CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

ROSS STREET

Two constructions on Lax functors

Cahiers de topologie et géométrie différentielle catégoriques, tome 13, nº 3 (1972), p. 217-264

<http://www.numdam.org/item?id=CTGDC_1972__13_3_217_0>

CAHIERS DE TOPOLOGIE
ET GEOMETRIE DIFFERENTIELLE

Vol. XIII-3

TWO CONSTRUCTIONS ON LAX FUNCTORS

by Ross STREET

Introduction

Bicategories have been defined by Jean Benabou [B], and there are examples of bicategories which are not *2*-categories. In the theory of a bicategory it appears that all the definitions and theorems of the theory of a 2-category still hold except for the addition of (coherent?) isomorphisms in appropriate places. For example, in a bicategory one may speak of adjoint *1*-cells; indeed, in Benabou's bicategory **Prof** of categories, profunctors and natural transformations, those profunctors which arise from functors do have adjoints in this sense.

The category **Cat** of categories is a cartesian closed category [EK], and a **Cat**-category is a *2*-category. In this work *2*-functor, *2*-natural transformation and *2*-adjoint will simply mean **Cat**-functor, **Cat**-natural transformation [EK], and **Cat**-adjoint [Ke]. It has long been realized by John Gray [G2] that the simple minded application of the theory of closed categories (for example, the work of [DK]) does not disclose all that is of interest in the theory of *2*-categories (his «*2*-comma categories» give an enriched Kan extension which is more involved that the **Cat**-Kan extension). Except for Grothendieck's pseudofunctors [G1], it was not until the paper [B] of Benabou that other morphisms of *2*-categories besides *2*-functors were considered. The pseudo-functors (they preserve composition and identities only up to isomorphism) do not appear to have a very different theory to that of *2*-functors; again (coherent?) isomorphisms must be added.

Morphisms of bicategories [B], here called *lax functors*, seem fundamental even when the domain and codomain are *2*-categories. The «formal» categorical purpose for this paper is to provide in detail the constructions of two universal functors from a lax functor with domain a category and codomain **Cat** (some generalizations are outlined in the appendix). These constructions (also see the appendix) lead to «limits and colimits» for these types of lax functors into **Cat** (adjoints to appropriate diagonal functors).

Let **1** denote the category with one object and one arrow. In [B] it is remarked that a lax functor from **1** to **Cat** is a category together with a triple (monad) on that category. A functor from **1** to **Cat** is just a category. So the two constructions assign two categories to each triple on a category. The first construction is that of Kleisli [Kl], and the second construction is that of Eilenberg-Moore [EM].

So the second purpose of this paper is to provide a generalization of the theory of triples. Yet it is more than a generalization: it provides a framework for the presentation of some new (?) results on triples.

We believe that Theorems 3 and 4 are unknown even in the triples case $(\mathbf{A}=\mathbf{1})$. The *2*-categories $\overrightarrow{Lax}\,[\mathbf{1},\mathbf{Cat}]$ and $\overleftarrow{Lax}\,[\mathbf{1},\mathbf{Cat}]$ might well be called $\overrightarrow{\mathbf{Trip}}$ and $\overleftarrow{\mathbf{Trip}}$, and $Gen\,[\mathbf{1},\mathbf{Cat}]$ is **Cat**. So we have the results that the Kleisli construction is a left *2*-adjoint of the inclusion of **Cat** in $\overrightarrow{\mathbf{Trip}}$, and that the Eilenberg-Moore construction is a right *2*-adjoint of the inclusion of **Cat** in $\overleftarrow{\mathbf{Trip}}$. If $\mathbf{X}$ is a category and $\mathbf{Y}$ is a category supporting a triple T, then the following are isomorphisms of categories

$$\overrightarrow{\mathbf{Trip}}((\mathbf{Y},T),(\mathbf{X},\mathbf{1}))\cong[\mathbf{Y}_T,\mathbf{X}],\quad \overleftarrow{\mathbf{Trip}}((\mathbf{X},\mathbf{1}),(\mathbf{Y},T))\cong[\mathbf{X},\mathbf{Y}^T],$$

where $\mathbf{Y}_T$ denotes the category of Kleisli algebras with respect to T, $\mathbf{Y}^T$ denotes the category of Eilenberg-Moore algebras with respect to T, and square brackets denote the functor category. Now T induces a triple $[T,\mathbf{X}]$ on $[\mathbf{Y},\mathbf{X}]$, and $\overrightarrow{\mathbf{Trip}}((\mathbf{Y},T),(\mathbf{X},\mathbf{1}))$ is readily seen to be the category of algebras $[\mathbf{Y},\mathbf{X}]^{[T,\mathbf{X}]}$ with respect to this triple. Also T induces a triple $[\mathbf{X},T]$ on $[\mathbf{X},\mathbf{Y}]$ and $\overleftarrow{\mathbf{Trip}}((\mathbf{X},\mathbf{1}),(\mathbf{Y},T))$ is readily seen to be the category of algebras $[\mathbf{X},\mathbf{Y}]^{[\mathbf{X},T]}$ with respect to this triple. So we have isomorphisms of categories

$$[\mathbf{Y},\mathbf{X}]^{[T,\mathbf{X}]}=[\mathbf{Y}_T,\mathbf{X}],\quad [\mathbf{X},\mathbf{Y}]^{[\mathbf{X},T]},\cong[\mathbf{X},\mathbf{Y}^T],$$

and these commute with the underlying functors. Dubuc [Du] called the objects of $[\mathbf{X},\mathbf{Y}]^{[\mathbf{X},T]}$ functors together with actions, and he proved that these are in bijective correspondence with functors from $\mathbf{X}$ to $\mathbf{Y}^T$.

The treatment of structure and semantics using cartesian arrows in such a way as to admit a dual, also seems to be new even in the triples case. The simple duality between triples and cotriples corresponds to a reversal of 2-cells in **Cat**. Here we have a duality corresponding to a reversal of 1-cells in **Cat** which takes Kleisli algebras to Eilenberg-Moore algebras. Note also the amazing adjointness which Linton at the end of his paper [Li] attributes to Lawvere and which relates Kleisli and Eilenberg-Moore algebras and coalgebras.

The construction (due to Grothendieck) of a pseudo-functor $\mathbf{V}: \mathbf{B}^{op} \to \mathbf{Cat}$ from a fibration $P: \mathbf{E} \to \mathbf{B}$ may be found in [G1]: for $B \in \mathbf{B}$, $\mathbf{V}B = P^{-1}(B)$ is the fibre category over B, and, for $f: B \to B'$ in $\mathbf{B}$, $\mathbf{V}f: \mathbf{V}B' \to \mathbf{V}B$ is the inverse image functor. If P is a split fibration, then inverse images can be chosen so that $\mathbf{V}$ is a genuine functor. If a fibration P is also an opfibration (terminology of [G1']), then it is called a bifibration [BR], and, for each $f: B \to B'$, the direct image functor $\check{\mathbf{V}}f: \mathbf{V}B \to \mathbf{V}B'$ provides a left adjoint for $\mathbf{V}f$. For a bifibration $P: \mathbf{E} \to \mathbf{B}$, two pseudo-functors $\mathbf{V}: \mathbf{B}^{op} \to \mathbf{Cat}$, $\check{\mathbf{V}}: \mathbf{B} \to \mathbf{Cat}$ are obtained and on objects they have the same values. If P is split as a fibration, it need not be split as an opfibration; if $\mathbf{V}$ is a genuine functor, there may still be no way of choosing direct images so that $\check{\mathbf{V}}$ is a genuine functor. The usual examples of bifibrations (see [BR]) are split either as fibrations or opfibrations. This should justify the consideration in § 5 of functors $\mathbf{V}: \mathbf{A} \to \mathbf{Cat}$ such that, for each $f: A \to A'$ in $\mathbf{A}$, $\mathbf{V}f$ has a left adjoint. In fact we show that the second basic construction gives such a functor under mild conditions.

The work for this paper started out in an attempt to generalize the concept of triple and the algebra construction in the hope that many well-known categories besides equationally defined theories could be shown to be examples of the construction - categories of sheaves especially. The first generalization which we worked through to a «tripleability» theorem amounts to the case where the lax functor $\mathbf{W}: \mathbf{A} \to \mathbf{Cat}$ has the property that

- each set $\mathbf{A}(A, B)$ has exactly one arrow $\langle AB \rangle$;
- for each $A \in \mathbf{A}$, $\mathbf{W}A = \mathbf{X}$ for some fixed category $\mathbf{X}$.

Note that the functor category $[\mathbf{X},\mathbf{X}]$ is monoïdal with composition as its tensor product, and such a lax functor $\mathbf{W}$ amounts to an $[\mathbf{X},\mathbf{X}]$-category with its objects the same as the objects of $\mathbf{A}$. For example, if $\mathbf{K}$ is a category with the same objects as $\mathbf{A}$ and if $\mathbf{X}$ has copowers enough, then a lax functor $\mathbf{W}: \mathbf{A} \to \mathbf{Cat}$ is obtained by

$$\mathbf{W}A = \mathbf{X},\ \mathbf{W}\langle AB \rangle = \mathbf{K}(A,B) \otimes - : \mathbf{X} \to \mathbf{X},$$

and the other data is provided by compositions and identities in $\mathbf{K}$. Then for each $A \in \mathbf{K}$, the second construction $\mathbf{W}A$ is the functor category $[\mathbf{K},\mathbf{X}]$.

Benabou suggested consideration of the case where $\mathbf{A}$ is a general category. The hope was that this extra freedom would give subcategories of functor categories $[\mathbf{K},\mathbf{X}]$, for example, those full subcategories of functors which preserve a particular set of assigned limits. At this point we do not know whether this is the case.

The first generalization of triples for $[\mathbf{X},\mathbf{X}]$-categories was completed at Tulane University in New Orleans, and there also were the two basic constructions of § 2 found. The remainder of the work was done at Macquarie University in Sydney.

