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ARISTIDE DELEANU
ARMIN FREI
PETER HILTON

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GENERALIZED ADAMS COMPLETION

by

Aristide DELEANU, Armin FREI and Peter HILTON

0. Introduction.

In [3] Deleanu and Hilton showed how, in the stable homotopy category, localization and profinite completion, in the sense of Sullivan [12], could be subsumed under a general categorical completion process suggested by Adams. If $\mathcal{C}$ is the stable homotopy category and h a homology theory on $\mathcal{C}$, we consider the family $S = S(h)$ of morphisms of $\mathcal{C}$ rendered invertible by h and form the category of fractions $\mathcal{C}[S^{-1}]$. Then for a given object Y of $\mathcal{C}$ we define Y_h to be that object, if such exists, for which there exists a natural equivalence of contravariant functors from $\mathcal{C}$ to *Ens*, the category of sets,

$$(0.1) \qquad \mathcal{C}[S^{-1}](-, Y) \simeq \mathcal{C}(-, Y_h).$$

We call Y_h the *Adams completion* or *h-completion* of Y. If h is reduced homology with coefficients in $\mathbf{Z}_P$, the integers localized at the family of primes P, then Y_h exists and is just the (stable) P-localization of Y; thus, $Y_h = Y_P$. If h is reduced homology with coefficients in $\mathbf{Z}/p$, the integers modulo the prime p, then, again, Y_h exists and is the (stable) p-profinite completion of Y; thus $Y_h = \hat{Y}_p$.

In this paper we generalize our approach considerably. In the first place, we no longer work stably. Moreover, we do not, in developing the theory, even assume that we have a functor given on $\mathcal{C}$, but merely consider a family S of morphisms of $\mathcal{C}$, with respect to which we construct the category of fractions $\mathcal{C}[S^{-1}]$. We then study conditions under which, to a given object Y of $\mathcal{C}$, we may associate Z in $\mathcal{C}$ such that the analog of (0.1) holds,

$$(0.2) \qquad \mathcal{C}[S^{-1}](-, Y) \simeq \mathcal{C}(-, Z).$$

If Z exists, we call it the *S-completion* of Y. It turns out that the theory is very different according to whether we ask that a given Y admit such a Z (the *local* completion problem), or that every Y in $\mathcal{C}$ admit such a Z (the *global* completion problem). We devote Section 1 to a consideration of the local problem. Here we show that, provided S is saturated and admits a calculus of left fractions [6], then Z is the S-completion of Y if and only if there exists a morphism $e: Y \to Z$ in S with a certain co-universal property, generalizing part of Theorem 3.2 of [3]. There is also described a sufficient condition for S to admit a calculus of left fractions, which is applied in Section 3. Section 1 closes with a description of the somewhat different situation which arises when S admits a a calculus of right fractions.

Section 2 is concerned with the global problem. Here our results consist largely of a collation of relevant facts drawn from [5,6], but set in the context and language appropriate to our purposes. In this case it turns out that, if the global S-completion exists, and S is saturated, then S does admit a calculus of left fractions. Thus the hypotheses of our main theorem in Section 1 are seen to be entirely reasonable, since the question of the existence of a global S-completion is clearly central to the continuing investigation. It is further proved in Section 2 that the category of fractions $\mathcal{C}[S^{-1}]$ is equivalent to the category of S-complete objects, thus generalizing a theorem of Quillen [11] (see Example 3.2). We give two examples of global S-completions outside the context of Adams completions, reserving examples of the (non-stable) Adams completion to Section 3. A further example of global S-completion is to be found on p. 73 of [6], where S is the collection of *anodyne extensions* in the homotopy category of pointed semi-simplicial complexes. We close Section 2 with an example of the dual concept of a global *S-cocompletion*.

In Section 3 we specialize to the (non-stable) Adams completion, We obtain a non-stable version of the Deleanu-Hilton result in [3], identifying the Adams h-completion Y_h with Y_p, $\hat{Y}_p$ respectively, in case the homology theory h is reduced homology with coefficients in $\mathbf{Z}_p$, $\mathbf{Z}/p$ respectively, and $\mathcal{C}$ is the homotopy category of 1-connected based spaces of the homotopy type of a CW-complex. We note that Adams [14] and Deleanu [15] have proved independently that the Adams completion Y_h exists for any additive homology theory defined on the category $\mathcal{C}$ above (modulo a set-theoretical assumption which seems to be required on foundational grounds). Here we merely prove the encouraging result that, for any reasonable category to which one might want to apply a homology theory h, the family S of morphisms rendered invertible by h admits a calculus of left fractions.

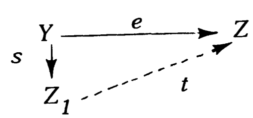
1. Local S-completions.

We consider a category $\mathcal{C}$ and suppose given a family S of morphisms of $\mathcal{C}$. Let $\mathcal{C}[S^{-1}]$ be the category of fractions with respect to S and let $F_S : \mathcal{C} \to \mathcal{C}[S^{-1}]$ be the canonical functor. We say that S is *saturated* if $f \in S$ whenever $F_S(f)$ is invertible. Notice that a saturated family always contains all invertible morphisms of S and is closed under composition. We will call a family having these last two properties *closed,* so that a saturated family is closed. The following proposition is evident.

