CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

ANDRÉE BASTIANI
CHARLES EHRESMANN

Multiple functors. I. Limits relative to double categories

Cahiers de topologie et géométrie différentielle catégoriques, tome 15, nº 3 (1974), p. 215-292

<http://www.numdam.org/item?id=CTGDC_1974__15_3_215_0>

CAHIERS DE TOPOLOGIE *Vol. XV-3*
ET GEOMETRIE DIFFERENTIELLE

MULTIPLE FUNCTORS

I. LIMITS RELATIVE TO DOUBLE CATEGORIES

by Andrée BASTIANI and Charles EHRESMANN

INTRODUCTION.

Double categories are sets equipped with two laws of categories satisfying the «axiom of permutability». This axiom was first exhibited in [E7] for the two laws on the set of natural transformations from a category C to itself and in [E8] for the two laws on the set of commutative squares of C. The general definition of a double category (and by induction of a multiple category) was given in [E2], as a category internal to the category $\mathcal{F}$ of categories or, more precisely, as a structured category relative to the faithful functor from $\mathcal{F}$ to the category of sets. 2-categories are those double categories whose identities for the second law are also identities for the first law (but they are most often defined as categories enriched in the cartesian closed category $\mathcal{F}$); they have been considered by many authors [GZ, G1, G2, G3, Bo, S] as well as the double categories of squares of a 2-category [GZ, G1, Pa]. Benabou's bicategories [B2] are a «laxification» of 2-categories (and double categories may be laxified in a similar way, as done in [Ch, M]).

While a substantial and extensive theory of 2-categories has been given by Gray [G1, 2, 3], no such theory exists for double categories. We are going to generalize here some of the numerous fine results of Gray in the frame of double categories, using a method outlined in [E2] and whose main idea is to associate to a category A and to a double category D a category $T(D, A)$ which plays the same role as the category of natural transformations (to which it reduces if D is the double category of commutative squares of a category).

In chapter 0 are gathered some complements about sketched structures (used in particular later on to construct internal multiple categories). In chapter I we study the functor $T(\cdot, A)$ from the category of double functors to $\mathcal{F}$; it associates to D the category formed by the functors from A to the first category underlying D, and whose law is deduced from the second law of D; it admits an adjoint $-\blacksquare A$. Free objects relative to the canonical functor from the first category of 1-morphisms of D toward $T(D,A)$ are called D-wise limits. The main theorem, proved in chapter II, asserts that, if D is representable (i.e. there exist D-wise limits indexed by **2**) and if the second category of 1-morphisms of D admits small limits, then all small D-wise limits exist. If D is the double category of up-squares of a representable 2-category, D is representable and the theorem reduces to a theorem of Gray, D-wise limits being cartesian quasi-limits of [G1].

This paper is the first part of a work whose other parts will appear in the following issues of the «Cahiers».

- In the second part, the present results are generalized to n-fold categories: the category of all multiple categories is equipped with a monoidal closed structure, whose internal Hom associates to the $n+m$-fold category M and to the m-fold category B the n-fold category $T(M, B)$ of generalized transformations; the tensor product $\blacksquare$ is only symmetrical «up to an interchange of the laws». As before, M-wise limits are defined and there is a similar theorem of existence of M-wise limits when there exist M-wise limits indexed by $\mathbf{2}^{\blacksquare n} = \mathbf{2} \blacksquare \dots \blacksquare \mathbf{2}$ (this theorem is proved using a result of Appelgate-Tierney [AT] and the fact that each n-fold category is generated from $\mathbf{2}^{\blacksquare n}$ by inductive limits).

- In the third part, we will describe different monoidal closed structures on the category of double functors: its cartesian closed structure (whose existence is proved in [BE]), whose internal Hom maps (D', D) on the double category of double functors from D to the 4-fold category of squares of squares of D'; two monoidal closed structures non symmetrical which occur when double natural transformations are laxified (and which generalize the monoidal closed structure on the category of 2-functors considered by Gray [G1]). These results will then be applied to the study of structures defined as realizations or lax realizations of «double sketches».

0. COMPLEMENTS ABOUT SKETCHED STRUCTURES

A. Notations.

1. We denote by $\mathfrak{U}$ a universe and a set is said *small* if it is an element of this universe. The category of maps between small sets is denoted by $\mathfrak{M}$.

A *small category* is a category whose set of morphisms is small.

2. Since we will have to consider several categories with the same set of morphisms, we will often denote a category by a symbol $A^{\cdot}$, where A is the set of its morphisms and «.» the symbol of its law of composition (i. e. the composite of (y, x) is written $y.x$). Then:

$\alpha^{\cdot}$, $\beta^{\cdot}$ and $\kappa^{\cdot}$ are its maps source, target and law of composition,
$A^{\cdot}_0$ is the set of its objects, $A^{\cdot} * A^{\cdot}$ the set of its composable pairs,
$A^{\cdot *}$ its dual category.

But often we also denote a category by a unique letter (an italic or a greek letter or, for «big» categories, a script letter). In that case, if C is a category, its set of morphisms is denoted by $\underline{C}$, its symbol of composition by «.», its set of objects by C_0, the dual category by C^*, and the set of morphisms from e to e' by $C(e', e)$ or by $e'.C.e$, and $x: e \to e'$ is read $x \in e'.C.e$. If the sets $C(e', e)$ are small, the *Hom* functor from $C \times C^*$ to $\mathfrak{M}$ is denoted by $C(\cdot, \cdot): C \times C^* \to \mathfrak{M}$.

3. A functor f from A to C is also denoted by (C, ϕ, A), where ϕ is the map from $\underline{A}$ to $\underline{C}$ defining it (sometimes we put $\underline{f} = \phi$). If f is constant on an object e of C, we write $f = e^{\wedge}$.

The category of functors between small categories (i. e. of small functors) is denoted by $\mathfrak{F}$, the composite functor:

$$A \xrightarrow{\ f\ } C \xrightarrow{\ f'\ } D$$

being written $f'.f$ or, more often, $f'f$.

There are two «canonical» functors from $\mathfrak{F}$ to $\mathfrak{M}$:

the faithful functor $p_{\mathfrak{F}}$ which associates to $f: A \to C$ the map $\underline{f}: \underline{A} \to \underline{C}$;
the functor $p^{\cdot}_{\mathfrak{F}}$ associating to $f: A \to C$ the map $f_0 : A_0 \to C_0$ restriction of f to the sets of objects.

The functor $\dot{p\mathcal{F}}$ admits an adjoint functor, mapping the small set M on the *discrete category on* M (each element of M is an identity) which will be denoted by M^0. It also admits a coadjoint which associates to M the *groupoid of pairs* $(M \times M)^{\circ}$.

The functor $p\mathcal{F}$ has no coadjoint (since it does not preserve coequalizers). But it admits an adjoint functor, which associates to M the category $\mathbf{2} \times M^0$, coproduct of M copies of the category $\mathbf{2}$, where

$$\mathbf{2} \quad \text{is} \quad 1 \xleftarrow{\;z\;} 0 \ .$$

4. If A and C are categories, we denote by C^A the category of natural transformations between functors from A to C. If $t = (f', \underline{t}, f)$ is the natural transformation from the functor f to f' defined by the map $\underline{t}$ from A_0 to C, we write $t(u) = \underline{t}(u)$ for each object u of A, and

$$t : f \to f' : A \rightrightarrows C \ , \quad \text{or} \quad t : A \rightrightarrows C .$$

If $t' : f' \to f''$ is another natural transformation, then

$$t' \square\!\square\, t : f \to f''$$

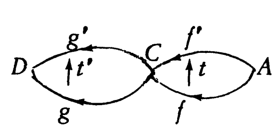
denotes their composite in C^A. Identical natural transformations are identified with functors.

On the set of all small natural transformations we have two laws:

$\mathfrak{N}^{\square\square}$ is the category coproduct of the categories C^A for all small categories A and C;

$\mathfrak{N}^{\cdot}$ is the category, admitting $\mathcal{F}$ as a sub-category, in which the composite of $t : f \to f' : A \rightrightarrows C$ and $t' : g \to g' : C \rightrightarrows D$ is

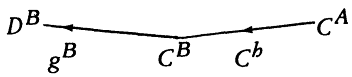
$t' . t : g f \to g' f' : A \rightrightarrows D$, where
$(t' . t)(u) = t'(f'(u)) . g(t(u))$,
for each object u of A.

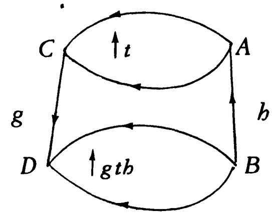


This composite is sometimes written $t' t$, especially when t or t' is an identical transformation. We have:

$$t' . t = (g' t) \square\!\square\, (t' f) = (t' f') \square\!\square\, (g t).$$

If $h : B \to A$ is a functor, the functor $t \to t h$ from C^A to C^B is denoted by C^b. In the same way, $g^A : C^A \to D^A$ is the functor associating gt to $t: A \rightrightarrows C$. Finally, $g^b: C^A \to D^B$ is the composite functor $g^B C^b$:

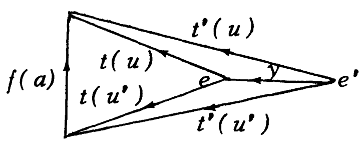


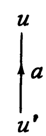


5. Let $f: A \to C$ be a functor. A natural transformation $t: e^{\wedge} \to f$, where $e^{\wedge}$ is a constant functor, is called *a projective cone indexed by* A, with vertex e and basis f. If $y: e' \to e$ is a morphism of C, we denote by ty the cone with basis f and vertex e' such that

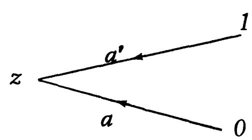
$$(ty)(u) = t(u).y \quad \text{for each object } u \text{ of } A.$$

If t is a limit-cone and t' a projective cone with basis f, the unique y such that $ty = t'$ is called *the factor of* t' *relative to* t.

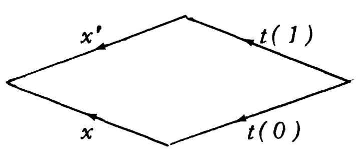




In particular, let us take for A the category



and for f the functor mapping a and a' onto x and x'. If t is a projective limit-cone with basis f, we also say that



is a pullback P of (x, x'). If t' is a projective cone with basis f (i. e. if $x.t'(0) = x'.t'(1)$), the factor y of t' relative to t is denoted by

$[t'(0), t'(1)]$ and called the factor of $(t'(0), t'(1))$ relative to P.

Similar notations are used for inductive cones $\bar{t}: f \to \bar{e}^{\wedge}$.

B. Sketched structures.

1. We recall [BE] that a (projective) limit-bearing category σ is a category Σ equipped with a set Γ of distinguished (projective) limit-cones; the set of the indexing categories of the cones $\gamma \in \Gamma$ is called the set of indexing categories of σ.

If Σ' is a category, a *σ-structure in* Σ' is a functor $\phi: \Sigma \to \Sigma'$ such that $\phi\gamma$ is a limit-cone for each $\gamma \in \Gamma$. We denote by Σ'^{σ} the category of σ-morphisms in Σ', which is the full sub-category of Σ'^{Σ} whose objects are the σ-structures in Σ'.

If $\psi: \Sigma \to \Sigma'^{*}$ is a σ-structure in the dual Σ'^{*} of Σ', then the dual functor $\psi^{*}: \Sigma^{*} \to \Sigma'$ is called a *σ-costructure in* Σ'.

σ-structures are called *sketched structures* (this terminology is justified by Proposition 8-I [BE]).

PROPOSITION 1. *If σ is a projective limit-bearing category (Σ, Γ) and Σ' a category, there exists a functor $\theta: \Sigma'^{\sigma} \times \Sigma'^{*} \to \mathfrak{M}^{\sigma}$ associating to an object (ϕ, ω) the σ-structure $\Sigma'(-, \omega)\phi: \Sigma \to \mathfrak{M}$.*

Δ. We consider the following functors:

the insertion $\iota: \Sigma'^{\sigma} \to \Sigma'^{\Sigma}$,

the Yoneda embedding $Y': \Sigma'^{*} \to \mathfrak{M}^{\Sigma'}$,

the «composition functor» $\lambda: \Sigma'^{\Sigma} \times \mathfrak{M}^{\Sigma'} \to \mathfrak{M}^{\Sigma}$ which associates to the pair (τ, τ') of natural transformations their composite $\tau'.\tau$.

The composite functor θ':

$$\Sigma'^{\sigma} \times \Sigma'^{*} \xrightarrow{\iota \times Y'} \Sigma'^{\Sigma} \times \mathfrak{M}^{\Sigma'} \xrightarrow{\lambda} \mathfrak{M}^{\Sigma}$$

maps the pair (ϕ, ω), where ϕ is a σ-structure in Σ' and ω an object of Σ', on the functor

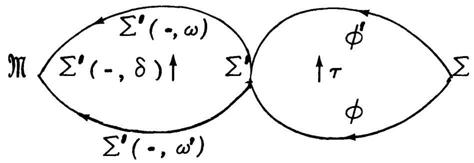
$$\lambda(\phi, Y'(\omega)) = \Sigma'(-, \omega)\phi: \Sigma \to \mathfrak{M}$$

which is a σ-structure in $\mathfrak{M}$, since $\Sigma'(-, \omega)$ preserves projective limits.

Hence θ' takes its values in $\mathfrak{M}^\sigma$ and it admits as a restriction

$$\theta : \Sigma'^{\sigma} \times \Sigma'^{*} \longrightarrow \mathfrak{M}^\sigma .$$

If $\tau : \phi \to \phi'$ is a σ-morphism in Σ' and $\delta : \omega \to \omega'$ a morphism in Σ', then $\theta(\tau, \delta) = \Sigma'(-, \delta) . \tau$:



∇

2. Let σ be a projective limit-bearing category (Σ, Γ) and Σ' a category admitting projective limits indexed by the indexing categories of σ. For each object ω of Σ, let $\nu_\omega : \mathfrak{M}^\sigma \to \mathfrak{M}$ be the functor «value in ω», which maps $\tau : \Sigma \rightrightarrows \mathfrak{M}$ onto $\tau(\omega)$.

PROPOSITION 2. *1° $(\mathfrak{M}^\sigma)^{\Sigma'^*}$ and $(\mathfrak{M}^{\Sigma'^*})^\sigma$ are isomorphic.*

2° Σ'^σ is equivalent to the full sub-category $\mathfrak{R}$ of $(\mathfrak{M}^\sigma)^{\Sigma'^*}$ whose objects are the functors $\psi : \Sigma'^* \to \mathfrak{M}^\sigma$ such that $\nu_\omega \psi : \Sigma'^* \to \mathfrak{M}$ is representable, for each $\omega \in \Sigma_0$.

Δ. 1° We denote by μ the canonical isomorphism

$$\mu : (\mathfrak{M}^{\Sigma'^*})^{\Sigma} \to (\mathfrak{M}^{\Sigma})^{\Sigma'^*}$$

and by $\nu'_{\omega'} : \mathfrak{M}^{\Sigma'^*} \to \mathfrak{M}$ the functor value in $\omega' \in \Sigma'_0$. Let $\phi : \Sigma \to \mathfrak{M}^{\Sigma'^*}$ be a functor. We have $\nu'_{\omega'} \phi = \mu(\phi)(\omega')$. If $\gamma : I \rightrightarrows \Sigma$ is a limit-cone, limits in $\mathfrak{M}^{\Sigma'^*}$ being computed termwise, $\phi\gamma$ is a limit-cone iff

$$\nu'_{\omega'} \phi\gamma = \mu(\phi)(\omega')\gamma : I \rightrightarrows \mathfrak{M}$$

is a limit-cone for each $\omega' \in \Sigma'_0$. Hence ϕ is a σ-structure iff

$$\mu(\phi)(\omega') \text{ is a } \sigma\text{-structure in } \mathfrak{M}, \text{ for each } \omega' \in \Sigma'_0 ,$$

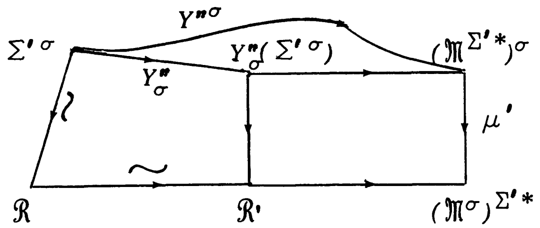
i. e. iff $\mu(\phi)$ takes its values in $\mathfrak{M}^\sigma$. So μ admits as a restriction an isomorphism $\mu' : (\mathfrak{M}^{\Sigma'^*})^\sigma \to (\mathfrak{M}^\sigma)^{\Sigma'^*}$.

2° Let $Y'' : \Sigma' \to \mathfrak{M}^{\Sigma'^*}$ be the Yoneda embedding. It gives an isomorphism $Y''_\sigma : \Sigma'^\sigma \to Y''(\Sigma')^\sigma \xrightarrow{\sim} Y''_\sigma(\Sigma'^\sigma) =$ sub-category of $(\mathfrak{M}^{\Sigma'^*})^\sigma$, the insertion $Y''(\Sigma') \hookrightarrow \mathfrak{M}^{\Sigma'^*}$ preserving projective limits. The isomorphism

μ' maps $Y''_\sigma(\Sigma'^\sigma)$ onto the full sub-category $\mathfrak{R}'$ of $(\mathfrak{M}^\sigma)^{\Sigma'^*}$ whose objects are the functors $\psi: \Sigma'^* \to \mathfrak{M}^\sigma$ such that $\mu'^{-1}(\psi): \Sigma \to \mathfrak{M}^{\Sigma'^*}$ takes its values in $Y''(\Sigma')$, i.e. such that

$$\mu'^{-1}(\psi)(\omega) = \nu_\omega \psi: \Sigma'^* \to \mathfrak{M}$$

is an object of $Y''(\Sigma')$ for each $\omega \in \Sigma_0$. Hence Σ'^σ is isomorphic with $\mathfrak{R}'$. As $Y''(\Sigma')$ is equivalent to the full sub-category of $\mathfrak{M}^{\Sigma'^*}$ whose objects are the representable functors, $\mathfrak{R}'$ is equivalent to the category $\mathfrak{R}$ defined in the Proposition. So Σ'^σ is equivalent to $\mathfrak{R}$.



∇

3. *Projective closure of a set.*

Let σ be a limit-bearing category (Σ, Γ) and Ω a sub-set of Σ_0. We define by induction a transfinite increasing sequence of full sub-categories Σ_ξ of Σ as follows:

Σ_0 is the full sub-category of Σ admitting Ω as its set of objects;

$\Sigma_\xi = \bigcup_{\zeta < \xi} \Sigma_\zeta$, if ξ is an ordinal without a predecessor;

if Σ_ξ is defined, then $\Sigma_{\xi+1}$ is the full sub-category of Σ whose objects are the vertices of the distinguished cones $\gamma \in \Gamma$ whose bases take their values in Σ_ξ, and the objects of Σ_ξ.

DEFINITION. We say that Σ is the Γ*-closure of* Ω if there exists an ordinal δ such that $\Sigma = \Sigma_\delta$; then $(\Sigma_\xi)_{\xi \leqslant \delta}$ is said to Γ-generate Σ.

If Σ is the Γ-closure of Ω, it is also the Γ'-closure of Ω, for each set Γ' of limit-cones including Γ.

PROPOSITION 3. *Let* σ *be a projective limit-bearing category* (Σ, Γ) *and* Σ' *a category admitting projective limits indexed by the indexing categories of* σ. *If* Σ *is the* Γ*-closure of a sub-set* Ω *of* Σ_0, *then* Σ'^σ

is equivalent to the full sub-category of $(\mathfrak{M}^{\sigma})^{\Sigma'*}$ *whose objects are the functors* ψ *such that* $\nu_\omega \psi$ *is representable fo each* $\omega \in \Omega$, *where* ν_ω *is the functor value in* ω *from* $\mathfrak{M}^{\sigma}$ *to* $\mathfrak{M}$.

Δ. Let Ω_ξ be the set of objects of the sub-category Σ_ξ of Σ defined above and δ the smallest ordinal such that $\Sigma = \Sigma_\delta$. Then the union of the transfinite sequence of sets $(\Omega_\xi)_{\xi \leqslant \delta}$ is Σ_0. In view of Proposition 2, it suffices to prove that, if $\psi: \Sigma'^* \to \mathfrak{M}^{\sigma}$ is a functor such that $\nu_\omega \psi$ be representable for each $\omega \in \Omega = \Omega_0$, the set Π of objects ω' of Σ such that $\nu_{\omega'}\psi$ be representable is equal to Σ_0. This will be proved by induction:

Ω_0 is included in Π.

If ξ has no predecessor and if Ω_ζ is included in Π for each ordinal $\zeta < \xi$, then the union Ω_ξ of $(\Omega_\zeta)_{\zeta < \xi}$ is included in Π.

Now let us suppose that Ω_ξ is included in Π for some ordinal $\xi < \delta$ and that $\omega' \in \Omega_{\xi+1} \setminus \Omega_\xi$. So ω' is the vertex of a cone $\gamma \in \Gamma$ whose basis ρ takes its values in Σ_ξ. Let ϕ be the σ-structure in $\mathfrak{M}^{\Sigma'*}$ associated to ψ by the isomorphism μ'^{-1} of Proposition 2. The cone $\phi\gamma$ is a limit-cone in $\mathfrak{M}^{\Sigma'*}$ with vertex $\phi(\omega') = \nu_{\omega'}\psi$ and the induction hypothesis implies that its basis $\phi\rho$ takes its values in the sub-category of $\mathfrak{M}^{\Sigma'*}$, whose objects are the representable functors. A projective limit of representable functors being a representable functor, this sub-category is closed for projective limits, so that the vertex $\nu_{\omega'}\psi$ of $\phi\gamma$ belongs to it. Therefore $\omega' \in \Pi$. It follows that $\Omega_{\xi+1}$ is included in Π.

By induction this proves that $\Pi = \Sigma_0$. ∇

DEFINITION. Let Σ be a category and Ω a sub-set of Σ_0. We say that Σ is the *projective* (resp. *inductive*, resp. *connected projective*) *closure* of Ω if Σ is the L-closure of Ω, where L is the set of all small limit-cones in Σ which are projective (resp. inductive, resp. projective and indexed by a connected category).

PROPOSITION 4. *Let* σ *be a projective limit-bearing category* (Σ, Γ) *and* $Y: \Sigma^* \to \mathfrak{M}^{\Sigma}$ *the Yoneda embedding.*

1° Y admits as a restriction an injective σ*-costructure* $\overline{Y}$ *in* $\mathfrak{M}^{\sigma}$ *and* *each σ-structure ϕ in $\mathfrak{M}$ is equivalent to $\mathfrak{M}^\sigma(\phi,\cdot)\overline{Y}^*$.*

2° $\mathfrak{M}^\sigma$ is the inductive closure of $Y(\Sigma_0)$.

3° If Σ is the Γ-closure of a sub-set Ω of Σ_0, then $\mathfrak{M}^\sigma$ is the inductive closure of $Y(\Omega)$.

Δ. 1° For each $\omega \in \Sigma_0$, the functor

$$Y(\omega) = \Sigma(\cdot, \omega): \Sigma \to \mathfrak{M},$$

which preserves projective limits, is a σ-structure in $\mathfrak{M}$, so that $Y(\Sigma)$ is included in the full sub-category $\mathfrak{M}^\sigma$ of $\mathfrak{M}^\Sigma$. The restriction

$$\overline{Y}: \Sigma^* \longrightarrow \mathfrak{M}^\sigma \quad \text{of } Y$$

is a σ-costructure, since Y sends projective limit-cones belonging to Γ on inductive limit-cones in $\mathfrak{M}^\sigma$, according to a result of [Lm]. Hence $\overline{Y}$ is a σ-costructure in $\mathfrak{M}^\sigma$.

2° Let ϕ be a σ-structure in $\mathfrak{M}$. The Yoneda lemma asserts that ϕ is equivalent to

$$\mathfrak{M}^\Sigma(\phi,\cdot)\,Y^*: \Sigma \xrightarrow{\ Y^*\ } (\mathfrak{M}^\Sigma)^* \xrightarrow{\ \mathfrak{M}^\Sigma(\phi,\cdot)\ } \mathfrak{M},$$

which is equal to $\mathfrak{M}^\sigma(\phi,\cdot)\overline{Y}^*$ since $\mathfrak{M}^\sigma$ is a full sub-category of $\mathfrak{M}^\Sigma$. On the other hand, in $\mathfrak{M}^\Sigma$ the object ϕ is the inductive limit of the functor $Y h^*$:

$$H^* \xrightarrow{\ \ h^*\ \ } \Sigma^* \xrightarrow{\ \ Y\ \ } \mathfrak{M}^\Sigma,$$

where $h: H \to \Sigma$ is the discrete fibration (or «hypermorphism functor» [E1]) associated to $\phi: \Sigma \to \mathfrak{M}$. The functor $Y h^*$ admits as a restriction a functor $k: H^* \to \mathfrak{M}^\sigma$ which takes its values in $Y(\Sigma)$. The sub-category $\mathfrak{M}^\sigma$ being full, its object ϕ is also the inductive limit of k. Hence $\mathfrak{M}^\sigma$ is the inductive closure of $Y(\Sigma_0)$. (In fact, the closure operation takes only one step in this case.)