Table of Contents

1. Definitions.

Suppose $\mathbf{A}$ is a category. A *lax functor* $\mathbf{W}: \mathbf{A} \to \mathbf{Cat}$ consists of the following data:

for each object A of $\mathbf{A}$, a category $\mathbf{W}A$,

for each arrow $f: A \to B$ in $\mathbf{A}$, a functor $\mathbf{W}f: \mathbf{W}A \to \mathbf{W}B$,

for each composable pair of arrows $f: A \to B$, $g: B \to C$ in $\mathbf{A}$, a natural transformation $\alpha_{g,f}: \mathbf{W}g \,.\, \mathbf{W}f \to \mathbf{W}(gf)$,

for each object A of $\mathbf{A}$, a natural transformation $\omega_A : 1_{\mathbf{W}A} \to \mathbf{W}1_A$;

such that the following diagams commute:

$$\begin{array}{ccc} \mathbf{W}h.\mathbf{W}g.\mathbf{W}f & \xrightarrow{(\mathbf{W}h)\omega_{gf}} & \mathbf{W}h.\mathbf{W}(gf) \\ \downarrow{\scriptstyle \omega_{h,g}(\mathbf{W}f)} & & \downarrow{\scriptstyle \omega_{h,gf}} \\ \mathbf{W}(hg).\mathbf{W}f & \xrightarrow[\omega_{hg,f}]{} & \mathbf{W}(hgf) \end{array} \qquad \begin{array}{ccccc} & & \mathbf{W}f & & \\ & \swarrow{\scriptstyle \omega_B(\mathbf{W}f)} & \| & \searrow{\scriptstyle (\mathbf{W}f)\omega_A} & \\ \mathbf{W}1_B.\mathbf{W}f & \xrightarrow[\omega_{1,f}]{} & \mathbf{W}f & \xrightarrow[\omega_{f,1}]{} & \mathbf{W}f.\mathbf{W}1_A . \end{array}$$

For lax functors $\mathbf{W}, \mathbf{W}' : \mathbf{A} \to \mathbf{Cat}$, a *left lax transformation* $L : \mathbf{W} \to \mathbf{W}'$ consists of:

for each $A \in \mathbf{A}$, a functor $L_A : \mathbf{W}A \to \mathbf{W}'A$,

for each arrow $f: A \to B$ in $\mathbf{A}$, a natural transformation

$$L_f : (\mathbf{W}'f) L_A \to L_B(\mathbf{W}f) ,$$

such that the following diagrams commute:

$$\begin{array}{ccccc} \mathbf{W}'g.\mathbf{W}'f.L_A & \xrightarrow{\omega_{g,f}L_A} & \mathbf{W}'(gf).L_A & & \\ \downarrow{\scriptstyle (\mathbf{W}'g)L_f} & & & \searrow{\scriptstyle L_{gf}} & \\ & & & & L_C\mathbf{W}(gf) \\ & & & \nearrow{\scriptstyle L_C\omega_{g,f}} & \\ \mathbf{W}'g.L_B.\mathbf{W}f & \xrightarrow[L_g(\mathbf{W}f)]{} & L_C.\mathbf{W}g.\mathbf{W}f & & \end{array}$$

$$\begin{array}{ccc} L_A & \xrightarrow{\omega_A L_A} & (\mathbf{W}'1_A)L_A \\ & \searrow{\scriptstyle L_A\omega_A} & \downarrow{\scriptstyle L_{1_A}} \\ & & L_A(\mathbf{W}1_A) \end{array}$$

The data for the left lax transformation $L:\mathbf{W}\to\mathbf{W}'$ is contained in the diagram

$$\begin{array}{ccc} \mathbf{W}A & \xrightarrow{\;L_A\;} & \mathbf{W}'A \\ {\scriptstyle \mathbf{W}f}\downarrow & \;\;{\scriptstyle L_f}\swarrow & \downarrow{\scriptstyle \mathbf{W}'f} \\ \mathbf{W}B & \xrightarrow[\;L_B\;]{} & \mathbf{W}'B\ ; \end{array}$$

the 2-cell points left. The data for a *right lax transformation* $R:\mathbf{W}\to\mathbf{W}'$ comes in a diagram

$$\begin{array}{ccc} \mathbf{W}A & \xrightarrow{\;R_A\;} & \mathbf{W}'A \\ {\scriptstyle \mathbf{W}f}\downarrow & \;\;{\scriptstyle R_f}\nearrow & \downarrow{\scriptstyle \mathbf{W}'f} \\ \mathbf{W}B & \xrightarrow[\;R_B\;]{} & \mathbf{W}'B \end{array}$$

the 2-cell points right; and the appropriate changes must be made in the two conditions.

For left lax transformations $L, M:\mathbf{W}\to\mathbf{W}'$, *a morphism* $s:L\to M$ *of left lax transformations* is a function which assigns to each object A of $\mathbf{A}$ a natural transformation $s_A:L_A\to M_A$ such that the following square commutes:

$$\begin{array}{ccc} (\mathbf{W}'f)L_A & \xrightarrow{\;(\mathbf{W}'f)s_A\;} & (\mathbf{W}'f)M_A \\ {\scriptstyle L_f}\downarrow & & \downarrow{\scriptstyle M_f} \\ L_B(\mathbf{W}f) & \xrightarrow[\;s_B(\mathbf{W}f)\;]{} & M_B(\mathbf{W}f)\ . \end{array}$$

A morphism $s:R\to S$ *of right lax transformations* consists of natural transformations $s_A:R_A\to S_A$ satisfying:

$$\begin{array}{ccc} R_A(\mathbf{W}f) & \xrightarrow{\;s_A(\mathbf{W}f)\;} & S_A(\mathbf{W}f) \\ {\scriptstyle R_f}\downarrow & & \downarrow{\scriptstyle S_f} \\ (\mathbf{W}'f)R_B & \xrightarrow[\;(\mathbf{W}'f)s_B\;]{} & (\mathbf{W}'f)S_B\ . \end{array}$$

The composite $L'L:\mathbf{W}\to\mathbf{W}''$ *of two left lax transformations* $L:\mathbf{W}\to\mathbf{W}'$, $L':\mathbf{W}'\to\mathbf{W}''$ is the left lax transformation given by

$$(L'L)_A = L'_A L_A,\quad (L'L)_f = (L'_B L_f).(L'_f L_A).$$

This composition is associative with identities.

There are two compositions for morphisms of left lax transformations. If $L, M, N : \mathbf{W} \to \mathbf{W}'$ are left lax transformations, and $s : L \to M$, $t : M \to N$ are morphisms of them, then the composite $ts : L \to N$ is the morphism given by $(ts)_A = t_A s_A$. This composition is associative and has identities. If $L, M : \mathbf{W} \to \mathbf{W}'$ and $L', M' : \mathbf{W}' \to \mathbf{W}''$ are left lax transformations and $s : L \to M$, $s' : L' \to M'$ are morphisms of them, then the composite

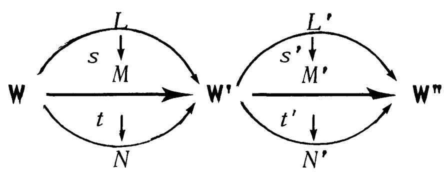
$s's : L'L \to M'M$ is the morphism given by

$$(s's)_A = s'_A s_A = (s'_A M_A).(L'_A s_A) = (M'_A s_A).(s'_A L_A);$$

then $s's$ is a morphism since the following diagram commutes.

$$\begin{array}{ccccc}
(\mathbf{W}''f)L'_A L_A & \xrightarrow{(\mathbf{W}''f)L'_A s_A} & (\mathbf{W}''f)L'_A M_A & \xrightarrow{(\mathbf{W}''f)s'_A M_A} & (\mathbf{W}''f)M'_A M_A \\
{\scriptstyle L'_f L_A}\downarrow & & \downarrow{\scriptstyle L'_f M_A} & & \downarrow{\scriptstyle M'_f M_A} \\
L'_B(\mathbf{W}'f)L_A & \xrightarrow[L'_B(\mathbf{W}'f)s_A]{} & L'_B(\mathbf{W}'f)M_A & \xrightarrow[s'_B(\mathbf{W}'f)M_A]{} & M'_B(\mathbf{W}'f)M_A \\
{\scriptstyle L'_B L_f}\downarrow & & \downarrow{\scriptstyle L'_B M_f} & & \downarrow{\scriptstyle M'_B M_f} \\
L'_B L_B(\mathbf{W}f) & \xrightarrow[L'_B s_B(\mathbf{W}f)]{} & L'_B M_B(\mathbf{W}f) & \xrightarrow[s'_B M_B(\mathbf{W}f)]{} & M'_B M_B(\mathbf{W}f)
\end{array}$$

Moreover, in the diagram



the equation

$$(t't).(s's) = (t's').(ts)$$

is satisfied since it holds for natural transformations and the compositions were defined componentwise.

Compositions may similarly be defined with left replaced by right.

Summarizing then, we have a 2-category $\overleftarrow{Lax}\,[\mathbf{A}, \mathbf{Cat}]$ whose 0-cells are lax functors from $\mathbf{A}$ to $\mathbf{Cat}$, whose 1-cells are left lax transformations, and whose 2-cells are morphisms of left lax transformations;

and also, by replacing left by right, a 2-category $\overrightarrow{Lax}\ [\mathbf{A}, \mathbf{Cat}]$. For lax functors $\mathbf{W}$, $\mathbf{W}' : \mathbf{A} \to \mathbf{Cat}$, we put

$$[\overleftarrow{\mathbf{W}, \mathbf{W}'}] = \overleftarrow{Lax}\ [\mathbf{A}, \mathbf{Cat}]\,(\mathbf{W}, \mathbf{W}') \quad \text{and} \quad [\overrightarrow{\mathbf{W}, \mathbf{W}'}] = \overrightarrow{Lax}\ [\mathbf{A}, \mathbf{Cat}]\,(\mathbf{W}, \mathbf{W}').$$

A functor $\mathbf{W} : \mathbf{A} \to \mathbf{Cat}$ may be regarded as a lax functor which has all the natural transformations $\omega_{g,f}$, ω_A identities. If $\mathbf{W}, \mathbf{W}'$ are functors, then a natural transformation $N : \mathbf{W} \to \mathbf{W}'$ may be regarded as a left and right lax transformation with all the natural transformations N_f identities. Let $[\mathbf{W}, \mathbf{W}']$ denote the full subcategory of $[\overleftarrow{\mathbf{W}, \mathbf{W}'}]$ whose objects are the natural transformations from $\mathbf{W}$ to $\mathbf{W}'$; it is also a full subcategory of $[\overrightarrow{\mathbf{W}, \mathbf{W}'}]$. Let $Gen\ [\mathbf{A}, \mathbf{Cat}]$ denote the 2-category whose objects are genuine functors from $\mathbf{A}$ to $\mathbf{Cat}$ and $Gen\ [\mathbf{A}, \mathbf{Cat}]\,(\mathbf{V}, \mathbf{V}') = [\mathbf{V}, \mathbf{V}']$, so that $Gen\ [\mathbf{A}, \mathbf{Cat}]$ is a sub-2-category of both $\overleftarrow{Lax}\ [\mathbf{A}, \mathbf{Cat}]$ and $\overrightarrow{Lax}\ [\mathbf{A}, \mathbf{Cat}]$, and both the inclusions are locally full.

A left adjoint of a left lax transformation $L : \mathbf{W} \to \mathbf{W}'$ is a right lax transformation $R : \mathbf{W}' \to \mathbf{W}$ such that, for each $A \in \mathbf{A}$, R_A is a left adjoint of L_A and the natural transformations L_f and R_f correspond under the natural isomorphism

$$[\mathbf{W}A, \mathbf{W}'B]\,(\mathbf{W}'f . L_A, L_B . \mathbf{W}f) \cong [\mathbf{W}'A, \mathbf{W}B]\,(R_B . \mathbf{W}'f, \mathbf{W}f . R_A)$$

which comes from the adjunctions $R_A \dashv L_A$, $R_B \dashv L_B$; the notation is $R \dashv L$.

THEOREM 1. *(a) A left lax transformation* $L : \mathbf{W} \to \mathbf{W}'$ *has a left adjoint if and only if each of the functors* $L_A : \mathbf{W}A \to \mathbf{W}'A$ *has a left adjoint.*

(b) If $L, M : \mathbf{W} \to \mathbf{W}'$ *are left lax transformations and* $R \dashv L$, $S \dashv M$, *then* $[\overleftarrow{\mathbf{W}, \mathbf{W}'}]\,(L, M) \cong [\overrightarrow{\mathbf{W}', \mathbf{W}}]\,(S, R)$.

(c) The left adjoint of $L : \mathbf{W} \to \mathbf{W}'$ *is unique up to isomorphism in* $[\overrightarrow{\mathbf{W}', \mathbf{W}}]$.

