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**The differential spectrum of a ring**

# THE DIFFERENTIAL SPECTRUM OF A RING

*by Paulo RIBENBOIM*

## Abstract

We continue the general study of higher-order derivations of rings, initiated in [2]. We consider filtered derivations of filtered rings, derivations of ringed spaces and show the existence of the ringed space of global differentials of a given ringed space. Then we specialize to the case of affine schemes. The category of rings with derivations as morphisms is isomorphic to the dual of the category of ringed spaces with local stalks, the morphisms being the derivations which induce filtered derivations on the stalks. It is shown that the presheaf of rings of differentials of an affine scheme is already a sheaf, which is called the differential spectrum of order $s$ of the corresponding ring. The paper concludes with a comparison between the differential spectra of $A$ and of $A^*$.

## 1. Derivations of Filtered Rings

Let $\mathbf{N}$ be the set of natural numbers, $S = \mathbf{N}$ or $S = \{0, 1, \ldots, s\}$ with $0 \leqslant s = sup\ S \leqslant \infty$.

Let $A$ be an (associative commutative) ring (with unit element). A family of subgroups $(A^{(n)})_{n \geqslant 0}$ of $A$ defines a *filtration* on $A$ when

$$A = A^{(0)} \supseteq A^{(1)} \supseteq A^{(2)} \supseteq \ldots \quad \text{and} \quad A^{(m)} A^{(n)} \subseteq A^{(m+n)}.$$

Then each $A^{(n)}$ is an ideal of $A$. A ring $A$, together with a filtration $(A^{(n)})_{n \geqslant 0}$, is called a *filtered ring*. If $A$ and $B$ are two filtered rings, a homomorphism $h : A \to B$ such that $h(A^{(n)}) \subseteq B^{(n)}$ is called a filtered homomorphism. Every filtration on $A$ defines a topology, compatible with the ring operations; $(A^{(n)})_{n \geqslant 0}$ is a fundamental system of open neighborhoods of $0$; it follows that each $A^{(n)}$ is also a closed subset of $A$. $A$ is a Hausdorff space if and only if $\underset{n \geqslant 0}{\cap}\ A^{(n)} = 0$. A mapping $h : A \to B$ between filtered rings is continuous when for every $m \geqslant 0$ there exists $n \geqslant 0$ such that $h(A^{(n)}) \subseteq B^{(m)}$; every filtered homomorphism is continuous.

DEFINITION 1. If $A$, $B$ are filtered rings, a derivation $d \in Der_s(A, B)$ is called a *filtered derivation* when $d_\nu(A^{(n)}) \subseteq B^{(n-\nu)}$ for $n \geqslant \nu \geqslant 0$, $\nu \in S$.

If $d \in Der_s(A, B)$, $d' \in Der_s(B, C)$ are filtered derivations then $d' \circ d = (d''_\nu)_{\nu \in S} \in Der_s(A, C)$ is also a filtered derivation (we recall that $d''_\nu = \underset{\lambda + \mu = \nu}{\Sigma} d'_\lambda \circ d_\mu$ for every $\nu \in S$).

If $h \in Hom(A, B)$ is a filtered homomorphism and $d = (d_\nu)_{\nu \in S}$, $d_0 = h$, $d_\nu = 0$, then $d \in Der_s(A, B)$ is a filtered derivation.

If $I$ is an ideal of $A$, $J$ an ideal of $B$ and $d \in Der_s(A, B)$ is such that $d_0(I) \subseteq J$, then $d$ is a filtered derivation (where $A$ has the $I$-adic filtration and $B$ has the $J$-adic filtration). Indeed, if $a_1, \ldots, a_n \in I$ then

$$d_\nu(a_1 \ldots a_n) = \underset{\lambda_1 + \ldots + \lambda_n = \nu}{\Sigma} d_{\lambda_1} a_1 \cdot d_{\lambda_2} a_2 \ldots d_{\lambda_n} a_n;$$

if $n \geqslant \nu$ then at least $n - \nu$ indices are equal to $0$, so each summand is

in $J^{n-\nu}$; this proves that $d_\nu(I^n) \subseteq J^{n-\nu}$.