3° Let Σ be the Γ-closure of Ω. The restriction $\overline{Y}$ of Y maps injectively a full sub-category of Σ onto a full sub-category of $\mathfrak{M}^\sigma$ and sends each cone of Γ onto an inductive limit-cone of $\mathfrak{M}^\sigma$. Hence $\overline{Y}$ maps the Γ-closure Σ of Ω into the inductive closure of $Y(\Omega)$ in $\mathfrak{M}^\sigma$, so that the inductive closure $\mathfrak{M}^\sigma$ of $Y(\Sigma_0)$ is also the inductive closure of $Y(\Omega)$. ∇

PROPOSITION 5. *If Σ is the projective connected closure of a sub-set Ω of Σ_0 and if Σ' is a category which is the projective connected closure of a sub-set Ω' of Σ'_0, then $\Sigma' \times \Sigma$ is the projective connected closure of $\Omega' \times \Omega$.*

Δ. Let $(\Sigma_\xi)_{\xi \leqslant \delta}$ and $(\Sigma'_\xi)_{\xi \leqslant \delta'}$ be the canonical increasing transfinite sequences of full sub-categories of Σ and Σ', where

$$\Sigma = \Sigma_\delta \quad \text{and} \quad \Sigma' = \Sigma'_{\delta'} \; ;$$

we may suppose that $\delta = \delta'$. Then we have an increasing transfinite sequence $(\Sigma'_\xi \times \Sigma_\xi)_{\xi \leqslant \delta}$ of full sub-categories of $\Sigma' \times \Sigma$ satisfying:

$$\Sigma' \times \Sigma = \Sigma'_\delta \times \Sigma_\delta \; .$$

If (ω', ω) is an object of $\Sigma'_{\xi+1} \times \Sigma_{\xi+1}$, there exist projective limit-cones γ in Σ and γ' in Σ', with vertices ω and ω', whose bases

$$\rho : I \to \Sigma \quad \text{and} \quad \rho' : I' \to \Sigma'$$

take their values in Σ_ξ and Σ'_ξ respectively, and whose indexing categories I and I' are connected. The product functor

$$\rho' \times \rho : I' \times I \to \Sigma' \times \Sigma$$

takes its values in $\Sigma'_\xi \times \Sigma_\xi$ and it admits (ω', ω) as its projective limit; its indexing category $I' \times I$ is connected, I and I' being connected. This proves that the connected projective closure Π of $\Omega' \times \Omega$ in $\Sigma' \times \Sigma$ contains $\Sigma'_{\xi+1} \times \Sigma_{\xi+1}$ as soon as it contains $\Sigma'_\xi \times \Sigma_\xi$. By induction it follows that Π contains $\Sigma'_\delta \times \Sigma_\delta = \Sigma' \times \Sigma$; whence $\Pi = \Sigma' \times \Sigma$. ∇

4. *Tensor product of cone-bearing categories.*

Let $\sigma = (\Sigma, \Gamma)$ and $\sigma' = (\Sigma', \Gamma')$ be two projective cone-bearing categories. Conduché [C] and Lair [L] have proved that there exists a cone-bearing category $\sigma' \otimes \sigma$ on $\Sigma' \times \Sigma$ satisfying the universal property:

Let H be a category admitting projective limits indexed by the indexing categories of σ and of σ'. Then the canonical isomorphism

$$(H^{\Sigma'})^{\Sigma} \xrightarrow{\sim} H^{\Sigma' \times \Sigma}$$

admits as a restriction an isomorphism from $(H^{\sigma'})^{\sigma}$ onto $H^{\sigma' \otimes \sigma}$.

They have given the following explicit construction of $\sigma'\otimes\sigma$:

- The underlying category is $\Sigma'\times\Sigma$.

If $\omega'\in\Sigma'_0$ and if $\gamma\in\Gamma$ is a cone with basis $\phi: I\to\Sigma$ and vertex ω, let γ'' be the cone $[\omega'^{\wedge},\gamma]: I\to\Sigma'\times\Sigma$, with basis $[\omega'^{\wedge},\phi]$, vertex (ω',ω), and such that

$$\gamma''(i)=(\omega',\gamma(i)) \quad \text{for each} \quad i\in I_0 .$$

If I is connected, this cone is a limit-cone, when γ is a limit-cone.

We define in a similar way the cone $[\gamma',\omega^{\wedge}]$, where

$$\gamma'\in\Gamma' \quad \text{and} \quad \omega\in\Sigma_0 .$$

- The set $\Gamma'\otimes\Gamma$ of cones is formed by all the cones $[\omega'^{\wedge},\gamma]$ and $[\gamma',\omega^{\wedge}]$, for $\gamma\in\Gamma$, $\gamma'\in\Gamma'$, $\omega'\in\Sigma'_0$ and $\omega\in\Sigma_0$.

If all the indexing categories of σ and of σ' are connected, then $\sigma'\otimes\sigma$ is a limit-bearing category, when so are σ and σ'.

DEFINITION. $\sigma'\otimes\sigma$ is called *the tensor product of* (σ',σ).

If $(\sigma_i)_{i<n}$ is a finite sequence of cone-bearing categories, their tensor product, denoted by

$$\underset{i<n}{\otimes}\sigma_i \quad \text{or} \quad \sigma_0\otimes\ldots\otimes\sigma_{n-1} ,$$

is defined by induction from the formula:

$$\underset{i<m+1}{\otimes}\sigma_i=(\underset{i<m}{\otimes}\sigma_i)\otimes\sigma_m \quad \text{for each} \quad m<n\text{-}1 .$$

If $\sigma_i=\sigma$ for each $i<n$, then $\underset{i<n}{\otimes}\sigma_i$ is also written $\overset{n}{\otimes}\sigma$.

The underlying category of $\underset{i<n}{\otimes}\sigma_i$ is the category $\underset{i<n}{\mathrm{X}}\Sigma_i$, defined by induction from the formula:

$$\underset{i<m+1}{\mathrm{X}}\Sigma_i=(\underset{i<m}{\mathrm{X}}\Sigma_i)\times\Sigma_m \quad \text{for each} \quad m<n\text{-}1 .$$

The word «tensor product» is well justified. Indeed, Lair proves in [L] that the category of morphisms between cone-bearing categories is equipped with a symmetrical monoidal closed structure, whose tensor product maps (σ',σ) onto $\sigma'\otimes\sigma$. From the general properties of symmetrical monoidal closed categories, we get:

PROPOSITION 6. *Let* $(\sigma_i)_{i<n}$ *be a sequence of projective limit-bearing categories and let* H *be a category admitting projective limits indexed by the indexing categories of* σ_i, *for each integer* $i<n$. *For each permutation* f *of* $\{0,\dots,n-1\}$ *and each sequence*

$$0 = n_0 < n_1 < \dots < n_m < n_{m+1} = n$$

of integers, there exists a canonical isomorphism

$$H^{\sigma_0 \otimes \dots \otimes \sigma_{n-1}} \xrightarrow{\sim} (\dots((H^{\sigma'_{n_0}})^{\sigma'_{n_1}})\dots)^{\sigma'_{n_m}},$$

where $\sigma'_{n_j} = \sigma_{f(n_j)} \otimes \dots \otimes \sigma_{f(n_{j+1}-1)}$.

PROPOSITION 7. *Let* n *be an integer,* $\sigma = (\Sigma, \Gamma)$ *a projective limit-bearing category whose indexing categories are connected,* $\sigma' = (\Sigma', \Gamma') = \overset{n}{\otimes}\sigma$ *and* Ω *a sub-set of* Σ_0.

1° *If* Σ *is the connected projective closure of* Ω, *then* Σ' *is the connected projective closure of* $\Omega' = \overset{n}{\times}\Omega$.

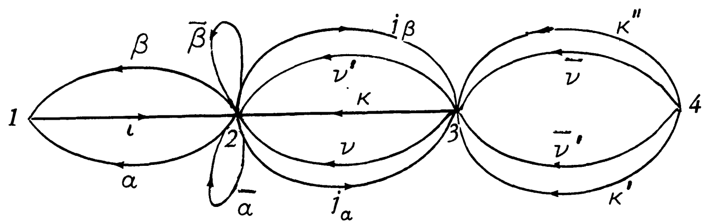
2° *If* Σ *is the* Γ*-closure of* Ω, *then* Σ' *is the* Γ'*-closure of* Ω' *and* $\mathfrak{M}^{\sigma'}$ *is the inductive closure of* $Y'(\Omega')$ *where* Y' *is the Yoneda embedding.*

Δ. By induction, part 1 follows from Proposition 5, part 2 from Proposition 4, since $(\Sigma_0 \times \Sigma_0, \dots, \Sigma_\delta \times \Sigma_0, \Sigma \times \Sigma_1, \dots, \Sigma \times \Sigma_\delta)$ is Γ'-generating Σ' for $n=2$, if $(\Sigma_\xi)_{\xi \leq \delta}$ is Γ-generating Σ. ∇

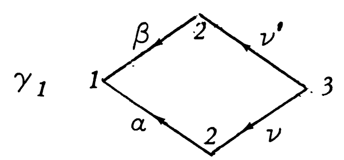
C. Internal categories.

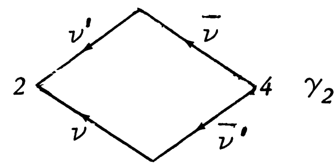
1. We denote by $\sigma\mathcal{F} = (\Sigma\mathcal{F}, \Gamma\mathcal{F})$ the sketch of categories [BE] which is the following limit-bearing category:

$\Sigma\mathcal{F}$ is the dual of the full sub-category of the simplicial category Δ whose objects are the natural integers $1, 2, 3$ and 4; its main morphisms are denoted according to the following diagram, where $\bar{\alpha} = \iota.\alpha$, $\bar{\beta} = \iota.\beta$:

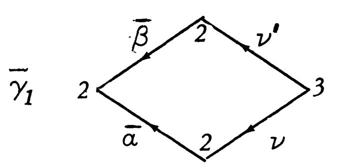


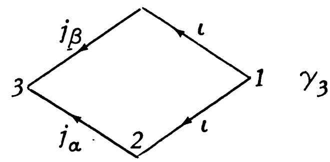
The only distinguished cones are the two pullbacks:





Since ι is a right inverse of α, it is an absolute equalizer of the pairs $(2, \bar\alpha)$ and $(\bar\beta, 2)$, and we have the pullbacks





in $\Sigma_{\mathcal{F}}$. We write $\bar\Gamma_{\mathcal{F}} = \{\bar\gamma_1, \gamma_2, \gamma_3\}$ and $\bar\sigma_{\mathcal{F}} = (\Sigma_{\mathcal{F}}, \bar\Gamma_{\mathcal{F}})$. Then $\Sigma_{\mathcal{F}}$ is the $\bar\Gamma_{\mathcal{F}}$-closure of $\{2\}$. So Propositions 3, 4, 7 may be applied to $\bar\sigma_{\mathcal{F}}$.

2. Let $\mathcal{H}$ be a category with pullbacks. A $\sigma_{\mathcal{F}}$-structure in $\mathcal{H}$ is called a *category internal to* (or *in*) $\mathcal{H}$; other names: category object in $\mathcal{H}$ for [Gr], «catégorie structurée généralisée dans $\mathcal{H}$» for [E3].

A $\sigma_{\mathcal{F}}$-morphism in $\mathcal{H}$ is called a *functor internal to* (or *in*) $\mathcal{H}$. We denote by $\mathcal{F}(\mathcal{H})$ the category $\mathcal{H}^{\sigma_{\mathcal{F}}}$ of the functors in $\mathcal{H}$. It is equal to the category $\mathcal{H}^{\bar\sigma_{\mathcal{F}}}$; indeed, if $\phi: \Sigma_{\mathcal{F}} \to \mathcal{H}$ is a functor, $\phi\gamma_3$ is a pullback, γ_3 being an absolute pullback, and, $\phi(\iota)$ being a monomorphism, $\phi\bar\gamma_1$ is a pullback iff $\phi\gamma_1$ is a pullback in $\mathcal{H}$.

If ψ is a category in the dual of $\mathcal{H}$, the dual functor $\psi^*: \Sigma^*_{\mathcal{F}} \to \mathcal{H}$ of ψ is called a *cocategory in* $\mathcal{H}$.

There exists a unique category δ in $\Sigma_{\mathcal{F}}$ mapping ι and κ on themselves and interchanging α and β, ν and ν'. If ϕ is a category in $\mathcal{H}$, then $\phi\delta$ is also a category in $\mathcal{H}$; we denote it by ϕ_* and call it the *dual* of ϕ. We get the «duality isomorphism» from $\mathcal{F}(\mathcal{H})$ onto $\mathcal{F}(\mathcal{H})$ by sending ϕ onto ϕ_* and the functor (ϕ', τ, ϕ) in $\mathcal{H}$ onto (ϕ'_*, τ, ϕ_*).

3. The categories $\mathcal{F}$ and $\mathcal{F}(\mathcal{M})$ are equivalent [E3, BE]. We will use the following canonical equivalences:

a) If C is a small category, there exists a unique category in $\mathfrak{M}$, denoted by $\eta_1(C)$ and called *the category in* $\mathfrak{M}$ *associated to* C, which transforms the pullbacks γ_1 and γ_2 into canonical pullbacks in $\mathfrak{M}$ and which maps α, β, κ and ι respectively onto the maps source, target, law of composition of C, and insertion from C_0 into C.

If $f: A \to C$ is a functor, $\eta_1(f)$ will denote the unique natural transformation (or functor internal to $\mathfrak{M}$)

$$\eta_1(f): \eta_1(A) \to \eta_1(C) \quad \text{such that} \quad \eta_1(f)(2) = \underline{f}.$$

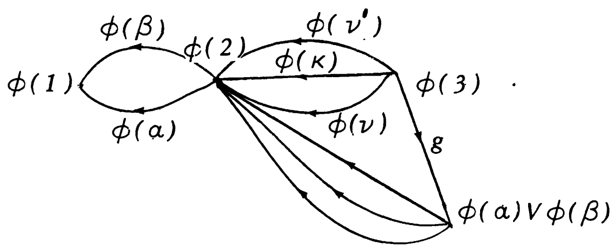
In this way, we get an equivalence $\eta_1: \mathcal{F} \to \mathcal{F}(\mathfrak{M})$. This equivalence admits as a restriction an isomorphism from $\mathcal{F}$ onto the full sub-category of $\mathcal{F}(\mathfrak{M})$ whose objects are the categories in $\mathfrak{M}$ mapping γ_1 and γ_2 on canonical pullbacks in $\mathfrak{M}$ and ι on an insertion.

b) On the other hand, we have an equivalence ζ_1 from $\mathcal{F}(\mathfrak{M})$ onto $\mathcal{F}$, which maps:

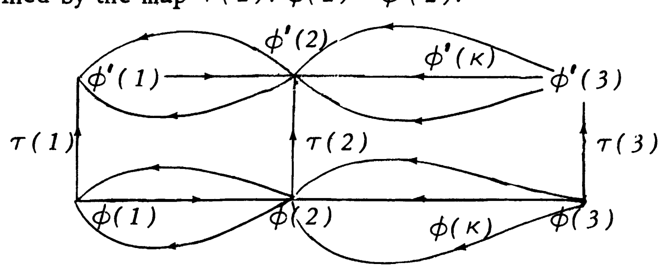
the category ϕ in $\mathfrak{M}$ on the category $\zeta_1(\phi)$, called the *category associated to* ϕ, whose underlying set is $\phi(2)$ and whose law of composition if $\phi(\kappa).g^{-1}$, where g is the bijection:

$$x \mapsto (\phi(\nu)(x), \phi(\nu')(x))$$

from $\phi(3)$ onto the canonical pullback of $(\phi(\alpha), \phi(\beta))$,



the functor $\tau: \phi \to \phi'$ internal to $\mathfrak{M}$ on the functor from $\zeta_1(\phi)$ to $\zeta_1(\phi')$ defined by the map $\tau(2): \phi(2) \to \phi'(2)$.



In particular, if M is a small set, the constant functor $M^{\wedge}: \Sigma_{\mathcal{F}} \to \mathfrak{M}$ is a category in $\mathfrak{M}$ whose associated category is the discrete category M^0.

4. *The functor toward $\mathcal{F}$ associated to a category in $\mathcal{H}$.*

PROPOSITION 8. *Let $\mathcal{H}$ be a category admitting pullbacks. The category $\mathcal{F}(\mathcal{H})$ is equivalent to the full sub-category $\mathcal{K}$ of $\mathcal{F}^{\mathcal{H}^*}$ whose objects are the functors $\phi: \mathcal{H}^* \to \mathcal{F}$ whose composite $p_{\mathcal{F}}\phi$ with the forgetful functor $p_{\mathcal{F}}: \mathcal{F} \to \mathfrak{M}$ is representable.*

Δ. Since $\Sigma_{\mathcal{F}}$ is the $\overline{\Gamma}_{\mathcal{F}}$-closure of $\{2\}$ and $\mathcal{H}^{\sigma\mathcal{F}} = \mathcal{H}^{\overline{\sigma}\mathcal{F}}$, Proposition 3 asserts that $\mathcal{F}(\mathcal{H}) = \mathcal{H}^{\sigma\mathcal{F}}$ is isomorphic with the full sub-category $\mathcal{R}$ of $\mathcal{F}(\mathfrak{M})^{\mathcal{H}^*}$ whose objects are the functors $\psi: \mathcal{H}^* \to \mathcal{F}(\mathfrak{M})$ such that $\nu\psi$ is representable, $\nu: \mathcal{F}(\mathfrak{M}) \to \mathfrak{M}$ denoting the functor value in 2 which sends τ onto $\tau(2)$. If $\zeta_1: \mathcal{F}(\mathfrak{M}) \to \mathcal{F}$ is the equivalence constructed in 3 above, the composite functor

$$\mathcal{F}(\mathfrak{M}) \xrightarrow{\zeta_1} \mathcal{F} \xrightarrow{p_{\mathcal{F}}} \mathfrak{M}$$

is equal to ν, so that $\nu\psi$ is representable iff $p_{\mathcal{F}}\zeta_1\psi$ is representable. The equivalence

$$\zeta_1^{\mathcal{H}^*}: \mathcal{F}(\mathfrak{M})^{\mathcal{H}^*} \to \mathcal{F}^{\mathcal{H}^*}$$

associating $\zeta_1\psi$ to $\psi: \mathcal{H}^* \to \mathcal{F}(\mathfrak{M})$, it admits as a restriction an equivalence from $\mathcal{R}$ onto the full sub-category $\mathcal{K}$ of $\mathcal{F}^{\mathcal{H}^*}$. Hence $\mathcal{F}(\mathcal{H})$ and $\mathcal{K}$ are equivalent. ∇

5. *The canonical cocategory in $\mathcal{F}$.*

If n is a natural integer, the composite functor

$$\Sigma_{\mathcal{F}} \hookrightarrow \Delta^* \xrightarrow{\Delta(n,\cdot)} \mathfrak{M}$$

is a category in $\mathfrak{M}$, since the pullbacks γ_1 and γ_2 in $\Sigma_{\mathcal{F}}$ are also pullbacks in Δ^*. Its associated category is the category **n** defining the canonical order of the ordinal $n = \{0, \dots, n\text{-}1\}$; the morphisms of **n** are the pairs (m', m) of integers such that $m \leqslant m' < n$.

If $f: n \to m$ is a morphism of $\Sigma_{\mathcal{F}}$, i.e. if f defines an increasing map from $(n, \leqslant)$ to $(m, \leqslant)$, the composite natural transformation

$$\Sigma_{\mathcal{F}} \hookrightarrow \Delta^* \overset{\Delta(f,\cdot)}{\Rightarrow} \mathfrak{M}$$

is a functor internal to $\mathfrak{M}$, to which is associated the functor $\mathbf{f}$:

$$(j, i) \to (f(j), f(i)) \quad \text{from } \mathbf{n} \text{ to } \mathbf{m}$$

(defined by the map $\Delta(f, 2)$).

PROPOSITION 9. *There exists a cocategory in* $\mathcal{F}$ *admitting as a restriction an isomorphism from* $\Sigma^*_{\mathcal{F}}$ *onto the full sub-category* $\overline{\Sigma}^*_{\mathcal{F}}$ *of* $\mathcal{F}$ *whose objects are* **1, 2, 3** *and* **4**. $\mathcal{F}$ *is the inductive closure of* $\{\mathbf{2}\}$.

Δ. From Proposition 4, it follows that the Yoneda embedding Y_1 from $\Sigma^*_{\mathcal{F}}$ to $\mathfrak{M}^{\Sigma_{\mathcal{F}}}$ admits as a restriction a cocategory $\overline{Y}_1$ in $\mathcal{F}(\mathfrak{M})$ and that $\mathcal{F}(\mathfrak{M}) = \mathfrak{M}^{\sigma\overline{\mathcal{F}}}$ is the inductive closure of $\{Y_1(2)\}$. As ζ_1 is an equivalence, $\mathcal{F}$ is the inductive closure of $\{\zeta_1 \overline{Y}_1(2)\}$ and the composite $\zeta_1 \overline{Y}_1$:

$$\Sigma^*_{\mathcal{F}} \xrightarrow{\overline{Y}_1} \mathcal{F}(\mathfrak{M}) \xrightarrow{\zeta_1} \mathcal{F}$$

is a cocategory in $\mathcal{F}$. It admits as a restriction an isomorphism from $\Sigma^*_{\mathcal{F}}$ onto the full sub-category of $\mathcal{F}$ whose objects are the categories

$$\zeta_1 \overline{Y}_1(n), \quad \text{where} \quad n \in \{1, 2, 3, 4\}.$$

So, it remains only to prove that the category $\zeta_1 \overline{Y}_1(n)$ is identical with $\mathbf{n}$. Indeed, this category is the category associated to the category in $\mathfrak{M}$:

$$Y_1(n) = \Sigma_{\mathcal{F}}(\cdot, n) : \Sigma_{\mathcal{F}} \to \mathfrak{M}.$$

Since $\Sigma_{\mathcal{F}}$ is a full sub-category of Δ^*, we have $Y_1(n)$ equal to the composite functor:

$$\Sigma_{\mathcal{F}} \hookrightarrow \Delta^* \xrightarrow{\Delta(n,\cdot)} \mathfrak{M},$$

to which is associated, by definition, the category $\mathbf{n}$. ∇

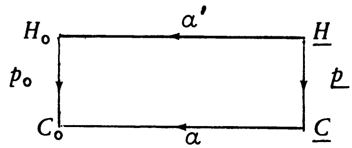
REMARK. The above constructed cocategory in $\mathcal{F}$ is a restriction of the canonical embedding of the simplicial category Δ into $\mathcal{F}$, which defines $\mathcal{F}$ as a category admitting as models the categories $\mathbf{n}$, for all the integers n. The corresponding «singular functor» from $\mathcal{F}$ to the category $\mathcal{S}$ of simplicial maps sends a category C onto the corresponding simplicial object; the homology of this simplicial object is called the homology of C [Gr]. The singular functor admits an adjoint, the realization functor, which associates to a simplicial object F the category canonically associated to F; the groupoid projection of this category is the fundamental groupoid of F (see [GZ]).

D. Internal discrete fibrations.

1. It is known [E1] that the three following notions are equivalent, where C is a category:

a) A functor from C to the category $\mathfrak{M}$ of maps.

b) A discrete fibration (or hypermorphism functor [E1]) over C, which is a functor $p: H \to C$ such that

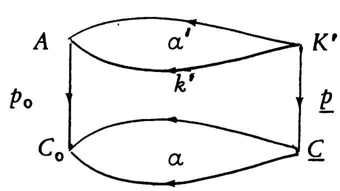


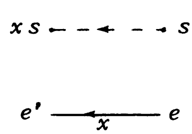
is a pullback, where α and α' are the source maps of C and H and p_0 the restriction of p to the objects; this means that, if s is an object of H and $x: p(s) \to e'$ a morphism in C, there exists one and only one morphism y in H admitting s as its source and satisfying $p(y) = x$.

c) A left action k' of C on a set A, also called a category action (or an operator category on A, or a species of structures in [E1]): then k' is a map $(x, s) \mapsto xs$ from a sub-set K' of $\underline{C} \times \underline{A}$ to A satisfying the following axioms: there exists a map $p_0 : A \to C_0$ such that K' is the canonical pullback of (α, p_0) and that:

$es = s$ if $s \in A$ and $e = p_0(s)$,

$x'(xs) = (x'.x)s$ if $x'.x$ exists in C and if $(x, s) \in K'$

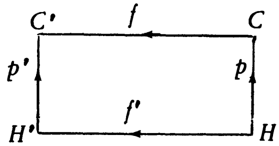




(the map p_0 is uniquely determined by these conditions, which imply that $p_0(xs)$ is the target of x). The associated discrete fibration is the functor $p: C*A \to C$, where $C*A$ is the category on K' such that:

$$(x', s').(x, s) = (x'.x, s) \quad \text{iff } x'.x \text{ exists in } C \text{ and } s' = xs.$$

2. We denote by $\square\mathcal{F}$ the horizontal category of commutative squares (or quartets [E1]) of the category $\mathcal{F}$ of small functors whose objects are the small functors, the morphisms from p to p' being the commutative squares (p', f, f', p).



We denote by $\mathcal{A}$ the full sub-category of $\square\mathcal{F}$ whose objects are the discrete fibrations; its morphisms are called morphisms between discrete fibrations.

The category $\mathcal{A}$ is equivalent to the category of covariant maps between category actions (see [E1]).

We denote by $\mathrm{p}_{\mathcal{A}}$ and $\mathrm{p}_{\mathcal{A}}^{0}$ the functors from $\mathcal{A}$ to $\mathcal{M}$ sending the morphism (p', f, f', p) respectively onto the map $\underline{f}$ defining f and onto the map $f'_0: H_0 \to H'_0$ restriction of f' to the objects.

$\mathcal{F}$ will be identified with the full sub-category of $\mathcal{A}$ whose objects are the identical fibrations.