PROOF. (a) Let $R_A : \mathbf{W}'A \to \mathbf{W}A$ be a left adjoint of L_A . If $R : \mathbf{W}' \to \mathbf{W}$ is to be a left adjoint of L , then the definition of R_f is forced. The naturality of the isomorphisms takes the conditions on the data L_A, L_f which make L a left lax transformation into the conditions on R_A, R_f which give a right lax transformation R. Then $R \dashv L$.

(b) From the adjunctions $R_A \dashv L_A$, $S_A \dashv M_A$ we have natural isomorphisms

$$[\mathbf{W}A, \mathbf{W}'A](L_A, M_A) \cong [\mathbf{W}'A, \mathbf{W}A](S_A, R_A)$$

under which the diagrams of morphisms of left lax transformations go to those for morphisms of right.

(c) If $R \dashv L$ and $R' \dashv L$, then the identity morphism from L to L gives an isomorphism between R and R' using part (b).

Given a lax functor $\mathbf{V} : \mathbf{A} \to \mathbf{Cat}$ and, for each object A of $\mathbf{A}$, an adjunction $\varepsilon_A, \eta_A : J_A \dashv E_A : (\mathbf{V}A, \mathbf{X}_A)$, then the following data defines a lax functor $\mathbf{W} : \mathbf{A} \to \mathbf{Cat}$:

$$\mathbf{W}A = \mathbf{X}_A, \quad \mathbf{W}f = (\mathbf{X}_A \xrightarrow{J_A} \mathbf{V}A \xrightarrow{\mathbf{V}f} \mathbf{V}B \xrightarrow{E_B} \mathbf{X}_B);$$

$$\omega_{g,f} = (E_C(\mathbf{V}g)J_B E_B(\mathbf{V}f)J_A \xrightarrow{E_C(\mathbf{V}g)\,\varepsilon_B(\mathbf{V}f)J_A} E_C(\mathbf{V}g)(\mathbf{V}f)J_A \xrightarrow{E_C\omega_{g,f}J_A} E_C\mathbf{V}(gf)J_A);$$

$$\omega_A = (1 \xrightarrow{\eta_A} E_A J_A \xrightarrow{E_A\omega_A J_A} E_A(\mathbf{V}1_A)J_A).$$

Moreover, the following data defines a left lax transformation $E : \mathbf{V} \to \mathbf{W}$:

$$E_A : \mathbf{V}A \to \mathbf{X}_A, \quad E_f = E_B(\mathbf{V}f)\,\varepsilon_A : E_B(\mathbf{V}f)J_A \to E_B(\mathbf{V}f);$$

and the following data defines a right lax transformation $J : \mathbf{W} \to \mathbf{V}$:

$$J_A : \mathbf{X}_A \to \mathbf{V}A, \quad J_f = \varepsilon_B(\mathbf{V}f)J_A : J_B E_B(\mathbf{V}f)J_A \to (\mathbf{V}f)J_A.$$

Then $J \dashv E$. (The proof of these assertions is left up to the reader and is recommended as an exercise in the new definitions.) Under these circumstances we say that $\mathbf{W}$ is *the lax functor generated by* $\mathbf{V}$ *and the adjunction* $J \dashv E$, and we write $\mathbf{W} = E\mathbf{V}J$.

2. The two basic constructions.

Suppose $\mathbf{W} : \mathbf{A} \to \mathbf{Cat}$ is a lax functor. A genuine functor $\widetilde{\mathbf{W}} : \mathbf{A} \to \mathbf{Cat}$ is defined as follows. For $A \in \mathbf{A}$, $\widetilde{\mathbf{W}}A$ is the category whose objects are pairs (u, X), where $u: A' \to A$ is an arrow of $\mathbf{A}$ and X is an object of $\mathbf{W}A'$, whose arrows are pairs $(h, \phi): (u, X) \to (u', X')$ where $h: A'' \to A'$ is an arrow of $\mathbf{A}$ such that $u' = uh$ and $\phi: X \to (\mathbf{W}h)X'$ is an arrow of $\mathbf{W}A'$, and whose composition is given by

$$\begin{array}{ccc} (u,X) & \xrightarrow{\;(hh',\, \omega_{h,h'}X''.(\mathbf{W}h)\phi'.\phi)\;} & (u'',X'') \\ & {\scriptstyle (h,\phi)}\searrow \quad\quad \nearrow{\scriptstyle (h',\phi')} & \\ & (u',X') & \end{array}$$

It should be checked that composition is associative and that the identity of (u, X) is $(1_{A'}, \omega_{A'}X)$. For $f: A \to B$ in $\mathbf{A}$, $\widetilde{\mathbf{W}}f: \widetilde{\mathbf{W}}A \to \widetilde{\mathbf{W}}B$ is the functor given by

$$(\widetilde{\mathbf{W}}f)(u, X) = (fu, X), \ (\widetilde{\mathbf{W}}f)(h, \phi) = (h, \phi).$$

For each $A \in \mathbf{A}$, define $\widetilde{E}_A: \widetilde{\mathbf{W}}A \to \widetilde{\mathbf{W}}A$ by

$$\widetilde{E}_A(u, X) = (\mathbf{W}u)X, \ E_A(h, \phi) = \omega_{u,h}X'.(\mathbf{W}u)\phi.$$

Then

$$\widetilde{E}_A(1_{A'}, \omega_{A'}X) = \omega_{u,1}X.(\mathbf{W}u)\omega_{A'}X = 1_{\mathbf{W}u},$$

and the following diagram completes the proof that $\widetilde{E}_A$ is a functor.

$$\begin{array}{ccccc} & (\mathbf{W}u)(\mathbf{W}h)(\mathbf{W}h')X'' & \xrightarrow{(\mathbf{W}u)\omega_{h,h'}X''} & (\mathbf{W}u)\mathbf{W}(hh')X'' & \\ {\scriptstyle (\mathbf{W}u)(\mathbf{W}h)\phi'.\phi}\nearrow & & & & \searrow{\scriptstyle \omega_{u,hh'}X''} \\ (\mathbf{W}u)X & & & & (\mathbf{W}u'')X'' \\ {\scriptstyle (\mathbf{W}u)\phi}\searrow & & & & \nearrow{\scriptstyle \omega_{u',h'}X''} \\ & (\mathbf{W}u)(\mathbf{W}h)X' \xrightarrow[\omega_{u,h}X']{} & (\mathbf{W}u')X' & \xrightarrow[(\mathbf{W}u')\phi']{} (\mathbf{W}u')(\mathbf{W}h')X'' & \end{array}$$

Also define $\widetilde{J}_A: \mathbf{W}A \to \widetilde{\mathbf{W}}A$ by

$$\widetilde{J}_A X = (1_A, X), \ \widetilde{J}_A x = (1_A, \omega_A X'.x) \text{ for } x: X \to X' \text{ in } \mathbf{W}A.$$

For each $(u, X) \in \widetilde{\mathbf{W}}A$, let

$$\widetilde{\varepsilon}_A(u, X) = (u, 1_{(\mathbf{W}u)X}): (1_A, (\mathbf{W}u)X) \to (u, X)$$

in $\mathbf{W}A$. These arrows are the components of a natural transformation $\widetilde{\varepsilon}_A: \widetilde{J}_A \widetilde{E}_A \to 1$. For each $X \in \mathbf{W}A$, let

$$\widetilde{\eta}_A X = \omega_A X : X \to (\mathbf{W}\, 1_A) X$$

in $\mathbf{W}A$. These arrows are the components of a natural transformation $\eta_A : 1 \to E_A J_A$. Commutativity of

$$\begin{array}{ccc}
(1_A, X) & \xrightarrow{(1_A,\, \omega_A(\mathbf{W}1_A)X \,.\, \omega_A X)} & (1_A, (\mathbf{W}\, 1_A) X) \\
 & \diagdown\!\!\diagdown & \downarrow (1_A, 1_{(\mathbf{W}1_A)X}) \\
 & & (1_A, X)
\end{array}$$

implies $\widetilde{\varepsilon}_A \widetilde{J}_A \,.\, \widetilde{J}_A \widetilde{\eta}_A = 1_{\widetilde{J}_A}$. Commutativity of

$$\begin{array}{ccc}
(\mathbf{W}u) X & \xrightarrow{\omega_A (\mathbf{W}u) X} & (\mathbf{W}\, 1_A)(\mathbf{W}u) X \\
 & \diagdown\!\!\diagdown & \downarrow \omega_{1_A, u} \\
 & & (\mathbf{W}u) X
\end{array}$$

implies $\widetilde{E}_A \widetilde{\varepsilon}_A \,.\, \widetilde{\eta}_A \widetilde{E}_A = 1_{\widetilde{J}_A}$. So for each $A \in \mathbf{A}$ we have an adjunction

$$\widetilde{\varepsilon}_A, \widetilde{\eta}_A : \widetilde{J}_A \dashv \widetilde{E}_A : (\widetilde{\mathbf{W}} A, \mathbf{W} A).$$

For $f : A \to B$ in A,

$$\widetilde{E}_B (\widetilde{\mathbf{W}} f) \widetilde{J}_A X = \widetilde{E}_B (\widetilde{\mathbf{W}} f)(1_A, X) = \widetilde{E}_B (f, X) = (\mathbf{W} f) X,$$

and

$$\widetilde{E}_B (\widetilde{\mathbf{W}} f) \widetilde{J}_A x = \widetilde{E}_B (\widetilde{\mathbf{W}} f)(1_A, \omega_A X' . x) = \widetilde{E}_B (1_A, \omega_A X' . x) =$$

$$\omega_{f, 1_A} X' . (\mathbf{W} f)(\omega_A X' . x) = \omega_{f, 1_A} X' . (\mathbf{W} f) \omega_A X' . (\mathbf{W} f) x = (\mathbf{W} f) x .$$

So $\mathbf{W} f = \widetilde{E}_B (\widetilde{\mathbf{W}} f) \widetilde{J}_A$. For $f : A \to B$, $g : B \to C$ in $\mathbf{A}$,

$$\widetilde{E}_C (\widetilde{\mathbf{W}} g) \widetilde{\varepsilon}_B (\widetilde{\mathbf{W}} f) J_A X = \widetilde{E}_C (\widetilde{\mathbf{W}} g) \widetilde{\varepsilon}_B (\widetilde{\mathbf{W}} f)(1_A, X) =$$

$$\widetilde{E}_C (\widetilde{\mathbf{W}} g) \widetilde{\varepsilon}_B (f, X) = \widetilde{E}_C (\widetilde{\mathbf{W}} g)(f, 1_{(\mathbf{W}f)X}) = \widetilde{E}_C (f, 1_{(\mathbf{W}f)X}) = \omega_{g, f} X .$$

So put

$$\widetilde{E}_f (u, X) = \widetilde{E}_B (\mathbf{W} f) \widetilde{\varepsilon}_A (u, X) = \widetilde{E}_B (\widetilde{\mathbf{W}} f)(u, 1_{(\mathbf{W}u)X}) =$$

$$\widetilde{E}_B (u, 1_{(\mathbf{W}u)X}) = \omega_{f, u} X : (\mathbf{W} f)(\mathbf{W} u) X \to \mathbf{W}(f u) X,$$

and

$$\widetilde{J}_f X = \widetilde{\varepsilon}_B (\widetilde{\mathbf{W}} f) \widetilde{J}_A X = \widetilde{\varepsilon}_B (\widetilde{\mathbf{W}} f)(1_A, X) = \widetilde{\varepsilon}_B (f, X) =$$

$$=(f, 1_{(\mathbf{W}f)X}):(1_B, (\mathbf{W}f)X)\to(f, X).$$

Then $\tilde{E}:\tilde{\mathbf{W}}\to\mathbf{W}$ is a left lax transformation, $\tilde{J}:\mathbf{W}\to\tilde{\mathbf{W}}$ is a right lax transformation, $\tilde{J}\dashv\tilde{E}$, and $\mathbf{W}=\widetilde{E\,\mathbf{W}J}$.

