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ON A MECHANISM OF DEFINING MORPHISMS IN CONCRETE CATEGORIES

by L. KUČERA *and* A. PULTR

A concrete category (K, U) (i.e., a category K together with a faithful functor $U: K \to Set$) is fully described if we know, for every two objects a and b, which mappings from the set $U(a)$ into the set $U(b)$ are admissible (i.e., carry morphisms) and which of them are not. The present paper deals with the question of a mechanism for picking up the admissible mappings.

We shall show that every (K, U) satisfying certain conditions (the condition (E) formulated in § 2, satisfied e.g. in every (K, U) such that for every cardinal m there is only a set of non-isomorphic objects a with $card\ U(a) = m$ and that

(R) every morphism α of K can be written as $\mu \circ \varepsilon$ with $U(\mu)$ one-to-one and $U(\varepsilon)$ onto)

can be looked upon in the following way:

A functor $F: Set \to Set$ is given, the objects of K are some couples (X, r), where X is a set and $r \subset F(X)$, and the admissible mappings from (X, r) into (Y, s) are exactly those $f: X \to Y$ for which $F(f)(r) \subset s$.

In particular, every concrete category resulting from a Bourbaki structure construction ([1] IV.2.1) and satisfying (R) above can be described like that. For this case we show that the functors F may be chosen such that cardinality of $F(X)$ does not exceed the *exp exp exp* of the cardinality of the last step in the structure construction on X.

This mechanism of choice of morphisms among mappings between underlying sets has been already studied in several papers ([3], [4], [5]) and many everyday life concrete categories have been observed to be subjected to it. The aim of this paper is to show its universality. In fact, we prove that the mentioned condition (E) is necessary and sufficient, the necessity being almost trivial, though.

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1. Preliminaries.

In this paragraph we recall some definitions from quoted papers and prove simple lemmas. The category of sets and all mappings is denoted by *Set*, the functors from *Set* into *Set* are referred to as set functors.

1.1. DEFINITION. Let F be a set functor. The category $S(F)$ is defined as follows:

- The objects are couples (X, r) with X a set and $r \subset F(X)$, the morphisms from (X, r) into (Y, s) are triples $((X, r), f, (Y, s))$ such that $f: X \to Y$ is a mapping with $F(f)(r) \subset s$,

- The composition is given by the formula

$$((X, r), f, (Y, s)) \; ((Z, t), g, (X, r)) = ((Z, t), fg, (Y, s)).$$

Throughout this paper, $S(F)$ will be considered as a concrete category endowed with the functor sending (X, r) into X and $((X, r), f, (Y, s))$ into f.

1.2. DEFINITION. Let (K, U), (L, V) be concrete categories. (K, U) is said to be *w-realizable in* (L, V), if there is a full and faithful $\Phi : K \to L$ such that $V \circ \Phi = U$. Such a Φ is called a *w-realization of* (K, U) in (L, V).

If it is a full embedding (i.e. full and one-to-one), we speak about *realization* and realizability.

REMARK. The realizability of (K, U) in (L, V) means that (K, U) is equivalent to some $(L', V / L')$, where L' is a full subcategory of L. If (K, U) is w-realizable in (L, V) and some mechanism is able to determine the morphisms of (L, V) among the mappings of underlying sets, it will do the job in (K, U), too.

1.3. DEFINITION. A concrete category (K, U) is said *to have the property* (I) if

$$(U(\alpha) = 1 \text{ and } \alpha \text{ is an isomorphism}) \Longrightarrow \alpha = 1.$$

1.4. REMARK. $S(F)$ obviously has (I).

1.5. LEMMA. *Let Φ be a w-realization of (K, U) in (L, V), let (L, V) have (I). Then Φ is a realization if and only if (K, U) has (I).*

PROOF. If Φ is one-to-one and $U(\alpha) = V \circ \Phi(\alpha) = 1$ for an isomorphism α, we have $\Phi(\alpha) = 1$ and hence $\alpha = 1$. If Φ is not one-to-one, take objects $a \neq b$ with $\Phi(a) = \Phi(b)$. Since Φ is full, there is an isomorphism $\alpha : a \to b$ with $\Phi(\alpha) = 1_{\Phi(a)}$. Thus, $U(\alpha) = V \circ \Phi(\alpha) = 1$.