Let C be a small category. $\mathcal{A}$ admits as a «non-full» sub-category the category $\mathcal{A}_C$ of morphisms over C, whose elements are the morphisms (p', f, f', p) such that f is the identity of C (such a morphism identifies with the triangle (p', f', p)). There exists an equivalence from $\mathcal{M}^C$ toward $\mathcal{A}_C$ which sends a functor $\phi: C \to \mathcal{M}$ onto the discrete fibration h_ϕ, from H_ϕ to C, associated to it (the morphisms of H_ϕ are the pairs

$$(x, s), \quad \text{where } x \in C \text{ and } s \in \phi(\alpha(x)),$$

and $h_\phi(x, s) = x$).

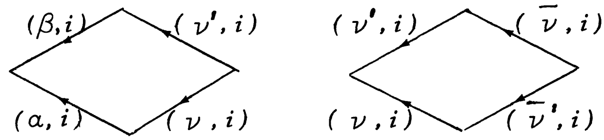
$\mathcal{A}_C$ is also equivalent to the category of covariant maps over C.

3. The sketch of discrete fibrations.

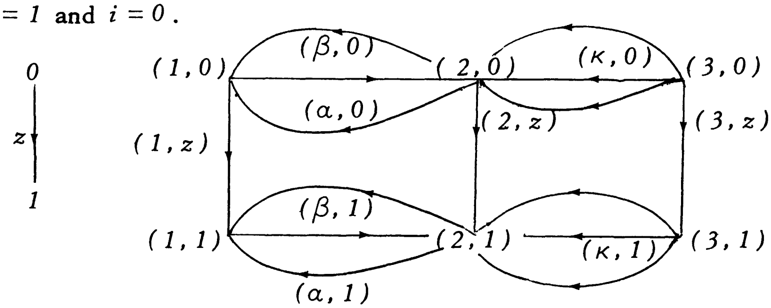
We denote by **2** the category

$$1 \xleftarrow{\;z\;} 0$$

as well as the limit-bearing category on **2** without any cone. The tensor product $\sigma\mathcal{F}\otimes\mathbf{2}$ is the category $\Sigma\mathcal{F}\times\mathbf{2}$ equipped with the pullbacks



for $i = 1$ and $i = 0$.



PROPOSITION 10. *There is a canonical equivalence which is surjective*

$$\zeta' : \mathfrak{M}^{\sigma\mathcal{F}\otimes\mathbf{2}} \xrightarrow{\sim} \square\mathcal{F}.$$

Δ. Let $\psi : \Sigma\mathcal{F}\times\mathbf{2} \to \mathfrak{M}$ be a $\sigma\mathcal{F}\otimes\mathbf{2}$-structure in $\mathfrak{M}$. Then

$$\psi(-,1) : \Sigma\mathcal{F} \to \mathfrak{M} \quad \text{and} \quad \psi(-,0) : \Sigma\mathcal{F} \to \mathfrak{M}$$

are categories in $\mathfrak{M}$; let C and H be the associated categories. The map $\psi(2,z)$ defines a functor $\zeta'(\psi)$ from H to C.

If $\tau : \psi \to \psi'$ is a $\sigma\mathcal{F}\otimes\mathbf{2}$-morphism, then

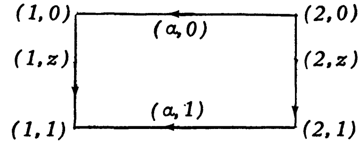
$$\zeta'(\psi) : H \to C \quad \text{and} \quad \zeta'(\psi') : H' \to C'$$

are functors and the maps $\tau(2,1)$ and $\tau(2,0)$ define, respectively, functors $f : C \to C'$ and $f' : H \to H'$. Then $\zeta'(\tau)$ is the commutative square

$$(\zeta'(\psi'), f, f', \zeta'(\psi)). \quad \nabla$$

DEFINITION. We define the *sketch of discrete fibrations* as the limit-bearing category $\sigma_\phi = (\Sigma_\phi, \Gamma_\phi)$ got by equipping the category $\Sigma_\phi = \Sigma\mathcal{J} \times \mathbf{2}$ with the set $\Gamma\mathcal{J} \otimes \emptyset$ of the distinguished cones of $\sigma\mathcal{J} \otimes \mathbf{2}$ and the pullback

γ_4 :



Let $\overline{\Gamma}_\phi$ be the set $(\overline{\Gamma}\mathcal{J} \otimes \emptyset) \cup \{\gamma_4\}$ of 7 cones, among them the absolute pullbacks $[\gamma_3, 0^{\hat{}}]$ and $[\gamma_3, 1^{\hat{}}]$, and $\overline{\sigma}_\phi = (\Sigma_\phi, \overline{\Gamma}_\phi)$. Then Σ_ϕ is the $\overline{\Gamma}_\phi$-closure of $\{(2,1),(1,0)\}$, since $\Sigma\mathcal{J}$ is the $\overline{\Gamma}\mathcal{J}$-closure of $\{2\}$ and γ_4 is a pullback. Moreover $\mathfrak{M}^{\sigma_\phi} = \mathfrak{M}^{\overline{\sigma}_\phi}$.

PROPOSITION 11. *The category $\mathfrak{A}$ is equivalent to $\mathfrak{M}^{\sigma_\phi}$ and it is the inductive closure of $\{\mathbf{1}, \mathit{2}\}$, where $\mathit{2}$ is the void fibration from $\emptyset$ to $\mathbf{2}$.*

Δ. 1° If $\psi : \Sigma\mathcal{J} \times \mathbf{2} \to \mathfrak{M}$ is a $\sigma\mathcal{J} \otimes \mathbf{2}$-structure in $\mathfrak{M}$, it is a σ_ϕ-structure iff it maps γ_4 on a pullback in $\mathfrak{M}$, i.e. iff the functor $\zeta'(\psi)$ is a discrete fibration, where ζ' is the equivalence defined in Proposition 10. Hence ζ' admits as a restriction an equivalence ζ'' from the full sub-category $\mathfrak{M}^{\sigma_\phi}$ of $\mathfrak{M}^{\sigma\mathcal{J} \otimes \mathbf{2}}$ onto the full sub-category $\mathfrak{A}$ of $\square\square\mathcal{F}$.

2° Since $\mathfrak{M}^{\sigma_\phi} = \mathfrak{M}^{\overline{\sigma}_\phi}$ and Σ_ϕ is the $\overline{\Gamma}_\phi$-closure of $\{(1,0),(2,1)\}$, by Proposition 4, the category $\mathfrak{M}^{\sigma_\phi}$ is the inductive closure of

$$\{ Y(1,0),\ Y(2,1)\}, \text{ where } Y : \Sigma_\phi^* \to \mathfrak{M}^{\sigma_\phi}$$

is the Yoneda embedding. Using the equivalence ζ'', we deduce that $\mathfrak{A}$ is the inductive closure of $\{\zeta'' Y(1,0),\ \zeta'' Y(2,1)\}$.

As $(1,0)$ is an initial object of Σ_ϕ, it is mapped by Y on a final object of $\mathfrak{M}^{\sigma_\phi}$, and by $\zeta'' Y$ on a final object of $\mathfrak{A}$. Hence $\zeta'' Y(1,0)$ is isomorphic with the identical fibration $\mathbf{1}$.

The category associated to $Y(2,1)(-,1) : \Sigma\mathcal{J} \to \mathfrak{M}$ is $\mathbf{2} \times \{1\}$, for

$$Y(2,1)(m,1) = \Sigma_\phi((m,1),(2,1)) = \Sigma\mathcal{J}(m,2) \times \{1\}$$

for each $m \in \{1,2,3,4\}$. In the same way, the category associated to $Y(2,1)(-,0) : \Sigma\mathcal{J} \to \mathfrak{M}$ is void. Therefore, $\zeta'' Y(2,1)$ is the discrete fibration from the void category to $\mathbf{2} \times \{1\}$ (isomorphic to $\mathbf{2}$). This fibration is isomorphic in $\mathfrak{A}$ with the fibration $\mathbf{2}$.

If follows that $\mathfrak{A}$ is the inductive closure of $\{\mathbf{1}, \mathbf{2}\}$. ∇

4. *Discrete fibrations in a category* $\mathcal{H}$.

We suppose that $\mathcal{H}$ is a category admitting pullbacks. A σ_ϕ-structure in $\mathcal{H}$ is called *a discrete fibration in* $\mathcal{H}$. We denote by $\mathfrak{A}(\mathcal{H})$ the category of σ_ϕ-morphisms in $\mathcal{H}$, which is equal to $\mathcal{H}^{\overline{\sigma}\phi}$.

PROPOSITION 12. *1° $\mathfrak{A}(\mathcal{H})$ is equivalent to the full sub-category $\mathfrak{R}$ of $\mathfrak{A}^{\mathcal{H}^*}$ whose objects are the functors $\rho : \mathcal{H}^* \to \mathfrak{A}$ such that $\mathfrak{p}_{\mathfrak{A}}\rho$ and $\mathring{\mathfrak{p}}_{\mathfrak{A}}\rho$ are representable (where $\mathfrak{p}_{\mathfrak{A}}$ and $\mathring{\mathfrak{p}}_{\mathfrak{A}}$ are the forgetful functors from $\mathfrak{A}$ to $\mathfrak{M}$ defined in 2).*

2° If ψ and ψ' are two discrete fibrations in $\mathcal{H}$ such that

$$\psi(\cdot, 1) = \psi'(\cdot, 1) : \Sigma_{\mathfrak{F}} \to \mathcal{H}, \ \psi'\gamma_4 = \psi\gamma_4, \ \psi(\beta, 0) = \psi'(\beta, 0),$$

then ψ and ψ' are isomorphic in $\mathfrak{A}(\mathcal{H})$.

Δ. 1° As $\mathfrak{A}(\mathcal{H}) = \mathcal{H}^{\overline{\sigma}\phi}$ and Σ_ϕ is the $\overline{\Gamma}_\phi$-closure of $\{(1,0), (2,1)\}$, Proposition 3 asserts that $\mathfrak{A}(\mathcal{H})$ is equivalent to the full sub-category $\mathfrak{R}'$ of $\mathfrak{A}(\mathfrak{M})^{\mathcal{H}^*}$ whose objects are the functors $\rho' : \mathcal{H}^* \to \mathfrak{A}(\mathfrak{M})$ such that $\mathfrak{q}\rho'$ and $\mathfrak{q}^0\rho'$ are representable, where $\mathfrak{q}^0$ and $\mathfrak{q}$ are the value functors from $\mathfrak{A}(\mathfrak{M})$ to $\mathfrak{M}$ associating to τ respectively $\tau(1,0)$ and $\tau(2,1)$. If ζ'' is the canonical equivalence (Proposition 11), then

$$\mathfrak{q} \text{ is the composite functor } \mathfrak{A}(\mathfrak{M}) \xrightarrow{\zeta''} \mathfrak{A} \xrightarrow{\mathfrak{p}_{\mathfrak{A}}} \mathfrak{M},$$

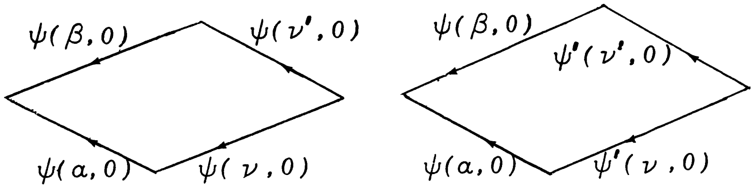
$$\mathfrak{q}^0 \text{ is the composite functor } \mathfrak{A}(\mathfrak{M}) \xrightarrow{\zeta''} \mathfrak{A} \xrightarrow{\mathring{\mathfrak{p}}_{\mathfrak{A}}} \mathfrak{M}.$$

It follows that a functor $\rho' : \mathcal{H}^* \to \mathfrak{A}(\mathfrak{M})$ is an object of $\mathfrak{R}'$ iff the functor $\zeta''\rho'$ is such that $\mathfrak{p}_{\mathfrak{A}}\zeta''\rho'$ and $\mathring{\mathfrak{p}}_{\mathfrak{A}}\zeta''\rho'$ are representable, i.e. iff $\zeta''\rho'$ is an object of the category $\mathfrak{R}$ defined in the Proposition. Hence the equivalence $\zeta''^{\mathcal{H}^*} : \mathfrak{A}(\mathfrak{M})^{\mathcal{H}^*} \to \mathfrak{A}^{\mathcal{H}^*}$ admits as a restriction an equivalence from $\mathfrak{R}'$ to $\mathfrak{R}$. Finally, $\mathfrak{A}(\mathcal{H})$ is equivalent to $\mathfrak{R}$.

2° Let ψ and ψ' be discrete fibrations in $\mathcal{H}$ satisfying

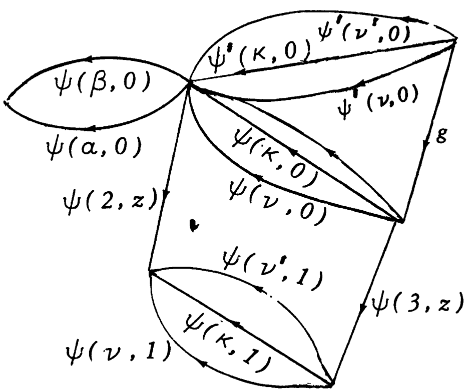
$$\psi(\cdot, 1) = \psi'(\cdot, 1), \qquad \psi\gamma_4 = \psi'\gamma_4, \qquad \psi(\beta, 0) = \psi'(\beta, 0).$$

Since



are two pullbacks, there exists an unique isomorphism g of $\mathcal{H}$ such that:

$$\psi(\nu_i, 0).\, g = \psi'(\nu_i, 0) \text{ for } \nu_0 = \nu \text{ and } \nu_1 = \nu'.$$



From the equalities

$$(\nu_i, 1).(3, z) = (2, z).(\nu_i, 0)$$

for $i = 0$ and $i = 1$, we deduce

$$\psi'(\nu_i, 1).\psi'(3, z) = \psi'(2, z).\psi'(\nu_i, 0) = \psi'(2, z).\psi(\nu_i, 0).\, g =$$
$$= \psi(2, z).\psi(\nu_i, 0).\, g = \psi'(\nu_i, 1).\psi(3, z).\, g$$

for i equal to 0 and 1. This implies (unicity of the factor relative to a pullback):

$$\psi'(3, z) = \psi(3, z).\, g.$$

In the same way, from

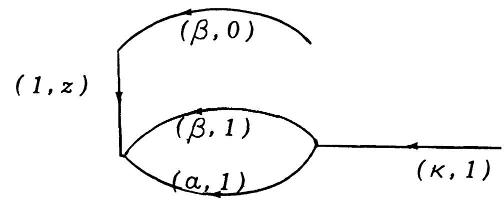
$$(\alpha, 0).(\kappa, 0) = (\alpha, 0).(\nu, 0) \text{ and } (2, z).(\kappa, 0) = (\kappa, 1).(3, z),$$

we get

$$\psi'(\kappa, 0) = \psi(\kappa, 0).\, g,$$

the functors ψ and ψ' taking the same values on $(\alpha, 0)$, on $(2, z)$ and on $(\kappa, 1)$. It follows that the categories $\psi(-, 0)$ and $\psi'(-, 0)$ in $\mathcal{H}$ are equivalent, whence ψ and ψ' are equivalent, i.e. isomorphic in $\mathcal{Q}(\mathcal{H})$. ∇

The preceding proof shows that σ_ϕ admits as its «idea»



Hence a discrete fibration ψ in $\mathcal{H}$ is determined up to an isomorphism by $(\psi(-,1), \psi(1,z), \psi(\beta,0))$. This leads to the following definition:

DEFINITION. We say that *(ϕ, h, k') is a category action in* $\mathcal{H}$ if:

1° ϕ is a category in $\mathcal{H}$,

2° h and k' are morphisms of $\mathcal{H}$,

3° there exists a discrete fibration ψ in $\mathcal{H}$ such that

$$\psi(-,1) = \phi, \quad \psi(1,z) = h, \quad \psi(\beta,0) = k'.$$

If ϕ is a category in $\mathcal{H}$, let ϕ^* be its dual (section C-2). If we have a category action (ϕ_*, h, k') in $\mathcal{H}$, we also say that (k', h, ϕ) is *a right category action in* $\mathcal{H}$.

EXEMPLE. Category actions were introduced in [E4] as an axiomatization of the notion of a fiber-bundle. Indeed, topological (resp. r-differentiable) fiber-bundles correspond exactly to the category actions in the category $\mathcal{T}$ of continuous maps (resp. $\mathfrak{D}^r$ of r-differentiable maps between manifolds) such that the operating topological (resp. differentiable) category be a locally trivial groupoid [E4, 5].

5. *Distributors in* $\mathcal{H}$.

If B and C are categories, the following notions are equivalent:

a) A distributor from B to C, which is defined [B1] as a functor from $C^* \times B$ to $\mathfrak{M}$.

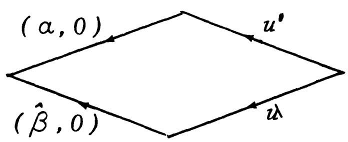
b) A pair of category actions on a set (introduced in [E1] under the name of «couple de catégories d'opérateurs»), i. e. a pair

$$((B, A, \kappa'), (C^*, A, \kappa''))$$

of category actions such that

$(xs)x' = x(sx')$ whenever the composites
$xs = \kappa'(x, s)$ and $sx' = \kappa''(x', s)$ are defined.

This last notion is easily «internalized» in a category $\mathcal{H}$ admitting pullbacks by defining a limit-bearing category σ_δ whose realizations in $\mathfrak{M}$ are pairs of category actions on a set. We will not formally construct σ_δ here; its description is given in [V]. Intuitively, it is got by gluing together along $(1, 0)$ the sketch of a discrete fibration and the sketch of a discrete fibration over the dual of a category (in which (μ, y) is replaced by $(\hat{\mu}, y)$), and by adding the pullback

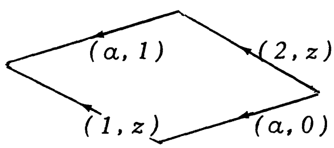


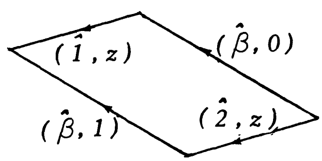
,

and the factors

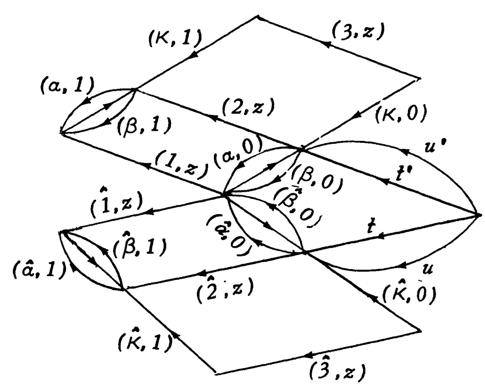
$$t' = [(2, z).u', (\hat{\alpha}, 0).u] \quad \text{and} \quad t = [(\beta, 0).u', (\hat{2}, z).u]$$

relative to the pullbacks





of $((\alpha, 1), (1, z))$ and $((\hat{1}, z), (\hat{\beta}, 1))$,
the axiom $(\hat{\alpha}, 0).t = (\beta, 0).t'$.



The realizations of σ_δ in $\mathcal{H}$ are called [B1] distributors in $\mathcal{H}$:

DEFINITION. Let $\mathcal{H}$ be a category admitting pullbacks. A *distributor in* $\mathcal{H}$ is defined as a sextuple $(\phi', b', k'', k', b, \phi)$, where:

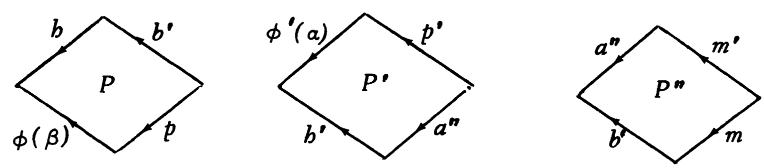
1° (ϕ', b', k'') is a category action in $\mathcal{H}$,

2° (k', b, ϕ) is a right category action in $\mathcal{H}$,

3° $k''.l' = k'.l$, where

$$l = [k''.m', p.m] \quad \text{and} \quad l' = [p'.m', k'.m]$$

are the factors relative to the pullbacks respectively P of $(b, \phi(\beta))$ and P' of $(\phi'(\alpha), b')$, where we have the pullbacks:



$\mathfrak{D}$ denoting the category $\mathfrak{M}^{\sigma_\delta}$ of morphisms between distributors, it follows from Propositions 2 and 12 that $\mathcal{H}^{\sigma_\delta}$ is equivalent to the full subcategory of $\mathfrak{D}^{\mathcal{H}^*}$ whose objects are the $\psi: \mathcal{H}^* \to \mathfrak{D}$ such that $\psi(-)(\omega)$ is representable, for $\omega \in \{(\hat{2},1), (2,1), (1,0)\}$.

REMARKS. 1° To a distributor $\delta: C^* \times B \to \mathfrak{M}$ is associated a functor Δ, from B to $\mathfrak{M}^{C^*}$ and, since $\mathfrak{M}^{C^*}$ and $\mathcal{Q}_{C^*}$ are equivalent, a functor from B to $\mathcal{Q}_{C^*}$. More generally, problems in Differential Geometry and in Analysis led to consider functors from a category B to $\mathcal{Q}$. Such a functor associates to each $e \in B_0$ a category action (C_e, A_e, k'_e); then B operates on the category sum of the C_e and on the set A sum of the A_e. This situation is easily internalized in $\mathcal{H}$ and enriched by giving supplementary structures on the A_e. In fact, it was this more general notion (suggested by that of a sheaf of operators on a sheaf) which was first introduced (in [AB] to define distributions on infinite dimensionnal vector spaces) under the name of «catégorie de catégories d'opérateurs» and which is studied in [E1,5] (and called espèce de structures dominée par des applications covariantes).

2° Distributors are the 1-morphisms of a bicategory (see [B1]), for a law which can be suggested by that of the category of atlases of a category defined in [E6].

1. THE CATEGORY OF DOUBLE FUNCTORS

A. Double categories.

1. In this section, we recall the initial «naive» definition of double categories, as it is given in [E2].

DEFINITION. *A double category* is defined as a pair $(\Sigma^{\circ}, \Sigma^{\cdot})$ of categories with the same set of morphisms, satisfying the following conditions:

1° The maps source and target of $\Sigma^{\cdot}$ define functors from Σ° onto a sub-category of Σ°.

2° The law of composition of $\Sigma^{\cdot}$ defines a functor toward Σ° from the sub-category of $\Sigma^{\circ} \times \Sigma^{\circ}$ formed by the pairs of morphisms composable in the category $\Sigma^{\cdot}$.

$(\Sigma^{\circ}, \Sigma^{\cdot})$ is then called a double category *on* Σ, and the categories Σ° and $\Sigma^{\cdot}$ are respectively its *first category* and its *second category.* A double category on Σ is said *small* if Σ is a small set.

In [E2] it is shown that the axioms 1 and 2 are equivalent to the following ones, where α, β and α°, β° denote the maps source and target in $\Sigma^{\cdot}$ and in Σ° respectively:

1° For each $d \in \Sigma$, we have

$$\alpha(\alpha^{\circ}(d)) = \alpha^{\circ}(\alpha(d)), \quad \alpha(\beta^{\circ}(d)) = \beta^{\circ}(\alpha(d)),$$
$$\beta(\alpha^{\circ}(d)) = \alpha^{\circ}(\beta(d)), \quad \beta(\beta^{\circ}(d)) = \beta^{\circ}(\beta(d)).$$

2° If the composite $d' \circ d$ exists in Σ°, then

$$\alpha(d' \circ d) = \alpha(d') \circ \alpha(d) \text{ and } \beta(d' \circ d) = \beta(d') \circ \beta(d);$$

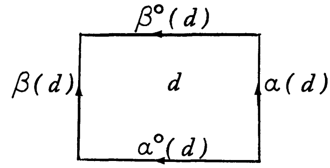
if the composite $\hat{d}.d$ exists in $\Sigma^{\cdot}$, then

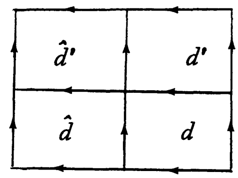
$$\alpha^{\circ}(\hat{d}.d) = \alpha^{\circ}(\hat{d}).\alpha^{\circ}(d) \text{ and } \beta^{\circ}(\hat{d}.d) = \beta^{\circ}(\hat{d}).\beta^{\circ}(d).$$

3° Permutability axiom: If the composites $d' \circ d$, $\hat{d}' \circ \hat{d}$, $\hat{d}.d$, $\hat{d}'.d'$ are defined, then the composites

$$(\hat{d}' \circ \hat{d}).(d' \circ d) \text{ and } (\hat{d}'.d') \circ (\hat{d}.d)$$

are defined and both are equal.





This set of axioms being symmetrical relative to $\Sigma^{\cdot}$ and to Σ^o, it follows that $(\Sigma^o, \Sigma^{\cdot})$ is a double category iff $(\Sigma^{\cdot}, \Sigma^o)$ is a double category; these two double categories are said *symmetrical.*

2. *Notations.*

A double category $(\Sigma^o, \Sigma^{\cdot})$ is generally denoted by a unique italic letter, for example D. In that case:

The underlying set Σ is denoted by $\underline{D}$.

The first category Σ^o is also denoted by D^1, its symbol of composition by $\circ_1$ (instead of $\circ$), its mappings source, target and law of composition by α^1, β^1 and κ^1.