This is the first basic construction. Its characterizing properties, along with those of the second construction, will be discussed in the next section.

Suppose again $\mathbf{W}:\mathbf{A}\to\mathbf{Cat}$ is any lax functor. A genuine functor $\hat{\mathbf{W}}:\mathbf{A}\to\mathbf{Cat}$ is defined as follows. For $A\in\mathbf{A}$, the objects of the category $\hat{\mathbf{W}}A$ are pairs (F,ξ), where F is a function which assigns to each arrow $u:A\to B$ of $\mathbf{A}$ an object $Fu\in\mathbf{W}B$, and ξ is a function which assigns to each composable pair $u:A\to B$, $v:B\to C$ of arrows of $\mathbf{A}$ an arrow $\xi_{v,u}:(\mathbf{W}v)Fu\to F(vu)$ in $\mathbf{W}C$ such that the following diagrams commute:

$$\begin{array}{ccc} \mathbf{W}w.\mathbf{W}v.Fu & \xrightarrow{(\mathbf{W}w)\xi_{v,u}} & \mathbf{W}w.F(vu) \\ \downarrow{\scriptstyle \omega_{w,v}Fu} & & \downarrow{\scriptstyle \xi_{w,vu}} \\ \mathbf{W}(wv)Fu & \xrightarrow[\xi_{wv,u}]{} & F(wvu) \end{array} \qquad \begin{array}{ccc} Fu & \xrightarrow{\omega_B Fu} & (\mathbf{W}1)Fu \\ & \searrow^{=} & \downarrow{\scriptstyle \xi_{1,u}} \\ & & Fu \end{array}$$

An arrow $\alpha:(F,\xi)\to(F',\xi')$ in $\hat{\mathbf{W}}A$ is a function which assigns to each arrow $u:A\to B$ of $\mathbf{A}$ an arrow $\alpha_u:Fu\to F'u$ of $\mathbf{W}B$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{W}v.Fu & \xrightarrow{(\mathbf{W}v)\alpha_u} & \mathbf{W}v.F'u \\ \downarrow{\scriptstyle \xi_{v,u}} & & \downarrow{\scriptstyle \xi'_{v,u}} \\ F(vu) & \xrightarrow[\alpha_{vu}]{} & F'(vu) \end{array}$$

The composition in $\hat{\mathbf{W}}A$ is simply given by $(\alpha'\alpha)_u=\alpha'_u\alpha_u$. For $f:A\to A'$ in $\mathbf{A}$, the functor $\hat{\mathbf{W}}f:\hat{\mathbf{W}}A\to\hat{\mathbf{W}}A'$ is defined by

$$(\hat{\mathbf{W}}f)(F,\xi)=(F',\xi'),\ \text{ where } F'u'=F(u'f),\ \xi'_{v',u'}=\xi_{v',u'f},$$
$$\text{and } ((\hat{\mathbf{W}}f)\alpha)_{u'}=\alpha_{u'f} \text{ for } u':A'\to B',\ v':B'\to C' \text{ in } \mathbf{A}.$$

For each $A\in\mathbf{A}$, define $\hat{E}:\hat{\mathbf{W}}A\to\mathbf{W}A$ by

$$\hat{E}_A(F,\xi)=F1_A \quad\text{and}\quad \hat{E}_A\alpha=\alpha_{1_A},$$

and define $\hat{J}_A : \mathbf{W}A \to \hat{\mathbf{W}}A$ by

$$\hat{J}_A X = (F, \xi) \quad \text{and} \quad \hat{J}_A x = \alpha,$$

where $Fu = (\mathbf{W}u)X$, $\xi_{v,u} = \omega_{v,u} X$ and $\alpha_u = (\mathbf{W}u)x$. The family of arrows $\omega_A X : X \to (\mathbf{W}1_A)X$, $X \in \mathbf{W}A$, are the components of a natural transformation $\hat{\eta}_A : 1 \to \hat{E}_A \hat{J}_A$. The family of arrows

$$\xi_{-,1_A} : (\mathbf{W}(-)F1_A, \omega F1_A) \to (F, \xi),\ (F, \xi) \in \hat{\mathbf{W}}A,$$

are the components of a natural transformation $\hat{\varepsilon}_A : \hat{J}_A \hat{E}_A \to 1$; the arrows are in $\hat{\mathbf{W}}A$ since

$$\begin{array}{ccc} \mathbf{W}v.\mathbf{W}u.F1_A & \xrightarrow{(\mathbf{W}v)\xi_{u,1_A}} & \mathbf{W}v.Fu \\ {\scriptstyle \omega_{v,u}F1_A}\downarrow & & \downarrow{\scriptstyle \xi_{v,u}} \\ \mathbf{W}(vu)F1_A & \xrightarrow[\xi_{vu,1_A}]{} & F(vu) \end{array}$$

commutes, and the family is natural since, for any arrow $\alpha : (F,\xi) \to (F',\xi')$ in $\hat{\mathbf{W}}A$, the diagram

$$\begin{array}{ccc} (\mathbf{W}u)F1_A & \xrightarrow{\xi_{u,1_A}} & Fu \\ {\scriptstyle (\mathbf{W}u)\alpha_{1_A}}\downarrow & & \downarrow{\scriptstyle \alpha_u} \\ (\mathbf{W}u)F'1_A & \xrightarrow[\xi'_{u,1_A}]{} & F'u \end{array}$$

commutes. The commutative diagrams

$$\begin{array}{ccc} F1_A & \xrightarrow{\omega_A F1_A} & (\mathbf{W}1_A)F1_A \\ & \searrow^{=} & \downarrow{\scriptstyle \xi_{1_A,1_A}} \\ & & F1_A \end{array} \qquad \begin{array}{ccc} (\mathbf{W}u)X & \xrightarrow{(\mathbf{W}u)\omega_A} & (\mathbf{W}u)(\mathbf{W}1_A)X \\ & \searrow^{=} & \downarrow{\scriptstyle \omega_{u,1_A}} \\ & & (\mathbf{W}u)X \end{array}$$

imply

$$\hat{E}_A \hat{\varepsilon}_A \cdot \hat{\eta}_A \hat{E}_A = 1_{\hat{E}_A} \quad \text{and} \quad \hat{\varepsilon}_A \hat{J}_A \cdot \hat{J}_A \hat{\eta}_A = 1_{\hat{J}_A}.$$

So for each $A \in \mathbf{A}$ we have an adjunction

$$\hat{\varepsilon}_A, \hat{\eta}_A : \hat{J}_A \dashv \hat{E}_A : (\hat{\mathbf{W}} A, \mathbf{W} A).$$

For $f : A \to A'$ in $\mathbf{A}$,

$$\hat{E}_{A'}(\hat{\mathbf{W}} f)\hat{J}_A X = \hat{E}_{A'}(\hat{\mathbf{W}} f)(\mathbf{W}(-)X, \omega X) = \hat{E}_{A'}(\mathbf{W}(-f)X, \omega_{-,-f}X)$$
$$= \mathbf{W}(1_{A'}, f)X = (\mathbf{W} f)X,$$

and

$$\hat{E}_{A'}(\hat{\mathbf{W}} f)\hat{J}_A x = \hat{E}_{A'}(\hat{\mathbf{W}} f)\mathbf{W}(-)x = \hat{E}_{A'}\mathbf{W}(-f)x = (\mathbf{W} f)x.$$

For $f : A \to A'$, $g : A' \to A''$ in $\mathbf{A}$,

$$\hat{E}_{A''}(\hat{\mathbf{W}} g)\,\hat{\varepsilon}_{A'}(\hat{\mathbf{W}} f)\hat{J}_A X = \hat{E}_{A''}(\hat{\mathbf{W}} g)\,\hat{\varepsilon}_{A'}(\hat{\mathbf{W}} f)\,(\mathbf{W}(-)X, \omega X)$$
$$= \hat{E}_{A''}(\hat{\mathbf{W}} g)\,\hat{\varepsilon}_{A'}(\mathbf{W}(-f)X, \omega_{-,-f}X) = \hat{E}_{A''}(\hat{\mathbf{W}} g)\,\omega_{-,f}X$$
$$= E_{A''}\omega_{-g,f}X = \omega_{g,f}X.$$

So put

$$\hat{E}_f(F, \xi) = \hat{E}_{A'}(\hat{\mathbf{W}} f)\,\varepsilon_A(F, \xi) = \hat{E}_{A'}(\hat{\mathbf{W}} f)\xi_{-,1_A}$$
$$= \hat{E}_{A'}\xi_{-f,1_A} = \xi_{f,1_A} : (\mathbf{W} f)F\,1_A \to Ff,$$

and

$$\hat{J}_f X = \hat{\varepsilon}_{A'}(\hat{\mathbf{W}} f)\hat{J}_A X = \hat{\varepsilon}_{A'}(\hat{\mathbf{W}} f)(\mathbf{W}(-)X, \omega_{-,-}X)$$
$$= \hat{\varepsilon}_{A'}(\mathbf{W}(-)X, \omega_{-,-f}X) = \omega_{-,f}X : \mathbf{W}(-)(\mathbf{W} f)X \to \mathbf{W}(-f)X.$$

Then $\hat{E} : \hat{\mathbf{W}} \to \mathbf{W}$ is a left lax transformation; $\hat{J} : \mathbf{W} \to \hat{\mathbf{W}}$ is a right lax transmation; $\hat{J} \dashv \hat{E}$, and $\mathbf{W} = \hat{E}\hat{\mathbf{W}}\hat{J}$.

THEOREM 2. *For every lax functor* $\mathbf{W} : \mathbf{A} \to \mathbf{Cat}$ *there exists a genuine functor* $\mathbf{V} : \mathbf{A} \to \mathbf{Cat}$, *a left lax transformation* $E : \mathbf{V} \to \mathbf{W}$ *and a right lax transformation* $J : \mathbf{W} \to \mathbf{V}$ *such that* J *is the left adjoint of* E *and* $\mathbf{W} = E\mathbf{V}J$.

3. Universal properties.

The basic constructions are characterized in this section as *2*-adjoints of two simple inclusion *2*-functors. All properties (up to isomormorphism) of the two constructions must be deducible from these characterizations. However, we do not choose to enter into this game; we use the explicit formulae wherever necessary. This is why the constructions are given in a separate section and are not included in the proofs of the existence of the adjoints.

THEOREM 3. *The inclusion of Gen* $[\mathbf{A},\mathbf{Cat}]$ *in* $\overrightarrow{Lax}\,[\mathbf{A},\mathbf{Cat}]$ *has a left 2-adjoint. The 2-reflection of the lax functor* $\mathbf{W}:\mathbf{A}\to\mathbf{Cat}$ *is the right lax transformation* $\widetilde{J}:\mathbf{W}\to\widetilde{\mathbf{W}}$.