If $A$ is a filtered ring, $B$ a ring, $d \in Der_s(A, B)$ such that $B = Im(D)$ (subring generated by $\bigcup_{\nu \in S} d_\nu(A)$), then the filtration on $A$ determines a filtration on $B$ in the following way: For every $n \geqslant 0$ let $G^{(n)}$ be the union (for $s \geqslant 1$) of the sets $d_{k_1}(A^{(n_1+k_1)}), \ldots, d_{k_s}(A^{(n_s+k_s)})$, where $\sum_{i=1}^{s} n_i = n$ and $k_i \geqslant 0$. Let $B^{(n)}$ be the ideal generated by $G^{(n)}$; then

$$B = B^{(0)} \supseteq B^{(1)} \supseteq B^{(2)} \supseteq \ldots \quad \text{and} \quad B^{(m)} \cdot B^{(n)} \subseteq B^{(m+n)}.$$

With respect to this filtration $d$ is a filtered derivation, because

$$d_\nu(A^{(n)}) \subseteq G^{(n-\nu)} \subseteq B^{(n-\nu)} \quad (\text{for } n \geqslant \nu, \ \nu \in S).$$

Let $\Omega_s(A) = A^*$ be the ring of differentials of order $s$ of $A$ and $\delta \in Der_s(A, \Omega_s(A))$ the universal derivation; by definition, for every ring $B$ and derivation $d \in Der_s(A, B)$ there exists a unique homomorphism $h : A^* \to B$ such that $h \circ \delta = d$.

If $A$ is a filtered ring, since $Im(d) = A^*$ then $A^*$ has also a filtration induced by that of $A$ and $d$ is a filtered derivation. Moreover, for every filtered ring $B$ and filtered derivation $d \in Der_s(A, B)$ there exists a unique homomorphism $h \in Hom(A^*, B)$ such that $h \circ \delta = d$. Indeed, we know already that there exists a unique homomorphism $h \in Hom(A^*, B)$ such that $h \circ \delta = d$ and we shall prove that $h$ is filtered, that is

$$h(A^{*(n)}) \subseteq B^{(n)} \quad (\text{for every } n \geqslant 0).$$

Every element of $A^{*(n)}$ is in the ideal of $A^*$ generated by $\bigcup_{k \in S} \delta_k(A^{(n+k)})$; since $h(\delta_k(a)) = d_k(a) \in B^{(n)}$ for every $a \in A^{(n+k)}$ then $h(A^{*(n)}) \subseteq B^{(n)}$.

## 2. Derivations of Ringed Spaces

Let $(X, \mathcal{O}_X)$ be a presheaf of rings over the topological space $X$. If $U \supseteq V$ are open sets of $X$, $\rho_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ denotes the restriction homomorphism and $\mathcal{O}_{X,x}$ the stalk of $\mathcal{O}_X$ at the point $x \in X$.

DEFINITION 2. A derivation of order $s$ from the presheaf of rings $(X, \mathcal{O}_X)$ to the presheaf of rings $(Y, \mathcal{O}_Y)$ is a couple

$$(\phi, d = (d^U)_{U \text{ open in } Y}), \quad \text{where}$$

1º    $\phi : X \rightarrow Y$ is a continuous map,

2º    $d^U \in Der_s(\mathcal{O}_Y(U), \mathcal{O}_X(\phi^{-1}(U)))$ for every open set $U$ of $Y$,

3º    if $U \supseteq V$ are open sets of $Y$ then

$$\rho_{\phi^{-1}(U), \phi^{-1}(V)} \circ d^U = d^V \circ \rho_{U, V}.$$

Let $d_n = (d_n^U)_{U \ open \ in \ Y}$, for every $n \in S$. Then $(\phi, d_0)$ is a morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$.