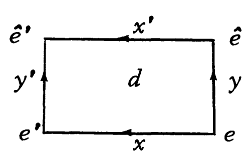
The second category $\Sigma^{\cdot}$ is denoted by D^2, its symbol of composition by $\circ_2$ (instead of $.$), its mappings source, target and law of composition by α^2, β^2 and κ^2.

The set of objects of D^1 defines a sub-category of D^2, which is denoted by D^1_o and called the *second category of 1-morphisms of* D.

The set of objects of D^2 defines a sub-category of D^1, which is denoted by D^2_o and called *the first category of 1-morphisms of* D.

The categories D^1_o and D^2_o have the same set of objects, which is written D_{oo} and called *the set of vertices of* D. The elements of $\underline{D}$ which are not objects for D^1 nor D^2 are called *2-blocks of* D.

Let d be a 2-block of D. As a morphism of D^1, it admits a source $x = \alpha^1(d)$ and a target $x' = \beta^1(d)$ and we write $d: x \to x'$. As a morphism of D^2, it admits a source $y = \alpha^2(d)$ and a target $y' = \beta^2(d)$ and we write $d: y \rightrightarrows y'$.



3. *Examples.*

a) 2-categories are defined as the double categories D such that the objects of D^2 are also objects of D^1, so that $D_{oo} = D^2_o \subset D^1_o$; a 2-block of D is then called a 2-cell. For example, we have the 2-category of natural transformations (between small categories), denoted by $(\mathfrak{N}^{\square\square}, \mathfrak{N}^{\cdot})$, or $\mathfrak{N}$, whose second category of 1-morphisms is $\mathcal{F}$.

b) If Σ° is a category and Σ^0 the discrete category on Σ, then the pair (Σ°, Σ^0) is a double category, called the *discrete double category on* Σ°. Similarly, (Σ^0, Σ°) is a 2-category, called the *discrete 2-category on* Σ°.

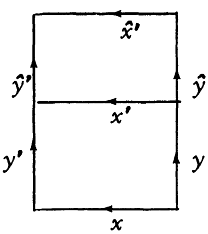
c) If A is a category, (A, A) is a double category iff A is a commutative category, i. e. a category coproduct of commutative monoids.

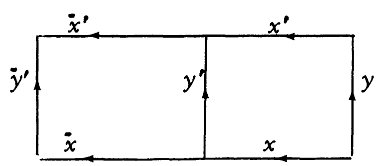
If D is a double category such that D^1_o and D^2_o are discrete categories, then $D^1 = D^2$.

d) Let A be a category. We denote by $\square A$ *the double category of commutative squares of* A. Its underlying set is the set of commutative squares (or quartets) of A, which are the 4-tuples (y', x', x, y) such that the composites $y'.x$ and $x'.y$ are defined and equal.

Its first category, denoted by $\boxminus A$, is called the *vertical category of squares*; its law of composition is:

$$(\hat{y}', \hat{x}', \hat{x}, \hat{y}) \boxminus (y', x', x, y) = (\hat{y}'.y', \hat{x}', x, \hat{y}.y) \text{ iff } x' = \hat{x}.$$





Its second category, denoted by $\square\square A$, is called the *horizontal category of squares*; its law of composition is:

$$(\bar{y}', \bar{x}', \bar{x}, \bar{y}) \square\square (y', x', x, y) = (\bar{y}', \bar{x}'.x', \bar{x}.x, y) \text{ iff } \bar{y} = y'.$$

There is an isomorphism $(y', x', x, y) \mapsto (x', y', y, x)$ from $\square\square A$ onto $\boxminus A$.

With similar laws, the set of all (non commutative) squares of A also becomes a double category.

e) Let D be a double category. If $\underline{C}$ is a sub-set of $\underline{D}$ which defines a sub-category C^1 of D^1 and a sub-category C^2 of D^2, then (C^1, C^2) is a double category C, called the *double sub-category of* D defined by $\underline{C}$ (or by C^1, or by C^2).

In particular, among all the double sub-categories of D which are 2-categories, there is a greatest one, namely that defined by the full sub-category of D^2 whose objects are the vertices of D.

The full sub-category of D^1 whose objects are all the vertices of D also defines a double sub-category of D, whose symmetrical double category is the greatest sub-2-category of the symmetrical of D.

f) Let D be a double category. Then,

$$(D^{1*}, D^2), \quad (D^1, D^{2*}) \text{ and } (D^{1*}, D^{2*})$$

are double categories, called respectively *the first dual*, the *second dual* and the *dual of* D.

4. *Double functors.*

DEFINITION. We say that (D, ϕ, C) is a *double functor* if C and D are double categories and if ϕ is a map from $\underline{C}$ to $\underline{D}$ defining a functor from C^1 to D^1 and a functor from C^2 to D^2.

A double functor (D, ϕ, C) will often be denoted by an italic letter f. In that case:

the map ϕ is also denoted by $\underline{f}$,
the functor (D^1, ϕ, C^1) by f^1,
the functor (D^2, ϕ, C^2) by f^2.

Moreover, we say that

$$f: C \to D \quad \text{is a double functor,}$$

or that ϕ defines a double functor from C to D.

EXAMPLES. a) The double functors between 2-categories are called 2-functors.

b) Let Σ° and Σ'° be categories. A map $\phi : \Sigma \to \Sigma'$ defines a functor $f: \Sigma^{\circ} \to \Sigma'^{\circ}$ iff it defines a double functor from the discrete double category $(\Sigma^{\circ}, \Sigma^{0})$ on Σ° toward the discrete double category on Σ'°. In that case, there exists a double functor from the double category of commutative squares $\square\Sigma^{\circ}$ to $\square\Sigma'^{\circ}$ defined by the restriction « to the commutative squares» of the product map $\phi\times\phi\times\phi\times\phi$. This double functor is denoted by $\square f$.

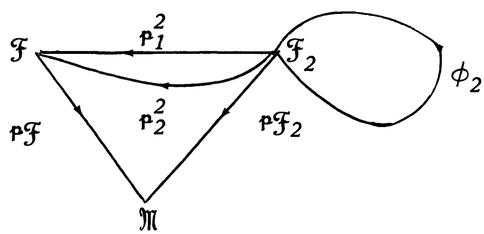
The double functors between small double categories are the morphisms of the *category* $\mathcal{F}_2$ *of (small) double functors*, whose objects are the small double categories.

This category is equipped with the following forgetful functors:

$p_1^2 : \mathcal{F}_2 \to \mathcal{F}$, which associates to the double functor f the functor f^1,

$p_2^2 : \mathcal{F}_2 \to \mathcal{F}$, which associates f^2 to f,

$p_{\mathcal{F}_2} : \mathcal{F}_2 \to \mathcal{M}$, which associates to f the underlying map $\underline{f}$.



Moreover, there is an isomorphism $\phi_2 : \mathcal{F}_2 \to \mathcal{F}_2$, which is its own inverse, mapping the double category D on its symmetrical one (denoted D^{21}) and associating

$$(D^{21}, \phi, C^{21}) \text{ to } (D, \phi, C) \in \mathcal{F}_2 .$$

We have the equality $p_1^2 \phi_2 = p_2^2$.

B. Double categories as sketched structures.

Double categories may be considered both as categories in $\mathcal{F}$ or as $\sigma_{\mathcal{F}} \otimes \sigma_{\mathcal{F}}$-structures in $\mathcal{M}$ (called double categories in $\mathcal{M}$).

1. *Categories in* $\mathcal{F}$.

Let $\mathcal{F}(\mathcal{F})$ be the category of functors in(ternal to) $\mathcal{F}$ (this category has been defined in 0-C).

PROPOSITION 1. *The category $\mathcal{F}_2$ of double functors is equivalent to the category $\mathcal{F}(\mathcal{F})$ and isomorphic to a full sub-category of $\mathcal{F}(\mathcal{F})$.*

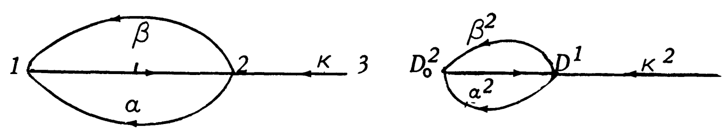
Δ. We are going to construct two canonical equivalences, which will be used later on.

1° a) Let D be a small double category. There exists a unique functor $\eta_{11}(D): \Sigma_{\mathcal{F}} \to \mathcal{F}$ mapping the two distinguished cones γ_1 and γ_2 of $\sigma_{\mathcal{F}}$ on canonical pullbacks in $\mathcal{F}$ and associating to the morphisms ι, α, β and κ of $\sigma_{\mathcal{F}}$ respectively:

the insertion from the sub-category D^2_o of D^1 into D^1,

the functors from D^1 to D^2_o defined by the mappings source and target α^2 and β^2 of D^2.

the functor defined by the law of composition κ^2 of D^2 from the sub-category $(D^2 * D^2)^1$ of $D^1 \times D^1$ on the set of composable pairs of D^2 to D^1.



Hence, $\eta_{11}(D)$ is the unique category in $\mathcal{F}$ such that $\eta_{11}(D)(2)$ is the first category D^1 and that $p_{\mathcal{F}}\,\eta_{11}(D)$ is the category $\eta_1(D^2)$ in $\mathfrak{M}$ associated to the second category D^2 (cf. 0-C-3).

b) If $f: C \to D$ is a double functor, we have a unique functor

$$\eta_{11}(f): \eta_{11}(C) \to \eta_{11}(D)$$

internal to $\mathcal{F}$ such that $\eta_{11}(f)(2) = f^1$.

c) We have so defined a functor η_{11}:

$$f \mapsto \eta_{11}(f) \text{ from } \mathcal{F}_2 \text{ to } \mathcal{F}(\mathcal{F}).$$

It satisfies $p_{\mathcal{F}}^{\sigma_{\mathcal{F}}}\,\eta_{11} = \eta_1$, where $p_{\mathcal{F}}^{\sigma_{\mathcal{F}}}$ is the functor:

$$\tau \mapsto p_{\mathcal{F}}\,\tau \text{ from } \mathcal{F}(\mathcal{F}) \text{ to } \mathcal{F}(\mathfrak{M}).$$

Since η_1 admits as a restriction an isomorphism from $\mathcal{F}$ onto a full sub-category of $\mathcal{F}(\mathfrak{M})$, the functor η_{11} admits as a restriction an isomorphism from $\mathcal{F}_2$ onto the full sub-category $\mathcal{F}(p_{\mathcal{F}})$ of $\mathcal{F}(\mathcal{F})$ whose objects are the categories ϕ in $\mathcal{F}$ such that $p_{\mathcal{F}}\,\phi$ is the category $\eta_1(\Sigma)$ in $\mathfrak{M}$ associated to a category Σ (such a category in $\mathcal{F}$ is called a $p\mathcal{F}$-structured category in [E2]).

2° We define now a functor ζ_{11} from $\mathcal{F}(\mathcal{F})$ onto $\mathcal{F}_2$.

a) Let $\phi: \Sigma_{\mathcal{F}} \to \mathcal{F}$ be a category in $\mathcal{F}$. Then $\phi(2)$ is a category Σ° and $p\mathcal{F}\phi$ is a category in $\mathfrak{M}$; the associated category $\zeta_1(p\mathcal{F}\phi)$ (defined in 0-C-3) is denoted by $\Sigma^{\cdot}$. The pair $(\Sigma^{\circ}, \Sigma^{\cdot})$ is a double category, whose image by η_{11} is a category in $\mathcal{F}$ equivalent to ϕ (by its construction). In particular, noting $(\Sigma^{\circ}, \Sigma^{\cdot})$ by $\zeta_{11}(\phi)$, we have

$$\zeta_{11}(\eta_{11}(C)) = C \quad \text{for each double category } C.$$

b) If $\tau: \phi \to \phi'$ is a functor in $\mathcal{F}$, then

$$\zeta_{11}(\tau) = (\zeta_{11}(\phi'), \tau(2), \zeta_{11}(\phi))$$

is a double functor; in this way we have defined a surjective functor ζ_{11} from $\mathcal{F}(\mathcal{F})$ to $\mathcal{F}$. The composite functor

$$\zeta_{11}\eta_{11}: \mathcal{F}_2 \xrightarrow[\eta_{11}]{} \mathcal{F}(\mathcal{F}) \xrightarrow[\zeta_{11}]{} \mathcal{F}_2$$

is an identity functor, while

$$\eta_{11}\zeta_{11}: \mathcal{F}(\mathcal{F}) \longrightarrow \mathcal{F}(\mathcal{F})$$

is equivalent to an identity. ∇

COROLLARY. *$\mathcal{F}_2$ is equivalent to the category* $\mathfrak{M}^{\sigma\mathcal{F}\otimes\sigma\mathcal{F}}$.

Δ. From Proposition 6-0, we know that $\mathfrak{M}^{\sigma\mathcal{F}\otimes\sigma\mathcal{F}}$ is isomorphic with $(\mathfrak{M}^{\sigma\mathcal{F}})^{\sigma\mathcal{F}}$; this last category is equivalent to $\mathcal{F}^{\sigma\mathcal{F}} = \mathcal{F}(\mathcal{F})$, and therefore to $\mathcal{F}_2$, according to the Proposition. ∇

DEFINITION. If D is a double category, $\eta_{11}(D)$ is called *the category in $\mathcal{F}$ associated to D*. If ϕ is a category in $\mathcal{F}$, then $\zeta_{11}(\phi)$ is called *the double category associated to ϕ*.

2. *The sketch of double categories.*

Since $\mathcal{F}_2$ is equivalent to $\mathfrak{M}^{\sigma\mathcal{F}\otimes\sigma\mathcal{F}}$, it is natural to give the

DEFINITION. The tensor product $\sigma\mathcal{F}\otimes\sigma\mathcal{F}$ is called *the sketch of double* *categories*; it is denoted by $\sigma\mathcal{F}_2$ and its underlying category by $\Sigma\mathcal{F}_2$.- A $\sigma\mathcal{F}_2$-structure (resp. -morphism) in a category $\mathcal{H}$ is called a *double category* (resp. a double *functor) in* $\mathcal{H}$.

The category $\mathcal{H}^{\sigma\mathcal{F}_2}$ of double functors in $\mathcal{H}$ is denoted by $\mathcal{F}_2(\mathcal{H})$. It is equal to $\mathcal{H}^{\bar\sigma\mathcal{F}\otimes\bar\sigma\mathcal{F}}$, since $(\mathcal{H}^{\bar\sigma\mathcal{F}})^{\bar\sigma\mathcal{F}} = (\mathcal{H}^{\sigma\mathcal{F}})^{\sigma\mathcal{F}}$. Proposition 7-0 asserts that $\Sigma\mathcal{F}_2$ is the $\bar\Gamma\mathcal{F}\otimes\bar\Gamma\mathcal{F}$-closure of $\{(2,2)\}$.

PROPOSITION 2. *There exist a surjective equivalence* $\zeta_2 : \mathcal{F}_2(\mathfrak{M}) \to \mathcal{F}_2$ *and an equivalence* $\eta_2 : \mathcal{F}_2 \to \mathcal{F}_2(\mathfrak{M})$ *such that* $\zeta_2\eta_2$ *be an identity.*

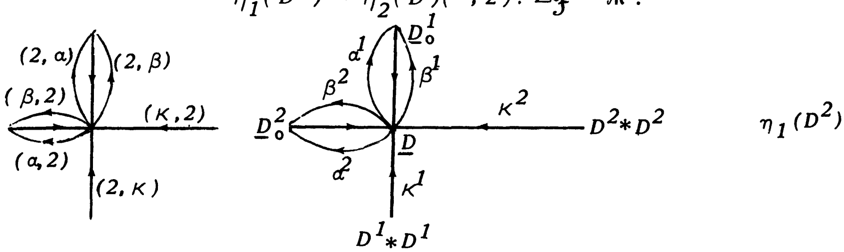
Δ. From Proposition 1 and from 0-C-3, we get the equivalence η_2:

$$\mathcal{F}_2 \xrightarrow{\eta_{11}} \mathcal{F}(\mathcal{F}) \xrightarrow{\eta_{1}^{\sigma\mathcal{F}}} (\mathfrak{M}^{\sigma\mathcal{F}})^{\sigma\mathcal{F}} \xrightarrow{\sim} \mathfrak{M}^{\sigma\mathcal{F}\otimes\sigma\mathcal{F}} = \mathcal{F}_2(\mathfrak{M}).$$

which is constructed as follows:

If D is a small double category, $\eta_2(D)$ is the unique double category in $\mathfrak{M}$ mapping the distinguished cones of $\sigma\mathcal{F}_2$ on canonical pullbacks in $\mathfrak{M}$, mapping the morphisms (ι, n) and (n, ι), for $n \in \{1, 2, 3, 4\}$, on insertions and such that

$$\eta_1(D^1) = \eta_2(D)(2, \cdot) : \Sigma\mathcal{F} \to \mathfrak{M},$$
$$\eta_1(D^2) = \eta_2(D)(\cdot, 2) : \Sigma\mathcal{F} \to \mathfrak{M}.$$



If $f: C \to D$ is a double functor, $\eta_2(f)$ is the unique double functor

$$\tau : \eta_2(C) \to \eta_2(D) \text{ in } \mathfrak{M}$$

such that $\tau(2,2)$ is the map $\underline{f}$ defining f.

We construct now a surjective equivalence $\zeta_2 : \mathcal{F}_2(\mathfrak{M}) \to \mathcal{F}_2$:

If $\phi : \Sigma\mathcal{F}_2 \to \mathfrak{M}$ is a double category in $\mathfrak{M}$, then

$$\phi_1 = \phi(2, \cdot) : \Sigma\mathcal{F} \to \mathfrak{M} \quad \text{and} \quad \phi_2 = \phi(\cdot, 2) : \Sigma\mathcal{F} \to \mathfrak{M}$$

are categories in $\mathfrak{M}$, and the pair of their associated categories

$$\zeta_2(\Phi) = (\zeta_1(\Phi_1), \zeta_1(\Phi_2))$$

is a double category on $\Phi(2,2)$.

If $\tau : \Phi \to \Phi'$ is a double functor in $\mathfrak{M}$, the map $\tau(2,2)$ defines a double functor $\zeta_2(\tau): \zeta_2(\Phi) \to \zeta_2(\Phi')$.

We have so defined the functor $\zeta_2 : \mathfrak{F}_2(\mathfrak{M}) \to \mathfrak{F}_2$.

Since $\zeta_1 \eta_1 : \mathfrak{F} \to \mathfrak{F}$ is an identity, the functor $\zeta_2 \eta_2$:

$$\mathfrak{F}_2 \xrightarrow{\eta_2} \mathfrak{F}_2(\mathfrak{M}) \xrightarrow{\zeta_2} \mathfrak{F}_2$$

is an identity, and $\eta_2(\mathfrak{F}_2)$ defines a full sub-category of $\mathfrak{F}_2(\mathfrak{M})$, isomorphic with $\mathfrak{F}_2$. ∇

3. General results about σ-structures in $\mathfrak{M}$ may be applied to the category $\mathfrak{F}_2$, according to Proposition 2. In particular:

PROPOSITION 3. *1° $\mathfrak{F}_2$ is a category admitting small projective limits and small inductive limits.*

2° The forgetful functor toward $\mathfrak{M}$ as well as the two forgetful functors p_1^2 and p_2^2 toward $\mathfrak{F}$ preserve projective limits and filtered inductive limits.

3° The forgetful functor toward $\mathfrak{M}$ admits quasi-quotient structures, i. e. [E1] if D is a small double category on $\underline{D}$ and r an equivalence on the set $\underline{D}$, there exists a small double category quasi-quotient of D by r.

These results are deduced in [BE1] from general theorems about internal categories (which would also apply to $\mathfrak{F}_2(\mathfrak{H})$).

C. Categories of generalized natural transformations.

If D is a double category and A a category, the functors from A to the first category of 1-morphisms of D are the objects of a category, denoted by $T(D,A)$, whose law is deduced from that of the second category D^2 underlying D. Functors from a category B to $T(D,A)$ may be identified with double functors toward D from the «square product» $B \blacksquare A$.

1. *The functor T_{11}.*

PROPOSITION 4. *There exists a functor $T_{11}: \mathfrak{F}_2 \times \mathfrak{F}^* \to \mathfrak{F}$ mapping the ob-*

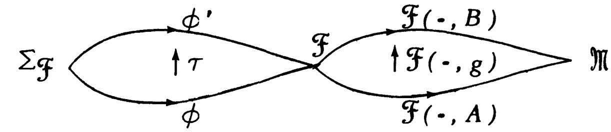
ject (D, A) *onto the category* $T(D, A)$ *got by equipping the set of functors from* A *to* D^1 *with the law:*

$t' \circ_2 t$ *is defined iff* $\alpha^2 \underline{t}' = \beta^2 \underline{t}$ *and is then equal to the functor*
$a \mapsto t'(a) \circ_2 t(a)$ *from* A *to* D^1.

Δ. 1° From Proposition 1-0 there exists a functor

$$\theta : \mathfrak{F}(\mathfrak{F}) \times \mathfrak{F}^* \to \mathfrak{F}(\mathfrak{M})$$

mapping $(\tau, g) \in \mathfrak{F}(\mathfrak{F}) \times \mathfrak{F}$ onto the natural transformation $\mathfrak{F}(\cdot, g) . \tau$:



(where $g : B \to A$). We denote by $T_{1\,1}$ the composite functor:

$$\mathfrak{F}_2 \times \mathfrak{F}^* \xrightarrow{\eta_{1\,1} \times \mathfrak{F}^*} \mathfrak{F}(\mathfrak{F}) \times \mathfrak{F}^* \xrightarrow{\theta} \mathfrak{F}(\mathfrak{M}) \xrightarrow{\zeta_1} \mathfrak{F} .$$

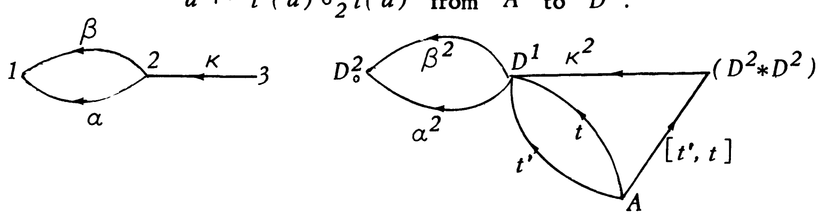
2° Let D be a small double category and A a small category. Then $T_{1\,1}(D, A)$ is the category $T(D, A)$ associated to $\mathfrak{F}(\cdot, A)\ \phi : \Sigma_{\mathfrak{F}} \to \mathfrak{M}$, where ϕ is the category in $\mathfrak{F}$ associated to D:

- Its set of morphisms is $\mathfrak{F}(\phi(2), A) = \mathfrak{F}(D^1, A) = L$.
- Its law $\mathfrak{F}(\phi(\kappa), A)$ is defined on the pullback

$$\mathfrak{F}(\phi(\alpha), A) \vee \mathfrak{F}(\phi(\beta), A) = \{ (t', t) \in L \times L \mid \alpha^2 \underline{t}' = \beta^2 \underline{t} \},$$

and it maps (t', t) onto the functor $\phi(\kappa) . [t', t]$:

$$a \mapsto t'(a) \circ_2 t(a) \text{ from } A \text{ to } D^1 .$$

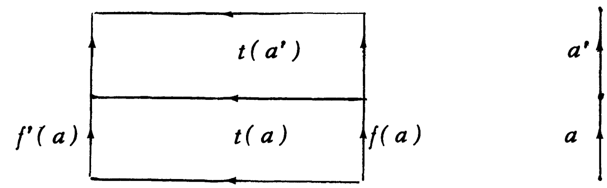


3° Let $h : D \to E$ be a double functor and $g : B \to A$ a functor. $T_{1\,1}(h, g)$ is the functor from $T(D, A)$ to $T(E, B)$ defined by the map

$$\mathfrak{F}(\cdot, g)\, \eta_{1\,1}(h)(2) = \mathfrak{F}(\cdot, g)(h^1) = \mathfrak{F}(h^1, g),$$

which associates $h^1 t g$ to $t \in \mathfrak{F}(D^1, A)$. ∇

DEFINITION. The category $T(D, A)$ defined above is called *the category of D-wise transformations from A to D*. A functor $t: A \to D^1$ is called a *D-wise transformation from f to f'*, if f is its source and f' its target in $T(D, A)$.



$\mathcal{F}(D_o^2, A)$ is the set of objects of $T(D, A)$. This definition, given in [E2] (where $T(D, A)$ was constructed directly), has been inspired by the following example:

EXAMPLES. 1° Let B be a category, $\square B$ the double category of its commutative squares. If A is a category, $T(\square B, A)$ is identified with the category B^A of natural transformations, by identifying a functor from A to $\boxminus B$ (i. e. a $\square B$-wise transformation) with a natural transformation between functors from A to B.

2° For any double category D, the category $T(D, \mathbf{2})$ is isomorphic with D^2.