PROOF. Suppose $\mathbf{V}:\mathbf{A}\to\mathbf{Cat}$ is a genuine functor.
A functor $\Sigma:[\overrightarrow{\mathbf{W},\mathbf{V}}]\to[\widetilde{\mathbf{W}},\mathbf{V}]$ will be defined. For an object R of $[\overrightarrow{\mathbf{W},\mathbf{V}}]$, the natural transformation $\Sigma(R):\widetilde{\mathbf{W}}\to\mathbf{V}$ is given by:

$$\Sigma(R)_A(u,X)=(\mathbf{V}u)R_{A'}X,$$

$$\Sigma(R)_A(h,\phi)=(\mathbf{V}u)(R_hX'.R_{A'}\phi).$$

Then

$$\Sigma(R)_A(1_{A'},\omega_{A'}X)=(\mathbf{V}u)(R_{1_{A'}}X.R_{A'}\omega_{A'}X)=1_{(\mathbf{V}u)R_{A'}X},$$

and $\Sigma(R)_A(hh',\omega_{h,h'}X''.(\mathbf{W}h)\phi'.\phi)=$

$$=(\mathbf{V}u)(R_{hh'}X''.R_{A'}(\omega_{h,h'}X''.(\mathbf{W}h)\phi'.\phi))$$

$$=(\mathbf{V}u)(R_{hh'}X''.R_{A'}\omega_{h,h'}X''.R_{A'}(\mathbf{W}h)\phi'.R_{A'}\phi)$$

$$=(\mathbf{V}u)((\mathbf{V}h)R_{h'}X''.R_h(\mathbf{W}h')X''.R_{A'}(\mathbf{W}h)\phi'.R_{A'}\phi)$$

$$=(\mathbf{V}u)((\mathbf{V}h)R_{h'}X''.(\mathbf{V}h)R_{A''}\phi'.R_hX'.R_{A'}\phi)$$

$$=(\mathbf{V}u')(R_{h'}X''.R_{A''}\phi').(\mathbf{V}u)(R_hX'.R_{A'}\phi)$$

$$=\Sigma(R)_A(h',\phi').\Sigma(R)_A(h,\phi).$$

So $\Sigma(R)_A:\widetilde{\mathbf{W}}A\to\mathbf{V}A$ is a functor. Suppose $f:A\to B$; then

$$\Sigma(R)_B(\widetilde{\mathbf{W}}f)(u,X)=\Sigma(R)_B(fu,X)=\mathbf{V}(fu)R_{A'}X$$

$$=(\mathbf{V}f)\Sigma(R)_A(u,X),$$

and

$$\Sigma(R)_B(\widetilde{\mathbf{W}}f)(h,\phi)=\Sigma(R)_B(h,\phi)=\mathbf{V}(fu)(R_hX'.R_{A'}\phi)$$

$$=(\mathbf{V}f)\Sigma(R)_A(h,\phi);$$

so $\Sigma(R)_B.\widetilde{\mathbf{W}}f=(\mathbf{V}f)\Sigma(R)_A$. Thus $\Sigma(R)$ is natural, as asserted. For an arrow $s:R\to S$ of $[\overrightarrow{\mathbf{W},\mathbf{V}}]$, the arrow $\Sigma(s):\Sigma(R)\to\Sigma(S)$ of $[\widetilde{\mathbf{W}},\mathbf{V}]$ is given by:

$$\Sigma(s)_A(u,X)=(\mathbf{V}u)s_{A'}X:(\mathbf{V}u)R_{A'}X\to(\mathbf{V}u)S_{A'}X.$$

Then

$$\Sigma(S)_A(h,\phi).\Sigma(s)_A(u,X)=(\mathbf{V}u)(S_hX'.S_{A'}\phi).(\mathbf{V}u)s_A{,}X$$
$$=(\mathbf{V}u)(S_hX'.S_A{,}\phi.s_A{,}X)$$
$$=(\mathbf{V}u)(S_hX'.s_A{,}(\mathbf{W}h)X'.R_A{,}\phi)$$
$$=(\mathbf{V}u)(\mathbf{V}h)s_A{,,}X'.R_hX'.R_A{,}\phi)$$
$$=(\mathbf{V}u')s_A{,,}X'.\Sigma(R)_A(h,\phi)$$
$$=\Sigma(s)_A(u',X').\Sigma(R)_A(h,\phi),$$

so that $\Sigma(s)_A:\Sigma(R)_A\to\Sigma(S)_A$ is natural. Also

$$(\Sigma(s)_B.\widetilde{\mathbf{W}}f)(u,X)=\Sigma(s)_B(fu,X)=\mathbf{V}(fu)s_A{,}X$$
$$=(\mathbf{V}f)\Sigma(s)_A(u,X),$$

so $\Sigma(s)_B.\widetilde{\mathbf{W}}f=(\mathbf{V}f)\Sigma(s)_A$, and $\Sigma(s)$ is an arrow of $[\widetilde{\mathbf{W}},\mathbf{V}]$. This clearly makes Σ a functor.

In order to show that Σ is an isomorphism we construct its inverse $\Sigma^{-1}:[\widetilde{\mathbf{W}},\mathbf{V}]\to[\overrightarrow{\mathbf{W},\mathbf{V}}]$. For a natural transformation $N:\widetilde{\mathbf{W}}\to\mathbf{V}$, the object $\Sigma^{-1}(N)$ of $[\overrightarrow{\mathbf{W},\mathbf{V}}]$ is given by:

$$\Sigma^{-1}(N)_A\phi=N_A(1_A,X),\ \Sigma^{-1}(N)_A\phi=N_A(1_A,\omega_AX'.\phi),$$
$$\Sigma^{-1}(N)_fX=N_B(f,1_{(\mathbf{W}f)X}):N_B(1_B,(\mathbf{W}f)X)\to N_B(f,X).$$

Many things must be checked. First

$$\Sigma^{-1}(N)_A1_X=N_A1_{(1_A,X)}=1_{N_A(1_A,X)},$$

and

$$\Sigma^{-1}(N)_A(\phi'\phi)=N_A(1_A,\omega_AX''.\phi'.\phi)=N_A(1_A,(\mathbf{W}1_A)\phi'.\omega_AX'.\phi)$$
$$=N_A(1_A,\omega_{1_A,1_A}X''.(\mathbf{W}1_A)(\omega_AX''.\phi').\omega_AX'.\phi)$$
$$=N_A((1_A,\omega_AX''.\phi')(1_A,\omega_AX'.\phi))$$
$$=(\Sigma^{-1}(N)_A\phi')(\Sigma^{-1}(N)_A\phi),$$

so that $\Sigma^{-1}(N)_A:\mathbf{W}A\to\mathbf{V}A$ is a functor. Then note

$$(\mathbf{V}f)\Sigma^{-1}(N)_A\phi.\Sigma^{-1}(N)_fX=N_B(1_A,\omega_AX'.\phi).N_B(f,1_{(\mathbf{W}f)X})$$
$$=N_B((1_A,\omega_AX'.\phi)(f,1_{(\mathbf{W}f)X}))$$

$$=N_B(f,(\mathbf{W}f)\phi)=N_B((f,1_{(\mathbf{W}f)X'})(1_B,\omega_A(\mathbf{W}f)X'.(\mathbf{W}f)\phi))$$
$$=N_B(f,1_{(\mathbf{W}f)X'}).N_B(1_B,\omega_A(\mathbf{W}f)X'.(\mathbf{W}f)\phi)$$
$$=\Sigma^{-1}(N)_fX'.\Sigma^{-1}(N)_B(\mathbf{W}f)\phi,$$

so $\Sigma^{-1}(N)_f:\Sigma^{-1}(N)_B(\mathbf{W}f)\to(\mathbf{V}f)\Sigma^{-1}(N)_A$ is natural. By applying N_C to the equation

$$(gf,1_{\mathbf{W}(gf)X})(1_C,\omega_C\mathbf{W}(gf)X.\omega_{g,f}X)=$$
$$=(gf,\omega_{1_C,gf}X.(\mathbf{W}1_C)1_{\mathbf{W}(gf)X}.\omega_C\mathbf{W}(gf)X.\omega_{g,f}X)$$
$$=(gf,\omega_{g,f}X)=(gf,\omega_{g,f}X.(\mathbf{W}g)1_{(\mathbf{W}f)X}.1_{(\mathbf{W}g)(\mathbf{W}f)X})$$
$$=(f,1_{(\mathbf{W}f)X})(g,1_{(\mathbf{W}g)(\mathbf{W}f)X});$$

by applying N_A to the equation

$$(1_A,1_{(\mathbf{W}1_A)X})(1_A,\omega_A\mathbf{W}(1_A)X.\omega_AX)=$$
$$=(1_A,\omega_{1_A,1_A}X.(\mathbf{W}1_A)1_{(\mathbf{W}1_A)X}.\omega_A\mathbf{W}(1_A)X.\omega_AX)=1_{(1_A,X)},$$

and by using the fact that N_C, N_A are functors, we obtain

$$\Sigma^{-1}(N)_{gf}.\Sigma^{-1}(N)_C\,\omega_{g,f}=(\mathbf{V}g)\Sigma^{-1}(N)_g(\mathbf{W}f)$$

and $\Sigma^{-1}(N)_{1_A}.\Sigma^{-1}(N)_A\,\omega_A=1_{\Sigma^{-1}(N)_A}$; so $\Sigma^{-1}(N)$ is an object of $[\overrightarrow{\mathbf{W},\mathbf{V}}]$. For an arrow $r:N\to P$ of $[\widetilde{\mathbf{W}},\mathbf{V}]$, the arrow

$$\Sigma^{-1}(r):\Sigma^{-1}(N)\to\Sigma^{-1}(P)$$

of $[\overrightarrow{\mathbf{W},\mathbf{V}}]$ is given by

$$\Sigma^{-1}(r)_AX=r_A(1_A,X):N_A(1_A,X)\to P_A(1_A,X).$$

Each r_A is natural, so $\Sigma^{-1}(r)_A$ is natural. Moreover,

$$\Sigma^{-1}(P)_fX.\Sigma^{-1}(r)_B(\mathbf{W}f)X=P_B(f,1_{(\mathbf{W}f)X}).r_B(1_B,(\mathbf{W}f)X)$$
$$=r_B(f,X).N_B(f,1_{(\mathbf{W}f)X})$$
$$=(\mathbf{V}f)r_A(1_A,X).N_B(f,1_{(\mathbf{W}f)X})$$
$$=(\mathbf{V}f)\Sigma^{-1}(r)_AX.\Sigma^{-1}(N)_fX,$$

so $\Sigma^{-1}(r)$ is an arrow of $[\overrightarrow{\mathbf{W},\mathbf{V}}]$. Clearly Σ^{-1} is a functor.