1.6. LEMMA. *For every concrete category (K, U) there is a w-realization Φ in (L, V) such that (L, V) has (I) and Φ is onto.*

PROOF: Write $a \sim b$ if there is an isomorphism $\alpha : a \to b$ with $U(\alpha) = 1$. The relation $\sim$ is obviously an equivalence. For every equivalence class C choose a representant a_C and put $\Phi(b) = a_C$ for $b \in C$. Let L be the full subcategory of K generated by the objects $\Phi(b)$, put $V = U \mid L$. Denote by α_a the isomorphism $\alpha_a : a \to \Phi(a)$ with $U(\alpha_a) = 1$ and put

$$\Phi(\phi) = \alpha_b \circ \phi \circ \alpha_a^{-1} \text{ for } \phi : a \to b.$$

We see easily that Φ is a w-realization.

2. The condition (E) and others.

2.1. DEFINITION. Let (K, U) be a concrete category. For an object $a \in K$ and a one-to-one mapping $m : X \to U(a)$ denote by $S(m, a)$ (more exactly, $S(m, a, K, U)$) the class

$$\{(c, f) \mid \exists\, \mu : c \to a \text{ in } K \text{ with } U(\mu) = m \circ f\}.$$

A *U-image* of a morphism $\phi : a \to b$ is an $S(m, b)$ such that $U(\phi) = m \circ p$ for some surjective p.

2.2. LEMMA. *For a one-to-one mapping f,*

$$(c, g) \in S(m \circ f, a) \iff (c, f \circ g) \in S(m, a).$$

PROOF. Follows immediately by definition.

2.3. LEMMA. *For a one-to-one mapping f,*

$$S(m, a) = S(n, b) \implies S(m \circ f, a) = S(n \circ f, b).$$

PROOF. This is an immediate consequence of 2.2.

2.4. DEFINITION. Morphisms α and β are said to be *parallel* (notation $\alpha // \beta$) if they have a common U-image.

2.5. LEMMA. *Identities 1_a and 1_b are parallel iff a is isomorphic to b.*

PROOF. If there is an isomorphism $\alpha : a \to b$, we see easily that

$$S(1_{U(a)}, a) = S(U(\alpha), b).$$

On the other hand, let $S(m, a) = S(n, b)$ be a common U-image of 1_a and 1_b. There are surjections p and q such that $m \circ p = U(1_a)$, $n \circ q = U(1_b)$. But then necessarily also $p \circ m$ and $q \circ n$ are identities. We have $(a, p) \in S(m, a)$ and hence $(a, p) \in S(n, b)$, so that there is an $\alpha : a \to b$ with $U(\alpha) = n \circ p$. Similarly there is a $\beta : b \to a$ with $U(\beta) = m \circ q$. Thus

$$U(\alpha\beta) = n \circ p \circ m \circ q = 1, \quad U(\beta\alpha) = m \circ q \circ n \circ p = 1.$$

Hence, α is an isomorphism.

2.6. LEMMA. *$//$ is an equivalence relation.*

PROOF. Let $S(m, a) = S(m', b)$ be a common U-image of $\alpha : a' \to a$, $\beta : b' \to b$, $S(n, b) = S(n', c)$ a common image of β and $\gamma : c' \to c$. We find easily an invertible mapping i with $m' = n \circ i$. By 2.3, hence,

$$S(m, a) = S(m', b) = S(n \circ i, b) = S(n' \circ i, c)$$

which is a U-image of γ. Thus, $//$ is transitive. The reflexivity and symmetry is evident.

2.7. LEMMA. *Let $\alpha // \beta$, let $S(m, a)$ be a U-image of α, $m : X' \to U(a)$, $S(n, b)$ a U-image of β, $n : Y \to U(b)$. Then there exists an invertible $f : X \to Y$ such that*

$$(c, g) \in S(m, a) \iff (c, f \circ g) \in S(n, b).$$

PROOF. Let $S(m', a) = S(n', b)$ be a common U-image of α and β. We find easily invertible i and j with $m' \circ i = m$, $n \circ j = n'$. Put $f = j \circ i$. By 2.3.

$$S(n \circ f, b) = S(n \circ j \circ i, b) = S(n' \circ i, b) = S(m' \circ i, a) = S(m, a).$$

Thus, the statement follows by 2.2.