PROPOSITION 1.1. *A family S of morphisms of $\mathcal{C}$ is saturated if and only if there exists a functor $F : \mathcal{C} \to \mathfrak{D}$ such that S is the collection of morphisms f such that Ff is invertible.*

Fix an object Y in $\mathcal{C}$. We say that Y is *S-completable* if the contravariant functor $\mathcal{C}[S^{-1}](\cdot, Y) : \mathcal{C} \to Ens$ is representable. If Z is the representing object, we call Z *the completion of* Y. Of course Z, if it exists, is determined up to a canonical equivalence. If Z is canonically equivalent to Y itself, we say that *Y is S-complete.* We prove (see [3]):

THEOREM 1.2. *Let S be a saturated family of morphisms of $\mathcal{C}$ admitting a calculus of left fractions. Then the object Z is the S-completion of the object Y if and only if there exists $e : Y \to Z$ in S which is co-universal: given $s : Y \to Z_1$ in S, there exists a unique $t : Z_1 \to Z$ such that $ts = e$.*



Notice that, since S is saturated, t is automatically in S. We may also express the co-universality of e by saying that e is terminal in the category $\mathcal{C}(Y;S)$ whose objects are morphisms of S with source Y, and whose morphisms $t: s_1 \to s_2$ are morphisms (of S) with $ts_1 = s_2$.

$$\begin{array}{ccc} & Y & \\ {\scriptstyle s_1}\swarrow & & \searrow{\scriptstyle s_2} \\ Z_1 & \xrightarrow{\;t\;} & Z_2 \end{array}$$

PROOF OF THEOREM 1.2. Assume that Z is the S-completion of Y, so that there is a natural equivalence of functors

$$\tau: \mathcal{C}[S^{-1}](-, Y) \simeq \mathcal{C}(-, Z).$$

Set $e = \tau(1_Y): Y \to Z$. Given $s: Y \to Z_1$, we have the commutative diagram (1)

$$\begin{array}{ccc} \mathcal{C}[S^{-1}](Y,Y) & \overset{\tau}{\simeq} & \mathcal{C}(Y,Z) \\ \uparrow{\scriptstyle s^{\dagger}} & & \uparrow{\scriptstyle s^{*}} \\ \mathcal{C}[S^{-1}](Z_1,Y) & \overset{\tau}{\simeq} & \mathcal{C}(Z_1,Z). \end{array}$$

But s is invertible in $\mathcal{C}[S^{-1}]$, so that $s^{\dagger}$ is bijective; so therefore is s^*, thus proving the co-universality of e.

It remains to show that e is in S. Since S is saturated, it suffices to show that $F_S(e)$ is invertible. Define $\alpha \in \mathcal{C}[S^{-1}](Z,Y)$ by $\tau(\alpha) = 1_Z$; we will show that α is inverse to $F_S(e)$. To this end, consider the commutative diagram

$$\begin{array}{ccc} \mathcal{C}[S^{-1}](Y,Y) & \overset{\tau}{\simeq} & \mathcal{C}(Y,Z) \\ \uparrow{\scriptstyle e^{\dagger}} & & \uparrow{\scriptstyle e^{*}} \\ \mathcal{C}[S^{-1}](Z,Y) & \overset{\tau}{\simeq} & \mathcal{C}(Z,Z) \end{array}$$

Then $e^* \tau(\alpha) = e$, while $\tau e^{\dagger}(\alpha) = \tau(\alpha \circ F_S(e))$. Thus

$$\tau(1_Y) = e = \tau(\alpha \circ F_S(e)),$$

(1) Here, and subsequently, we write f^* for $\mathcal{C}(f, Z)$ and $f^{\dagger}$ for $\mathcal{C}[S^{-1}](F_S(f), Y)$.

so that

$$\alpha \circ F_S(e) = 1_Y .$$

Finally we must show that $F_S(e) \circ \alpha = 1_Z$; it is at this point that we require the hypothesis that S admits a calculus of left fractions. For this hypothesis implies that α may be written as

$$\alpha = F_S(s)^{-1} F_S(g) ,$$
$$Z \xrightarrow{g} U \xleftarrow{s} Y .$$

We may thus construct the diagram

$$\begin{array}{ccc} \mathcal{C}[S^{-1}](Y,Y) & \overset{\tau}{\simeq} & \mathcal{C}(Y,Z) \\ \uparrow s^{\dagger} & & \uparrow s^* \\ \mathcal{C}[S^{-1}](U,Y) & \overset{\tau}{\simeq} & \mathcal{C}(U,Z) \\ \downarrow g^{\dagger} & & \downarrow g^* \\ \mathcal{C}[S^{-1}](Z,Y) & \overset{\tau}{\simeq} & \mathcal{C}(Z,Z) \end{array} .$$

Define $\beta \in \mathcal{C}[S^{-1}](U,Y)$ by $s^{\dagger}(\beta) = 1_Y$ and let $\tau(\beta) = h$. Then $g^{\dagger}(\beta) = \alpha$, so

$$s^*(h) = e,\ g^*(h) = 1_Z ,\ \text{ or } hs = e,\ hg = 1_Z .$$

It follows that $F_S(h) = F_S(e) F_S(s)^{-1}$ so

$$F_S(e)\alpha = F_S(e) F_S(s)^{-1} F_S(g) = F_S(h) F_S(g) = F_S(hg) = 1_Z .$$

Conversely, suppose given $e : Y \to Z$ in S which is co-universal. Since e is in S,

$$e_{\dagger} : \mathcal{C}[S^{-1}](-,Y) \simeq \mathcal{C}[S^{-1}](-,Z),$$

so it remains to show that F_S induces an equivalence

$$F_{S*} : \mathcal{C}(-,Z) \to \mathcal{C}[S^{-1}](-,Z).$$

First, F_{S*} is surjective; for if $\alpha \in \mathcal{C}[S^{-1}](X,Z)$, then we may represent α by $X \xrightarrow{g} Z_1 \xleftarrow{s} Z$.

However, the co-universality of e readily implies (Proposition 3.4 of [3]) the existence of t in S, $t : Z_1 \to Z$ with $ts = 1$. Thus

$$F_S(tg) = F_S(t) F_S(g) = F_S(s)^{-1} F_S(g) = \alpha .$$

Second, F_{S*} is injective; for if $F_S(f_1)=F_S(f_2)$, where $f_1, f_2 \in \mathcal{C}(X,Y)$, then, since S admits a calculus of left fractions, we infer the existence of s in S with $sf_1 = sf_2$. But, again, $s: Z \to Z_1$ admits a left inverse t, so that $f_1 = f_2$, and the theorem is proved.