Take R in $[\overrightarrow{\mathbf{W},\mathbf{V}}]$. Then

$$\Sigma^{-1}(\Sigma(R))_AX=\Sigma(R)_A(1_A,X)=(\mathbf{V}1_A)R_AX=R_AX,$$

$$\Sigma^{-1}(\Sigma(R))_A\phi = \Sigma(R)_A(1_A, \omega_A X'.\phi)$$
$$= (\mathbf{V}1_A)(R_{1_A}X'.R_A(\omega_A X'.\phi))$$
$$= R_A\phi,$$
$$\Sigma^{-1}(\Sigma(R))_f X = \Sigma(R)_B(f, 1_{(\mathbf{W}f)X}) =$$
$$= \mathbf{V}(1_B)(R_f X.R_B 1_{(\mathbf{W}f)X}) = R_f X;$$

so $\Sigma^{-1}(\Sigma(R)) = R$. Now take an arrow $s : R \to S$ of $[\overrightarrow{\mathbf{W},\mathbf{V}}]$. Then

$$\Sigma^{-1}(\Sigma(s))_A X = \Sigma(s)_A(1_A, X) = (\mathbf{V}1_A)s_A X = s_A X.$$

So $\Sigma^{-1}\Sigma = 1$.

Take N in $[\widetilde{\mathbf{W}},\mathbf{V}]$. Then

$$\Sigma(\Sigma^{-1}(N))_A(u,X) = (\mathbf{V}u)\Sigma^{-1}(N)_{A'}X = (\mathbf{V}u)N_{A'}(1_{A'}, X)$$
$$= N_A(\widetilde{\mathbf{W}}u)(1_{A'}, X) = N_A(u, X),$$
$$\Sigma(\Sigma^{-1}(N))_A(h,\phi) = (\mathbf{V}u)(\Sigma^{-1}(N)_h X'.\Sigma^{-1}(N)_{A'}\phi)$$
$$= (\mathbf{V}u)(N_{A'}(h, 1_{(\mathbf{W}h)X'}).N_{A'}(1_{A'}, \omega_{A'}(\mathbf{W}h)X'.\phi))$$
$$= N_A(\widetilde{\mathbf{W}}u)(h, \omega_{1_{A'},h}X'.(\mathbf{W}1_{A'})1_{(\mathbf{W}h)X'}.\omega_{A'}(\mathbf{W}h)X'.\phi)$$
$$= N_A(h,\phi),$$

so $\Sigma(\Sigma^{-1}(N)) = N$. Take $r : N \to P$ in $[\widetilde{\mathbf{W}},\mathbf{V}]$. Then

$$\Sigma(\Sigma^{-1}(r))_A(u,X) = (\mathbf{V}u)\Sigma^{-1}(r)_{A'}X = (\mathbf{V}u)r_{A'}(1_{A'}, X)$$
$$= r_A(\widetilde{\mathbf{W}}u)(1_{A'}, X) = r_A(u, X).$$

So $\Sigma\Sigma^{-1} = 1$.

It remains to prove that Σ is 2-natural in $\mathbf{V}$. Suppose $N : \mathbf{V} \to \mathbf{V}'$ is a natural transformation. We must show that

$$\begin{array}{ccc}
[\overrightarrow{\mathbf{W},\mathbf{V}}] & \xrightarrow{\quad\Sigma\quad} & [\widetilde{\mathbf{W}},\mathbf{V}] \\
{\scriptstyle [\overrightarrow{\mathbf{W},\mathbf{N}}]}\downarrow & & \downarrow{\scriptstyle [\widetilde{\mathbf{W}},\mathbf{N}]} \\
[\overrightarrow{\mathbf{W},\mathbf{V}'}] & \xrightarrow[\quad\Sigma\quad]{} & [\widetilde{\mathbf{W}},\mathbf{V}']
\end{array}$$

commutes. So take R in $[\overrightarrow{\mathbf{W},\mathbf{V}}]$. Then

$$\Sigma(NR)_A(u,X)=(\mathbf{V}'u)(NR)_A.X=(\mathbf{V}'u)N_A.R_A.X$$
$$=N_A(\mathbf{V}u)R_A.X=N_A\Sigma(R)_A(u,X)=(N\Sigma(R))_A(u,X),$$

and

$$\Sigma(NR)_A(h,\phi)=(\mathbf{V}'u)((NR)_h X'.(NR)_A.\phi)$$
$$=(\mathbf{V}'u)N_A.(R_h X'.R_A.\phi)=N_A\Sigma(R)_A(h,\phi)=(N\Sigma(R))_A(h,\phi).$$

So $\Sigma(NR)=N\Sigma(R)$ and we have the commutativity on objects. Suppose $s:R\to S$ is an arrow of $[\overrightarrow{\mathbf{W},\mathbf{V}}]$. Then

$$\Sigma(Ns)_A(u,X)=(\mathbf{V}'u)(Ns)_A.X=N_A(\mathbf{V}u)s_A.X$$
$$=N_A\Sigma(s)_A(u,X)=(N\Sigma(s))_A(u,X),$$

so $\Sigma(Ns)=N\Sigma(s)$. So the square commutes. This proves ordinary naturality of Σ. For 2-naturality, we must show that

$$\begin{array}{ccc}
[\overrightarrow{\mathbf{W},\mathbf{V}}](R,S) & \xrightarrow{\ \Sigma\ } & [\widetilde{\mathbf{W}},\mathbf{V}](\Sigma(R),\Sigma(S)) \\
{\scriptstyle [\overrightarrow{\mathbf{W},r}]}\downarrow & & \downarrow{\scriptstyle [\widetilde{\mathbf{W}},r]} \\
[\overrightarrow{\mathbf{W},\mathbf{V}'}](NR,PS) & \xrightarrow[\ \Sigma\ ]{} & [\widetilde{\mathbf{W}}\ \mathbf{V}'](\Sigma(NR),\Sigma(PS))
\end{array}$$

commutes for any arrow $r:N\to P$ of $[\mathbf{V},\mathbf{V}']$. So take $s:R\to S$ in $[\overrightarrow{\mathbf{W},\mathbf{V}}]$. Then

$$\Sigma(rs)_A(u,X)=(\mathbf{V}'u)(rs)_A.X=(\mathbf{V}'u)r_A.s_A.X=r_A(\mathbf{V}u)s_A X$$
$$=r_A\Sigma(s)_A(u,X)=(r\Sigma(s))_A(u,X)\ .$$

It follows that the assignment $\mathbf{W}\to\widetilde{\mathbf{W}}$ is the object function of a unique 2-functor from $\overrightarrow{Lax}\,[\mathbf{A},\mathbf{Cat}]$ to $Gen\,[\mathbf{A},\mathbf{Cat}]$ such that, for each functor $\mathbf{V}$, Σ is 2-natural in $\mathbf{W}$; and this 2-functor is the required left 2-adjoint. The 2-reflection of $\mathbf{W}$ is the image of the identity of $\widetilde{\mathbf{W}}$ under $\Sigma^{-1}:[\widetilde{\mathbf{W}},\widetilde{\mathbf{W}}]\to[\overrightarrow{\mathbf{W},\widetilde{\mathbf{W}}}]$. From the definitions of Σ^{-1} and $\widetilde{J}$ one readily see that $\Sigma^{-1}(1_{\widetilde{\mathbf{W}}})=\widetilde{J}$.

COROLLARY. *Suppose the lax functor* $\mathbf{W}:\mathbf{A}\to\mathbf{Cat}$ *is generated by the adjunction* $J\dashv E:(\mathbf{V},\mathbf{W})$, *where* $\mathbf{V}:\mathbf{A}\to\mathbf{Cat}$ *is a genuine functor. Then there exists a unique natural transformation* $N:\widetilde{\mathbf{W}}\to\mathbf{V}$ *such that* $N\widetilde{J}=J$;

moreover, this N *also satisfies* $E N = \widetilde{E}$.

PROOF. The existence and uniqueness of N satifying $NJ = J$ is immediate from the theorem. Then

$$\begin{aligned} N_A\, \widetilde{\varepsilon}_A(u, X) &= N_A\, \widetilde{\varepsilon}_A(\widetilde{\mathbf{W}} u)(1_{A'}, X) = N_A\, \widetilde{\varepsilon}_A(\widetilde{\mathbf{W}} u)\widetilde{J}_{A'} X = N_A \widetilde{J}_u X \\ &= (N\widetilde{J})_u X = J_u X = \varepsilon_A(\mathbf{V} u) J_{A'} X = \varepsilon_A(\mathbf{V} u) N_{A'} \widetilde{J}_{A'} X \\ &= \varepsilon_A\, N_A(\widetilde{\mathbf{W}} u)\widetilde{J}_{A'} X = \varepsilon_A\, N_A(u, X), \end{aligned}$$

so $N_A\, \widetilde{\varepsilon}_A = \varepsilon_A\, N_A$. If follows that $E_A\, N_A = \widetilde{E}_A$. Then

$$\begin{aligned} E_f N_A &= E_B(\mathbf{V} f)\, \varepsilon_A\, N_A = E_B(\mathbf{V} f) N_A\, \widetilde{\varepsilon}_A = E_B N_B(\widetilde{\mathbf{W}} f)\, \widetilde{\varepsilon}_A \\ &= \widetilde{E}_A(\widetilde{\mathbf{W}} f)\widetilde{\varepsilon}_A = \widetilde{E}_f. \end{aligned}$$

So $E N = \widetilde{E}$.

THEOREM 4. *The inclusion of Gen* $[\mathbf{A}, \mathbf{Cat}]$ *in* $\overleftarrow{Lax}\,[\mathbf{A}, \mathbf{Cat}]$ *has a right 2-adjoint. The 2-coreflection of the lax functor* $\mathbf{W}: \mathbf{A} \to \mathbf{Cat}$ *is the left lax transformation* $\hat{\mathbf{E}}: \hat{\mathbf{W}} \to \mathbf{W}$.

PROOF. Suppose $\mathbf{V}: \mathbf{A} \to \mathbf{Cat}$ is a genuine functor.

A functor $\Pi: [\overleftarrow{\mathbf{V}, \mathbf{W}}] \to [\mathbf{V}, \hat{\mathbf{W}}]$ will be defined. For an object L of $[\overleftarrow{\mathbf{V}, \mathbf{W}}]$, the natural transformation $\Pi(L): \mathbf{V} \to \hat{\mathbf{W}}$ is given by:

for $A \in \mathbf{A}$ and $H \in \mathbf{V} A$, $\Pi(L)_A H = (F, \xi)$, where $F u = L_B(\mathbf{V} u) H$ and $\xi_{v,u} = L_v(\mathbf{V} u) H$,

and for $h: H \to H'$ in $\mathbf{V} A$, $(\Pi(L)_A h)_u = L_B(\mathbf{V} u) h$.