2.8. DEFINITION. Let m be a cardinal number, (K, U) a concrete category. Denote by $O_m(K, U)$ the class of all objects a of K with $card\, U(a) < m$, by $M_m(K, U)$ the class of all morphisms $\alpha : a \to b$ of K such that $card\, U(\alpha)(U(a)) < m$. (K, U) is said to have the property:

(S) if there is a class A of objects of K such that for every object of K there is an isomorphic one in A, and that $A \cap O_m(K, U)$ is a set for every m.

(E) if there is a class A of morphisms of K such that for every morphism of K there is a parallel one in A, and that $A \cap M_m(K, U)$ is a set for every m.

2.9. REMARK. By 2.5 we see easily that (E) implies (S).

2.10. LEMMA. *Let Φ be a w-realization of (K, U) in (L, V), let Φ map K onto L. If (K, U) has the property (S) ((E) resp.), then (L, V) has (S) ((E) resp.).*

PROOF. It suffices to take the image of the class A under Φ.

2.11. DEFINITION. (K, U) is said to have the property:

(R) If for every $\phi : a \to b$ in K there are $\varepsilon : a \to c$ and $\mu : c \to b$ such that $U(\varepsilon)$ is onto, $U(\mu)$ one-to-one and $\phi = \mu \circ \varepsilon$.

(P) If for every $\phi : a \to b$ in K and for any two mappings $f : U(a) \to X$, $g : X \to U(b)$ with $U(\phi) = g \circ f$ there are morphisms $\alpha : a \to c$ and $\beta : c \to b$ such that $U(\alpha) = f$ and $U(\beta) = g$.

(M) If $\{ a \mid U(a) = X \}$ is a set for every set X.

REMARK. Obviously, (R) and (P) are preserved under a w-realization from 2.10.

2.12. LEMMA. *(S) and (I) imply (M).*

PROOF. Let (K, U) have (S) and (I) and let $\{ a \mid U(a) = X \}$ be a proper class. Thus, there is an object b and an $N \subset \{ a \mid U(a) = X \}$ such that $card\, N > card\, X^X$ and that for every $a \in N$ there is an isomorphism $\alpha(a) : a \to b$. Hence, for some distinct a_1, a_2, $U(\alpha(a_1)) = U(\alpha(a_2))$, so that $U(\alpha(a_2)^{-1} \circ \alpha(a_1)) = 1$, which is a contradiction.

2.13. LEMMA. *Let a be isomorphic to $b \in O_m(K, U)$. Then $a \in O_m(K, U)$.*

Let α be parallel to $\beta \in M_m(K, U)$. Then $\alpha \in M_m(K, U)$.

PROOF is trivial.

2.14. LEMMA. *Let (K, U) be w-realizable in (L, V), let (L, V) have (S) (resp. (E)). Then (K, U) has (S) (resp. (E)).*

PROOF. The statement on (S) is evident. Let us prove the other one. Let $\Phi : K \to L$ be a w-realization, let A be the class of morphisms of L from (E). For every $\alpha \in A$ choose an α' in K with $\Phi(\alpha') // \alpha$, if there is any. Denote by A' the class of thus obtained morphisms. $A' \cap M_m(K, U)$ is always a set, by 2.13. Let $\Phi(\phi) // \Phi(\psi)$, $\phi : a \to b$, $\psi : c \to d$; let $S(m, \Phi(b), L, V) = S(n, \Phi(d), L, V)$ be their common image. Let p, q be surjections such that

$$V \circ \Phi(\phi) = m \circ p, \quad V \circ \Phi(\psi) = n \circ q.$$

Thus, $U(\phi) = m \circ p$, $U(\psi) = n \circ q$. We see easily that

$$(x, f) \in S(m, b, K, U) \quad \text{iff} \quad (\Phi(x), f) \in S(m, \Phi(b), L, V)$$

and similarly for n, d, so that $S(m, b, K, U) = S(n, d, K, U)$. Thus, $\phi // \psi$. Now, take a general ϕ and find an $\alpha \in A$ parallel to $\Phi(\phi)$, take the $\alpha' \in A'$ with $\Phi(\alpha') // \alpha$. By 2.6, $\Phi(\phi) // \Phi(\alpha')$ and hence $\phi // \alpha'$.