REMARKS. 1. In the course of the proof we constructed $e: Y \to Z$ out of $\tau: \mathcal{C}[S^{-1}](-,Y) \simeq \mathcal{C}(-,Z)$, and τ out of e. These two processes are mutually inverse. That $e \mapsto \tau \mapsto e$ is obvious. To show that $\tau \mapsto e \mapsto \tau$ observe that if $\bar{\tau}$ is the natural equivalence $F_{S*}^{-1} \circ e_{\dagger}$ determined by e and if α is represented by

$$X \xrightarrow{g} Z_1 \xleftarrow{s} Y,$$

then $\bar{\tau}(\alpha) = tg$, where $t: Z_1 \to Z$ is given by the co-universal property to satisfy $ts = e$. Consider the diagram

$$\begin{array}{ccc} \mathcal{C}[S^{-1}](X,Y) & \overset{\tau}{\simeq} & \mathcal{C}(X,Z) \\ \uparrow g^{\dagger} & & \uparrow g^{*} \\ \mathcal{C}[S^{-1}](Z_1,Y) & \overset{\tau}{\simeq} & \mathcal{C}(Z_1,Z) \\ \uparrow s^{\dagger} & & \uparrow s^{*} \\ \mathcal{C}[S^{-1}](Y,Y) & \overset{\tau}{\simeq} & \mathcal{C}(Y,Z) \end{array}$$

Let $\beta = F_S(s)^{-1}$. Then $s^{\dagger}(\beta) = 1$, $g^{\dagger}(\beta) = \alpha$. Thus

$$s^* \tau(\beta) = \tau(1) = e = s^*(t),$$

so $\tau(\beta) = t$. Then

$$\tau(\alpha) = \tau g^{\dagger}(\beta) = g^* \bar{\tau}(\beta) = g^*(t) = tg,$$

so that $\tau = \bar{\tau}$, as required.

2. In the course of the proof we saw that

$$F_{S*}: \mathcal{C}(-,Z) \simeq \mathcal{C}[S^{-1}](-,Z).$$

This shows that *an S-completion is S-complete,* under the hypotheses of the theorem.

Theorem 1.2 suggests that we should seek useful criteria for S to admit a calculus of left fractions. The following yields a condition verified in applications to general homology theories.

THEOREM 1.3. *Let* S *be a closed family of morphisms of* $\mathcal{C}$ *satisfying:*

(i) if $uv \in S$ *and* $v \in S$, *then* $u \in S$;

(ii) every diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{\ s\ } & \cdot \\ {\scriptstyle f}\big\downarrow & & \\ \cdot & & \end{array}$$

with $s \in S$, *may be embedded in a weak push-out diagram*

$$\begin{array}{ccc} \cdot & \xrightarrow{\ s\ } & \cdot \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \cdot & \xrightarrow[\ t\ ]{} & \cdot \end{array}$$

with $t \in S$.
Then S *admits a calculus of left fractions.*

Notice that a saturated family automatically satisfies (i).

PROOF. Since S is closed, we have only to verify that, given

$$\xrightarrow{\ s\ } \ \underset{g}{\overset{f}{\rightrightarrows}}$$

with $s \in S$, $fs = gs$, we can find $t \in S$ with $tf = tg$.

Form a weak push-out

$$(1.1) \qquad \begin{array}{ccc} \cdot & \xrightarrow{\ s\ } & \cdot \\ {\scriptstyle fs}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ \cdot & \xrightarrow[\ u\ ]{} & \cdot \end{array}$$

with $u \in S$. Since (1.1) is a weak push-out, we may find v with $vu = 1$, $vh = f$. Again, since $fs = gs$, we may find w with $wu = 1$, $wh = g$. By (i) we know that v and w are in S. Now form a commutative square

$$\begin{array}{ccc} \cdot & \xrightarrow{\ v\ } & \cdot \\ {\scriptstyle w}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \cdot & \xrightarrow[\ q\ ]{} & \cdot \end{array}$$

with p (and hence q) in S. Then

$$p = pvu = qwu = q,$$

and

$$pf = pvh = qwh = qg = pg.$$

We turn our attention now to families S admitting a calculus of right fractions. The result corresponding to theorem 1.2 then takes a somewhat different form.

THEOREM 1.4. *Let S be a saturated family of morphisms of $\mathcal{C}$ admitting a calculus of right fractions. Then the object Y is the S-completion of the object Z if and only if there exists $e : Y \to Z$ in S and $\mathcal{C}(s, Z)$ is bijective for every s in S.*

Notice that the conclusion certainly implies that e is co-universal. We may express the fact that $\mathcal{C}(s, Z)$ is bijective for every s in S by saying that Z is *left-closed* for S (see p. 19 of [6]).

PROOF. Assume that Z is the S-completion of Y. Then we refer to the proof of the corresponding part of Theorem 1.2 and observe, first, that a very slight generalization of the argument shows $s^* = \mathcal{C}(s, Z)$ to be bijective and, second, that we again find $\alpha F_S(e) = 1$. Thus it remains to prove that $F_S(e)\alpha = 1$. Now since S admits a calculus of right fractions we may write $\alpha = F_S(g) F_S(s)^{-1}$,

$$Z \xleftarrow{\ s\ } U \xrightarrow{\ g\ } Y,$$

and form the commutative diagram

$$\begin{array}{ccc} \mathcal{C}[S^{-1}](Y, Y) & \overset{\tau}{\simeq} & \mathcal{C}(Y, Z) \\ \downarrow g^{\dagger} & & \downarrow g^* \\ \mathcal{C}[S^{-1}](U, Y) & \overset{\tau}{\simeq} & \mathcal{C}(U, Z) \\ \uparrow s^{\dagger} & & \uparrow s^* \\ \mathcal{C}[S^{-1}](Z, Y) & \overset{\tau}{\simeq} & \mathcal{C}(Z, Z) \end{array}$$

Then $s^{\dagger -1} g^{\dagger}(1) = \alpha$, so $s^{\dagger}(\alpha) = F_S(g)$. But

$$\tau F_S(g) = g^*(e) = eg, \quad \text{and} \quad \tau s^{\dagger}(\alpha) = s^*(1) = s.$$

Thus $eg = s$, whence

$$F_S(e) F_S(g) = F_S(s), \quad \text{and} \quad F_S(e) F_S(g) F_S(s)^{-1} = 1.$$

We now prove the converse. Again, it is a matter of proving that

$$F_{S*}: \mathcal{C}(-,Z) \to \mathcal{C}[S^{-1}](-,Z)$$

is an equivalence.