For every $x \in X$ the derivation $(\phi, d)$ induces a derivation $d^x \in Der_s(\mathcal{O}_{Y, \phi(x)}, \mathcal{O}_{X, x})$. This follows from [2], page 261.

In order to define a derivation it is sufficient to give the derivations $d^U$, for open sets $U$, belonging to a basis of open sets of $Y$. More precisely, let $(U_i)_{i \in I}$ be a basis of open sets of $Y$, let $\phi : X \rightarrow Y$ be a continuous map and for every $i \in I$ let $d^{(i)} \in Der_s(\mathcal{O}_Y(U_i), \mathcal{O}_X(\phi^{-1}(U_i)))$ such that if $U_i \supseteq U_j$ then

$$\rho_{\phi^{-1}(U_i), \phi^{-1}(U_j)} \circ d^{(i)} = d^{(j)} \circ \rho_{U_i, U_j}.$$

Then there exists a unique derivation of order $s$, $(\phi, (d^U)_{U \ open \ in \ Y})$ from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ such that $d^{U_i} = d^{(i)}$ (for every $i \in I$). The proof is lengthy but easy.

If $(\phi, d) \in Der_s((X, \mathcal{O}_X), (Y, \mathcal{O}_Y))$ and $(\psi, e) \in Der_s((Y, \mathcal{O}_Y), (Z, \mathcal{O}_Z))$, we define a derivation $(\eta, f) \in Der_s((X, \mathcal{O}_X), (Z, \mathcal{O}_Z))$ as follows :

$$\eta = \psi \circ \phi, \quad f = (f^U)_{U \ open \ in \ Z} \quad \text{with} \quad f^U = d^{\psi^{-1}(U)} \circ e^U ;$$

explicitly for every $n \in S$ :

$$f_n^U = \sum_{l + m = n} d_l^{\psi^{-1}(U)} \circ e_m^U .$$

It is easy to verify that $(\eta, f)$ is indeed a derivation from presheaves of rings. With this composition of derivations, we have the category of presheaves of rings with morphisms being the derivations of order $s$.

Every homomorphism $(\phi, h) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ may be viewed as a derivation of order $s$, namely $(\phi, d)$, with $d_0^U = h^U$, $d_n^U = 0$ (for $0 < n \in S$) for every open set $U$ of $Y$.

Hence we may consider the composition of a derivation and a ho-

momorphism.

A presheaf of rings $(X, \mathcal{O}_X)$ is called a *ringed space* if it is a a sheaf of rings. We denote by $sh(X, \mathcal{O}_X)$ the sheaf of rings canonically associated with the presheaf of rings $(X, \mathcal{O}_X)$; it is the sheaf of continuous sections $\Gamma(U, \mathcal{O}_X)$ of $(X, \mathcal{O}_X)$.

If $(\phi, \delta) \in Der_s((X, \mathcal{O}_X), (Y, \mathcal{O}_Y))$ it induces a derivation

$$\Gamma(\phi, \delta) = (\phi, \Gamma\delta) \in Der_s(sh(X, \mathcal{O}_X), sh(Y, \mathcal{O}_Y)).$$

Explicitly for every open set $U$ of $Y$,

$$(\Gamma\delta)^U \in Der_s(\Gamma(U, \mathcal{O}_Y), \Gamma(\phi^{-1}(U), \mathcal{O}_X))$$

is defined by

$$\left[ ((\Gamma\delta)^U_n(\sigma)) \right](x) = \delta^x_n(\sigma(\phi(x))) \quad (\text{for every } n \in S),$$

where $\phi(x) \in U$, $\sigma \in \Gamma(U, \mathcal{O}_Y)$, $\delta^x \in Der(\mathcal{O}_{Y, \phi(x)}, \mathcal{O}_{X, x})$.