2.15. LEMMA. *$S(F)$ has (E) for every set functor F.*

PROOF. Define A as the class of all $1_{(m,s)}$ such that m is a cardinal. Thus,

$$A \cap O_m(S(F)) = \{ 1_{(b,s)} \mid b \leqslant m, \ s \subset F(b) \}$$

is a set. Let $((X, r), f, (Y, s))$ be a morphism of $S(F)$. Put $Z = f(X)$, $t = F(j)^{-1}(s)$, where j is the embedding of Z into Y. We have

$$((Z, q), g) \in S(j, (Y, s)) \iff F(j)F(g)(q) \subset s \iff$$

$$\iff F(g)(q) \subset F(j)^{-1}(s) = t \iff ((Z, q), g) \in S(1_Z, (Z, t)),$$

so that $S(j, (Y, s)) = S(1_Z, (Z, t))$. Thus $((X, r), f, (Y, s)) // 1_{(Z,t)}$. Take, now, an object (m, u) isomorphic to (Z, t). We obtain

$$((X, r), f, (Y, s)) // 1_{(m,u)}$$

by 2.5 and 2.6.

3. Main Theorem.

3.1. LEMMA. *Every (K, U) having the property (E) is realizable in a category with properties (S) and (R).*

PROOF. For (K, U) define a concrete category $(\bar{K}, \bar{U})$ as follows:
- The objects are triples (X, m, a) where X is a set, a an object of K and $m: X \to U(a)$ a one-to-one mapping such that there is a $\phi: b \to a$ with U-image $S(m, a)$.
- The morphisms from (X, m, a) into (Y, n, b) are triples $((X, m, a), f, (Y, n, b)$ such that $f: X \to Y$ is a mapping satisfying the implication:

$$(c, g) \in S(m, a) \Longrightarrow (c, f \circ g) \in S(n, b).$$

- The morphisms are composed in the obvious way.
-

$$\bar{U}(X, m, a) = X, \quad \bar{U}((X, m, a), f, (Y, n, b)) = f.$$

For an object a of K put $\Phi(a) = (U(a), 1, a)$; for a morphism ϕ put $\Phi(\phi) = (\Phi(a), U(\phi), \Phi(b))$. We see easily that we define a one-to-one functor $\Phi: K \to \bar{K}$ with $\bar{U} \circ \Phi = U$. If $(\Phi(a), f, \Phi(b))$ is a morphism of K, we have a $\phi: a \to b$ such that $U(\phi) = 1 \circ f \circ 1 = f$, since $(a, 1) \in S(1, a)$. Thus, Φ is a realization. Let $((X, m, a), f, (Y, n, b))$ be in K. Decompose f into $f_1 \circ f_2$ with $f_1: Z \to Y$ one-to-one and f_2 onto. $(Z, n \circ f_1, b)$ is an object of $\bar{K}$, since there is a $\phi: c \to a$ and $p: U(c) \to X$ surjective such that $U(\phi) = m \circ p$ and hence there is a $\psi: c \to b$ with

$$\bar{U}(\psi) = n \circ f \circ p = (n \circ f_1) \circ (f_2 \circ p).$$

By 2.2, $(c, g) \in S(n \circ f_1, b)$ iff $(c, f_1 \circ g) \in S(n, b)$, so that $((Z, n \circ f_1, b), f_1, (Y, n, b))$ is a morphism.

If $(c, h) \in S(m, a)$, then

$$(c, f_1 \circ f_2 \circ h) = (c, f \circ h) \in S(n, b)$$

and hence, again by 2.2, $(c, f_2 \circ h) \in S(n . f_1, b)$. Hence, also $((X, m, a), f_2, (Z, n \circ f_1, b))$ is a morphism. Thus, $(\bar{K}, \bar{U})$ has (R). Now, let (K, U) have (E). Take the class of morphisms A from (E) and construct a class A' of objects of K taking for every $\alpha \in A$ an (X, m, a) such that $S(m, a)$ is the U-image of α. By 2.7 and the definition of U, A' has the properties required in (S).

3.2. LEMMA. *Let (K, U) have the properties (S) and (R). Then it is realizable in a category with the properties (S) and (P).*

PROOF. For (K, U) define a concrete category (K', U') as follows:

- The objects are triples (X, a, p), where X is a set, a an object of K and p a mapping of a subset $X' \subset X$ onto $U(a)$ (we shall use the symbols $D(p)$ for X' and $j(p, X)$ for the embedding of $D(p)$ into X).