- To show that F_{S*} is surjective, represent $\alpha \in \mathcal{C}[S^{-1}](X,Z)$ by

$$X \xleftarrow{\ s\ } Z_1 \xrightarrow{\ g\ } Z.$$

Since $s^*: \mathcal{C}(X,Z) \to \mathcal{C}(Z_1,Z)$ is bijective, we find a unique $h: X \to Z$, such that $hs = g$. Then α is also represented by h (since $F_S(h) = F_S(g)F_S(s)^{-1} = \alpha$) so that $F_{S*}(h) = \alpha$.

- Finally we show that F_{S*} is injective. If $F_S(f_1) = F_S(f_2)$, where $f_1, f_2: X \to Z$, then, since S admits a calculus of right fractions, we infer the existence of $s: W \to X$ in S with $f_1 s = f_2 s$. But since $s^*: \mathcal{C}(X,Z) \to \mathcal{C}(W,Z)$ is bijective, this implies that $f_1 = f_2$, completing the proof of the theorem.

It is convenient to exhibit the contrast between the situation when S admits a calculus of left fractions and that in which S admits a calculus of right fractions in the following way. Consider the following three statements about the object Y in $\mathcal{C}$.

A: Z is the S-completion of Y;

B: there exists $e: Y \to Z$ in S and Z is left-closed for S;

C: there exists $e: Y \to Z$ in S terminal in $\mathcal{C}(Y;S)$.

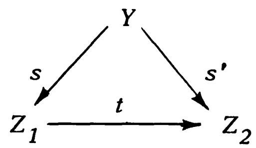
Then we have proved

THEOREM 1.5. *(i) If S is saturated and admits a calculus of left fractions, then* A $\Longleftrightarrow$ B $\Longleftrightarrow$ C;

(ii) if S is saturated and admits a calculus of right fractions then A $\Longleftrightarrow$ B $\Longrightarrow$ C.

The following theorem makes no explicit mention of a calculus of fractions but is concerned with statement C above.

THEOREM 1.6. *Let S be saturated and let $P_Y: \mathcal{C}(Y;S) \to \mathcal{C}$ be the functor given by $P_Y(s) = Z_1$, for $s: Y \to Z_1$ in S, $P_Y(t) = t$ for*



in $\mathcal{C}(Y;S)$. *Then* $\mathcal{C}(Y;S)$ *admits a terminal object if and only if:*

$$\varinjlim P_Y,\ \varinjlim F_S P_Y \text{ exist and } \varinjlim F_S P_Y = F_S(\varinjlim P_Y).$$

Moreover, if $e: Y \to Z$ *is terminal and* $e = \lambda_s s$, $\lambda_s : Z_1 \to Z$, *then*

$$\varinjlim P_Y = \{Z; \lambda_s\}.$$

PROOF. Of course if a category I admits a terminal object e, then, for for any functor $F: I \to D$, $\varinjlim F$ exists and is $\{Fe; F\phi_s\}$, where ϕ_s : $s \to e$ in I is the morphism to e. Thus if $\mathcal{C}(Y;S)$ admits a terminal object $e: Y \to Z$, then

$$\varinjlim P_Y = \{Z;\lambda_s\},\quad \varinjlim F_S P_Y = \{Z; F_S \lambda_s\} = F_S(\varinjlim P_Y).$$

Now suppose that $\varinjlim F_S P_Y = F_S(\varinjlim P_Y)$, and let $\varinjlim P_Y = \{Z;\lambda_s\}$, for some $\lambda_s : Z_1 \to Z$ in $\mathcal{C}$, where $s: Y \to Z_1$ in S. Then $\lambda_{s'} t = \lambda_s$ if $ts = s'$ in S. Also

$$F_S(\varinjlim P_Y) = \{Z; F_S \lambda_s\},\ \text{so}\ \varinjlim F_S P_Y = \{Z; F_S \lambda_s\}.$$

Since $\mathcal{C}(Y;S)$ has an initial object 1_Y, and since $F_S P_Y t$ is invertible, it follows that $F_S \lambda_s$ is invertible for each s, so that λ_s is in S. Let $\lambda_{1_Y} = e$, where $e: Y \to Z$. Then

$$\lambda_s s = e, \text{ for all } s: Y \to Z_1 \text{ in } S.$$

We infer that $\lambda_e \lambda_s = \lambda_s$ for all s, so that, $\{Z;\lambda_s\}$ being a colimit, $\lambda_e = 1_Z$. Thus, if $ts = e$, $t = \lambda_s$, so that e is terminal in $\mathcal{C}(Y;S)$.

We will revert to this theorem in Section 2. It is plainly significant for the S-completability of Y if S admits a calculus of left (or right) fractions, in view of Theorem 1.5.

Remarks analogous to the two which followed Theorem 1.2 are valid in the context of families S admitting a calculus of right fractions. The analog of Theorem 1.3 follows immediately by duality:

THEOREM 1.3*. *Let* S *be a closed family of morphisms of* $\mathcal{C}$ *satisfying:*

(i) if $vu \in S$ and $v \in S$, then $u \in S$;

(ii) every diagram

$$\begin{array}{ccc} & & \\ & & \downarrow f \\ & \xrightarrow{\ s\ } & \end{array}$$

with $s \in S$, may be embedded in a weak pull-back diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{\ t\ } & \cdot \\ g \downarrow & & \downarrow f \\ \cdot & \xrightarrow{\ s\ } & \cdot \end{array}$$

with $t \in S$.
Then S admits a calculus of right fractions.

As we shall see in the next two sections, saturated families S admitting a calculus of left fractions arise naturally in the context of the Adams completion with respect to a homology theory. We close this section by describing an interesting family S which is seen to admit a calculus of right fractions by fulfilling the hypotheses of Theorem 1.3*.

