to A$ in $(\Sigma/A)^{o}$ we have that

$$\Sigma^2(id, s) = \underline{A}(A, B) \text{ and } R(id, s) = \underline{A}(A, s).$$

Then f, g satisfy $R(id, s)(f) = R(id, s)(g)$. Since the limit is filtered, there is a $t: E_1 \to A$ in $(\Sigma/A)^{o}$ and



in $(\Sigma/A)^o$ such that $\Sigma^2(t, id)(f) = \Sigma^2(t, id)(g)$. Hence $ft = gt$.

(2) implies (1) is clear.

COROLLARY 1.16. *Let $\underline{V}$ be cocomplete with pullbacks such that $V: \underline{V} \to Sets$ commute with filtered colimits. Let $\Sigma \subset \underline{A}_0$ admit a $\underline{V}$-calculus of right fractions such that $(\Sigma/A)^o$ for each A is filtered. Then Σ admits a calculus of fractions relative to Sets and $\underline{A}[\Sigma^{-1}]_o = \underline{A}_o[\Sigma^{-1}]$.*

If $\underline{V}$ reflects filtered colimits then the converse is true.

PROOF. It is clear that $\Sigma^2 \xrightarrow{R} \underline{A}_0$ is left covering in this situation and consequently by 1.15 and [1] Σ admits a calculus of fractions relative to *Sets*. It is easily checked that relative to *Sets* the conditions of 1.7 are satisfied for $\Phi_0 : \underline{A}_o \to \underline{A}[\Sigma^{-1}]_o$.

The converse follows from 1.14.

For $\underline{V}$ = abelian groups it is well known that $V: \underline{V} \to Sets$ preserves and reflects filtered colimits. Hence in this case an additive localization and a set localization are the same by 1.16.

Most of the results about fractional categories which appear in [1] and [11] go over to the $\underline{V}$-case. For full details see [22]. The one result that we need is the following.

PROPOSITION 1.17. *Let $\underline{V}$ be complete and cocomplete and $\underline{A}$ be a small $\underline{V}$-category. Let $\Phi: \underline{A} \to \underline{A}[\Sigma^{-1}]$ be a $\underline{V}$-right fractional categogory. Let $F, G: \underline{A}[\Sigma^{-1}] \to \underline{B}$ be two $\underline{V}$-functors. Then Φ induces an isomorphism: $[F, G] \cong [F\Phi, G\Phi]$, i.e. the functor*

$$\Phi' : [\underline{A}[\Sigma^{-1}], \underline{B}] \to [\underline{A}, \underline{B}]$$

is $\underline{V}$-full and faithful.

PROOF. Clear.

As an example of how $\underline{V}$-calculus of fractions can be used we briefly indicate how one can extend to $\underline{V}$-theory the notions of Grothendieck topologies. Our context is the following. $\underline{V}$ is complete and cocomplete. We further assume that $\underline{V}$ has a fixed $(\mathcal{E}, \mathfrak{M})$ factorization as discussed in [4] or [8] and is $\mathfrak{M}$-well powered. Let $\underline{A}$ be a small

$\underline{V}$-category. An $\mathfrak{M}$-crible is a $\underline{V}$-functor $\hat{R}: \underline{A}^o \to \underline{V}$ for which there exists a $\underline{V}$-natural transformation $\hat{R} \to \underline{A}(\cdot, A)$ each of whose components is in $\mathfrak{M}$. Let $\mathfrak{B}$ be the $\underline{V}$-full subcategory of $[\underline{A}^o, \underline{V}]$ whose objects are $\mathfrak{M}$-cribles. By a $\underline{V}$-topology on $\underline{A}$ we mean the following: for each A let $J(A)$ be a set of cribles with codomain $\underline{A}(\cdot, A)$ such that $id \in J(A)$. Let Σ be the subcategory of the underlying category of $\mathfrak{B}$ generated by the $J(A)$. Σ is called a *$\underline{V}$-topology* if it admits a $\underline{V}$-calculus of right fractions. A $\underline{V}$-functor $F: \underline{A}^o \to \underline{V}$ is a *$\underline{V}$-sheaf* if for each $i: \hat{R} \to \underline{A}(\cdot, A)$ in $J(A)$ the canonical morphism $[\underline{A}(\cdot, A), F] \to [\hat{R}, F]$ is an isomorphism.

In analogy to the case $V = Sets$, we prove the following

PROPOSITION. *Given a $\underline{V}$-topology Σ on $\underline{A}$ then there exist a $\underline{V}$-functor $\hat{R}: [\underline{A}^o, \underline{V}] \to [\underline{A}^o, \underline{V}]$ and a $\underline{V}$-natural transformation $\delta: 1 \to \hat{R}$ such that:*

(1) $\delta \hat{R} = \hat{R} \delta$.