- The morphisms from (X, a, p) into (Y, b, q) are triples $((X, a, p), f, (Y, b, q))$ such that $f: X \to Y$ is a mapping for which there are $f': D(p) \to D(q)$ and $\mu: a \to b$ with

$$j(q, Y) \circ f' = f \circ j(p, X), \quad U(\mu) \circ p = q \circ f'.$$

The morphisms are composed in the obvious way.

$$U'(X, a, p) = X, \quad U'((X, a, p), f, (Y, b, q)) = f.$$

Define $\Phi: K \to K'$ by

$$\Phi(a) = (U(a), a, 1_{U(a)}), \quad \Phi(\phi) = (\Phi(a), U(\phi), \Phi(b)) \text{ for } \phi: a \to b.$$

We see easily that Φ is a one-to-one functor with $U' \circ \Phi = U$. Let $(\Phi(a), f, \Phi(b))$ be a morphism. Thus, there are f' and $\mu: a \to b$ such that

$$1 \circ f' = f \circ 1, \quad U(\mu) \circ 1 = 1 \circ f',$$

so that $(\Phi(a), f, \Phi(b)) = \Phi(\mu)$. Hence, Φ is a realization. Now, let (K, U) satisfy (R). Let $((X, a, p), f, (Y, b, q))$ be a morphism of K', let $g: X \to Z$ and $h: Z \to Y$ be such that $f = h \circ g$; let $f': D(p) \to D(q)$ and $\mu: a \to b$ satisfy

$$j(q, Y) \circ f' = f \circ j(p, X) \text{ and } U(\mu) \circ p = q \circ f'.$$

Put $Z' = g(D(p))$ and denote by j the embedding of Z' into Z. The formulas $g'(x) = g(x)$, $h'(x) = h(x)$ evidently define mappings $g': D(p) \to Z'$, $h': Z' \to D(q)$ with

$$j \circ g' = g \circ j(p, X), \quad j(q, Y) \circ h' = h \circ j, \quad f' = h' \circ g',$$

g' being onto. By (R) there are morphisms $\alpha: a \to c$, $\beta: c \to b$ such that $U(\alpha)$ is onto, $U(\beta)$ one-to-one and $\mu = \beta \circ \alpha$. If $g'(x) = g'(y)$, we have

$$U(\beta) U(\alpha) p(x) = q h' g'(x) = q h' g'(y) = U(\beta) U(\alpha) p(y)$$

and hence $U(\alpha)p(x)=U(\alpha)p(y)$. Thus, there is a surjective $r: Z' \to U(c)$ such that $U(\alpha)\circ p = r\circ g'$. Further,

$$U(\beta)\circ r\circ g' = U(\beta\circ\alpha)\circ p = q\circ f' = q\circ h'\circ g',$$

so that $U(\beta)\circ r = q\circ h'$, since g' is onto. Thus, $((X,a,p),g,(Z,c,r))$, $((Z,c,r),f,(Y,b,q))$ are morphisms required in (P).
Finally if (K,U) has (S), so has (K',U'): take the class A of objects of K from (S) and define a class A' of objects of K' as the class of all (m,a,p), where $a\in A$ and m is a cardinal.

3.3. THEOREM. *The following statements are equivalent:*

(a) (K,U) *has the property* (E).

(b) (K,U) *is realizable in a category with* (S) *and* (R).

(c) (K,U) *is realizable in a category with* (S) *and* (P).

(d) *There exists a set functor* F *such that* (K,U) *is* w*-realizable in* $S(F)$.

PROOF. (a) $\Longrightarrow$ (b) by 3.1, (b) $\Longrightarrow$ (c) by 3.2.
(c) $\Longrightarrow$ (d): If (K,U) is realizable in a category with (S) and (P), it is, by 1.6 and 2.10, w-realizable in a category with (S), (P) and (I). Thus, by 2.12, it suffices to show that a category (K,U) with (M) and (P) is w-realizable in $S(F)$. Thus, let (K,U) have (M) and (P). For a set X put

$$F(X)=\{A \mid A\subset \{a \mid U(a)=X\}, ((a\in A,\ \alpha: a\to a',\ U(\alpha)=1_X) \Longrightarrow a'\in A)\}$$

for a mapping $f: X\to Y$ define $F(f): F(X)\to F(Y)$ by

$$F(f)(A)=\{b \mid U(b)=Y,\ \exists a\in A,\ \exists \phi: a\to b,\ U(\phi)=f\}$$