EXAMPLE 1.7 Let $\mathfrak{N}$ be the category of nilpotent groups, and let P be a family of primes. Describe a homomorphism $\phi : G \to H$ in $\mathfrak{N}$ as a *P-isomorphism* if (i) the kernel of ϕ consists of elements of finite order prime to P, and (ii) for each $y \in H$, there exists n prime to P such that $y^n \in$ $\in im\,\phi$. It is easy to see that the family S of P-isomorphisms is closed. To show that it satisfies condition (i) of Theorem 1.3*, we must invoke the following fact about nilpotent groups (see Corollary 6.2 of [8]):

LEMMA 1.8. *If G is nilpotent of class c, and $a, b \in G$ with $b^n = 1$, then $(ab)^{n^c} = a^{n^c}$.*

For suppose we have $G \xrightarrow{u} H \xrightarrow{v} K$ in $\mathfrak{N}$, with vu and v P-isomorphisms. We prove that u is a P-isomorphism. First $ker\,u \subseteq ker\,vu$, so condition (i) on a P-isomorphism is satisfied. Second, let $y \in H$. Then, since vu is a P-isomorphism, there exists n prime to P with $v(y^n) = vu(x)$, $x \in G$. Since v is a P-isomorphism, we infer that $y^n =$ $= u(x)z$, $z \in ker\,v$, so that $z^m = 1$ with m prime to P. By Lemma 1.8, $y^{m^c n} = u(x^{m^c})$, where $nil\,H = c$, and $m^c n$ is prime to P. Thus u is a P-isomorphism.

We now verify condition (ii) of Theorem 1.3*. We suppose given

$$\begin{array}{ccc} & & G_1 \\ & & \downarrow f \\ G_2 & \xrightarrow{s} & H \end{array}$$

in $\mathfrak{N}$, where s is a P-isomorphism. We form the pull-back in the category of groups,

$$(1.2) \qquad \begin{array}{ccc} G & \xrightarrow{t} & G_1 \\ \downarrow g & & \downarrow f \\ G_2 & \xrightarrow{s} & H \end{array}$$

Now G is a subgroup of $G_1 \times G_2$. Since G_1, G_2 are nilpotent, so is $G_1 \times G_2$ and so therefore is G. Thus (1.2) is certainly a pull-back in $\mathfrak{N}$, so it remains to show that t is a P-isomorphism. Since $\ker t = \ker s$, condition (i) on a P-isomorphism is certainly satisfied. Now let $y \in G_1$. Since s is a P-isomorphism, there exists n prime to P such that

$$f(y^n) = s(x), \quad x \in G_2.$$

But then $(y^n, x) \in G$ and $t(y^n, x) = y^n$, so that t is a P-isomorphism.

In fact (see Section 6 of [8]), the P-isomorphisms of $\mathfrak{N}$ are precisely the homomorphisms whose *P-localizations* (see Example 2.10) are isomorphisms. Thus we may, in fact, treat this example by the general theory of Section 2, to infer that S also admits a calculus of left fractions.

2. Global S-completions.

In this section we consider, as before, a family S of morphisms of a category $\mathcal{C}$; now, however, assume that every object Y in $\mathcal{C}$ admits an S-completion Z. The situation is now considerably simplified, due to the following observations.

PROPOSITION 2.1. (Proposition 10 of [5]). *Let $F: \mathcal{C} \to \mathfrak{D}$ be a functor and suppose that, for each Y in $\mathfrak{D}$, there exists Z in $\mathcal{C}$ such that*

(2.1) $\qquad \mathcal{C}(X, Z) \cong \mathfrak{D}(FX, Y)$, *naturally in X.*

Then $Y \mapsto Z$ *determines a functor* $G : \mathfrak{D} \to \mathcal{C}$ *such that (2.1) expresses an adjunction* $F \dashv G$.

COROLLARY 2.2. *Every object* Y *in* $\mathcal{C}$ *admits an S-completion if and only if* $F_S : \mathcal{C} \to \mathcal{C}[S^{-1}]$ *possesses a right adjoint* G_S*; and the S-completion of* Y *is then* $G_S(Y)$.

PROPOSITION 2.3. *If* $F_S : \mathcal{C} \to \mathcal{C}[S^{-1}]$ *has a right adjoint* G_S*, then* G_S *is full and faithful.*

PROOF. This is essentially contained in Proposition 1.3 of [6], except that Gabriel-Zisman assume S to be saturated. On the other hand, it is plain that this assumption is not necessary for the conclusion. For let $\bar{S}$ be the saturation of S, that is, $\bar{S}$ is the family of morphisms f of $\mathcal{C}$ such that $F_S(f)$ is invertible. Then there is clearly an equivalence of categories $\Omega : \mathcal{C}[S^{-1}] \cong \mathcal{C}[\bar{S}^{-1}]$ such that $\Omega F_S = F_{\bar{S}}$.

PROPOSITION 2.4. *If* $F : \mathcal{C} \to \mathfrak{D}$, $G : \mathfrak{D} \to \mathcal{C}$, $F \dashv G$, *and* G *is full and faithful, then the unit* $e : 1 \to GF$ *of the adjunction belongs to* S*, where* S *is the family of morphisms of* $\mathcal{C}$ *rendered invertible by* F.

PROOF. This is to be found on p. 8 of [6], but we will give the easy proof. Since G is full and faithful, the co-unit $\delta : FG \to 1$ of the adjunction is an equivalence. But $\delta F \circ Fe = 1$, so Fe is an equivalence.

PROPOSITION 2.5. *If* $F : \mathcal{C} \to \mathfrak{D}$, $G : \mathfrak{D} \to \mathcal{C}$, $F \dashv G$, *and* G *is full and faithful, then* S *admits a calculus of left fractions, where* S *is the family of morphisms of* $\mathcal{C}$ *rendered invertible by* F. *Moreover, the unit* $e : Y \to GFY$ *is then terminal in* $\mathcal{C}(Y; S)$.

PROOF. The first part is to be found on p. 15 of [6], but we again give the easy proof. Of course, S is saturated so we have only to prove properties (a), (b) below. Notice that, by Proposition 2.4, e is in S.