PROPOSITION 1. *If $(X, \mathcal{O}_X)$ is a ringed space, there exists a ringed space $(X^*, \mathcal{O}^*_{X^*})$ and a derivation $(\phi^*, \delta^*) \in Der_s((X^*, \mathcal{O}^*_{X^*}), (X, \mathcal{O}_X))$ such that for every ringed space $(Y, \mathcal{O}_Y)$ and every derivation $(\psi, \varepsilon) \in Der_s((Y, \mathcal{O}_Y), (X, \mathcal{O}_X))$ there exists a unique homomorphism $(\theta, h):$ $(Y, \mathcal{O}_Y) \to (X^*, \mathcal{O}^*_{X^*})$ such that $(\phi^*, \delta^*) \circ (\theta, h) = (\psi, \varepsilon)$. Moreover, $(X^*, \mathcal{O}^*_{X^*})$ and $(\phi^*, \delta^*)$ are unique (up to a unique isomorphism) with the above property.*

PROOF. We take $X^* = X$, and for every open set $U$ of $X$ let $\mathcal{O}^*_X(U) = \Gamma(U, \mathcal{O}_X)^*$ (the ring of differentials of order $s$ of the ring $\Gamma(U, \mathcal{O}_X)$). Let $\delta^U \in Der_s(\Gamma(U, \mathcal{O}_X), \mathcal{O}^*_X(U))$ be the universal derivation. If $U \supseteq V$ are open sets of $X$, there exists a unique homomorphism $\rho^*_{U, V}$ such that the diagram commutes:

$$
\begin{array}{ccc}
\Gamma(U, \mathcal{O}_X) & \xrightarrow{\delta^U} & \mathcal{O}^*_X(U) \\
\downarrow{\scriptstyle \rho_{U, V}} & \quad \rho^*_{U, V} & \downarrow \\
\Gamma(V, \mathcal{O}_X) & \xrightarrow{\delta^V} & \mathcal{O}^*_X(V)
\end{array}
$$

Thus $(\mathcal{O}^*_X(U))_{U \text{ open in } X}$ is a presheaf of rings over $X$. Let $(X, \mathcal{O}^*_X)$ be the sheaf associated with this presheaf, so $\Gamma(U, \mathcal{O}^*_X) = \mathcal{O}^*_X(U) = \Gamma(U, \mathcal{O}_X)^*$. It is now straightforward to verify that $(X, \mathcal{O}^*_X)$ and the induced derivation $\Gamma(id_X, \delta) = (id_X, \delta^*)$ from $(X, \mathcal{O}^*_X)$ to $(X, \mathcal{O}_X)$ satisfy the universal property of the statement. ∎

The ringed space $(X, \mathcal{O}^*_X)$ constructed above is called the ringed space of *global differentials* of order $s$ of $(X, \mathcal{O}_X)$; $(id_X, \delta^*)$ is the universal derivation of $(X, \mathcal{O}_X)$.

## 3. Derivations of affine schemes

We consider the category whose objects are (commutative) rings (with unit element) and whose morphisms are derivations of order $s$.

Let $Spec(A) = (X, \mathcal{O}_X)$, $Spec(B) = (Y, \mathcal{O}_Y)$ be the affine schemes belonging to the rings $A$, $B$.

If $\varepsilon \in Der_s(A, B)$ it induces a derivation $(\breve{\varepsilon}_0, \tilde{\varepsilon}) \in Der_s(Spec B, Spec A)$, where $\breve{\varepsilon}_0 : Y \to X$ is defined by $\breve{\varepsilon}_0(y) = x$ when $\varepsilon_0^{-1}(Q_y) = P_x$ ($Q_y$ prime ideal of $B$, $P_x$ prime ideal of $A$); it is a continuous map. To define $\tilde{\varepsilon}$ it suffices to define $\tilde{\varepsilon}^U(f)$ where $f \in A$ and