2. *The square product of categories.*

We are going to construct an adjoint to the «partial» functor

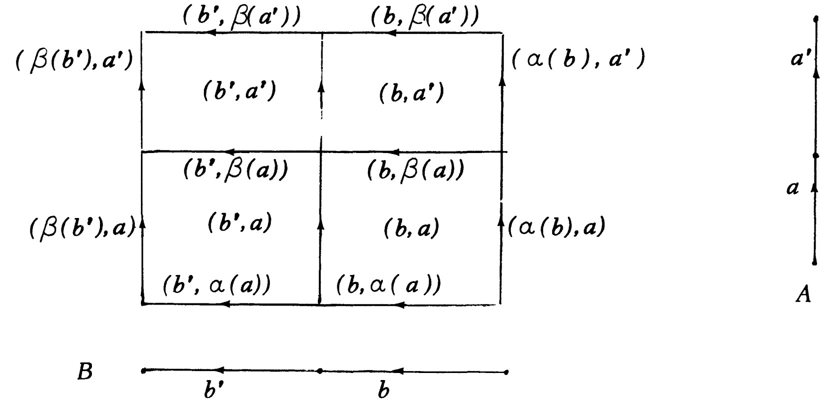
$$T_{11}(-, A): \mathcal{F}_2 \to \mathcal{F}, \text{ for each small category } A.$$

DEFINITION. Let A and B be categories. We call the *square product of* (B, A), denoted by $B \blacksquare A$, the double category $(\underline{B}^0 \times A, B \times \underline{A}^0)$ (where $\underline{A}^0$ and $\underline{B}^0$ are the discrete categories on the sets of morphisms of A and B respectively).

$B \blacksquare A$ is a double category, since it is the product in $\mathcal{F}_2$ of the double categories $(\underline{B}^0, B)$ and $(A, \underline{A}^0)$. Its laws are:

$(b', a') \circ_1 (b, a) = (b, a'.a)$ iff $b' = b$ and $a'.a$ exists in A,

$(b', a') \circ_2 (b, a) = (b'.b, a)$ iff $a' = a$ and $b'.b$ exists in B.



REMARK. If we identify the block (b, a) (sometimes written $b \blacksquare a$) with its frame

$$(\beta(b) \blacksquare a,\ b \blacksquare \beta(a),\ b \blacksquare \alpha(a),\ \alpha(b) \blacksquare a),$$

we get an isomorphism from $B \blacksquare A$ onto a double sub-category of the double category $\square(B \times A)$.

DEFINITION. We say that $(D, \phi, (B, A))$ is an *alternative double functor,* or that ϕ defines an alternative double functor *from* (B, A) *to* D if:

1° A and B are categories on $\underline{A}$ and $\underline{B}$;

2° D is a double category on $\underline{D}$ and $\phi : \underline{B} \times \underline{A} \to \underline{D}$ a map;

3° the partial map $\phi(b, \cdot) : \underline{A} \to \underline{D}$ defines a functor from A to D^1 for every b in B;

4° the partial map $\phi(\cdot, a) : \underline{B} \to \underline{D}$ defines a functor from B to D^2 for every a in A.

PROPOSITION 5. *Let A and B be categories on $\underline{A}$ and $\underline{B}$. The double category $B \blacksquare A$ is characterized by each of the following conditions:*

1° If D is a double category, a map $\phi : \underline{B} \times \underline{A} \to \underline{D}$ defines an alternative double functor from (B, A) to D iff ϕ defines a double functor from $B \blacksquare A$ to D.

2° $B \blacksquare A$ is a free object associated to B relative to the partial functor $T_{11}(\cdot, A) : \mathcal{F}_2 \to \mathcal{F}$.

Δ. 1° Let D be a double category and $\phi : \underline{B} \times \underline{A} \to \underline{D}$ a map.

a) The category $\underline{B}^0 \times A$ being the coproduct category $\coprod_{b \in B} \{b\} \times A$, the map ϕ defines a functor from $\underline{B}^0 \times A$ to D^1 iff the map

$$\phi(b, -): a \mapsto \phi(b, a) \text{ from } \underline{A} \text{ to } \underline{D}$$

defines a functor from A to D^1, for each b in B. In the same way, since $B \times \underline{A}^0 = \coprod_{a \in A} B \times \{a\}$, the map ϕ defines a functor from $B \times \underline{A}^0$ to D^2 iff $\phi(-, a)$ defines a functor from B to D^2 for each a in A. Hence $B \blacksquare A$ satisfies the first property.

b) Suppose that $\phi(b, -)$ defines a functor $f'(b): A \to D^1$ for each b in B. The map

$$f': b \mapsto f'(b) \text{ from } B \text{ to } \mathcal{F}(D^1, A)$$

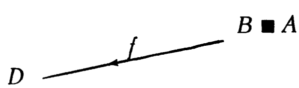
defines a functor from B to $T(D, A)$ iff:

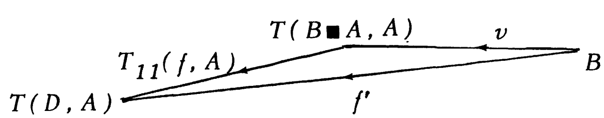
- For each object e of B, $f'(e)$ is an object of $T(D, A)$, which means that $f'(e)(a) = \phi(e, a)$ is an object of D^2, for any a in A.
- For each composite $b' . b$ in B, we have $f'(b' . b) = f'(b') \circ_2 f'(b)$, i. e. $\phi(b' . b, a) = \phi(b', a) \circ_2 \phi(b, a)$ for each a in A.

These conditions are equivalent to say that $\phi(-, a)$ defines a functor from B to D^2 for each a in A. In view of Part a, they are verified iff ϕ defines a double functor from $B \blacksquare A$ to D.

2° By the preceding method, we associate to the identity of $B \blacksquare A$ a functor $v: B \to T(B \blacksquare A, A)$ such that $v(b)$ be the functor

$$a \mapsto (b, a) \text{ from } A \text{ to } (B \blacksquare A)^1, \text{ for each } b \in B.$$





If $f': B \to T(D, A)$ is a functor, it follows from Part 1-b that the map ϕ:

$$(b, a) \mapsto f'(b)(a) \text{ from } \underline{B} \times \underline{A} \text{ to } \underline{D}$$

defines a double functor $f: B \blacksquare A \to D$. Then the functor $T_{11}(f, A).v$, from B to $T(D, A)$, maps b onto the functor

$$T_{11}(f, A)(v(b)) = f^1 v(b) : A \to D^1 .$$

which associates $f(b, a) = f'(b)(a)$ to $a \in A$, and hence is equal to $f'(b)$. So v defines $B \blacksquare A$ as a free object associated to B relative to the functor $T_{11}(\cdot, A)$. ∇

COROLLARY 1. *Let A, B and C be categories. There are bijections*

$$\mathcal{F}_2(\Box C, B \blacksquare A) \to \mathcal{F}(C^A, B) \to \mathcal{F}(C, B \times A).$$

Δ. This results from Proposition 5, since C^A is isomorphic with the category $T(\Box C, A)$. The canonical composite bijection maps $f: B \blacksquare A \to \Box C$ onto the functor g:

$$(b, a) \mapsto f(b, \beta(a)). f(\alpha(b), a) \text{ from } B \times A \text{ to } C. \quad \nabla$$

COROLLARY 2. *Let A and B be categories. If D is a double category, there are canonical bijections:*

$$\mathcal{F}(T(D^{21}, B), A) \xrightarrow{\sim} \mathcal{F}_2(D^{21}, A \blacksquare B) \xrightarrow{\sim} \mathcal{F}_2(D, B \blacksquare A) \xrightarrow{\sim} \mathcal{F}(T(D, A), B).$$

Δ. Since $(B \blacksquare A)^{21} = (B \times \underline{A}^0, \underline{B}^0 \times A)$, there exists an isomorphism

$$h: (b, a) \mapsto (a, b) \text{ from } (B \blacksquare A)^{21} \text{ onto } A \blacksquare B,$$

and

$$\mathcal{F}_2(D^{21}, h): \mathcal{F}_2(D^{21}, A \blacksquare B) \xrightarrow{\sim} \mathcal{F}_2(D^{21}, (B \blacksquare A)^{21})$$

is a bijection l. Now, by sending a double functor from $(B \blacksquare A)^{21}$ to D^{21} onto the functor from $B \blacksquare A$ to D defined by the same map we get a bijection

$$l': \mathcal{F}_2(D^{21}, (B \blacksquare A)^{21}) \xrightarrow{\sim} \mathcal{F}_2(D, B \blacksquare A).$$

From Proposition 5, there are canonical bijections

$$l'': \mathcal{F}(T(D^{21}, B), A) \xrightarrow{\sim} \mathcal{F}_2(D^{21}, A \blacksquare B),$$

$$l''': \mathcal{F}_2(D, B \blacksquare A) \xrightarrow{\sim} \mathcal{F}(T(D, A), B).$$

Composing all these bijections, we get the bijection

$$\gamma_{B,A} = l''' l' l l'': \mathcal{F}(T(D^{21}, B), A) \xrightarrow{\sim} \mathcal{F}(T(D, A), B),$$

which sends the functor $f': A \to T(D^{21}, B)$ onto the functor f'', from B to $T(D, A)$, such that $f''(b)$ be the functor

$$a \mapsto f'(a)(b) \text{ from } A \text{ to } D^1. \quad \nabla$$

COROLLARY 3. $T_{11}(D, \cdot): \mathcal{F}^* \to \mathcal{F}$ *is coadjoint to the dual of* $T_{11}(D^{21}, \cdot)$

for each double category D.

Δ. The canonical bijections $\gamma_{B,A}$ defined above determine an equivalence $\gamma: \mathfrak{F}(T_{11}(D^{21},=),\cdot) \to \mathfrak{F}^*(=,T_{11}(D,\cdot)): \mathfrak{F}^* \times \mathfrak{F}^* \to \mathfrak{M}$. ∇

3. *The functor* $\blacksquare: \mathfrak{F} \times \mathfrak{F} \to \mathfrak{F}_2$.

PROPOSITION 6. *There exists a functor* $\blacksquare$ *from* $\mathfrak{F}\times\mathfrak{F}$ *to* $\mathfrak{F}_2$ *such that the partial functor* $\cdot \blacksquare A$ *be an adjoint of* $T_{11}(\cdot, A)$ *for each small category* A. *If* $\mathfrak{F}_{\mathfrak{c}}$ *denotes the full sub-category of* $\mathfrak{F}$ *whose objects are the small connected categories, then* $\blacksquare$ *maps* $\mathfrak{F}_{\mathfrak{c}} \times \mathfrak{F}_{\mathfrak{c}}$ *onto a full sub-category of* $\mathfrak{F}_2$.

Δ. 1° If $g: A \to A'$ and $h: B \to B'$ are functors, the product map $\underline{h}\times\underline{g}$ defines a double functor $h \blacksquare g: B \blacksquare A \to B' \blacksquare A'$. We so define the functor

$$\blacksquare: (h,g) \mapsto h \blacksquare g \text{ from } \mathfrak{F}\times\mathfrak{F} \text{ to } \mathfrak{F}_2 .$$

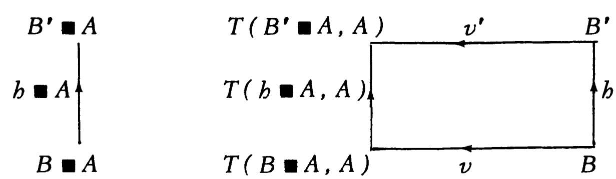
2° The «canonical» adjoint of $T_{11}(\cdot, A): \mathfrak{F}_2 \to \mathfrak{F}$ maps $h: B \to B'$ onto the double functor $h': B \blacksquare A \to B' \blacksquare A$ associated to the functor $v'h$, where

$$v: B \to T(B \blacksquare A, A) \text{ and } v': B' \to T(B' \blacksquare A, A),$$

are the functors defining $B \blacksquare A$ and $B' \blacksquare A$ as free objects. As $v'h$ maps $b \in B$ onto the functor

$$a \mapsto (h(b), a) \text{ from } A \text{ to } \underline{B}'^0 \times A,$$

the functor h' maps (b,a) onto $(h(b),a)$, and $h' = h \blacksquare A$. Hence the partial functor $\cdot \blacksquare A: \mathfrak{F} \to \mathfrak{F}_2$ is the canonical adjoint of $T_{11}(\cdot, A)$.



3° Let A, B, A' and B' be small connected categories and suppose that $f: B \blacksquare A \to B' \blacksquare A'$ is a double functor. Since A and A' are connected, the components

- of $\underline{B}^0 \times A$ are the sets $\{b\}\times\underline{A}$, where $b \in B$,
- of $\underline{B}'^0 \times A'$ are the sets $\{b'\}\times\underline{A}'$, where $b' \in B'$.

The functor $f^1: \underline{B}^0 \times A \to \underline{B}'^0 \times A'$ mapping a component into a component, for each $b \in B$, there exists a unique $b'_b \in B'$ such that

$$f(\{b\} \times \underline{A}) \subset \{b'_b\} \times \underline{A}'.$$

In the same way $\underline{f}$ defining a functor $f^2 : B \times \underline{A}^0 \to B' \times \underline{A}'^0$, for each $a \in A$ there exists a unique $a'_a \in A'$ such that

$$f(\underline{B} \times \{a\}) \subset \underline{B}' \times \{a'_a\} .$$

Hence

$$f(b, a) = (b'_b, a'_a) \text{ for each } (b, a) \in B \times A,$$

which implies $\underline{f} = \underline{h} \times \underline{g}$, where the map

$\underline{h} : \underline{B} \to \underline{B}'$ associates b'_b to $b \in B$,

$\underline{g} : \underline{A} \to \underline{A}'$ associates a'_a to $a \in A$.

These maps define functors $h : B \to B'$ and $g : A \to A'$, and $f = h \blacksquare g$. ∇

D. Some applications.

1. *The canonical double cocategory in* $\mathcal{F}_2$.

PROPOSITION 7. *1° There exists a double cocategory* ι_2 *in* $\mathcal{F}_2$ *which defines an isomorphism from* $\Sigma \mathcal{F}_2^*$ *onto the full sub-category* $\overline{\Sigma} \mathcal{F}_2^*$ *of* $\mathcal{F}_2$ *whose objects are the double categories* $\mathbf{m} \blacksquare \mathbf{n}$, *for* m *and* n *in* $\{1, 2, 3, 4\}$.

2° $\mathcal{F}_2$ *is the inductive closure of* $\{\mathbf{2} \blacksquare \mathbf{2}\}$.

Δ. 1° From Proposition 4-0, the restriction $\overline{Y}_2 : \Sigma \mathcal{F}_2^* \to \mathcal{F}_2(\mathfrak{M})$ of the Yoneda embedding is a double cocategory in $\mathcal{F}_2(\mathfrak{M})$. The composite

$$\iota_2 : \Sigma \mathcal{F}_2^* \xrightarrow{\overline{Y}_2} \mathcal{F}_2(\mathfrak{M}) \xrightarrow{\zeta_2} \mathcal{F}_2,$$

where ζ_2 is the canonical equivalence (Proposition 2), is a double cocategory in $\mathcal{F}_2$, and $\overline{Y}_2$ defines an isomorphism from $\Sigma \mathcal{F}_2^*$ onto a full sub-category of $\mathcal{F}_2(\mathfrak{M})$ which is mapped by the surjective equivalence ζ_2 onto a full sub-category of $\mathcal{F}_2$. The equivalence ζ_2 being faithful, so is ι_2.

2° We are going to prove that

$$\mathbf{m} \blacksquare \mathbf{n} = \iota_2(m, n), \text{ for } m \text{ and } n \text{ in } \{1, 2, 3, 4\}.$$

Indeed, $\Sigma \mathcal{F}_2 = \Sigma \mathcal{F} \times \Sigma \mathcal{F}$, so that $\overline{Y}_2(m, n) : \Sigma \mathcal{F}_2 \to \mathfrak{M}$ maps the pair

$$(\mu, \nu) \in \Sigma \mathcal{F}_2 \text{ onto}$$

$$\Sigma\mathcal{F}_2((\mu,\nu),(m,n)) = \Sigma\mathcal{F}(\mu,m)\times\Sigma\mathcal{F}(\nu,n) =$$
$$= Y_1(m)(\mu)\times Y_1(n)(\nu),$$

where $Y_1 : \Sigma\mathcal{F}^* \to \mathfrak{M}^{\Sigma\mathcal{F}}$ is the Yoneda embedding. From the construction of ζ_2 it follows that $\iota_2(m,n)$ is the double category

$$(\zeta_1(Y_1(m)(2)\times Y_1(n)(\cdot)),\ \zeta_1(Y_1(m)(\cdot)\times Y_1(n)(2))),$$

where $\zeta_1 : \mathcal{F}(\mathfrak{M}) \to \mathcal{F}$ is the canonical equivalence (Proposition 9-0). The set $Y_1(m)(2)$ is the set $\underline{\mathbf{m}}$ underlying $\mathbf{m}$ and $\zeta_1(Y_1(n))$ is the category $\mathbf{n}$ (see 0-C), so that

$$\zeta_1(Y_1(m)(2)\times Y_1(n)(\cdot)) = \underline{\mathbf{m}}^0\times\mathbf{n}.$$

In the same way

$$\zeta_1(Y_1(m)(\cdot)\times Y_1(n)(2)) = \mathbf{m}\times\underline{\mathbf{n}}^0.$$

Hence

$$\iota_2(m,n) = (\underline{\mathbf{m}}^0\times\mathbf{n},\ \mathbf{m}\times\underline{\mathbf{n}}^0) = \mathbf{m}\blacksquare\mathbf{n}.$$

3° The preceding results imply that ι_2 maps $\Sigma\mathcal{F}_2$ onto the full subcategory of $\mathcal{F}_2$ whose objects are the double categories $\mathbf{m}\blacksquare\mathbf{n}$. As

$$\mathbf{m}\blacksquare\mathbf{n} = \mathbf{m}'\blacksquare\mathbf{n}' \text{ iff } m = m' \text{ and } n = n',$$

the faithful functor ι_2 is injective on the objects, whence injective.

4° $\Sigma\mathcal{F}$ being the $\overline{\Gamma}\mathcal{F}$-closure of $\{2\}$ (see 0-C), Proposition 7-0 asserts that $\mathcal{F}_2(\mathfrak{M}) = \mathfrak{M}^{\sigma\mathcal{F}\otimes\sigma\mathcal{F}} = \mathfrak{M}^{\overline{\sigma}\mathcal{F}\otimes\overline{\sigma}\mathcal{F}}$ is the inductive closure of the set $\{\overline{Y}_2(2,2)\}$. The image $\mathcal{F}_2$ of $\mathcal{F}_2(\mathfrak{M})$ by the equivalence ζ_2 is then the inductive closure of the set whose unique element is

$$\zeta_2(\overline{Y}_2(2,2)) = \iota_2(2,2) = \mathbf{2}\blacksquare\mathbf{2}. \quad \nabla$$

2. *Generalized limits.*

By analogy with the usual definition of a limit for a functor we define limits relative to D for functors toward the first category of 1-morphisms of the double category D.

Let D be a double category and A a category. The category $T(D,A)$ of D-wise transformations (see C-1) admits for objects the functors from

A to the first category D_0^2 of 1-morphisms of D.

The alternative double functor

$$(x, a) \mapsto x \text{ from } (D_0^1, A) \text{ to } D$$

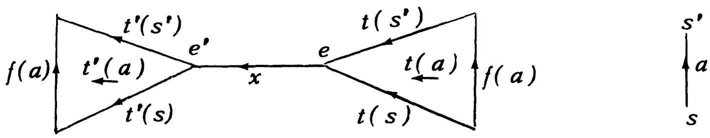
determines a functor $d_{DA}: D_0^1 \to T(D, A)$ (Proposition 5-C). This functor maps

- the vertex e of D onto the constant functor $e\hat{}: A \to D_0^2$,
- the morphism $x: e \rightrightarrows e'$ of D_0^1 onto the constant functor $x\hat{}: A \to D^1$,

which is a D-wise transformation form $e\hat{}$ to $e'\hat{}$.

As for natural transformations, we will use a more «geometrical» language: Let $f: A \to D_0^2$ be a functor.

- If $t: f \to e\hat{}$ is a D-wise transformation toward a constant functor, we say that t is an *inductive D-wise cone, indexed by A, with vertex e and basis f.*
- A D-wise transformation $t': e'\hat{} \to f$ is called *a projective D-wise cone with vertex e' and basis f.*



- Let $x: e \rightrightarrows e'$ be a morphism of D_0^1. If $t: f \to e\hat{}$ is an inductive D-wise cone, we denote by xt the inductive D-wise cone

$$x\hat{} \circ_2 t: f \to e'\hat{} \text{ such that } xt(a) = x \circ_2 t(a) \text{ for each } a \in A.$$

Dually, if $t': e'\hat{} \to f$ is a projective D-wise cone, then $t'x: e\hat{} \to f$ is the projective D-wise cone $t' \circ_2 x\hat{}$ such that

$$(t'x)(a) = t'(a) \circ_2 x \text{ for each } a \in A.$$

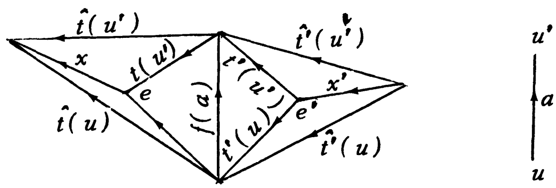
DEFINITION. Let $f: A \to D_0^2$ be a functor. If $t: f \to e\hat{}$ (resp. $t: e\hat{} \to f$) is a D-wise cone defining e as a free (resp. a cofree) object generated by f relative to the functor d_{DA}, then e is called an *inductive* (resp. a *projective*) *D-wise limit of f* and t is called an *inductive* (resp. a *projective*) *D-wise limit-cone.*

REMARKS. Limits relative to a double category were introduced by Ehresmann in [E2] and some general properties of these limits are given in [Le].

Quasi-limits of Gray [G1], analimits and catalimits of Bourn [Bo] are examples of such limits which will be studied later on.

Let $f: A \to D_o^2$ be a functor. The inductive D-wise cone t with vertex e and basis f is a D-wise limit-cone iff, for each inductive D-wise cone $\hat{t}$ with basis f, there exists a unique morphism x in D_o^1, called the *factor of* $\hat{t}$ *relative to* t, such that $\hat{t} = x\,t$.

The projective D-wise cone t' with basis f is a D-wise limit-cone iff, for each projective D-wise cone $\hat{t}'$ with basis f, there exists a unique morphism x' in D_o^1, called the *factor of* $\hat{t}'$ *relative to* t', such that $\hat{t}' = t'x'$.



The terminology is justified by the following examples.

EXAMPLES. 1° If B is a category and $\square B$ the double category of its commutative squares, a functor $f: A \to B$ admits a projective (resp. an inductive) $\square B$-wise limit e iff e is a (usual) projective (resp. inductive) limit of f. Indeed, if we identify $T(\square B, A)$ with B^A and B with the second category of 1-morphisms of $\square B$, the functor $d_{\square BA}$ is identified with the «diagonal» functor from B to B^A.

2° If I^0 is the discrete category on I, a projective D-wise cone t indexed by I^0 and with vertex e is identified with the family $(t(i))_{i \in I}$ of 1-morphisms $t(i): e \rightrightarrows e_i$ of D; hence t is a D-wise limit-cone iff e is a product of $(e_i)_{i \in I}$ in D_o^1, the $t(i)$'s being the projections.

3° If D^* is the double category (D^1, D^{2*}) which is the second dual of D, then $T(D^*, A)$ is the dual of $T(D, A)$, so that a D-wise cone t is an inductive D-wise limit-cone iff t is a projective D^*-wise limit-cone.

DEFINITION. We say that D_o^2 *admits inductive* (resp. *projective*) *D-wise A-limits* if d_{DA} admits an adjoint (resp. a coadjoint), which is then called a *D-wise A-limit functor.* If D_o^2 admits inductive (resp. projective) D-wise limits for each small (or finite,...) category A, we say that D_o^2 *admits inductive* (resp. *projective) D-wise small* (or *finite,...*) *limits.*

2. REPRESENTABLE DOUBLE CATEGORIES

We are going to study the double categories D whose first category of 1-morphisms D_0^2 admits D-wise **2**-limits. For them, the existence of D-wise limits reduces to the existence of «enough» usual limits in D_0^1. Fundamental examples of such double categories are the double categories of squares of a representable (in the sense of Gray) 2-category.

In all this chapter, we denote by D a double category, by « $\circ$» and «.» respectively its first and its second law.

A. Representation of a 1-morphism.

DEFINITION. The double category D is said *representable* (resp. *corepresentable*) if D_0^2 admits projective (resp. inductive) D-wise **2**-limits.

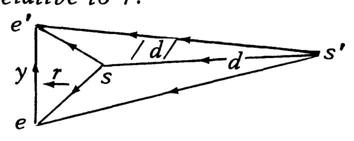
D is corepresentable iff its second dual is representable.

Let $v: T(D, \mathbf{2}) \to D^2$ be the canonical isomorphism mapping t onto $t(z)$, where z always denotes the morphism from 0 to 1 in **2**. The composite functor

$$D_0^1 \xrightarrow{d_{D\mathbf{2}}} T(D, \mathbf{2}) \xrightarrow{v} D^2$$

is the insertion into D^2 of its sub-category D_0^1. Hence D is representable (resp. corepresentable) iff the insertion $D_0^1 \hookrightarrow D^2$ admits a coadjoint (resp. an adjoint). In particular, for 2-categories, these definitions are equivalent to that given by Gray [G2].

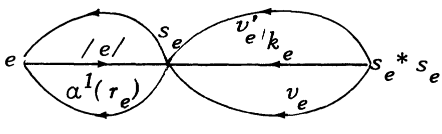
Let D be a representable double category. If $y: e \to e'$ is a morphism of D_0^2 and if $r: s \rightrightarrows y$ is a 2-block of D defining s as a cofree object generated by y relative to the insertion $D_0^1 \hookrightarrow D^2$, we call r *a representation of y in D*. If $d: s' \rightrightarrows y$ is a 2-block, there exists a unique 1-morphism $x: s' \rightrightarrows s$ such that $r.x = d$; this x is denoted by $/d/$ and called the *factor of d relative to r*.