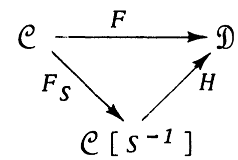
Property (a) states that $\begin{array}{ccc} \cdot & \xrightarrow{s} & \cdot \\ {\scriptstyle f}\downarrow & & \\ \cdot & & \end{array}$ $s \in S$, may be embedded in

$\begin{array}{ccc} \cdot & \xrightarrow{s} & \cdot \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \cdot & \xrightarrow{t} & \cdot \end{array}$, with $t \in S$. Now let v' be adjoint to $F(s)^{-1}$. Then $v's = e$, so we may take

$$g = GFf \circ v', \quad t = e.$$

Property (b) states that if $fs = gs$, $s \in S$, then there exists $t \in S$ with $tf = tg$. But, since Fs is invertible, $Ff = Fg$, so that $ef = eg$.

To prove the final statement of the proposition, observe that we have a commutative diagram



and that, since F admits a fully faithful right adjoint G, it follows from Proposition 1.3 of [6] that H is an equivalence of categories. Thus GH is right adjoint to F_S, with unit $e: 1 \to GF = G_S F_S$, where $G_S = GH$. Thus the statement follows from Corollary 2.2 and Theorem 1.2, since we know that S admits a calculus of left fractions.

THEOREM 2.6. *Let S be a saturated family of morphisms of $\mathcal{C}$ and suppose that every object of $\mathcal{C}$ has an S-completion. Then S admits a calculus of left fractions.*

PROOF. By Corollary 2.2, F_S has a right adjoint G_S, which is full and faithful by Proposition 2.3. We now apply Proposition 2.5 with

$$\mathfrak{D} = \mathcal{C}[S^{-1}], \quad F = F_S, \quad G = G_S,$$

noting that, since S is saturated, S is indeed the family of morphisms of $\mathcal{C}$ rendered invertible by F_S.

Theorem 1.2 retains little of its potency in this global context, since we have utilized the conclusion (Proposition 2.4) that e belongs to S to prove that S admits a calculus of left fractions; notice however that the unit of the adjunction $F_S \dashv G_S$ is, in fact, the co-universal morphism of Theorem 1.2.

Suppose now that, as in Theorem 2.6, the functor $F_S: \mathcal{C} \to \mathcal{C}[S^{-1}]$ has a right adjoint G_S. Let $\mathcal{C}_S$ be the full subcategory of $\mathcal{C}$ generated by the objects $G_S Y = G_S F_S Y$ for all Y in $\mathcal{C}$. We call $\mathcal{C}_S$ the category of

S-complete objects of $\mathcal{C}$ or *of S-completions.*

PROPOSITION 2.7. *$\mathcal{C}_S$ is a full reflective subcategory of $\mathcal{C}$, equivalent to $\mathcal{C}[S^{-1}]$.*

PROOF. The functor G_S factors as $\mathcal{C}[S^{-1}] \xrightarrow{Q} \mathcal{C}_S \xrightarrow{E} \mathcal{C}$, where E is the embedding. Q is full and faithful (since G_S is) and surjective on objects; hence it is an equivalence.

Now Q has a «quasi-inverse» $R: \mathcal{C}_S \to \mathcal{C}[S^{-1}]$ which is a two-sided adjoint to Q. Denote QF_S by L. Then $L: \mathcal{C} \to \mathcal{C}_S$, and S is precisely the family of morphisms of $\mathcal{C}$ rendered invertible by L; moreover,

$$L = QF_S \dashv G_S R = EQR \cong E,$$

so that

$$(2.2) \qquad L \dashv E, \quad L: \mathcal{C} \to \mathcal{C}_S, \quad E: \mathcal{C}_S \subseteq \mathcal{C}.$$

Then the S-completion of Y is $ELY = G_S F_S Y$; we write Y_S for ELY.

In the following statement we attempt a comprehensive picture of global S-completions.

THEOREM 2.8. *Let S be a saturated family of morphisms of $\mathcal{C}$ and let $\mathcal{F}_S$ be the family of functors F with domain $\mathcal{C}$ such that S is precisely the collection of morphisms rendered invertible by F. Then the following four statements are equivalent:*

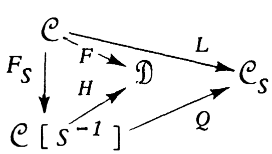
(i) Every object Y in $\mathcal{C}$ admits an S-completion;

(ii) F_S has a right adjoint G_S;

(iii) $\mathcal{F}_S$ contains a reflector $L: \mathcal{C} \to \mathcal{C}_S$, left adjoint to a full embedding $E: \mathcal{C}_S \subseteq \mathcal{C}$;

(iv) There is an F in $\mathcal{F}_S$ admitting a fully faithful right adjoint G. Moreover, any of the above implies

(v) There are unique equivalences H, Q such that the diagram



commutes;

(vi) Any functor of the form VF_S*, with* V *an equivalence, belongs to* $\mathfrak{F}_S$ *and has a fully faithful right adjoint;*

(vii) S *admits a calculus of left fractions.*

PROOF. (i) <==> (ii) by Corollary 2.2;
(ii) ==> (iii) by the immediately preceding argument;
(iii) ==> (iv) trivially;
(iv) ==> (ii) by the argument proving the last part of Proposition 2.5;
(ii) ==> (v) again by the argument of the last part of Prop. 2.5, since F and L have fully faithful right adjoints (see Proposition 1.3 of [6]);
(ii) ==> (vi) obviously, since V and F_S both have fully faithful right adjoints;
(ii) ==> (vii) Theorem 2.6.

Now let $\overline{\mathfrak{F}}_S$ be the subfamily of $\mathfrak{F}_S$ consisting of those F admitting a fully faithful right adjoint. Then each of the first four statements of Theorem 2.8 is equivalent to the assertion that $\overline{\mathfrak{F}}_S$ is not empty. Moreover, the argument of the last part of Proposition 2.5 may again be invoked to point out that, when $\overline{\mathfrak{F}}_S$ is not empty, it consists precisely of the functors VF_S of (vi), where V is an equivalence. Of course, L is then a member of $\overline{\mathfrak{F}}_S$. Proposition 2.5 asserts effectively that S-completions may be obtained using any F in $\overline{\mathfrak{F}}_S$.