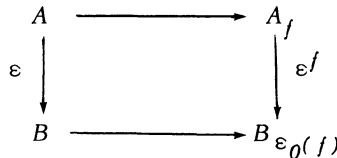$$U(f) = \{ x \in X \mid f \notin P_x \},$$

because $\{ U(f) \mid f \in A \}$ is a basis of open sets of $X$. But

$$\Gamma(U(f), \mathcal{O}_X) = A_f, \quad \breve{\varepsilon}_0^{-1}(U(f)) = U(\varepsilon_0(f))$$

and

$$\Gamma(U(\varepsilon_0(f)), \mathcal{O}_Y) = B_{\varepsilon_0(f)};$$

hence there exists a unique derivation $\varepsilon^f \in Der_s(A_f, B_{\varepsilon_0(f)})$ such that the diagram commutes (see [2] page 258):

$$
\begin{array}{ccc}
A & \longrightarrow & A_f \\
\varepsilon \downarrow & & \downarrow \varepsilon^f \\
B & \longrightarrow & B_{\varepsilon_0(f)}
\end{array}
$$

By the uniqueness, we conclude that, if $U(f) \supseteq U(g)$, then $\varepsilon^f$, $\varepsilon^g$ com-

mute with the canonical restriction homomorphisms. Thus, the derivations $(\varepsilon^f)_{f \in A}$ suffice to determine the derivation $(\check{\varepsilon}_0, \tilde{\varepsilon})$.

For every $y \in Y$, let $x = \check{\varepsilon}_0(y)$ so $\varepsilon_0^{-1}(Q_y) = P_x$. Then $\mathcal{O}_{X,x} = A_{P_x}$ and $\mathcal{O}_{Y,y} = B_{Q_y}$ are local rings (i.e. have only one maximal ideal) and there exists a unique derivation $\varepsilon_y$ such that the diagram commutes

$$
\begin{array}{ccc}
A & \longrightarrow & A_{P_x} \\
\varepsilon \downarrow & & \downarrow \varepsilon_y \\
B & \longrightarrow & B_{Q_y}
\end{array}
$$

Hence $(\varepsilon_y)_0 (A_{P_x} P_x) \subseteq B_{Q_y} Q_y$ and we deduce that, if $n \leqslant m, n \in S$, then

$$(\varepsilon_y)_n (A_{P_x} P_x^m) \subseteq B_{Q_y} Q_y^{m-n}.$$

So $\varepsilon_y$ is a filtered derivation.

This leads to the following

PROPOSITION 2. *The category of rings with derivations of order $s$ is isomorphic to the dual of the category of ringed spaces with local stalks, the morphisms being the derivations of order $s$, which induce filtered derivations on the stalks.*

PROOF. In view of the preceding considerations, it is clear that we have a contravariant functor from the first to the second category.

Let be given a derivation $(\phi, \delta) \in Der_s((Y, \mathcal{O}_Y), (X, \mathcal{O}_X))$ of ringed spaces with local stalks, which induces filtered derivations on the stalks.

Let $A = \Gamma(X, \mathcal{O}_X)$, $B = \Gamma(Y, \mathcal{O}_Y)$, and let

$$\varepsilon = \delta^X \in Der_s(\Gamma(X, \mathcal{O}_X), \Gamma(Y, \mathcal{O}_Y)).$$

We shall show that $(\check{\varepsilon}_0, \tilde{\varepsilon}) = (\phi, \delta)$. Letting $\delta_0 = (\delta_0^U)_{U \ open \ in \ X}$, then

$$(\phi, \delta_0) \in Hom((Y, \mathcal{O}_Y), (X, \mathcal{O}_X)) \quad \text{and} \quad \varepsilon_0 = \delta_0^X \in Hom(A, B).$$

By hypothesis for every $y \in Y$, $\delta_y$ is a filtered derivation from $A_{P_x}$ to $B_{Q_y}$ (where $x = \phi(y)$); in particular, $(\delta_y)_0 (A_{P_x} P_x) \subseteq B_{Q_y} Q_y$, so