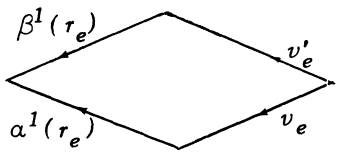
If $r_e : s_e \rightrightarrows e$ is a representation of a vertex e in D, it is also the representation of e in the greatest sub-2-category C of D; it follows that C is also representable. From Gray's results [G2], we know that, if D_0^1 (which is the category of 1-morphisms of C) admits pullbacks, there exists a category ϕ_e in D_0^1 mapping the morphisms $\alpha, \beta, \iota, \nu, \nu', \kappa$ of $\Sigma\mathcal{F}$ respectively on:

$a_e = \alpha^1(r_e)$, $b_e = \beta^1(r_e)$,

the factor $/e/$ relative to r_e,



the canonical projections v_e and v'_e of the pullback in D_0^1:



the factor $k_e = /(r_e . v_e) \circ (r_e . v'_e)/$ relative to r_e.

Indeed, let $D(e, \cdot)$ be the functor from the dual of D_0^1 to $\mathcal{F}$ which maps:

the vertex s of D on the sub-category of D^1 defined by $D^2(e, s)$,

the morphism $x : s \rightrightarrows s'$ of D_0^1 on the functor $c \mapsto c . x$ from $D(e, s')$ to $D(e, s)$.

The functor $p_{\mathcal{F}} D(e, \cdot)$ is equal to $D^2(e, \cdot)$ which, as r_e is a representation of e in D, is equivalent to $D_0^1(s_e, \cdot)$, whence representable. Then Proposition 8-0 associates to $D(e, \cdot)$ a category in D_0^1, which is ϕ_e.

PROPOSITION 1. *Let D be a representable double category such that D_0^1 admits pullbacks. If $y : e \to e'$ is a morphism of D^2 and if ϕ_e and $\phi_{e'}$ are the categories in D_0^1 associated above to e and e', there exists a distributor in D_0^1:*

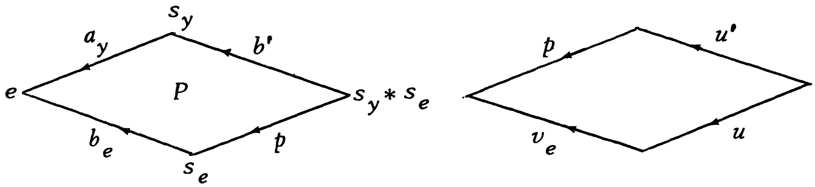
$$\delta_y = (\phi_{e'}, b_y, k'', k', a_y, \phi_e),$$

where $r : a_y \to b_y : s_y \rightrightarrows y$ is a representation of y.

Δ. 1° We get a right category action (k', a_e, ϕ_e) in D_0^1 by considering the factor relative to r:

$$k' = /(r . b') \circ (r_e . p)/$$

and the pullbacks in D_0^1



a) This action k' is unitary. Indeed, if i' is the factor $[s_y, /e/.\, a_y]$ relative to the pullback P, then $k'.\, i' = s_y$ follows from the equalities

$$r.\, k'.\, [s_y, /e/.\, a_y] = ((r.\, b') \circ (r_e.\, p)).\, [s_y, /e/.\, a_y] =$$

$$= r \circ (r_e./e/.a_y) = r \circ a_y = r$$

and from the unicity of the factor relative to r.

b) To show the associativity of the action, we consider the factors

$$c = [\, b'.\, u',\ k_e.\, u\,] \quad \text{and} \quad c' = [\, k'.\, u',\ v'_e.\, u\,]$$

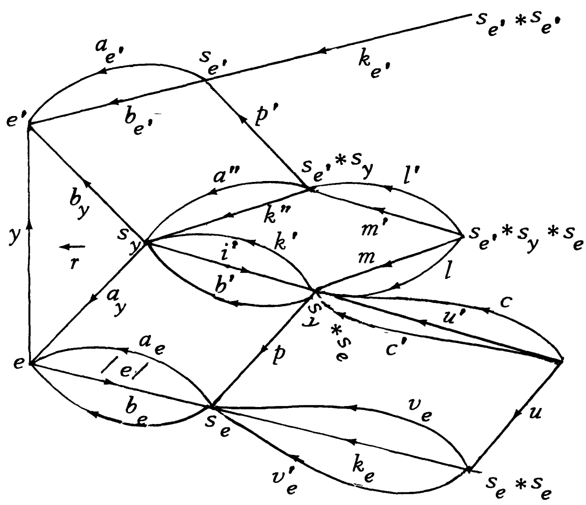
relative to the pullback P, and we have to prove that $k'.\, c = k'.\, c'$. This is deduced from the equalities:

$$r.\, k'.\, c = ((r.\, b') \circ (r_e.\, p)).\, [\, b'.\, u',\ k_e.\, u\,]$$

$$= (r.\, b'.\, u') \circ (r_e.\, k_e.\, u)$$

$$= (r.\, b'.\, u') \circ ((r_e.\, v_e) \circ (r_e.\, v'_e)).\, u$$

$$= (r.\, b'.\, u') \circ (r_e.\, v_e.\, u) \circ (r_e.\, v'_e.\, u)),$$



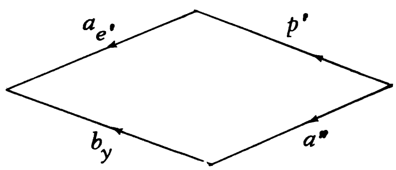
$$r.k'.c' = ((r.b') \circ (r_e . p)). [k'.u', v'_e . u]$$
$$= (r.k'.u') \circ (r_e . v'_e . u)$$
$$= (((r.b') \circ (r_e . p)).u') \circ (r_e . v'_e . u)$$
$$= (r.b'.u') \circ (r_e . v_e . u) \circ (r_e . v'_e . u) = r.k'.c,$$

since $p.u' = v_e . u$.

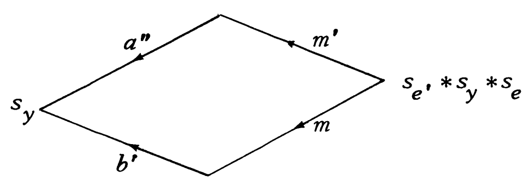
2° A similar proof shows that $(\phi_{e'}, b_y, k'')$ is a category action in D_0^1, where k'' is the factor

$$k'' = /(r_{e'} . p') \circ (r . a'')/$$

relative to r and where we have the pullback P' in D_0^1 :



3° For $(\phi_{e'}, b_y, k'', k', a_y, \phi_e)$ to be a distributor, it remains to prove the «compatibility» of the two actions, i.e. $k''.l' = k'.l$, where



is a pullback and where

$$l' = [p'.m', k'.m] \quad \text{and} \quad l = [k''.m', p.m]$$

are the factors relative to the pullbacks P' and P . Indeed, we get the equalities:

$$r.k''.l' = ((r_{e'} . p') \circ (r . a'')). [p'.m', k'.m]$$
$$= (r_{e'} . p'.m') \circ (r.k'.m)$$
$$= (r_{e'} . p'.m') \circ (((r.b') \circ (r_e . p)).m)$$
$$= (r_{e'} . p'.m') \circ (r.b'.m) \circ (r_e . p.m),$$

$$\begin{aligned} r.k'.l &= ((r.b')\circ(r_e.p)).\,[k''.m', p.m] \\ &= (r.k''.m')\circ(r_e.p.m) \\ &= (((r_{e'}.p')\circ(r.a'')).m')\circ(r_e.p.m) \\ &= (r_{e'}.p'.m')\circ(r.a''.m')\circ(r_e.p.m) = r.k''.l', \end{aligned}$$

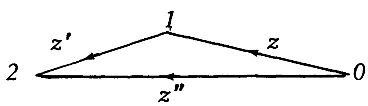
since $b'.m = a''.m'$. ∇

REMARK. δ_y may be defined as the distributor in D_0^1 associated (p. 24) to the canonical functor ψ from the dual of D_0^1 to the category $\mathfrak{M}^{\sigma}\delta$ which maps the vertex s on the distributor $(D(e',s), \beta_{y,s}, \kappa''_s, \kappa'_s, \alpha_{y,s}, D(e,s))$, where κ'_s and κ''_s are restrictions of the law of D^1 and where $\alpha_{y,s}$ and $\beta_{y,s}$ are the maps from $D^2(y,s)$ to $D_0^1(e,s)$ and $D_0^1(e',s)$ restrictions of α^1 and β^1. Indeed, $\psi(\cdot)(1,0)$ is represented by s_y, $\psi(\cdot)(\hat{2},1)$ and $\psi(\cdot)(2,1)$ by s_e and $s_{e'}$ respectively.

B. Existence of limits relative to a representable double category.

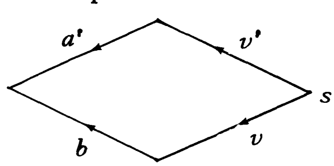
PROPOSITION 2. *If D is a representable double category and if D_0^1 admits pullbacks, then D_0^2 admits projective D-wise **3**-limits.*

Δ. We denote by z, z' and z'' the morphisms of the category **3**:



Functors from **3** to a category A are in bijection with pairs of composable morphisms of A. Let $f\colon \mathbf{3} \to D_0^2$ be a functor.

1° Let $r\colon a \to b$ and $r'\colon a' \to b'$ be representations of $f(z)$ and $f(z')$ in D. By hypothesis there exists a pullback

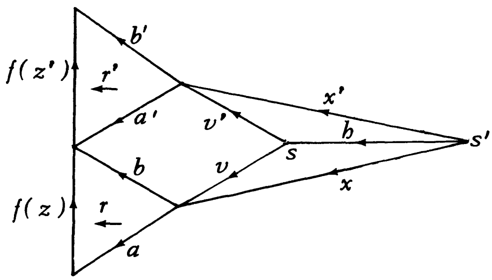


P of (a', b) in D_0^1. We put

$$t(z) = r.v : s \rightrightarrows f(z) \quad \text{and} \quad t(z') = r'.v' : s \rightrightarrows f(z').$$

Since

$$\alpha^1(t(z')) = \alpha^1(r'.v') = \alpha^1(r').v' = a'.v' = b.v = \beta^1(r.v) = \beta^1(t(z)),$$



there exists a composite $t(z'') = t(z') \circ t(z)$ in D^1. We have so defined a D-wise cone t with vertex s and basis f.

2° t is a limit-cone. Indeed, let t' be a projective D-wise cone with basis f and vertex s'. The 2-block $t'(z): s' \rightrightarrows f(z)$ admits a factor x relative to r and $t'(z'): s' \rightrightarrows f(z')$ admits a factor x' relative to r'. Since

$$a'.x' = \alpha^1(r').x' = \alpha^1(r'.x') = \alpha^1(t'(z')) = \beta^1(t'(z)) = b.x,$$

there exists a factor $h = [x', x]$ relative to P. From the equalities

$$t(z).h = r.v.h = r.x = t'(z),$$

$$t(z').h = r'.v'.h = r'.x' = t'(z'),$$

$$t(z'').h = (t(z') \circ t(z)).h = (t(z').h) \circ (t(z).h) =$$
$$= t'(z') \circ t'(z) = t'(z''),$$

we deduce that h is the unique morphism of D^1_0 satisfying $th = t'$. ∇

COROLLARY. *If D is a corepresentable double category and if D^1_0 admits pushouts, then D^2_0 admits D-wise inductive* **3***-limits.*

Δ. This results from Proposition 2 applied to the second dual of D, which is representable. ∇

PROPOSITION 3. *Let D be a representable double category and A a small (resp. a finite) category. If D^1_0 admits small (resp. finite) projective limits, then D^2_0 admits projective D-wise A-limits.*

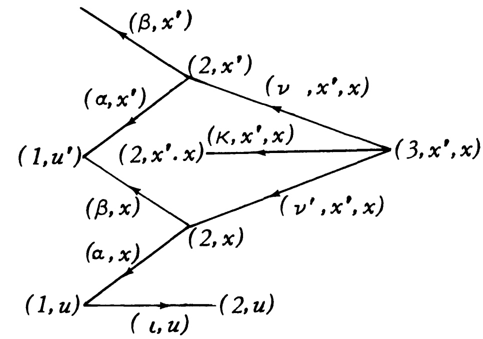
Δ. 1° *Some notations.*

a) The category H: Let $\psi_A: \Sigma\mathcal{J} \to \mathfrak{M}$ be the category in $\mathfrak{M}$ associated to A and ψ its restriction to the sub-category Σ of $\Sigma\mathcal{J}$ generated by $\{\alpha, \beta, \iota, \kappa, \nu, \nu'\}$. We denote by H the source of the discrete fibration $\eta: H \to \Sigma$ associated to ψ. Then H is generated by the morphisms:

(ν, x', x), (ν', x', x), (κ, x', x) from $(3, x', x)$ to: $(2, x')$, $(2, x)$ and $(2, x'.x)$ respectively, where (x', x) is any pair of composable morphisms of A,

(α, x) and (β, x) from $(2, x)$ to $(1, u)$ and $(1, u')$, where x is any morphism in A, from u to u',

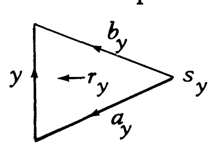
$(\iota, u): (1, u) \to (2, u)$, for any object u of A.



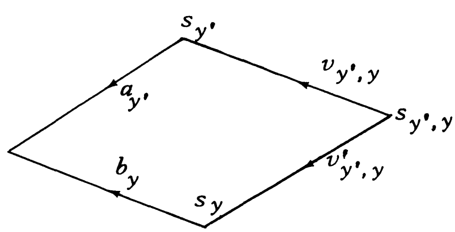
Since $\Sigma\mathcal{J}$ is a finite category, H is small or finite when so is A.

b) For each morphism y in D_0^2, we choose a representation of y in D:

$r_y: a_y \to b_y: s_y \rightrightarrows y$



and for each pair (y', y) of composable morphisms of D_0^1, we choose a pullback $P_{y',y}$ of $(a_{y'}, b_y)$ in D_0^1:



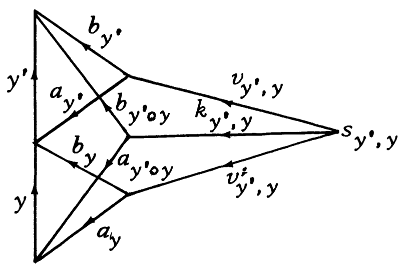
(This pullback exists, pullbacks being finite projective limits.) Since

$$\beta^1(r_y . v'_{y',y}) = \beta^1(r_y). v'_{y',y} = b_y . v'_{y',y} = a_{y'}. v_{y',y} = \alpha^1(r_{y'} . v_{y',y}),$$

there exists a composite

$$(r_{y'} . v_{y',y}) \circ (r_y . v'_{y',y}) : s_{y',y} \rightrightarrows y' \circ y$$

in D^1, and it admits a factor relative to $r_{y' \circ y}$, which will be denoted by $k_{y',y}$.



2° Let $f: A \to D^2_0$ be a functor. We are going to construct a projective D-wise cone t with basis f.

a) There exists a functor $p: H \to D^1_0$ defined as follows: it maps
(α, x) and (β, x) on $a_{f(x)}$ and on $b_{f(x)}$, for each x in A,
(ι, u) on the factor $/f(u)/$ relative to $r_{f(u)}$, for each $u \in A_0$,
(κ, x', x) on $k_{y',y}$, (ν, x', x) on $v_{y',y}$ and (ν', x', x) on $v'_{y',y}$,
for each pair (x', x) of composable morphisms of A, where $y = f(x)$ and $y' = f(x')$.

Since H is small (resp. finite), there exists a projective limit-cone l with basis p and vertex s (in the usual meaning).

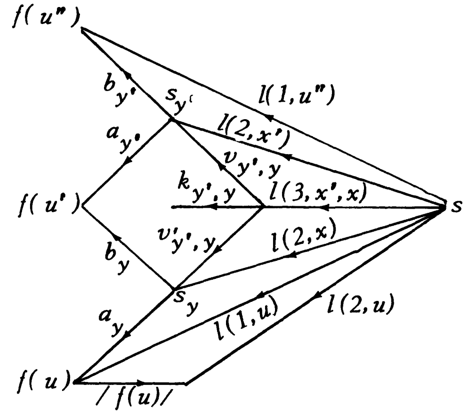
b) For each morphism $x: u \to u'$ in A, we define $t(x) = r_{f(x)} . l(2,x)$. The map associating $t(x)$ to x in A defines a functor $t: A \to D^1$. Indeed, if u is an object of A, we get

$$t(u) = r_{f(u)} . l(2,u) = r_{f(u)} ./f(u)/ . l(1,u) = l(1,u) \in D^1_0,$$

since, l being a cone with basis p, we have

$$l(2,u) = p(\iota, u). l(1,u) = /f(u)/ . l(1,u).$$

On the other hand, if $x: u \to u'$ and $x': u' \to u''$ are morphisms of A and if



$y = f(x)$ and $y' = f(x')$, the equalities

$$\alpha^1(t(x')) = \alpha^1(r_{y'}).l(2,x') = a_{y'}.l(2,x') = p(\alpha, x').l(2,x') = l(1,u')$$

and

$$\beta^1(t(x)) = b_y.l(2,x) = p(\beta, x).l(2,x) = l(1,u') = \alpha^1(t(x'))$$

imply that the composite $t(x') \circ t(x)$ is defined in D^1. From

$$l(2,x') = p(\nu, x', x).l(3,x',x) = v_{y',y}.l(3,x',x),$$

$$l(2,x) = p(\nu', x', x).l(3,x',x) = v'_{y',y}.l(3,x',x),$$

$$l(2,x'.x) = p(\kappa, x', x).l(3,x',x) = k_{y',y}.l(3,x',x)$$

and from the definition of $k_{y',y}$ as a factor relative to $r_{y',y}$, we deduce

$$t(x') \circ t(x) = (r_{y'}.l(2,x')) \circ (r_y.l(2,x)) =$$

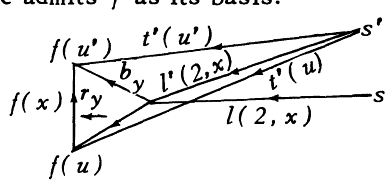
$$= ((r_{y'}.v_{y',y}) \circ (r_y.v'_{y',y})).l(3,x',x) = r_{y' \circ y}.k_{y',y}.l(3,x',x) =$$

$$= r_{y' \circ y}.l(2,x'.x) = t(x'.x).$$

Hence $t: A \to D^1$ is a D-wise cone. As

$$t(x) = r_{f(x)}.l(2,x): s \rightrightarrows f(x),$$

this D-wise cone admits f as its basis.



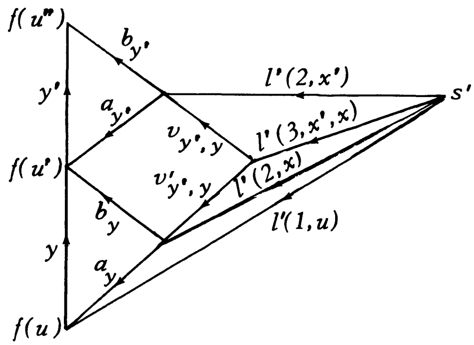
3° We are going to prove that t is a D-wise limit-cone. For this, we suppose that t' is a projective D-wise cone with vertex s' and basis f.

a) We first construct a (usual) cone l' with vertex s' and basis p as follows:

for each object u of A, we define $l'(1, u)$ as the 1-morphism

$$l'(1, u) = t'(u) : s' \rightrightarrows f(u);$$

$l'(2, x)$, for each morphism x of A, is the factor of the 2-block $t'(x) : s' \rightrightarrows f(x)$ relative to $r_{f(x)}$.



If $x : u \to u'$ and $x' : u' \to u''$ are composable morphisms of A and if $y = f(x)$ and $y' = f(x')$, we have

$$a_{y'} . l'(2, x') = \alpha^1 (r_{y'} . l'(2, x')) = \alpha^1 (t'(x')) =$$
$$= \beta^1 (t(x)) = b_y . l'(2, x),$$

so that there exists a factor $l'(3, x', x) = [l'(2, x'), l(2, x)]$ relative to the pullback $P_{y',y}$.

b) We prove now that in this way we get a cone l' with vertex s' and basis p. Indeed:

If $x : u \to u'$ in A, then

$$p(\alpha, x) . l'(2, x) = a_{f(x)} . l'(2, x) = \alpha^1 (t'(x)) = t'(u) = l'(1, u),$$
$$p(\beta, x) . l'(2, x) = b_{f(x)} . l'(2, x) = \beta^1 (t'(x)) = t'(u') = l'(1, u') .$$

If u is an object of A, we get

$$p(\iota, u) . l'(1, u) = /f(u)/ . t'(u) = /f(u) . t'(u)/ = \surd t'(u)/ = l'(2, u)$$

the factors being relative to $r_{f(u)}$.

Let $x : u \to u'$ and $x' : u' \to u''$ be composable morphisms of A and write $y = f(x)$ and $y' = f(x')$. By definition of $l'(3, x', x)$, we have

$$p(\nu', x', x) . l'(3, x', x) = v'_{y', y} . l'(3, x', x) = l(2, x),$$

$$p(\nu , x', x) . l'(3, x', x) = v_{y', y} . l'(3, x', x) = l(2, x').$$

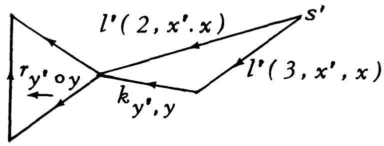
Finally, $p(\kappa, x', x) . l'(3, x', x)$ and $l'(2, x'.x)$ are equal, since both are factors relative to $r_{y' \circ y}$ of

$$r_{y' \circ y} . p(\kappa, x', x) . l'(3, x', x) = r_{y' \circ y} . k_{y', y} . l'(3, x', x) =$$

$$= ((r_{y'} . v_{y', y}) \circ (r_y . v'_{y', y})) . l'(3, x', x) =$$

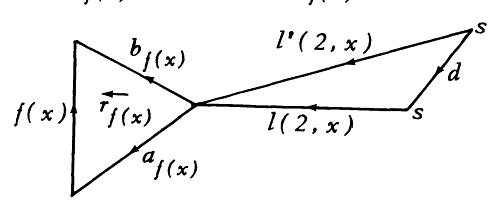
$$= (r_{y'} . l'(2, x')) \circ (r_y . l'(2, x)) = t'(x') \circ t'(x) =$$

$$= t'(x' . x) = r_{y' \circ y} . l'(2, x' . x) .$$



c) The projective usual cone l' with basis p admits a factor $d : s' \rightrightarrows s$ relative to the limit-cone l, so that $l' = l d$. For x in A, we have

$$t(x) . d = r_{f(x)} . l(2, x) . d = r_{f(x)} . l'(2, x) = t'(x).$$



Hence d is the unique morphism of D_0^1 satisfying the equality $t' = t d$. This ends the proof. ∇

More precisely, we have proved:

COROLLARY 1. *If D is a representable double category, A a category, and if D_0^1 admits pullbacks and projective H-limits (where H is the category defined in the preceding proof), then D_0^2 admits projective D-wise A-limits.*

COROLLARY 2. *If D is a representable double category and if D_0^1 admits* *small connected projective limits (resp. pullbacks and equalizers), then* D_0^2 *admits small (resp. finite) connected projective D-wise limits.*

Δ. If A is connected, $\Sigma_{\mathcal{F}}$ being connected it is easily seen that H is also connected. Now a category admitting pullbacks and equalizers has connected finite projective limits. So Corollary 1 implies Corollary 2. ∇

By duality, it follows from Proposition 3:

PROPOSITION 4. *If D is a corepresentable double category and if D_0^1 admits small (resp. finite) inductive limits, then D_0^2 admits small (resp. finite) inductive D-wise limits.*

COROLLARY. *If D is a corepresentable double category and if D_0^1 admits pushouts and cokernels, D_0^2 admits finite connected inductive D-wise limits.*

REMARK. 1° In the proof of Proposition 3, instead of H we could have used the source $\hat{H}$ of the discrete fibration associated to the category in $\mathfrak{M}$ associated to A. Indeed, the functor p constructed in this proof extends in a functor $\hat{p}: \hat{H} \to D_0^1$. As H is a cofinal sub-category of $\hat{H}$, the functor p has the same limit as $\hat{p}$, and this limit is the D-wise limit of f.

2° The preceding remark leads to a more abstract proof of Proposition 3 (which will be explicited later on for multiple categories). This proof proceeds as follows: Let Ω be the set of categories A such that D_0^2 admits D-wise A-limits. As $\mathcal{F}$ is the inductive closure of $\{\mathbf{2}\}$ and as $\mathbf{2}$ belongs to Ω (by definition of a representable double category), we will have $\Omega = \mathcal{F}_0$ if B belongs to Ω when B is the vertex of an inductive limit-cone $c: I \rightrightarrows \mathcal{F}$ whose basis w satisfies:

$$w(i) \in \Omega \quad \text{for each object } i \text{ of } I.$$

Indeed, the functor $T_{11}(D, \cdot): \mathcal{F}^* \to \mathcal{F}$, coadjoint to $\overline{T_{11}(D^{21}, \cdot)}^*$, transforms c into a projective limit-cone $\bar{c}$ with basis $\bar{w} = T_{11}(D, w\cdot): I^* \to \mathcal{F}$ and vertex $T(D, B)$. The canonical functor d_{DB} is the factor relative to $\bar{c}$ of the projective cone c' with basis $\bar{w}$ defined by:

$$c'(i) = d_{D\, w(i)} \quad \text{for each object } i \text{ of } I.$$

Since $c'(i)$ admits a coadjoint for each i, a theorem of Appelgate-Tierney [AT] asserts that the factor of c' also has a coadjoint; hence $B \in \Omega$.