The following result shows the dual role played by S-completion.

THEOREM 2.9. *Let* S *be a saturated family of morphisms of* $\mathcal{C}$*, and let every object of* $\mathcal{C}$ *admit an* S*-completion. Then the unit* $e: Y \to Y_S$ *belongs to* S *and is universal for morphisms to* S*-complete objects and co-universal for morphisms in* S.

PROOF. e is in S by Proposition 2.4; it is universal since it is the unit of $L \dashv E$, and co-universal by Theorems 1.2 and 2.6.

We now «globalize» Theorem 1.6. The theorem takes the following form:

THEOREM 2.10. *Let* S *be a saturated family of morphisms of* $\mathcal{C}$. *Then* *the following three statements are equivalent:*

(i) Every object Y in $\mathcal{C}$ admits an S-completion;

(ii) S admits a calculus of left fractions, $\varinjlim P_Y$ exists for all Y, where $P_Y : \mathcal{C}(Y;S) \to \mathcal{C}$, and F_S commutes with $\varinjlim P_Y$;

(iii) S admits a calculus of left fractions, $\varinjlim P_Y$ exists for all Y and F_S commutes with all colimits in $\mathcal{C}$.

PROOF. (i) $\Longrightarrow$ (iii): By Theorem 2.8, S admits a calculus of left fractions and F_S admits a right adjoint. Thus, by Theorem 1.5 (i), $\mathcal{C}(Y;S)$ admits a terminal object for all Y in $\mathcal{C}$ and so, by Theorem 1.6, $\varinjlim P_Y$ exists for all Y; and F_S, admitting a right adjoint, commutes with all colimits in $\mathcal{C}$.

(iii) $\Longrightarrow$ (ii): Trivial.

(ii) $\Longrightarrow$ (i): Theorem 1.6 and Theorem 1.5 (i).

Of course, we may combine Theorem 2.8, 2.10 to obtain an enlarged set of conditions equivalent to the existence of global S-completions.

We will be giving several examples of global S-completions in the next section, in connection with our study of Adams completions. Here we give two examples outside the context of topological homology theory.

EXAMPLE 2.9 [3, 7]. Let A be an abelian category with sufficient injectives and let $\mathcal{C}^+(A)$ be the category of positive cochain complexes over A and homotopy classes of cochain maps. Let S be the family of morphisms of $\mathcal{C}^+(A)$ inducing cohomology isomorphisms. Then with each object C of $\mathcal{C}^+(A)$ we may associate an object C_S of $\mathcal{C}^+(A)$, whose constituents are injective objects of A, and a morphism $e: C \to C_S$ in S. Indeed, C_S is the S-completion of C, since one may verify the co-universal property.

EXAMPLE 2.10. We take up again Example 1.7. Then we call a nilpotent group G *P-local* [9] if the function $x \mapsto x^n$, $x \in G$, is a bijection for all n prime to P. It may be shown (see, e.g., [8]) that every nilpotent group G admits a *P-localization* $e: G \to G_P$ and, in fact, that G_P is the S-completion of G, where S is the family of P-isomorphisms of $\mathfrak{N}$. Thus S admits a calculus of left fractions; of course, here one knows that S is saturated because it is precisely the family of morphisms rendered invertible by the P-localization functor.

Although, in this example, S admits a calculus of left fractions, it is by no means clear that it fulfills condition (ii) of Theorem 1.3.

Finally, we give an example of what may be regarded as the dual situation. We would say that the *S-cocompletion* of Y in $\mathcal{C}$ is an object Z which represents the functor

$$\mathcal{C}[S^{-1}](Y,\cdot):\mathcal{C}\to Ens.$$

It is plain how to formulate statements and results for cocompletions corresponding to those given for completions.

EXAMPLE 2.11. Let $\mathcal{C}$ be the category of based path-connected topological spaces and based homotopy classes of continuous maps, and let S be the family of morphisms of $\mathcal{C}$ inducing homotopy isomorphisms. Then each object Y of $\mathcal{C}$ admits an S-cocompletion Y_S; namely, Y_S is the geometrical realization of the singular complex of Y. The canonical map $e: Y_S \to Y$ is classical (see, e.g., [13]).

Of course, Example 2.9 may be dualized to provide an example of global S-cocompletion.

3. Adams Completions.

Let $\mathcal{T}$ be a full subcategory of the category of based topological spaces and based continuous maps. We suppose that $\mathcal{T}$ contains singletons and also that it contains entire based homotopy types. Let $f: X\to Y$ be in $\mathcal{T}$ and let i_f embed X in M_f, the mapping cylinder of f; then it follows that i_f is also in $\mathcal{T}$. Given a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \\ Z & & \end{array} \tag{3.1}$$

in $\mathcal{T}$ we form the push-out of g and i_f in the category of all spaces,

$$\begin{array}{ccc} X & \xrightarrow{i_f} & M_f \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k} \\ Z & \xrightarrow{u} & W \end{array} \tag{3.2}$$

and we call $\mathcal{T}$ admissible if (3.2) is also in $\mathcal{T}$. Note that, since $\mathcal{T}$ contains singletons, it contains all contractible spaces and hence all cones. It then follows that, if $\mathcal{T}$ is admissible, it contains all mapping cones. For the mapping cone of g is precisely the space W of (3.2) when Y is a singleton.

Let $\mathcal{T}$ be an admissible category in the above sense and let $\tilde{\mathcal{T}}$ be the homotopy category derived from $\mathcal{T}$. Let h be a (generalized) homology theory defined on $\mathcal{T}$ (or $\tilde{\mathcal{T}}$), so that $h_n : \tilde{\mathcal{T}} \to \mathfrak{Ab}$, $-\infty < n < +\infty$, and let $S = S(h)$ be the family of morphisms of $\tilde{\mathcal{T}}$ rendered invertible by h. If Y is in $\tilde{\mathcal{T}}$ and if Y admits an S-completion Y_h, we call Y_h the *Adams completion* or *h-completion* of Y. At this level of generality we do not attempt to enunciate precise conditions for the Adams completion to exist. However, we show that we are, in fact, in a position to apply Theorem 1.2.