(it is really an element of $F(Y)$: If $b\in F(f)(A)$, take an $a\in A$ and $\phi: a\to b$ with $U(\phi)=f$. If, for a $\beta: b\to b'$, $U(\beta)=1_Y$, we have $U(\beta\phi)=f$, $\beta\phi: a\to b'$).
Evidently, $F(1_X)(A)\supset A$. On the other hand, $a\in F(1_X)(A)$ means that there is an $a'\in A$ and $\alpha: a'\to a$ with $U(a)=1_X$. Then, however, $a\in A$. Thus, $F(1_X)=1_{F(X)}$.
Let $f: X\to Y$ and $g: Y\to Z$ be mappings. If $c\in F(g\circ f)(A)$ there is an $a\in A$ and $\phi: a\to c$ with $U(\phi)=g\circ f$. By (P) there are b, $\alpha: a\to b$ and

$\beta : b \to c$ with $Y = U(b)$, $f = U(\alpha)$ and $g = U(\beta)$. Thus, $b \in F(f)(A)$ and $c \in F(g)(F(f)(A))$, so that $F(g \circ f)(A) \subset F(g)(F(f)(A))$. Since obviously $F(g \circ f)(A) \supset F(g)(F(f)(A))$, we have $F(g \circ f) = F(g) \circ F(f)$. Thus, F is a set functor.

Now, construct a functor $\Phi : K \to S(F)$ as follows:

For an object $a \in K$ denote $M(a) = \{ A \in F(U(a)) \mid a \in A \}$ and put

$$\Phi(a) = (U(a), M(a)), \quad \Phi(\phi) = (\phi(a), U(\phi), \phi(b)) \text{ for } \phi : a \to b$$

(Φ really maps into $S(F)$: If $\phi : a \to b$ and $A \in M(a)$, we have $a \in A$ and $b \in F(U(\phi))(A)$, so that $F(U(\phi))(A) \in M(b)$). Φ is faithful, since U is. Thus, in order to prove that it is a w-realization, it remains to show that it is full. Let $F(f)(M(a)) \subset M(b)$ for an $f : U(a) \to U(b)$. In particular,

$$\{ a' \mid \exists \alpha : a \to a', \ U(\alpha) = 1 \} \in M(a),$$

so that

$$F(f)(\{ a' \mid \exists \alpha : a \to a', \ U(\alpha) = 1 \}) \ni b.$$

Hence, there are $\alpha : a \to a'$, $\phi : a' \to b$ with $U(\alpha) = 1$, $U(\phi) = f$. Thus, $f = U(\phi \alpha)$.

(d) $\Longrightarrow$ (a) follows immediately by 2.14 and 2.15.

3.4. THEOREM. *The following statements are equivalent:*

(a) (K, U) has the properties (E) and (I).

(b) (K, U) is realizable in a category with (S), (R) and (I).

(c) (K, U) is realizable in a category with (S), (P) and (I).

(d) There exists a set functor F such that (K, U) is realizable in $S(F)$.

PROOF. This is an immediate consequence of 3.3. and 1.5.

REMARK. The property (E) is in general not easy to check. In concrete cases we can decide mostly rather after the statements (b) of 3.3 or 3.4. It would be useful to find some more sufficient conditions of the kind of (S) and (R). (S) alone does not work, which can be shown on the following example: Take the subcategory K of *Set* consisting of all the invertible mappings and all the constants between sets of equal cardinality, let U be the embedding of K into *Set*. (K, U) has obviously the property (S). But it has not the property (E): Really, if in a concrete category $\alpha : a' \to a$ and $\beta : b' \to b$ are parallel, there is a $\gamma : a' \to b$ (let $S(m, a) = S(n, b)$ be the common U-image, let $U(\alpha) = m \circ p$. Hence $(a', p) \in S(m, a) = S(n, b)$ and therefore there is a $\gamma : a' \to b$ with $U(\gamma) = n \circ p$). Thus, in the case of our category (K, U) the cardinalities of ranges of parallel morphisms have to be equal, and hence a class A, containing for every morphism of K a parallel one, has to contain for every cardinal m a constant

$$\gamma(m) : X(m) \to Y(m) \text{ with } card\, X(m) = card\, Y(m) = m.$$

But then $A \cap M_1(K, U)$ is a proper class.

Thus, this (K, U) has (S), obviously (I) and (M) (moreover, $card\, \{ a \mid U(a) = X \} = 1$ for every X), but it is realizable in no $S(F)$.

3.6. REMARK. So far, we have spoken explicitly about realizability in $S(F)$ with a covariant F. By [6], however, a concrete category is realizable in $S(F)$ with a covariant F iff it is realizable in $S(F)$ with a contravariant F.