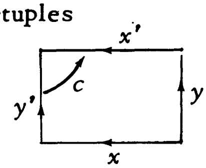
C. The double category of squares of a 2-category.

In this section we give a fundamental example of a representable double category.

We denote by C a 2-category, by «$\circ$» and by «.» the symbols of the laws of the categories C^1 and C^2.

1. *The double category of up-squares of* C is [GZ] the following double category, which is denoted by $Q(C)$:

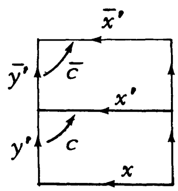
- Its 2-blocks, called *up-squares of* C, are the 5-tuples $q = (y', x', c, x, y)$, where (y', x', x, y) is a (non-commutative) square of C^1_0 and $c: y'.x \to x'.y$ a 2-cell of C.

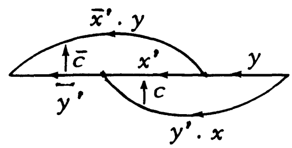


- The first law, said *vertical composition* and denoted by $\boxminus$, is

$$(\overline{y}', \overline{x}', \overline{c}, \overline{x}, \overline{y}) \boxminus (y', x', c, x, y) = (\overline{y}'.y', \overline{x}', (\overline{c}.y)\circ(\overline{y}'.c), x, \overline{y}.y)$$

iff $\overline{x} = x'$.

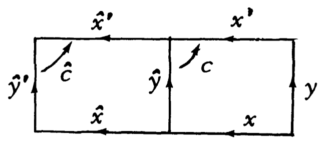




- The second law, the *horizontal composition*, denoted by $\square\!\square$, is:

$$(\hat{y}', \hat{x}', \hat{c}, \hat{x}, \hat{y}) \,\square\!\square\, (y', x', c, x, y) = (\hat{y}', \hat{x}'.x', (\hat{x}'.c)\circ(\hat{c}.x), \hat{x}.x, y)$$

iff $\hat{y} = y'$.

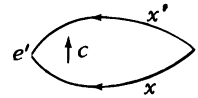


The first and the second categories underlying $Q(C)$ will be denoted by $Q(C)^{\boxminus}$ and $Q(C)^{\square\!\square}$. Both admit as objects the 1-morphisms of C. The identity of $Q(C)^{\boxminus}$ (resp. of $Q(C)^{\square\!\square}$) corresponding to the 1-morphism z will be denoted by $z^{\boxminus}$ (resp. by $z^{\square\!\square}$), or sometimes even by z. Hence we write

$$q: x^{\boxminus} \to x'^{\boxminus}, \quad q: y^{\square\!\square} \rightrightarrows y'^{\square\!\square}, \text{ or more simply } q: x \to x', \; q: y \rightrightarrows y'.$$

We may identify C with the greatest sub-2-category of $Q(C)$ by identifying

$$c : x \to x' : e \rightrightarrows e' \quad \text{with } (e', x', c, x, e).$$



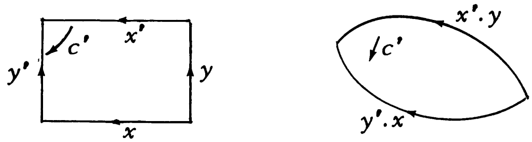
The double category $\Box C_0^1$ of commutative squares of C_0^1 is identified with the double sub-category of $Q(C)$ formed by the up-squares of the form $(y', x', y'.x, x', y)$ (i.e. the up-squares q such that c be a 1-morphism of C). In particular, this double sub-category is equal to $Q(C)$ iff $C_0^1 = C^1$, i.e. iff C is the discrete 2-category on C^2.

2. *The double category of down-squares of* C.

This double category, denoted by $Q_{\downarrow}(C)$, is defined as the double category of up-squares of the first dual (C^{1*}, C^2) of C. Hence its 2-blocks, called *down-squares of* C, are the 5-tuples (y', x', c', x, y), where

(y', x', x, y) is a (non-commutative) square of C_0^1,

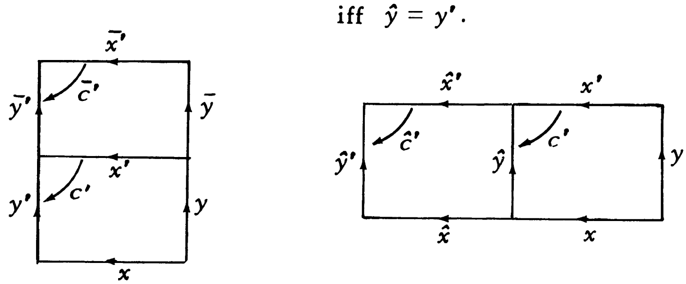
$c' : x'.y \to y'.x$ is a 2-cell of C.



The two laws, expressed with the laws of C only, are:

$$(\bar{y}', \bar{x}', \bar{c}', \bar{x}, \bar{y}) \boxminus (y', x', c', x, y) = (\bar{y}'.y', \bar{x}', (\bar{y}'.c') \circ (\bar{c}'.y), x, \bar{y}.y) \quad \text{iff } x' = \bar{x},$$

$$(\hat{y}', \hat{x}', \hat{c}', \hat{x}, \hat{y}) \,\Box\!\Box\, (y', x', c, x, y) = (\hat{y}', \hat{x}'.x', (\hat{c}'.x) \circ (\hat{x}'.c'), \hat{x}.x, y) \quad \text{iff } \hat{y} = y'.$$



The double category $\Box C_0^1$ is also identified with a double sub-category of $Q_{\downarrow}(C)$.

The bijection:

$$(y', x', c, x, y) \to (x', y', c, y, x)$$

from the set of up-squares of C onto the set of down-squares of C defines a canonical isomorphism from $Q(C)$, to the double category symmetrical of the double category $Q_{\downarrow}(C)$.

3. Let $f: C \to K$ be a 2-functor. We have double functors

$$Q(f): Q(C) \to Q(K) \quad \text{and} \quad Q_{\downarrow}(f): Q_{\downarrow}(C) \to Q_{\downarrow}(K)$$

associating $(f(y'), f(x'), f(c), f(x), f(y))$ to (y', x', c, x, y).

In this way are defined two functors $Q(\cdot)$ and $Q_{\downarrow}(\cdot)$ from the category of small 2-functors into the category $\mathcal{F}_2$ of small double functors.

4. *Limits in* $Q(C)^{\boxminus}$.

PROPOSITION 5. *If* C_0^1 *admits projective* A*-limits preserved by the insertion* $i: C_0^1 \hookrightarrow C^2$, *then* $\boxminus C_0^1$ *admits projective* A*-limits which are preserved by the insertions* j *and* j' *into* $Q(C)^{\boxminus}$ *and* $Q_{\downarrow}(C)^{\boxminus}$.

Δ. We denote by $\overline{\alpha}$ and $\overline{\beta}$ the functors from the category $K = \boxminus C_0^1$ to C_0^1 defined by the maps source and target of the vertical category $\boxminus C_0^1$ (whose objects are identified with 1-morphisms of C). Let F be a functor from A to K. We write:

$$f = \overline{\alpha} F \quad \text{and} \quad \hat{f} = \overline{\beta} F.$$

Since K is isomorphic with the category $(C_0)^2$, there exists a projective limit-cone T with basis F and the cones

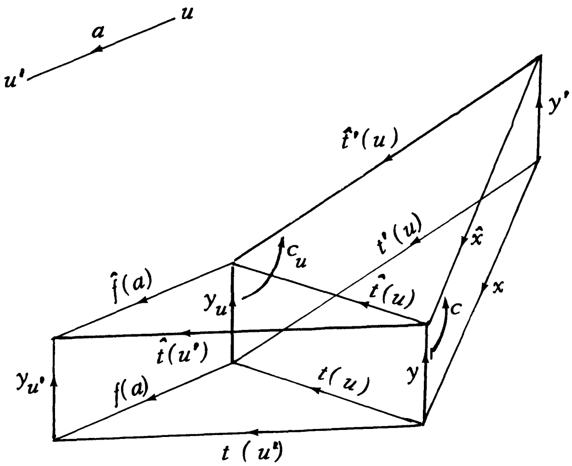
$$t = \overline{\alpha} T \quad \text{and} \quad \hat{t} = \overline{\beta} T$$

are limit-cones with bases f and $\hat{f}$.

1° jT is a limit-cone. Indeed, let $T': A \to Q(C)^{\boxminus}$ be a projective cone with basis jF. As $t' = \overline{\alpha} T'$ is a projective cone with basis f, there exists a factor x of t' relative to t. There also exists a factor $\hat{x}$ of $\hat{t}'$ relative to $\hat{t}$, where $\hat{t}' = \overline{\beta} T'$. We have $T'(u) = (y_u, \hat{t}'(u), c_u, t'(u), y')$, for each $u \in A_0$, where $c_u: y_u \cdot t'(u) \to \hat{t}'(u) \cdot y$ is a 2-cell. The equality

$$F(a) \boxminus T'(u) = T'(u') \quad \text{implies} \quad \hat{f}(a) \cdot c_u = c_{u'}$$

for each morphism $a: u \to u'$ in A. Hence, there exists a projective cone



$t'': A \rightrightarrows C^2$ with basis $i\hat{f}$ such that

$$t''(u) = c_u \text{ for each object } u \text{ of } A.$$

Since $\hat{t}$ is a limit-cone, the hypothesis asserts that $i\hat{t}$ is a projective limit-cone and there exists a factor c of t'' relative to $i\hat{t}$. From

$$\hat{t}(u).\,\alpha^1(c) = \alpha^1(\hat{t}(u).\,c) = \alpha^1(c_u) = y_u\,.\,t'(u) =$$
$$= y_u\,.\,t(u).\,x = \hat{t}(u).\,y.\,x,$$

we deduce, $\hat{t}$ being a limit-cone, $\alpha^1(c) = y.\,x$. Similarly, $\beta^1(c) = \hat{x}.\,y'$. It follows that $(y, \hat{x}, c, x, y')$ is an up-square q and, by its construction, it is the unique up-square satisfying

$$T(u) \square\!\square\, q = T'(u) \text{ for each object } u \text{ of } A.$$

2° The category $\underset{\downarrow}{Q}(C)^{\square\square}$ being identical with $Q(C^*)^{\square\square}$, where C^* is the first dual of C, the preceding proof applied to this dual shows that $j'T$ is also a limit-cone. ∇

COROLLARY 1. *If C_0^1 admits projective A-limits preserved by the insertion into C^2, then $\boxminus C_0^1$ admits projective A-limits which are preserved by the insertions into $Q(C)^{\boxminus}$ and $\underset{\downarrow}{Q}(C)^{\boxminus}$.*

△. This corollary is deduced from the proposition, via the canonical isomorphism from $Q(C)^{\square\square}$ onto $Q(C)^{\boxminus}$ (resp. from $\underset{\downarrow}{Q}(C)^{\square\square}$ onto $\underset{\downarrow}{Q}(C)^{\boxminus}$) which maps $\square\square\, C_0^1$ onto $\boxminus\, C_0^1$. ∇

COROLLARY 2. *If C_0^1 admits inductive A-limits preserved by the insertion into C^2, then $\square\square C_0^1$ admits inductive A-limits preserved by the insertions into $Q(C)^{\square\square}$ and $Q_{\downarrow}(C)^{\square\square}$.*

Δ. This results from Proposition 5 applied to the second dual of C. ∇

COROLLARY 2. *If C_0^1 admits I-products (resp. I-sums) preserved by the insertion into C^2, then $Q(C)^{\square\square}$, $Q_{\downarrow}(C)^{\square\square}$, $Q(C)^{\boxminus}$ and $Q_{\downarrow}(C)^{\boxminus}$ admit I-products (resp. I-sums).*

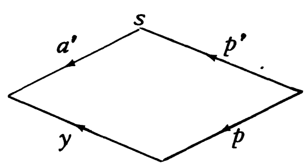
Δ. This comes from Proposition 5 and its corollaries, applied to the discrete category I^0 on I. ∇

REMARK. $Q(C)^{\square\square}$ does not always admit pullbacks, for up-squares which are not commutative squares.

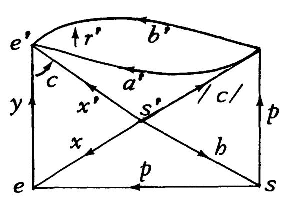
5. *Representability of $Q(C)$.*

PROPOSITION 6. *If C is a representable 2-category and if C_0^1 admits pullbacks, then $Q(C)$ is a representable double category.*

Δ. We consider an object of $Q(C)^{\square\square}$, identified with a 1-morphism y of C, where $y: e \rightrightarrows e'$. In the 2-category C, there exists a representation $r': \dot{a}' \to b'$ of e'; there exists also in C_0^1 a pullback P



of (a', y). Then $q = (y, b'.p', r'.p', p, s)$ is an up-square of C. We are going to prove that q is a representation of y in $Q(C)$. Indeed, let $q' = (y, x', c, x, s')$ be an up-square of C, where s' is a vertex of C. Since $c: s' \rightrightarrows e'$ is a 2-cell of C, it admits a factor $/c/$ relative to the representation r' of e' in C; we have



$$a'./c/ = \alpha^1(r')./c/ = \alpha^1(r'./c/) = \alpha^1(c) = y.x,$$

so that there exists a factor h of $(/c/, x)$ relative to P. As

$$b'.p'.h = b'./c/ = \beta^1(r'./c/) = \beta^1(c) = x',$$

$$r'.p'.h = r'./c/ = c \text{ and } p.h = x,$$

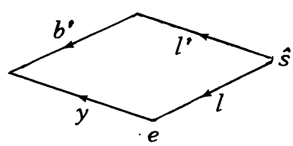
we get

$$q \,\square\!\square\, h^{\boxminus} = (y, b'.p'.h, r'.p'.h, p.h, s') = q'.$$

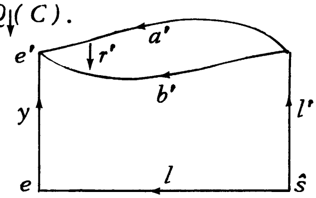
The unicity of the factors asserts the unicity of the 1-morphism h satisfying $q \,\square\!\square\, h^{\boxminus} = q'$. ∇

COROLLARY. *If C is a representable 2-category and if C_0^1 admits pullbacks, then $Q_{\downarrow}(C)$ is a representable double category.*

Δ. Since C is representable, so is its first dual. This dual admitting also C_0^1 as its category of 1-morphisms, the double category of its up-squares, which is $Q_{\downarrow}(C)$ by definition, is representable. More precisely, a representation of $y: e \rightrightarrows e'$ is constructed as follows: Let $r': a' \to b'$ be a representation of e' in C; then r' is also a representation of e' in the first dual of C, but its source in C^{1*} is b'. Let



be a pullback in C_0^1. The down-square $(y, a'.l', r'.l', l, \hat{s})$ of C is a representation of y in $Q_{\downarrow}(C)$.

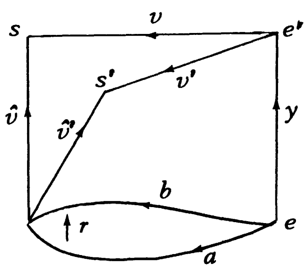


∇

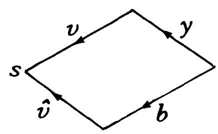
PROPOSITION 7. *If C is a corepresentable 2-category and if C_0^1 admits pushouts, then $Q(C)$ and $Q_{\downarrow}(C)$ are corepresentable double categories.*

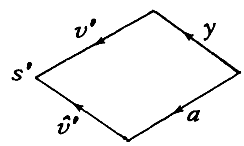
Δ. A proof similar to that of Proposition 6 and of its Corollary shows that the 1-morphism $y: e \rightrightarrows e'$ of C admits:

- as a corepresentation in $Q(C)$ the up-square $(s, v, \hat{v}.r, \hat{v}.a, y)$,



- as a corepresentation in $Q_{\downarrow}(C)$ the down-square $(s', v', \hat{v}'.r, \hat{v}'.b, y)$ where $r: a \rightarrow b$ is a corepresentation of e in C and





are pushouts in C_0^1. ∇

6. *Limits relative to the double category $Q(C)$.*

PROPOSITION 8. *If C is a representable 2-category such that C_0^1 admits connected (resp. small, resp. finite) projective limits, then C_0^1 admits connected (resp. small, resp. finite) projective $Q(C)$-wise and $Q_{\downarrow}(C)$-wise limits.*

Δ. This follows from Proposition 3, since $Q(C)$ and $Q_{\downarrow}(C)$ are both representable double categories (Proposition 7) whose second categories of 1-morphisms are isomorphic to C_0^1. ∇

COROLLARY. *If C is a corepresentable 2-category such that C_0^1 admits connected (resp. small, resp. finite) inductive limits, then C_0^1 admits connected (resp. small, resp. finite) inductive $Q(C)$-wise and $Q_{\downarrow}(C)$-wise limits.* ∇

We are going to look more closely to $Q(C)$-wise limits and to compare them with generalized limits introduced by Gray.

Let A be a category. The category $T(Q(C), A)$ of $Q(C)$-wise transformations indexed by A admits as objects the functors from A to C_0^1, since C_0^1 is canonically isomorphic with the first category of 1-morphisms of the double category $Q(C)$. So a $Q(C)$-wise transformation $t: f \rightarrow f'$

is equivalent to the following data:

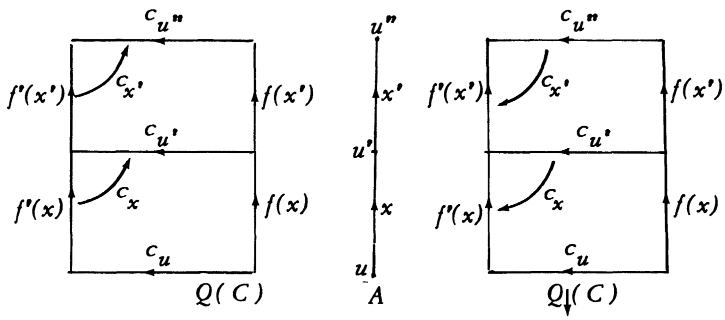
1° Functors f and f' from A to C_0^1.

2° For each object u of A a 1-morphism $c_u : f(u) \to f'(u)$ of C.

3° For each $x : u \to u'$ in A, a 2-cell $c_x : f'(x).c_u \to c_{u'}.f(x)$ such that, if $x' : u' \to u''$ in A, then $c_{x'.x} = (c_{x'}.f(x)) \circ (f'(x').c_x)$.

Indeed, these conditions mean that there exists a functor t from A to $Q(C)^{\boxminus}$ such that

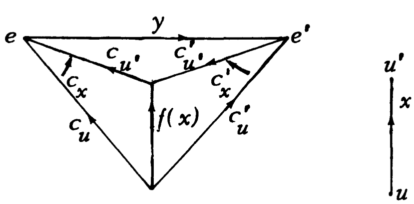
$$t(x) = (f'(x), c_{u'}, c_x, c_u, f(x)) \text{ for each } x : u \to u' \text{ in } A.$$



We have a similar description for $Q_\downarrow(C)$-wise transformations, except that c_x goes «down» instead of «up».

In other words, if $\hat{f}$ and $\hat{f}'$ are the 2-functors from the discrete 2-category on A toward C defined by f and f', the $Q_\downarrow(C)$-transformations from f to f' correspond to the quasi-natural transformations from $\hat{f}$ to $\hat{f}'$ defined by Gray [G1], called anadeses by Bourn [Bo], while the $Q(C)$-wise transformations from f to f' correspond to the quasi_d-natural transformations from $\hat{f}$ to $\hat{f}'$ of Gray or to the catadeses of Bourn. (The way the diagrams are drawn explains why we call «up» what these authors consider as being «down».)

Let $f : A \to C_0^1$ be a functor considered as an object of $T(Q(C), A)$. An inductive $Q(C)$-wise cone t with basis f and vertex e corresponds to



a family $(c_x)_{x \in A}$ of 2-cells of C such that:

$c_u : f(u) \rightrightarrows e$ is a 1-morphism of C, for each object u of A,

$c_x : c_u \to c_{u'} \cdot f(x)$ is a 2-cell, for $x : u \to u'$ in A,

$c_{x'.x} = (c_{x'} \cdot f(x)) \circ c_x$, if $x' : u' \to u''$ in A

(the corresponding cone t associates to $x : u \to u'$ the up-square:

$$t(x) = (e, c_{u'}, c_x, c_u, f(x))).$$

This family corresponds to an inductive $Q(C)$-wise limit-cone if, for each family $(c'_x)_{x \in A}$ satisfying the same conditions, there exists one and only one 1-morphism y of C such that $y \cdot c_x = c'_x$ for each x in A.

With this formulation, we see that the inductive (resp. projective) $Q(C)$-wise limits «are» the cartesian quasi-colimits (resp. quasi-limits) of Gray [G1] and also the inductive (resp. projective) catalimits of Bourn [Bo], for 2-functors from a discrete 2-category. Hence Proposition 8 has been announced by Gray [G2] and proved by Bourn [Bo] (in a more general case which will be considered later on).

D. Examples and Applications to sketched structures.

1. *Limits relative to the double category of quintets.*

The 2-category $\mathcal{N}$ of small natural transformations admits the category $\mathcal{F}$ of functors as its category of 1-morphisms. It is representable and corepresentable, a small category A admitting:

as a representation the natural transformation $r_A : \boxminus A \rightrightarrows A$ associated to the identity functor of $\boxminus A$,

as a corepresentation the natural transformation $r'_A : A \rightrightarrows A \times \mathbf{2}$, from $v = [-, 0^\frown]$ to $[-, 1^\frown]$ such that $r'_A(u) = (u, z)$ for each $u \in A_0$.

An up-square of $\mathcal{N}$ is called a *quintet* and we denote by $\mathcal{Q}$ the double category $\mathcal{Q}(\mathcal{N})$ of quintets (following [E2], where this double category was introduced, as well as its sub-2-category $\mathcal{N}$). Let $\mathcal{Q}_{\downarrow}$ be the double category of down-squares of $\mathcal{N}$.

As $\mathcal{F}$ admits small projective and inductive limits, preserved by the insertion into $\mathcal{N}^{\cdot}$ (which admits an adjoint and a coadjoint), Proposition 5 asserts that $\square\square\mathfrak{F}$ admits small projective and inductive limits preserved by the insertions into $\mathfrak{Q}^{\square\square}$ and into $\mathfrak{Q}_{\downarrow}^{\square\square}$.

REMARK. The category $\mathfrak{N}^{\cdot}$ is cartesian closed; it may be shown that $\mathfrak{Q}^{\square\square}$ is «partially» cartesian closed. More precisely:

Let $f: A \to B$ be a small functor and K a small category; if there exist left Kan extensions along f for functors from A to K, each small functor $g: H \to K$ admits a cofree object G relative to the partial product functor $\cdot\times f: \mathfrak{Q}^{\square\square} \to \mathfrak{Q}^{\square\square}$.

Indeed, G is the composite functor:

$$H^A \xrightarrow{g^A} K^A \xrightarrow{L} K^B ,$$

where L is the left Kan extension functor (adjoint to K^f). There is a similar result replacing $\mathfrak{Q}$ by $\mathfrak{Q}_{\downarrow}$ and left Kan extensions by right Kan extensions.

From Propositions 6 and 8, it follows:

PROPOSITION 9. *The double category $\mathfrak{Q}$ is representable and corepresentable and $\mathfrak{F}$ admits small projective and inductive $\mathfrak{Q}$-wise limits.*

In fact, Gray has given in [G1] an explicit construction of $\mathfrak{Q}$-wise limits: Let $F: A \to \mathfrak{F}$ be a functor, where A is a small category.

1° F admits as an inductive $\mathfrak{Q}$-wise limit the source $K(F)$ (denoted by $[1, F]$ in Gray) of the fibration $k_F: K(F) \to A$ associated to F. (The category $K(F)$ is called in [E1] the «catégorie produit croisé associée à l'espèce de morphismes» F.) The category $F(u)$, for each object u of A, is identified to a sub-category of $K(F)$. From a general result of Gray (the Yoneda-like lemma [G1]), it follows that, if $F': A \to \mathfrak{F}$ is a functor, the $\mathfrak{Q}$-wise transformations from F to F' are in a one-to-one correspondence (a restriction of the adjoint K of $d\mathfrak{Q}_A: \mathfrak{F} \to T(\mathfrak{Q}, A)$) with the functors $h: K(F) \to K(F')$ such that $k_{F'}.h = k_F$.

2° F admits as a projective $\mathfrak{Q}$-wise limit the sub-category $L(F)$ of $K(F)^A$ formed by the natural transformations $t: A \Rightarrow K(F)$ such that $k_F\, t$ is an identity. $L(F)$ is isomorphic with the category of crossed transformations, whose objects are the crossed homomorphisms (defined in [E1]);

the set of components of its greatest sub-groupoid is called in [E1] the *first non-abelian cohomology class of* F, by analogy with the case where A is a group and F a A-module. This remark might be helpful to define the higher order non-abelian cohomology classes of F (see also the Appendix of Bourn [Bo]).

2. *Limits relative to a sub-2-category.*

The following criterium is often useful in applications, for example we will use it in the next section.