THEOREM 3.1. *Let $\mathcal{T}$ be an admissible category and let h be a homology theory defined on $\mathcal{T}$. Let $S = S(h)$ be the family of morphisms of the homotopy category $\tilde{\mathcal{T}}$ rendered invertible by h. Then S is saturated and admits a calculus of left fractions.*

PROOF. It is obvious (see Proposition 1.1) that S is saturated. To prove that S admits a calculus of left fractions, we invoke Theorem 1.3. Thus we must embed the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ {\scriptstyle \sigma}\downarrow & & \\ Z & & \end{array}$$

in $\tilde{\mathcal{T}}$, with σ in S, in a weak push-out diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \rho} \\ Z & \overset{}{\underset{\psi}{\dashrightarrow}} & W \end{array}$$

in $\tilde{\mathcal{T}}$, with ρ in S.

Let f, g be in the classes ϕ, σ respectively. We may then form the push-out diagram (3.2)

(3.2)
$$\begin{array}{ccc} X & \xrightarrow{i_f} & M_f \\ g\downarrow & & \downarrow k \\ Z & \xrightarrow{u} & W \end{array}$$

in $\mathcal{T}$, together with maps $v: M_f \to Y$, $w: Y \to M_f$ such that

$$vw = 1, \quad wv \simeq 1, \quad vi_f = f.$$

Since (3.2) is a push-out in $\mathcal{T}$ (indeed, in the category of all spaces), and since i_f is a cofibration, it follows that u is a cofibration and the diagram (3.2) may be enlarged to

(3.3)
$$\begin{array}{ccccc} X & \xrightarrow{i_f} & M_f & \xrightarrow{l} & C \\ g\downarrow & & \downarrow k & & \| \\ Z & \xrightarrow{u} & W & \xrightarrow{m} & C \end{array}$$

where C is the cokernel of i_f and of u, and l, m are the projections. Moreover, C is the mapping cone of f and hence in $\mathcal{T}$, each horizontal row of (3.3) gives rise to an exact homology sequence, and the vertical maps $(g, k, 1)$ induce a homomorphism of one homology sequence to the other. Since $g \in \sigma \in S$, it follows that g induces homology isomorphisms (in the theory h). Thus, by the 5-lemma, so does k.

Set $\psi = class(u)$, $\rho = class(kw)$. Since w is a homotopy equivalence, kw induces homology isomorphisms, so ρ is in S. Also

$$kwf = kwvi_f \simeq ki_f = ug,$$

so the diagram

(3.4)
$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ \sigma\downarrow & & \downarrow \rho \\ Z & \xrightarrow[\psi]{} & W \end{array}$$

commutes in $\tilde{\mathcal{T}}$. It remains to show that it has the weak push-out property. But this was proved in [4] for the category of all spaces.

Theorem 3.1 not only leaves us free to apply Theorem 1.2; it also renders plausible, in the light of Theorem 2.6, the conjecture that the global Adams completion may exist for a fairly broad class of categories $\mathcal{T}$ and theories h. We give some examples.

EXAMPLE 3.2. Let $\mathcal{T}$ be the category of based spaces of the based homotopy type of 1-connected CW-complexes, and let h be reduced (ordinary) homology with coefficients in $\mathbf{Z}_P$, the integers localized at the family of primes P. Then Theorem 1.1 of [3] shows that the global Adams completion exists and Y_h is just Y_P, the P-localization of Y in $\mathcal{T}$, in the sense of Sullivan [12]. The category $\tilde{\mathcal{T}}_h$ of h-complete objects of $\tilde{\mathcal{T}}$ consists of those CW-complexes for which $\pi_*(Y)$ is a $\mathbf{Z}_P$-module. In this particular case, Proposition 2.7 yields a theorem of Quillen (Theorem 6.1b of [11]), and the adjunction $L \dashv E$, where

$$L : \tilde{\mathcal{T}} \to \tilde{\mathcal{T}}_h, \quad E : \tilde{\mathcal{T}}_h \xrightarrow{\subset} \tilde{\mathcal{T}},$$

generalizes a statement given by Mislin in [10] , who discussed the cases $P = \{ p \}$, $P = \emptyset$.

We may generalize this example to the case of *nilpotent* CW-complexes (see [1]).

EXAMPLE 3.3. Let $\mathcal{T}$ be as in Example 3.2, let $\mathcal{T}_0$ be the subcategory of $\mathcal{T}$ consisting of those Y whose homotopy groups are finitely generated, and let h be reduced homology with coefficients in $\mathbf{Z}/p$. Then the Adams completion exists globally on $\mathcal{T}_0$, and Y_h is precisely the p-profinite completion $\hat{Y}_p$ of Y in the sense of Sullivan [12] . An object Y is h-complete if

$$\pi_*(Y) \otimes \mathbf{Z} \to \pi_*(Y) \otimes \hat{\mathbf{Z}}_p$$

is an isomorphism, where $\hat{\mathbf{Z}}_p$ denotes the p-adic integers [12] .

EXAMPLE 3.4. Generalizing the previous two examples, let $\mathcal{T}$ now be the category of based spaces of the based homotopy type of connected CW-complexes and let h be reduced homology with coefficients in a solid ring R [2]. Then, if Y is R-good [2] , the h-completion of Y is $Y_{\hat{R}}$, the R-completion of Y in the sense of Bousfield-Kan [2]. We recover the earlier examples by taking $R = \mathbf{Z}_P$, $R = \mathbf{Z}/p$, respectively.

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Syracuse University, SYRACUSE, New-York, U.S.A.,

University of British Columbia, VANCOUVER, British Columbia, CANADA,

Battelle Seattle Research Center and University of Washington,
SEATTLE, Washington, U.S.A.