3.7. NOTATION. If (K, U) is a concrete category, we denote by $iso(K,U)$ the category of all isomorphisms of K endowed with the restriction of U.

3.8. THEOREM. *Let $iso(K, U)$ be w-realizable in $iso\,S(F)$. Let (K,U) have the property (R). Then (K, U) is w-realizable in an $S(G)$ with G such that*

$G(X)$ is finite if $F(X)$ is finite,

$card\, G(X) \leqslant exp\ exp\ exp\ card\, F(X)$ for infinite X.

If (K, U) has the property (P), the functor G may be chosen with

$$card\, G(X) \leqslant exp\ exp\ card\, F(X) \text{ for every } X.$$

PROOF. We see immediately that $iso(M, W)$ has (S) iff (M, W) has. Thus, (K, U) has (S) and (R), so that we can do the constructions from the proofs of 3.2 and 3.3. Assuming, without loss of generality, that (K, U) has (I) (see 1.6 and 1.5), we see that there is at most $exp\ card\, F(X)$ objects a of K with $U(a) = X$. Thus, if (K, U) has (P), we obtain from the construction of the functor in 3.3 that

$$card\ G(X) \leqslant exp\ exp\ card\ F(X).$$

If (K, U) has only (R) we have to construct first the (K, U) from 3.2. Replacing this by a category (K, U) with (I) by 1.6, we see easily that there is a finite number of objects a with $U(a) = X$ finite and at most $exp\ exp\ card\ F(X)$ objects with $U(a) = X$ infinite.

3.9. DEFINITION. A *Bourbaki-Ehresmann structure schema* (shortly, BESS) $\mathfrak{S}$ is a system

$$(m(\mathfrak{S}), n(\mathfrak{S}), (A(i,\mathfrak{S}))_{i=-m(\mathfrak{S}),\dots,-1}, ((a(i,\mathfrak{S}), b(i,\mathfrak{S}))_{i=1,\dots,n(\mathfrak{S})}),$$

where $m(\mathfrak{S})$, $n(\mathfrak{S})$ are natural numbers, $A(i, \mathfrak{S})$ sets and $a(i, \mathfrak{S})$, integers such that

$$-m-1 \leqslant a(i, \mathfrak{S}) \leqslant i-1, \quad 0 \leqslant b(i, \mathfrak{S}) \leqslant i-1.$$

If X is a set and $\mathfrak{S}$ a BESS, define $\mathfrak{S}(X)$ as the sequence

$$\mathfrak{S}(X, -m(\mathfrak{S})), \dots, \mathfrak{S}(X, n(\mathfrak{S}))$$

satisfying the following conditions (which, obviously, determine it):

1) for $i < 0$, $\mathfrak{S}(X, i) = A(i, \mathfrak{S})$,

2) $\mathfrak{S}(X, 0) = X$,

3) for $i > 0$ and $a(i, \mathfrak{S}) = -m(\mathfrak{S}) - 1$, $\mathfrak{S}(X, i) = \mathfrak{P}\,\mathfrak{S}(X, b(i, \mathfrak{S}))$

4) for $i > 0$ and $a(i,\mathfrak{S}) \geqslant -m(\mathfrak{S})$, $\mathfrak{S}(X,i) = \mathfrak{S}(X, a(i,\mathfrak{S})) \times \mathfrak{S}(X, b(i,\mathfrak{S}))$

($\mathfrak{P}$ designates the power set).

Write further $\bar{\mathfrak{S}}(X) = \mathfrak{S}(X, n(\mathfrak{S}))$. A $\mathfrak{S}$-*structure on* X is a subset of $\bar{\mathfrak{S}}(X)$. (In the definition in [1] or [2], an element is taken. This difference is, however, purely technical.)

If $\mathfrak{S}$ is a BESS and $f: X \to Y$ an invertible mapping, define $\mathfrak{S}(f)$ as the sequence of mappings $\mathfrak{S}(f, -m(\mathfrak{S})), \dots, \mathfrak{S}(f, n(\mathfrak{S}))$ such that

$$\mathfrak{S}(f, i) = 1_{A(i,\mathfrak{S})} \text{ for } i < 0, \quad \mathfrak{S}(f, 0) = 1_X,$$

$\mathfrak{S}(f, i)$ takes subsets to their images under $\mathfrak{S}(f, b(i, \mathfrak{S}))$ if $i > 0$ and $a(i, \mathfrak{S}) = -m(\mathfrak{S}) - 1$,

$\mathfrak{S}(f, i) = \mathfrak{S}(f, a(i, \mathfrak{S})) \times \mathfrak{S}(f, b(i, \mathfrak{S}))$ for $i > 0$ and $a(i, \mathfrak{S}) \geqslant -m(\mathfrak{S})$.