PROPOSITION 10. *Let* C *be a 2-category and* H *a full sub-2-category (i.e.* H^1 *and* H^2 *are full sub-categories of* C^1 *and* C^2*). If the insertion* $j: H^1_0 \hookrightarrow C^1_0$ *admits an adjoint (resp. a coadjoint) and if* C^1_0 *admits* $Q(C)$*-wise inductive (resp. projective)* A*-limits, then* H^1_0 *admits* $Q(H)$*-wise inductive (resp. projective)* A*-limits.*

Δ. Since H is a full sub-2-category of C, the double category $Q(H)$ of the up-squares of H is a full double sub-category of $Q(C)$, and the category $T(Q(H), A)$ is identified with a full sub-category of $T(Q(C), A)$. The hypotheses imply that the composite functor:

$$H^1_0 \longrightarrow C^1_0 \xrightarrow{\sim} Q(C)^{\Box\Box}_0 \xrightarrow{d_{Q(C)A}} T(Q(C), A)$$

admits an adjoint (resp. a coadjoint). This functor taking its values into the full sub-category $T(Q(H), A)$, we deduce that its restriction from H^1_0 to $T(Q(H), A)$ also admits an adjoint (resp. a coadjoint). Hence H^1_0 admits inductive (resp. projective) $Q(H)$-wise A-limits. ∇

3. *Limits relative to 2-categories of bimorphisms between sketches.*

In [BE] we have defined the category $\mathcal{F}m''$ of morphisms between small mixed cone-bearing categories, and its full sub-categories:

$\mathcal{F}m'$, whose objects are the presketches σ (i.e. two distinguished cones of σ have different bases),

$\mathcal{P}m'$, whose objects are the limit-bearing categories,

$\mathcal{P}m = \mathcal{F}m' \cap \mathcal{P}m'$, whose objects are the prototypes,

$\mathcal{F}m''^{\mathcal{I}\mathcal{J}}$ (resp. $\mathcal{F}^{\mathcal{I}\mathcal{J}}$), whose objects are the $(\mathcal{I}, \mathcal{J})$-cone-bearing categories (resp. the $(\mathcal{I},\mathcal{J})$-types), where $\mathcal{I}$ and $\mathcal{J}$ are small sets of small categories.

These different categories $\mathcal{X}$ admit small projective and inductive limits, and the following insertion functors admit adjoints:

$$\begin{array}{ccccc} & & \mathcal{P}_{m'} & & \\ & \nearrow & & \searrow & \\ \mathcal{P}_m & & & & \mathcal{F}_{m''} \\ & \searrow & & \nearrow & \\ & & \mathcal{F}_{m'} & & \end{array} \qquad\qquad \mathcal{F}^{\mathcal{I}\mathcal{J}} \hookrightarrow \mathcal{F}_{m''}{}^{\mathcal{I}\mathcal{J}}$$

$\mathcal{X}$ is the category of 1-morphisms of a 2-category $\mathcal{N}\mathcal{X}$, whose double category of up-squares will be denoted by $\mathcal{Q}\mathcal{X}$.

Proposition 18-2 [BE] asserts that $\mathcal{N}\mathcal{X}$ is a representable (except for $\mathcal{X} = \mathcal{F}_{m'}$) and corepresentable 2-category, so that we deduce from Proposition 8:

PROPOSITION 11. *$\mathcal{X}$ admits small projective (resp. inductive) $\mathcal{Q}\mathcal{X}$-wise limits, for $\mathcal{X} = \mathcal{F}_{m''}$, $\mathcal{P}_{m'}$, $\mathcal{P}_m$, $\mathcal{F}_{m''}{}^{\mathcal{I}\mathcal{J}}$, $\mathcal{F}^{\mathcal{I}\mathcal{J}}$ (resp. $\mathcal{F}^{\mathcal{I}\mathcal{J}}$ and $\mathcal{F}_{m'}$).*

Δ. Using the preceding results, we may give an explicit construction of some of these limits. Let $S: A \to \mathcal{X}$ be a functor, where A is a small category. We denote by F the functor from A to $\mathcal{F}$ got by composing S with the forgetful functor from $\mathcal{X}$ to $\mathcal{F}$. If $\mathcal{X}$ is a proper sub-category of $\mathcal{F}_{m''}$, we consider the composite functor $\hat{S}$:

$$A \xrightarrow{\;S\;} \mathcal{X} \hookrightarrow \mathcal{F}_{m''}.$$

1° If $\mathcal{X} = \mathcal{F}_{m''}$ or $\mathcal{F}_{m''}{}^{\mathcal{I}\mathcal{J}}$, then S admits:

as an inductive $\mathcal{Q}\mathcal{X}$-wise limit the cone-bearing category $K(S)$ got by equipping $K(F)$ (the inductive $\mathcal{Q}$-wise limit of F) with all the cones $i_u \gamma_u$, where $i_u: F(u) \to K(F)$ is the insertion and where γ_u is a distinguished cone of $S(u)$, for each object u of A;

as a projective $\mathcal{Q}\mathcal{X}$-wise limit the cone-bearing category $L(S)$ got by equipping $L(F)$ (the $\mathcal{Q}$-wise projective limit of F) with the cones T such that $v_u T$ be a distinguished cone of $S(u)$ for each object u of A, where $v_u: L(F) \to F(u)$ is the valuation functor, which maps the natural transformation $t: A \rightrightarrows K(F)$ onto $t(u)$.

2° If $\mathcal{X} = \mathcal{F}_m'$, then $K(\hat{S})$ is a presketch, which is the $\mathcal{Q}\mathcal{X}$-wise inductive limit of S. If $\mathcal{X} = \mathcal{P}_m$ or $\mathcal{P}_m'$ (resp. $= \mathcal{F}^{\mathcal{I}\mathcal{J}}$), it follows from part 1 and Proposition 10 that S admits as an inductive $\mathcal{Q}\mathcal{X}$-wise limit the limit-bearing category (resp. the $(\mathcal{I}, \mathcal{J})$-type) freely associated to $K(\hat{S})$.

3° The insertion functors i_u, for $u \in A_0$, preserve connected limits; if A defines a preorder on the set of its objects, they preserve all limits. Using these facts we deduce:

a) Let us suppose that $\mathcal{X} = \mathcal{P}_m'$ (resp. $\mathcal{P}_m$) and that the indexing categories of $S(u)$ are connected for each object u of A, or that A defines a preorder. Then $K(\hat{S})$ is a limit-bearing category (resp. a prototype), so that it is the inductive $\mathcal{Q}\mathcal{X}$-wise limit of S. Moreover the insertion from the category $L(F)$ into $K(F)^A$ reflecting limits, $L(\hat{S})$ is also a limit-bearing category (resp. a prototype), projective $\mathcal{Q}\mathcal{X}$-wise limit of S.

b) Finally, if $\mathcal{X} = \mathcal{F}^{\mathcal{I}\mathcal{J}}$ and if $\mathcal{I}$ and $\mathcal{J}$ are sets of connected categories, or if A defines a preorder, $L(\hat{S})$ is a $(\mathcal{I}, \mathcal{J})$-type, which is the $\mathcal{Q}\mathcal{X}$-wise projective limit of S. ∇

4. *Lax morphisms between sketched structures.*

In this section σ will be a projective limit-bearing category (Σ, Γ) and $\mathcal{I}$ is the set of its indexing categories.

DEFINITION. If D is a double category and if ϕ and ϕ' are σ-structures in the first category D_0^2 of 1-morphisms of D, a *D-wise σ-morphism from ϕ to ϕ'* is defined as a σ-structure τ in D^1 such that

$$\phi = \alpha^2 \tau \quad \text{and} \quad \phi' = \beta^2 \tau .$$

EXAMPLE. If B is a category and $\square B$ the double category of its commutative squares, the $\square B$-wise σ-morphisms are identified with σ-morphisms in B (by identifying a functor to $\boxminus B$ with a natural transformation to B).

PROPOSITION 12. *If D is a double category and if the functors*

$$\alpha^2 : D^1 \to D_0^2, \quad \beta^2 : D^1 \to D_0^2 \quad \text{and} \quad \kappa^2 : (D^2 * D^2)^1 \to D^1$$

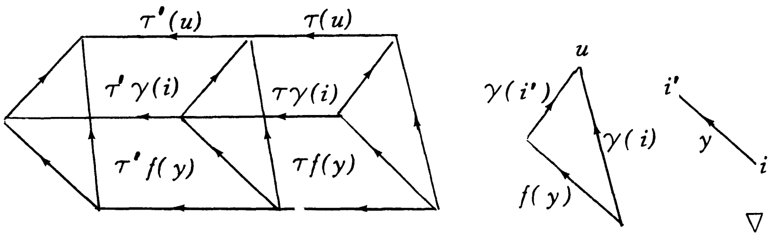
preserve projective limits indexed by elements of $\mathcal{I}$, then the D-wise σ-morphisms define a sub-category of $T(D, \Sigma)$.

Δ. Let τ be a D-wise σ-morphism from ϕ to ϕ' and τ' a D-wise

σ-morphism from ϕ' to ϕ''. They have a composite $\tau'' = \kappa^2\,[\tau', \tau]$ in $T(D, \Sigma)$, which is a D-wise transformation from ϕ to ϕ''. Let $\gamma \in \Gamma$ a cone with basis $f: I \to \Sigma$; the cone $\tau''\gamma = \kappa^2\,[\tau', \tau]\,\gamma$ is the image by κ^2 of the cone γ', with basis $[\tau' f, \tau f] : I \to (D^2 * D^2)^1$ such that

$$\gamma'(i) = (\tau'\gamma(i), \tau\gamma(i)) \text{ for each } i \in I_0.$$

Since $\tau\gamma$ and $\tau'\gamma$ are limit-cones in D^1, the cone γ' is a limit-cone in the category $(D^2 * D^2)^1$, which is the pullback of (α^2, β^2); its image by κ^2, which is $\tau''\gamma$, is a limit-cone in D^1. Hence τ'' is a σ-structure in D^1, i. e. a D-wise σ-morphism from ϕ to ϕ''.



∇

We consider now the case where D is the double category of up-squares of a 2-category C.

DEFINITION. Let C be a 2-category, ϕ and ϕ' two σ-structures in C_0^1. *A C-lax σ-morphism from ϕ to ϕ'* is defined as a $Q(C)$-wise σ-morphism τ from ϕ to ϕ' such that $\tau f(y)$ be a commutative square for any morphism y of I, if $f: I \to \Sigma$ is the basis of a cone $\gamma \in \Gamma$.

PROPOSITION 13. *If C is 2-category and if C_0^1 admits projective limits indexed by elements of $\mathfrak{J}$ and preserved by the insertion into C^2, then the C-lax σ-morphisms define a sub-category of $T(Q(C), \Sigma)$.*

Δ. Let τ and τ' be C-lax σ-morphisms from ϕ to ϕ' and from ϕ' to ϕ'', and τ'' their composite in $T(Q(C), \Sigma)$. If $f: I \to \Sigma$ is the basis of a cone $\gamma \in \Gamma$, since $\tau f(y)$ and $\tau' f(y)$ are commutative squares, so is

$$\tau'' f(y) = \tau' f(y) \boxminus \tau f(y) \text{ for each } y \text{ in } I.$$

C_0^1 admitting projective I-limits preserved by the insertion into C^2, the functor from I to $\boxminus C_0^1$ restriction of τf admits a projective limit which is also a projective limit of τf (Prop. 5); hence the limit-cone $\tau\gamma$ takes its values in $\boxminus C_0^1$, as well as $\tau' \gamma$. The composite

$$\tau'' \gamma(i) = \tau' \gamma(i) \,\square\!\square\, \tau\gamma(i), \quad \text{for each } i \in I_0,$$

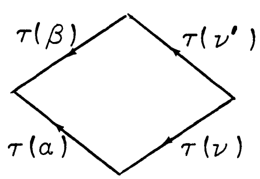
is a commutative square, so that $\tau''\gamma$ takes its values in $\boxminus C_0^1$ and, considered as a cone in $\boxminus C_0^1$, it is a limit-cone (limits in $\boxminus C_0 = (C_0^1)^2$ being computed pointwise). Hence (Proposition 5) $\tau''\gamma$ is a limit-cone in the category $Q(C)^{\boxminus}$. This proves that τ'' is a $Q(C)$-wise σ-morphism, i. e. a C-lax σ-morphism from ϕ to ϕ''. ∇

5. *Lax double functors.*

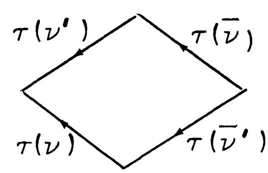
We apply here the preceding results to the sketch $\sigma\mathcal{F} = (\Sigma\mathcal{F}, \Gamma\mathcal{F})$ of categories.

DEFINITION. Let A and B be double categories, ϕ_A and ϕ_B the corresponding categories in $\mathcal{F}$. A $\mathfrak{N}$-lax $\sigma\mathcal{F}$-morphism from ϕ_A to ϕ_B is called *a lax double functor from A to B.*

The lax double functors from A to B are exactly the $\mathcal{Q}$-wise transformations (where $\mathcal{Q}$ is always the double category of quintets) τ such that $\tau(\mu)$ be a commutative square for $\mu \in \{\alpha, \beta, \nu, \nu', \bar{\nu}, \bar{\nu}'\}$. Indeed, these conditions imply that



and



are pullbacks in $\boxminus\mathcal{F}$ (and therefore in $\mathcal{Q}^{\boxminus}$), since pullbacks in $\boxminus\mathcal{F} = \mathcal{F}^2$ are computed pointwise and ϕ_A and ϕ_B are $\sigma\mathcal{F}$-structures.

It follows from Proposition 13 that the lax double functors between small double categories define a sub-category of $T(\mathcal{Q}, \Sigma\mathcal{F})$.

Let A and B be double categories, ϕ_A and ϕ_B the associated categories in $\mathcal{F}$; we denote by «$\circ$» the laws of A^1 and B^1, by «$\cdot$» those of A^2 and B^2, by a, b, i, k and a', b', i', k' respectively the images of $\alpha, \beta, \iota, \kappa$ by ϕ_A and ϕ_B.

PROPOSITION 14. *The lax double functors from A to B are in one-to-*

one correspondence with the 4-tuples (g_0, g, t, t'), *where*

1° $g: A^1 \to B^1$ *and* $g_0: A^2_0 \to B^2_0$ *are functors such that*

$$a' g = g_0\, a \quad \text{and} \quad b' g = g_0\, b.$$

This implies the existence of a functor g':

$$(x', x) \mapsto (g(x'), g(x)) \ \text{ from } \ (A^2 * A^2)^1 \ \text{ to } \ (B^2 * B^2)^1.$$

2° $t: i' g_0 \to g i$ *and* $t': k' g' \to g k$ *are natural transformations.*

3° *The following coherence axioms are satisfied:*

$$(u) \quad t'(x, e).(g(x) \circ t(e)) = g(x) = t'(e', x).(t(e') \circ g(x))$$

$$\text{for each } x: e \to e' \text{ in } A^2_0.$$

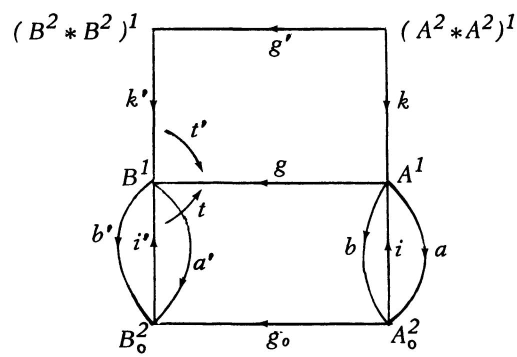
$$(a) \quad t'(x'', x' \circ x).(g(x'') \circ t'(x', x)) = t'(x'' \circ x', x).(t'(x'', x') \circ g(x))$$

$$\text{for each path } (x'', x', x) \text{ in } A^2_0.$$

Δ. Let τ be a lax double functor from A to B. We take for g and for g_0 the functors $\tau(2)$ and $\tau(1)$, for t and t' the natural transformations arising in the quintets $\tau(\iota)$ and $\tau(\kappa)$. Condition 1 is satisfied, $\tau(\alpha)$ and $\tau(\beta)$ being commutative squares. The two coherence axioms are respectively deduced by pointwise computation from the axioms

$$\tau(\kappa) \boxminus \tau(j_\alpha) = \tau(2) = \tau(\kappa) \boxminus \tau(j_\beta),$$

$$\tau(\kappa) \boxminus \tau(\kappa') = \tau(\kappa) \boxminus \tau(\kappa'').$$



2° If (g_0, g, t, t') is given, we construct as follows a lax double functor τ:

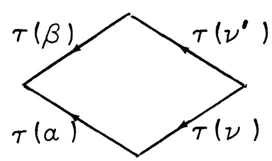
$\tau(\alpha)$ and $\tau(\beta)$ «are» the commutative squares

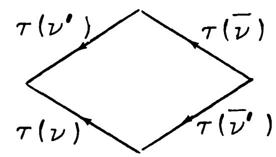
$$(a', g_0, g, a) \text{ and } (b', g_0, g, b),$$

and g' is their canonical pullback in $\boxminus\mathcal{F}$ (and also in $\mathfrak{Q}^{\boxminus}$),

$$\tau(\iota) = (i', g, t, g_0, i) \quad \text{and} \quad \tau(\kappa) = (k', g, t, g', k).$$

As $\Sigma_{\mathcal{F}}$ is «generated» by $\alpha, \beta, \iota, \kappa$, the other quintets $\tau(\lambda)$, for λ in $\Sigma_{\mathcal{F}}$, are then deduced as composites or factors relative to the pullbacks:



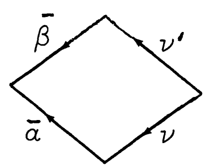


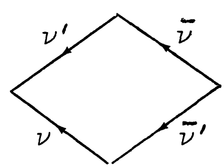
(in $\boxminus\mathcal{F}$). The axioms *(u)* and *(a)* imply that we have so defined a functor $\tau : \Sigma_{\mathcal{F}} \to \mathfrak{Q}^{\boxminus}$. ∇

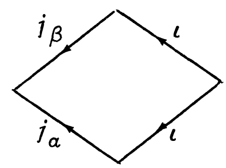
REMARKS. 1° Let A and B be 2-categories. The 4-tuples considered in Proposition 14 are then the morphisms of bicategories from the bicategory A to B defined by Bénabou [B2] (called pseudo-functors in [G1]); as a natural transformation toward the discrete category A_0^2 is an identity, any $\mathfrak{Q}$-wise $\sigma_{\mathcal{F}}$-morphism from ϕ_A to ϕ_B is a double functor.

2° By a process of «laxification» similar to that leading from 2-categories to bicategories and from 2-functors to morphisms of bicategories, Moreau [M] defines lax double functors between dicategories, i. e. categories equipped with a second law which is unitary and associative «up to isomorphisms»,which reduce for double categories to those considered here; he generalizes Proposition 14 to the case where A and B are dicategories.

3° The $\sigma_{\mathcal{F}}$-morphisms are identical to the $\bar{\sigma}_{\mathcal{F}}$-morphisms (see Part C-0), where $\bar{\sigma}_{\mathcal{F}}$ is the sketch $(\Sigma_{\mathcal{F}}, \bar{\Gamma}_{\mathcal{F}})$ in which the pullbacks are







But $\mathfrak{N}$-lax $\bar{\sigma}_{\mathcal{F}}$-morphisms are only those lax double functors τ corresponding to 4-tuples (g_0, g, t, t') such that t is an identity (since the factors $\tau(j_\alpha)$ and $\tau(j_\beta)$ must be commutative squares); they are said *unitary*.

Let A and B be double categories. We denote by

$$k_A : K(A) \to \Sigma\mathcal{F} \quad \text{and} \quad k_B : K(B) \to \Sigma\mathcal{F}$$

the fibrations corresponding to ϕ_A and ϕ_B. With the notations of [E1], a morphism of $K(A)$ is a triple $m = (z, \mu, s)$, where $\mu : \omega \to \omega'$ is a morphism of $\Sigma\mathcal{F}$, where s is an object of $\phi_A(\omega)$ and

$$z : s' \to s'' \text{ in } \phi_A(\omega'), \text{ if } s' = \phi_A(\mu)(s).$$

Identifying $\phi_A(\omega)$ to a sub-category of $K(A)$ and the «cartesian» morphism (s', μ, s) to (μ, s), we get $m = z.(\mu, s)$ in $K(A)$.

PROPOSITION 15. *There is a bijection from the set of lax double functors from A to B onto the set of functors $h : K(A) \to K(B)$ such that:*

$$(l) \qquad k_B h = k_A \text{ and } h(\mu, s) = (\mu, h(s)),$$

for each cartesian morphism (μ, s), where $\mu \in \{\alpha, \beta, \nu, \nu', \overline{\nu}, \overline{\nu}'\}$.

Δ. This bijection is a restriction of the bijection K' (considered after Proposition 9) from the set of $\mathcal{Q}$-wise transformations from ϕ_A to ϕ_B onto the set of functors from $K(A)$ to $K(B)$ commuting with the fibrations to $\Sigma\mathcal{F}$. Indeed K' maps τ onto the functor h whose restriction to $\phi_A(\omega)$ is $\tau(\omega)$ for each object ω of $\Sigma\mathcal{F}$ and such that

$$h(\mu, s) = (t_\mu(s), \mu, h(s)),$$

if t_μ is the natural transformation arising in the quintet $\tau(\mu)$. Hence, h satisfies the condition *(l)* iff t_μ is an identity (i.e. iff $\tau(\mu)$ is a commutative square) for $\mu \in \{\alpha, \beta, \nu, \nu', \overline{\nu}, \overline{\nu}'\}$. ∇

This proposition reduces the study of lax double functors to that of ordinary functors.

REFERENCES.

AT. APPELGATE-TIERNEY, Iterated cotriples, *Lecture Notes in Math.* 137, Springer (1970), 56 - 99.

AB. A. BASTIANI, *Applications différentiables de dimension infinie - Distructures,* Thèse (multigraphiée) Paris, 1962.

BE. BASTIANI - EHRESMANN, Sketched structures, *Cahiers Topo. et Géo. diff.* XIII-2 (1972), 105 - 214.

BE1. BASTIANI - EHRESMANN, Catégories de foncteurs structurés, *Cahiers Topo. et Géo. diff.* XI-3 (1969), 329 - 384.

B1. J. BENABOU, Les distributeurs (rédigé par Roisin), Un. Cath. Louvain, *Rapport* 33 (1973).

B2. J. BENABOU, Introduction to bicategories, *Lecture Notes* 47 (1967).

Bo. D. BOURN, Natural anadeses and catadeses, *Cahiers Topo. et Géo. diff.* XIV-4 (1973) .

Ch. D. CHAMAILLARD, Catégories structurées par des catégories non associatives, *Esquisses Math.* 6 (1970).

C. F. CONDUCHE, Sur les structures définies par limites projectives, *Esquisses Math.* 11, Paris (1971).

E1. C. EHRESMANN, *Catégories et Structures,* Dunod, Paris, 1965.

E2. C. EHRESMANN, Catégories structurées:
I et II, *Ann. Ec. Norm. Sup.* 80, Paris (1963), 349 - 426.
III, *Topo. et Géo. diff.* V, Paris (1963).

E3. C. EHRESMANN, Catégories structurées généralisées, *Cahiers Topo. et Géo. diff.* X-1 (1968), 139 - 168.

E4. C. EHRESMANN, Catégories topologiques et catégories différentiables, *Col. Géo. diff. globale,* Bruxelles (1958), 137 - 152.

E5. C. EHRESMANN, Sur les catégories différentiables, *Atti del Cong. Inter. Geo. diff.*, Bologna (1967).

E6. C. EHRESMANN, Catégories ordonnées, Holonomie et Cohomologie, *Ann. Inst. Fourier* 14-1, Grenoble (1964), 205 - 268.

E7. C. EHRESMANN, Catégorie des foncteurs types, *Rev. Un. Mat. Arg.* 20 (1960).

E8. C. EHRESMANN, Espèces de structures locales. Elargissement de catégories, *Topo. et Géo. diff.* III, Paris (1961).

GZ. GABRIEL - ZISMAN, *Calculus of fractions and homotopy theory,* Springer 1966.

Gr. A. GROTHENDIECK, Techniques de descente, *Séminaire Bourbaki* 195, Paris (1959 - 60).

G1. J. W. GRAY, Formal category Theory, *Lecture Notes* 391 (1974).

G2. J. W. GRAY, The Meeting of the Midwest Category Seminar in Zürich, *Lecture Notes* 195 (1971), and notes taken by Leroux at Gray's lectures at Paris 1971.

G3. J. W. GRAY, The categorical comprehension scheme, *Lecture Notes* 99 (1969).

L. C. LAIR, Etude générale de la catégorie des esquisses, *Esquisses Math.* 24 (1974).

Le. S. LEGRAND, Transformations naturelles généralisées, *Cahiers Topo. et Géo. diff.* X-3 (1968), 351 - 374.

Lm. J. LAMBEK, Completions of categories, *Lecture Notes* 24 (1966).

Ml. S. MAC LANE, *Categories for the working mathematician,* Springer, 1972.

M. G. MOREAU, *Catégories doubles à isomorphisme près,* à paraître.

Pa. P. H. PALMQUIST, The double category of adjoint squares, *Lecture Notes* 195 (1971), 123 - 153.

S. R. STREET, Two constructions on lax functors, *Cahiers Topo. et Géo. diff.* XIII-3 (1972), 217 - 264.

V. E. VAUGELADE, Application des bicatégories à l'étude des catégories internes, *Esquisses Math.* 21 (1974).

Théorie et Applications des Catégories
Faculté des Sciences
33 rue Saint-Leu
80039 AMIENS

et

U. E. R. de Mathématiques
Université Paris 7, Tour 55
2 Place Jussieu
75005 PARIS

CONTENTS.