The two diagrams which commute due to the fact that L is an object of $[\overleftarrow{\mathbf{V}, \mathbf{W}}]$ show that (F, ξ) is an object of $\hat{\mathbf{W}} A$, and the naturality of each L_v shows that $\Pi(L)_A h: \Pi(L)_A H \to \Pi(L)_A H'$ is an arrow of $\hat{\mathbf{W}} A$. For $f: A \to A'$, one readily checks that $\hat{\mathbf{W}} f . \Pi(L)_A = \Pi(L)_{A'} . \mathbf{V} f$, so that $\Pi(L): \mathbf{V} \to \mathbf{W}$ is a natural transformation. For an arrow $s: L \to M$ of $[\overleftarrow{\mathbf{V}, \mathbf{W}}]$, the arrow $\Pi(s): \Pi(L) \to \Pi(M)$ of $[\mathbf{V}, \hat{\mathbf{W}}]$ is given by:

$$(\Pi(s)_A H)_u = s_B(\mathbf{V} u) H : L_B(\mathbf{V} u) H \to M_B(\mathbf{V} u) H.$$

Then $\Pi(s)_A H: \Pi(L)_A H \to \Pi(M)_A H$ is an arrow of $\hat{\mathbf{W}} A$ and $\Pi(s)_A: \Pi(L)_A \to \Pi(M)_A$ is natural. From the calculation

$$(\hat{\mathbf{W}} f . \Pi(s)_A) H_{u'} = (\Pi(s)_A H)_{u' f} = s_{B'} \mathbf{V}(u' f) H$$

$$= s_{B'}(\mathbf{V}u')(\mathbf{V}f)H = (\Pi(s)_{A'}(\mathbf{V}f)H)_{u'},$$

it follows that $\Pi(s):\Pi(L)\to\Pi(M)$ is an arrow of $[\mathbf{V},\hat{\mathbf{W}}]$. If s is the identity, so is $\Pi(s)$; and the calculation

$$(\Pi(ts)_A H)_u = (ts)_B(\mathbf{V}u)H = (t_B s_B)(\mathbf{V}u)H$$
$$= t_B(\mathbf{V}u)H\,.\,s_B(\mathbf{V}u)H = (\Pi(t)_A H)_u\,.\,(\Pi(s)_A H)_u$$
$$= (\Pi(t)_A H\,.\,\Pi(s)_A H)_u = ((\Pi(t)\Pi(s))_A H)_u$$

completes the proof that Π is a functor.

We show that Π is an isomorphism by constructing its inverse $\Pi^{-1}:[\mathbf{V},\hat{\mathbf{W}}]\to[\overleftarrow{\mathbf{V},\mathbf{W}}]$. For an object N of $[\mathbf{V},\hat{\mathbf{W}}]$, the object $\Pi^{-1}(N)$ of $[\overleftarrow{\mathbf{V},\mathbf{W}}]$ is given by:

$$\Pi^{-1}(N)_A H = F1_A,\quad \Pi^{-1}(N)_A h = (N_A h)_{1_A},$$
$$\Pi^{-1}(N)_f H = \xi_{f,1_A} : \mathbf{W}f\,.\,F1_A \to Ff,$$

where $N_A H = (F,\xi)$. Each $\Pi^{-1}(N)_A$ is clearly a functor. Also $\hat{\mathbf{W}}f\,.\,N_A = N_{A'}\,.\,\mathbf{V}f$, so evaluating at H gives $N_{A'}(\mathbf{V}f)H = (F(\cdot f),\xi_{-,-f})$; so $\Pi^{-1}(N)_{A'}(\mathbf{V}f)H = Ff$ and thus

$$\Pi^{-1}(N)_f H : (\mathbf{W}f)\Pi^{-1}(N)_A H \to \Pi^{-1}(N)_{A'}(\mathbf{V}f)H.$$

Evaluating $\hat{\mathbf{W}}f\,.\,N_A = N_{A'}\,.\,\mathbf{V}f$ at h, we get

$$\Pi^{-1}(N)_{A'}(\mathbf{V}f)h = (N_{A'}(\mathbf{V}f)h)_{1_{A'}} = ((\hat{\mathbf{W}}f)(N_A h))_{1_{A'}} = (N_A h)_f;$$

and $N_A h$ is an arrow of $\hat{\mathbf{W}}A$, so

$$\begin{array}{ccc}
(\mathbf{W}f)\Pi^{-1}(N)_A H & \xrightarrow{\ \Pi^{-1}(N)_f H = \xi_{f,1}\ } & \Pi^{-1}(N)_{A'}(\mathbf{V}f)H = Ff \\
{\scriptstyle (\mathbf{W}f)\Pi^{-1}(N)_A h = (\mathbf{W}f)(N_A h)_{1_A}}\big\downarrow & & \big\downarrow{\scriptstyle \Pi^{-1}(N)_{A'}(\mathbf{V}f)h = (N_A h)_f} \\
(\mathbf{W}f)\Pi^{-1}(N)_{A'}H' & \xrightarrow[\ \Pi^{-1}(N)_f H' = \xi'_{f,1}\ ]{} & \Pi^{-1}(N)_{A'}(\mathbf{V}f)H' = F'f
\end{array}$$

commutes, exposing $\Pi^{-1}(N)_f : (\mathbf{W}f)\Pi^{-1}(N)_A \to \Pi^{-1}(N)_{A'}(\mathbf{V}f)$ as a natural transformation. From the diagrams for the object (F,ξ) of $\hat{\mathbf{W}}A$ come the diagrams which prove $\Pi^{-1}(N)$ is an object of $[\overleftarrow{\mathbf{V},\mathbf{W}}]$. For an arrow $r:N\to P$ of $[\mathbf{V},\hat{\mathbf{W}}]$, the arrow $\Pi^{-1}(r):\Pi^{-1}(N)\to\Pi^{-1}(P)$ of

$[\overleftarrow{\mathbf{V},\mathbf{W}}]$ is given by $\Pi^{-1}(r)_A H = (r_A H)_{1_A}$. The naturality of $\Pi^{-1}(r)_A$ follows from that of r_A. Also

$$((\Pi^{-1}(r)_A \cdot (\mathbf{V} f)) . \Pi^{-1}(N)_f) H = (r_A \cdot (\mathbf{V} f) H)_{1_A} \cdot \xi_{f,1_A}$$

$$= ((\mathbf{W} f) r_A H)_{1_A} \cdot \xi_{f,1_A} = (r_A H)_f \cdot \xi_{f,1_A} = \eta_{f,1_A} \cdot (\mathbf{W} f)(r_A H)_{1_A}$$

$$= \Pi^{-1}(P)_f H . (\mathbf{W} f) \Pi^{-1}(r)_A H = (\Pi^{-1}(P)_f \cdot (\mathbf{W} f) \Pi^{-1}(r)_A) H$$

where $P_A H = (G, \eta)$; so $\Pi^{-1}(r)$ is an arrow of $[\overleftarrow{\mathbf{V},\mathbf{W}}]$. Moreover, Π^{-1} is clearly a functor.

Take an object L of $[\overleftarrow{\mathbf{V},\mathbf{W}}]$ and let $\Pi(L)_A H = (F, \xi)$. Then

$$\Pi^{-1}(\Pi(L))_A H = F 1_A = L_A H,\ \Pi^{-1}(\Pi(L))_A h = (\Pi(L)_A h)_{1_A} = L_A h$$

and $\Pi^{-1}(\Pi(L))_f H = \xi_{f,1_A} = L_f H$. So $\Pi^{-1}(\Pi(L)) = L$. Take an arrow $s : L \to M$ of $[\overleftarrow{\mathbf{V},\mathbf{W}}]$. Then

$$\Pi^{-1}(\Pi(s))_A H = (\Pi(s)_A H)_{1_A} = s_A H.$$

So $\Pi^{-1}\Pi = 1$.

Take an object N of $[\mathbf{V}, \hat{\mathbf{W}}]$ and let $N_A H = (F, \xi)$. Then

$$\Pi(\Pi^{-1}(N))_A H = (\bar{F}, \bar{\xi})$$

is given by

$$\bar{F} u = \Pi^{-1}(N)_B (\mathbf{V} u) H = F u,$$

and

$$\bar{\xi}_{v,u} = \Pi^{-1}(N)_v (\mathbf{V} u) H = \xi_{v,1_A u} = \xi_{v,u}.$$

Also

$$(\Pi(\Pi^{-1}(N))_A h)_u = \Pi^{-1}(N)_B (\mathbf{V} u) h = (N_A h)_u,$$

so $\Pi(\Pi^{-1}(N)) = N$. Take an arrow $r : N \to P$ of $[\mathbf{V}, \hat{\mathbf{W}}]$. Then

$$\Pi(\Pi^{-1}(r)) H)_u = \Pi^{-1}(r)_B (\mathbf{V} u) H = (r_B (\mathbf{V} u) H)_{1_B}$$

$$= ((\hat{\mathbf{W}} u) r_A H)_{1_B} = (r_A H)_u ;$$

so $\Pi\Pi^{-1} = 1$.

In order to show that Π is natural in $\mathbf{V}$ we must prove that the diagram

$$\begin{array}{ccc} [\overleftarrow{\mathbf{V}',\mathbf{W}}] & \xrightarrow{\quad\Pi\quad} & [\mathbf{V}',\hat{\mathbf{W}}] \\ {\scriptstyle [\overleftarrow{N,\mathbf{W}}]}\downarrow & & \downarrow{\scriptstyle [N,\hat{\mathbf{W}}]} \\ [\overleftarrow{\mathbf{V},\mathbf{W}}] & \xrightarrow[\quad\Pi\quad]{} & [\mathbf{V},\hat{\mathbf{W}}] \end{array}$$

commutes for all natural transformations $N:\mathbf{V}\to\mathbf{V}'$. Take $L'\in[\overleftarrow{\mathbf{V}',\mathbf{W}}]$. Then $\Pi(L'N)_A H=(F,\xi)$ where

$$Fu=(L'N)_B(\mathbf{V}u)H=L'_B(\mathbf{V}'u)N_A H$$

and

$$\xi_{v,u}=(L'N)_v(\mathbf{V}u)H=(L'_vN_B)(\mathbf{V}u)H=L'_v(\mathbf{V}u)N_A H;$$

so $(F,\xi)=\Pi(L')_A N_A H$. Also

$$(\Pi(L'N)_A h)_u=(L'N)_B(\mathbf{V}u)h=L'(\mathbf{V}'u)N_A h.$$

So $\Pi(L'N)=\Pi(L')N$. Now take $s':L'\to M'$ in $[\overleftarrow{\mathbf{V}',\mathbf{W}}]$. Then

$$\begin{aligned}(\Pi(s'N)_A H)_u&=(s'N)_B(\mathbf{V}u)H=s'_BN_B(\mathbf{V}u)H\\ &=s'_B(\mathbf{V}'u)N_A H=(\Pi(s')_A N_A H)_u,\end{aligned}$$

so $\Pi(s'N)=\Pi(s')N$.

To show that Π is 2-natural in $\mathbf{V}$, we must show that

$$\begin{array}{ccc} [\overleftarrow{\mathbf{V}',\mathbf{W}}](L',M') & \xrightarrow{\quad\Pi\quad} & [\mathbf{V}',\hat{\mathbf{W}}](\Pi(L'),\Pi(M')) \\ {\scriptstyle [\overleftarrow{r,\mathbf{W}}]}\downarrow & & \downarrow{\scriptstyle [r,\hat{\mathbf{W}}]} \\ [\overleftarrow{\mathbf{V},\mathbf{W}}](L'N,M'P) & \xrightarrow[\quad\Pi\quad]{} & [\mathbf{V},\hat{\mathbf{W}}](\Pi(L'N),\Pi(M'P)) \end{array}$$

commutes for all arrows $r:N\to P$ of $[\mathbf{V},\mathbf{V}']$. Take $s':L'\to M'$ in $[\overleftarrow{\mathbf{V}',\mathbf{W}}]$. Then

$$\begin{aligned}(\Pi(s'r)_A H)_u&=(s'r)_B(\mathbf{V}u)H=(s'_BP_B\,.\,L'_Br_B)(\mathbf{V}u)H\\ &=s'_BP_B(\mathbf{V}u)H\,.\,L'_Br_B(\mathbf{V}u)H=s'_B(\mathbf{V}u)P_AH\,.\,L'_B(\mathbf{V}u)r_AH\\ &=(\Pi(s')_AP_AH)\,.\,(\Pi(L')_Ar_AH)_u=((\Pi(s')_AP_A\,.\,\Pi(L')_Ar_A)H)_u\\ &=((\Pi(s')r)_AH)_u;\end{aligned}$$

so $\Pi(s'r)=\Pi(s')r$. So Π is 2-natural in $\mathbf{V}$.