3.10. DEFINITION. A concrete category (K, U) is said to be *Bourbakian* (more exactly, Bourbakian of a type $\mathfrak{S}$) if there is a BESS $\mathfrak{S}$ and a correspondence $\mathfrak{H}$ associating with every object a of K a $\mathfrak{S}$-structure $\mathfrak{H}(a)$ on $U(a)$ such that for an invertible mapping $f: U(a) \to U(b)$ there is an isomorphism $\phi: a \to b$ with $U(\phi) = f$ iff $\bar{\bar{\mathfrak{S}}}(f)(\mathfrak{H}(a)) = \mathfrak{H}(b)$ (cf. [1] IV.2.1).

3.11. LEMMA. *Let (K, U) be a Bourbakian category of a type $\mathfrak{S}$. The $\bar{\bar{\mathfrak{S}}}$ from 3.9 can be extended to a set functor such that iso(K, U) is w-realizable in iso $S(\bar{\bar{\mathfrak{S}}})$.*

PROOF. Denote by C_A the constant set functor sending every X to A and every f to 1_A, by P^+ the set functor defined by

$$P^+(X) = \mathfrak{P}(X), \quad P^+(f)(M) = f(M) \text{ (the image of } M \text{ under } f);$$

for F, G set functors denote by $F \times G$ the set functor defined by

$$(F \times G)(X) = F(X) \times G(X), \quad (F \times G)(f) = F(f) \times G(f).$$

Given $\mathfrak{S}$, construct a sequence of set functors $F_{-m}, \dots, F_n$ as follows: for $i < 0$ $F_i = C_{A(i, \mathfrak{S})}$, F_0 is the identity functor; for $i > 0$, $a(i, \mathfrak{S}) = -m(\mathfrak{S}) - 1$, $F_i = P^+ \circ F_{b(i, \mathfrak{S})}$; for $i > 0$, $a(i, \mathfrak{S}) \geqslant -m(\mathfrak{S})$,

$$F_i = F_{a(i, \mathfrak{S})} \times F_{b(i, \mathfrak{S})}.$$

By definition 3.10, we see easily that we can put $\bar{\bar{\mathfrak{S}}} = F_{n(\mathfrak{S})}$.

3.12. COROLLARY. *Let (K, U) be a Bourbakian category of a type $\mathfrak{S}$. If it has the property (R), it is w-realizable in an $S(F)$ with F such that*

$F(X)$ is finite iff $\bar{\bar{\mathfrak{S}}}(X)$ is finite,

card $F(X) \leqslant$ exp exp exp card $\bar{\bar{\mathfrak{S}}}(X)$ for infinite X.

If it has the property (P), the functor F can be chosen such that

$$\text{card } F(X) \leqslant \text{exp exp card } \bar{\bar{\mathfrak{S}}}(X) \text{ for any } X.$$

REFERENCES

[1] N. BOURBAKI, *Théorie des Ensembles* (Eléments de Mathématique Livre 1), Hermann, Paris.

[2] C. EHRESMANN, Gattungen von lokalen Strukturen, *Jahresbericht der Deutschen Mathematiker-Verein,* Bd. 60,2 (1957).

[3] Z. HEDRLIN - A. PULTR, On categorical embeddings of topological structures into algebraic, *Comment. Math. Univ. Carolinae* 7 (1966), 377-400.

[4] Z. HEDRLIN - A. PULTR - V. TRNKOVA, Concerning a categorical approach to topological and algebraic theories, *General Topology and its Relations to Modern Analysis and Algebra* II, Proc. of the 2nd Prague Topological Symposium, 1966, 176-181.

[5] A. PULTR, On full embeddings of concrete categories with respect to forgetful functors, *Comment. Math. Univ. Carolinae* 9 (1968), 281-305.

[6] A. PULTR, On selecting of morphisms among all mappings between underlying sets of objects in concrete categories and realizations of these, *Comment. Math. Univ. Carolinae,* 8 (1967), 53-83.

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