(2) The following are equivalent:

(a) ϕ is a $\underline{V}$-sheaf.

(b) $\delta\phi$ is an isomorphism.

(c) For all $G: \underline{A}^o \to \underline{V}$, $[\delta G, \phi]$ is an isomorphism.

PROOF. Let $\Phi: \mathfrak{B} \to \mathfrak{B}[\Sigma^{-1}]$ be the canonical functor. Then Φ induces a functor which we denote also by $\Phi: [\mathfrak{B}[\Sigma^{-1}]^o, \underline{V}] \to [\mathfrak{B}^o, \underline{V}]$. Φ is $\underline{V}$-fully faithful and has a $\underline{V}$-left adjoint Ψ. Let $(\bar{\eta}, \bar{\varepsilon}): \Psi \dashv \Phi$ be the front and back adjunction. Furthermore the functor

$$U: [\underline{A}^o, \underline{V}] \to [\mathfrak{B}^o, \underline{V}] \quad \text{defined by} \quad U(G) = [I\text{-}, G],$$

where $I: \mathfrak{B} \to [\underline{A}^o, \underline{V}]$ is the inclusion, is $\underline{V}$-full and faithful and has a left adjoint F which is composing with $y^o: \underline{A}^o \to \mathfrak{B}^o$ where y is the Yoneda embedding. Let (η, ε) denote the front and back adjunction of $F \dashv U$.

Define $\hat{R}$ as the following $F\Phi\Psi U = \hat{R}$ and δ as the following composite

$$1 \xrightarrow{\varepsilon^{-1}} FU \xrightarrow{F\bar{\eta}U} F\Phi\Psi U = \hat{R}.$$

To prove 1 we have

$$\delta\hat{R} = F\bar{\eta}UF\Phi\Psi U . F\eta\Phi\Psi U = F\Phi\Psi\eta\Psi U . F\bar{\eta}\Phi\Psi U$$

$$= F\Phi\Psi\bar{\eta}\Psi U \,.\, F\Phi\Psi\eta U = F\Phi\Psi U F\bar{\eta} U \,.\, F\Phi\Psi\eta U$$

$$= F\Phi\Psi U F\bar{\eta} U \,.\, F\Phi\Psi U \varepsilon^{-1} = \hat{R}\,\delta$$

Hence $\delta\hat{R} = \hat{R}\,\delta$.

2. (a)$\Longrightarrow$(b) If ϕ is a $\underline{V}$-sheaf then $U(\phi) = [I\cdot, \phi]$ inverts the morphisms of Σ. Consequently η is an isomorphism and therefore δ is an isomorphism.

(b)$\Longrightarrow$(c) Define $\sigma : [G, \phi] \to [\hat{R}G, \phi]$ as the following composition

$$[G, \phi] \xrightarrow{\hat{R}} [\hat{R}G, \hat{R}\phi] \xrightarrow{[\hat{R}G,\, \delta\phi^{-1}]} [\hat{R}G, \phi] \,.$$

We claim that σ is the inverse of $[\delta G, \phi]$. One way is clear. Now

$$\begin{aligned}[\hat{R}G, \delta\phi^{-1}] \,.\hat{R}.\, [\delta G, \phi] &= [\hat{R}G, \delta\phi^{-1}] \,.\, [\hat{R}\,\delta G, \hat{R}\phi] \\ &= [\hat{R}G, \delta\phi^{-1}] \,.\, [\delta\hat{R}G, \hat{R}\phi] \\ &= [\hat{R}G, \delta\phi^{-1}] \,.\, [\hat{R}G, \delta\phi] \\ &= id.\end{aligned}$$

(c)$\Longrightarrow$(a) Since $[\delta\phi, \phi]$ is an isomorphism, there is a $\sigma : \hat{R}\phi \to \phi$ such that $\sigma\,.\,\delta\phi = id$. Then

$$\begin{aligned}U\sigma \,.\, \eta\Phi\Psi U\phi \,.\, \bar{\eta} U\phi &= U\sigma \,.\, UF\bar{\eta}U\phi \,.\, \eta U\phi \\ &= U\sigma \,.\, UF\bar{\eta}U\phi \,.\, U\varepsilon^{-1}\phi \\ &= U(\sigma \,.\, \delta\phi) \\ &= id.\end{aligned}$$

Since Φ is fully faithful, we get that $\bar{\eta}U\phi$ is an isomorphism, and consequently ϕ is a $\underline{V}$-sheaf.

Now using the above proposition and the methods of [20] one gets the following

THEOREM. *If R preserves $\underline{V}$-filtered colimits for some regular cardinal α, then the $\underline{V}$-sheaves form a $\underline{V}$-reflective subcategory.*

For further applications and examples we refer the reader to [23] and [25].

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