$( \delta_y )_0$ is a local homomorphism. But $( \delta_y )_0 = ( \delta_0 )_y$ by the uniqueness of homomorphism localizing $\varepsilon_0 = \delta_0^X : A \to B$. So, it is known that $\check{\varepsilon}_o = \phi$. From the uniqueness of derivation localizing $\varepsilon = \delta^X$, we have $\delta_y = \varepsilon_y$ for every $y \in Y$. This implies that the derivations $\delta$, $\tilde{\varepsilon}$ coincide, since they coincide on each stalk.

The proof is concluded by noting finally that the two correspondences $\varepsilon \mapsto (\check{\varepsilon}_o, \tilde{\varepsilon})$ and $(\phi, \delta) \mapsto \delta^X$ are reciprocal. ∎

## 4. The differential spectrum of a ring

Let $A$ be a ring, let $(X, \mathcal{O}_X)$ be the affine scheme corresponding to $A$. We may show :

PROPOSITION 3. *The presheaf $U \mapsto \Gamma(U, \mathcal{O}_X)^*$ (ring of differentials of order $s$ of the ring $\Gamma(U, \mathcal{O}_X)$) is a sheaf.*

PROOF. The presheaf is already defined over the basis of open sets $U(f) = \{ x \in X \,|\, f \notin P_x \}$ (for $f \in A$). To show that it is a sheaf, we have to verify the sheaf property on each open set of $X$; restricting the sheaf to the open set, we are reduced to prove: if $X = \bigcup_{i \in I} U(f_i)$, if $\sigma_i \in \Gamma(U(f_i), \mathcal{O}_X)^*$ for every $i \in I$, and if $U(f_k) \subseteq U(f_i) \cap U(f_j)$ implies

$$\rho^*_{U(f_i), U(f_k)}(\sigma_i) = \rho^*_{U(f_j), U(f_k)}(\sigma_j),$$

then there exists a unique $\sigma \in \Gamma(X, \mathcal{O}_X)^* = A^*$ such that $\rho^*_{X, U(f_i)}(\sigma) = \sigma_i$ (for every $i \in I$).

First we establish the uniqueness of $\sigma$. If $\sigma, \sigma' \in A^*$ satisfy the required condition, let $\tau = \sigma - \sigma'$, hence $\rho^*_{X, U(f_i)}(\tau) = 0$ for every $i \in I$. So there exists $n_i \geqslant 0$ integer such that $(\delta_0 f_i)^{n_i} \tau = 0$. The ideal of $A^*$ generated by all elements $(\delta_0 f_i)^{n_i}$ is equal to $A^*$; indeed, if $P^*$ is a prime ideal of $A^*$ containing every $(\delta_0 f_i)^{n_i} = \delta_0 f_i^{n_i}$, then $f_i^{n_i} \in \delta_0^{-1}(P^*) = P$ (a prime ideal of $A$), so $f_i \in P$ for every $i \in I$, hence $P \notin \bigcup_{i \in I} U(f_i) = X$, a contradiction. So we may write $1 = \sum_{i \in I} \alpha_i (\delta_0 f_i)^{n_i}$, with $\alpha_i \in A^*$, $\alpha_i = 0$ except for finitely many indices. Thus

$$\tau = \sum_{i \in I} \alpha_i (\delta_0 f_i)^{n_i} \tau = 0.$$

Now we prove the existence of $\sigma$. Since $X$ is quasi-compact there

exists a finite subset $J$ of $I$ such that $X = \bigcup_{j \in J} U(f_j)$. But

$$\sigma_j = \frac{z_j}{(\delta_0 f_j)^{m_j}} \quad \text{with} \quad z_j \in A^*$$

and $J$ is finite, hence we may assume that $m_j = m$ for all $j \in J$. We have $U(f_i f_j) \subseteq U(f_i) \cap U(f_j)$, hence $\sigma_i$, $\sigma_j$ have the same image in $A^*_{\delta_0(f_i f_j)}$, namely

$$\frac{z_i (\delta_0 f_j)^m}{\delta_0(f_i f_j)^m} = \frac{z_j (\delta_0 f_i)^m}{\delta_0(f_i f_j)^m}$$

(taking $i, j \in J$). So there exist integers $m_{ij} \geqslant 0$ (for $i, j \in J$) such that

$$\delta_0(f_i f_j)^{m_{ij}} [z_i (\delta_0 f_j)^m - z_j (\delta_0 f_i)^m] = 0$$

in $A^*$; since $J$ is finite, we may also assume that all $m_{ij}$ are equal, say to $n$. Letting $z'_i = z_i (\delta_0 f_i)^n$ we have

$$z'_i (\delta_0 f_j)^{m+n} = z'_j (\delta_0 f_i)^{m+n} .$$

Noting that $X = \bigcup_{j \in J} U(f_j)$ we deduce as before that

$$1 = \sum_{j \in J} \alpha_j (\delta_0 f_j)^{m+n} \quad \text{with} \quad \alpha_j \in A^*.$$

Thus we may take $\sigma = \sum_{j \in J} \alpha_j z'_j$. Indeed $\rho^*_{X, U(f_i)}(\sigma) = \sigma_i$ for every $i \in J$ because

$$(\delta_0 f_i)^{m+n} \sigma = \sum_{j \in J} \alpha_j z'_j (\delta_0 f_i)^{m+n} = \sum_{j \in J} \alpha_j (\delta_0 f_j) z'_i = z'_i .$$

Next, if $i \in I$, $i \notin J$ let $J' = \{i\} \cup J$ so $\{U(f_j)\}_{j \in J'}$ is still a covering of $X$ and we deduce the existence of $\sigma' \in A^*$ with a similar property. From $\rho^*_{X, U(f_j)}(\sigma') = \rho^*_{X, U(f_j)}(\sigma)$ for every $j \in J$, we conclude by the uniqueness that $\sigma' = \sigma$ and therefore for every $i \in I$ we have

$$\rho^*_{X, U(f_i)}(\sigma) = \sigma_i . \quad \blacksquare$$

DEFINITION 3. The sheaf of rings $U \mapsto \Gamma(U, \mathcal{O}_X)^*$ is called the *differential spectrum of order $s$ of the ring $A$*; we denote it by $Diff\ Spec_s(A)$.

We combine the previous results. If $\varepsilon \in Der_s(A, B)$, let $(\breve{\varepsilon}_0, \tilde{\varepsilon}) \in Der_s(Spec\ B, Spec\ A)$ be the induced derivation, let

$$( id, \delta^A ) \in Der_s (Diff \ Spec_s A, Spec A),$$

$$( id, \delta^B ) \in Der_s ( Diff \ Spec_s B, Spec B )$$

be the universal derivations. Then there exists a unique homomorphism

$$( \check{\varepsilon}_0, b ) : Diff \ Spec_s B \to Diff \ Spec_s A$$

such that the following diagram is commutative :



Moreover, $b$ induces filtered homomorphisms on the stalks.

We consider now the special case where $B = A^*$ ( the ring of differentials of order $s$ of $A$ ) and $\varepsilon$ is the universal derivation of $A$. We have the commutative diagram :



In this situation we have shown in [2], page 103, that the continuous map $\check{\varepsilon}_0 : X^* \to X$ is surjective.

## Bibliographie

[1] I.G. MACDONALD, *Algebraic Geometry, Introduction to Schemes,* Benjamin, 1968, New-York.

[2] P. RIBENBOIM, Higher Derivations of Rings I, II. *Revue Roumaine de Math.*, 16 (1971), p. 77-110 and 245-272.

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