Finally, note that the image of $1_{\hat{\mathbf{W}}}$ under the functor

$$\Pi^{-1} : [\hat{\mathbf{W}}, \hat{\mathbf{W}}] \to [\hat{\mathbf{W}}, \overleftarrow{\mathbf{W}}]$$

is the left lax transformation $\hat{E} : \hat{\mathbf{W}} \to \mathbf{W}$.

COROLLARY. *Suppose the lax functor* $\mathbf{W} : \mathbf{A} \to \mathbf{Cat}$ *is generated by the adjunction* $J \dashv E : (\mathbf{V}, \mathbf{W})$, *where* $\mathbf{V} : \mathbf{A} \to \mathbf{Cat}$ *is a genuine functor. Then there exists a unique natural transformation* $N : \mathbf{V} \to \hat{\mathbf{W}}$ *such that* $\hat{E} N = E$; *moreover, this* N *also satisfies* $N J = \hat{J}$.

PROOF. The existence and uniqueness of N such that $\hat{E} N = E$ follow from the theorem. Then

$$(\hat{\varepsilon}_A N_A H)_u = \hat{E}_u N_A H = (\hat{E} N)_u H = E_u H = E_B (\mathbf{V} u) \varepsilon_A H$$
$$= \hat{E}_B N_B (\mathbf{V} u) \varepsilon_A H = \hat{E}_B (\mathbf{W} u) N_A \varepsilon_A H = (N_A \varepsilon_A H)_u,$$

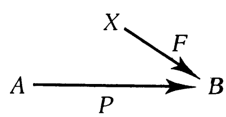
so $\hat{\varepsilon}_A N_A = N_A \varepsilon_A$. It follows that $N_A J_A = \hat{J}_A$. Then

$$N_B J_f = N_B \varepsilon_B (\mathbf{V} f) J_A = \hat{\varepsilon}_B N_B (\mathbf{V} f) J_A = \hat{\varepsilon}_B (\mathbf{W} f) N_A J_A$$
$$= \hat{\varepsilon}_B (\mathbf{W} f) J_A = \hat{J}_f ;$$

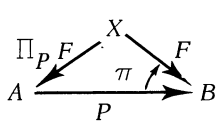
so $N J = \hat{J}$.

4. Structure and semantics, and a dual.

Given a diagram



of functors, a *right lifting of* F *along* P is a functor $\Pi_P F : X \to A$ and a natural transformation $\pi : P . \Pi_P F \to F$ such that any natural transformation $\theta : G \to \Pi_P F$ with codomain $\Pi_P F$ is uniquely determined by the composite $\pi . P \theta : P G \to P . \Pi_P F \to F$.



THEOREM 5. *If* $P : A \to B$ *is a functor with a right adjoint* $\hat{P} : B \to A$ *and* $\varepsilon : P \hat{P} \to 1$ *is the counit of the adjunction, then any functor* $F : X \to B$ *has*

a right lifting along P *given by the functor* $PF: X \to A$ *and the natural transformation* $\varepsilon F : P(\hat{P} F) \to F$.

Suppose $\mathbf{A}$ is a category and $\mathbf{X}$ is a family of categories $\mathbf{X}_A$ indexed by the objects A of $\mathbf{A}$. A *lax functor at* $\mathbf{X}$ is a lax functor $\mathbf{W}: \mathbf{A} \to \mathbf{Cat}$ such that $\mathbf{W}A = \mathbf{X}_A$ for all $A \in \mathbf{A}$. A *morphism* $\phi: \mathbf{W} \to \mathbf{W}'$ *of lax functors at* $\mathbf{X}$ is a function which assigns to each arrow $f: \mathbf{A} \to \mathbf{B}$ of $\mathbf{A}$ a natural transformation $\phi_f: \mathbf{W}f \to \mathbf{W}'f$ such that the following diagrams commute:

$$\begin{array}{ccccc}
\mathbf{W}g.\mathbf{W}f & \xrightarrow{\omega_{g,f}} & \mathbf{W}(gf) & & \\
\downarrow{\scriptstyle \phi_g(\mathbf{W}f)} & & & \searrow{\scriptstyle \phi_{gf}} & \\
 & & & & \mathbf{W}'(gf) \\
 & & & \nearrow{\scriptstyle \omega_{g,f}} & \\
\mathbf{W}'g.\mathbf{W}f & \xrightarrow[(\mathbf{W}'g)\phi_f]{} & \mathbf{W}'g.\mathbf{W}'f & &
\end{array}$$

$$\begin{array}{ccc}
1_{\mathbf{W}A} & \xrightarrow{\omega_A} & \mathbf{W}1_A \\
 & \searrow{\scriptstyle \omega_A} & \downarrow{\scriptstyle \phi_{1_A}} \\
 & & \mathbf{W}'1_A
\end{array}$$

Let $|\mathbf{A}|$ denote the subcategory of $\mathbf{A}$ with the same objects as $\mathbf{A}$ but with only the identity arrows. Objects, arrows and 2-cells of *Gen* $[\,|\mathbf{A}|, \mathbf{Cat}\,]$ are just families of categories, functors and natural transformations indexed by the objects of $\mathbf{A}$. The analysis of «structure and semantics» for the first basic construction involves partial fibration properties of the 2-functor $\vec{P}: \vec{Lax}\,[\mathbf{A}, \mathbf{Cat}] \to Gen\,[\,|\mathbf{A}|, \mathbf{Cat}\,]$ given by

$$\vec{P}(\mathbf{W}) = (\mathbf{W}A)_{A \in |\mathbf{A}|},\ \vec{P}(R) = (R_A)_{A \in |\mathbf{A}|},\ \vec{P}(s) = (s_A)_{A \in |\mathbf{A}|}.$$

Notice that $\vec{P}$ is faithful on 2-cells and so the fibre 2-categories are just categories - they have only identity 2-cells. For this reason it suffices to consider $\vec{P}$ as only a functor, neglecting its action on 2-cells. The fibre category $\vec{P}^{-1}(\mathbf{X})$ over $\mathbf{X}$ will be denoted by $Fib_{\mathbf{A}}(\mathbf{X})$; its objects are lax functors at $\mathbf{X}$ and its arrows are morphisms of lax functors at $\mathbf{X}$.

For $\mathbf{X} \in Gen\,[\,|\mathbf{A}|, \mathbf{Cat}\,]$, the comma category $(\mathbf{X}, \vec{P})$ has objects pairs $(\mathbf{V}, J)$ where $\mathbf{V}: \mathbf{A} \to \mathbf{Cat}$ is a lax functor and $J: \mathbf{X} \to \overrightarrow{P}\mathbf{V}$ is an arrow of $Gen\,[\,|\mathbf{A}|, \mathbf{Cat}\,]$, and has arrows $R:(\mathbf{V}, J) \to (\mathbf{V}', J')$, right lax transformations $R: \mathbf{V} \to \mathbf{V}'$ such that $\overrightarrow{P}(R).J = J'$. An object $(\mathbf{V}, J)$ of $(\mathbf{X}, \overrightarrow{P})$ is said to be *tractable* when there exists a cartesian arrow (with respect to the functor $\overrightarrow{P}$) over J which has codomain $\mathbf{V}$. This means that there exists a lax functor $J * \mathbf{V}$ at $\mathbf{X}$ and a right lax transformation $\bar{J}: J * \mathbf{V} \to \mathbf{V}$ with $\overrightarrow{P}(J) = J$ such that, if $R: \mathbf{W} \to \mathbf{V}$ is a right lax transformation with $\overrightarrow{P}(R) = J$, then there exists a unique morphism $\phi: \mathbf{W} \to J * \mathbf{V}$ of lax functors at $\mathbf{X}$ such that $R = J\phi$.

THEOREM 6. *Suppose $(\mathbf{V}, J)$ is an object of $(\mathbf{X}, \overrightarrow{P})$. Suppose that, for each $f: A \to B$ in $\mathbf{A}$, a diagram*

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\;J_A\;} & \mathbf{V}A \\ {\scriptstyle (\mathbf{V}J)f}\downarrow & \nearrow_{J_f} & \downarrow{\scriptstyle \mathbf{V}f} \\ \mathbf{X} & \xrightarrow[\;J_B\;]{} & \mathbf{V}B \end{array}$$

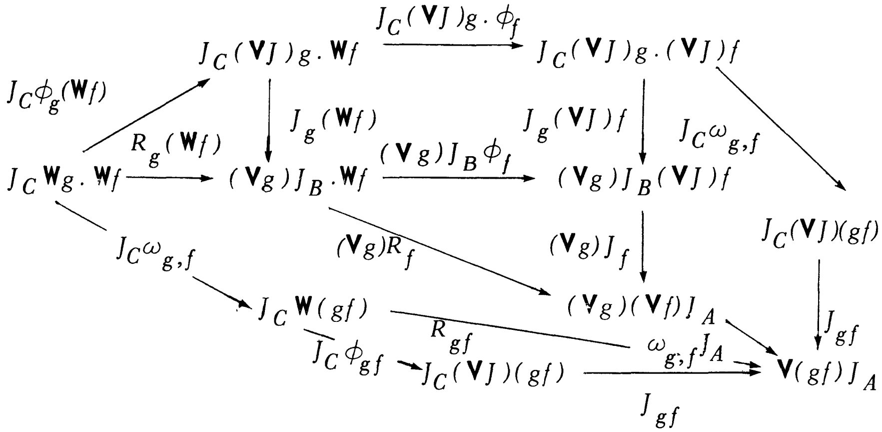
is given, in which the functor $(\mathbf{V}J)f: \mathbf{X}_A \to \mathbf{X}_B$ and the natural transformation $J_f: J_B.(\mathbf{V}J)f \to (\mathbf{V}f).J_A$ form a right lifting of $\mathbf{V}f.J_A$ along J_B. Then

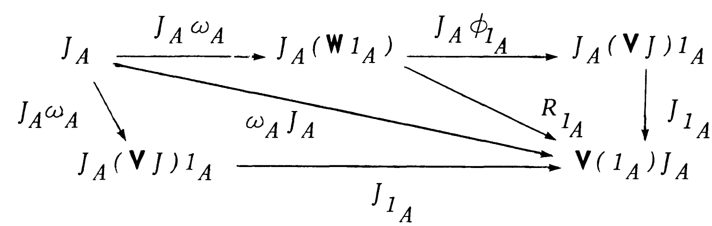
(a) the data $(\mathbf{V}J)_A = \mathbf{X}_A$, $(\mathbf{V}J)f$ can be uniquely enriched to a lax functor $\mathbf{V}J: A \to \mathbf{Cat}$ such that the data J_A, J_f form a right lax transformation $\mathbf{J}: \mathbf{V}J \to \mathbf{V}$;

(b) the right lax transformation $\bar{J}: \mathbf{V}J \to \mathbf{V}$ is a cartesian arrow over $J: \mathbf{X} \to \overrightarrow{P}\mathbf{V}$, so $(\mathbf{V}, J)$ is tractable.

PROOF. (a) The definitions of $\omega_{g,f}$, ω_A for $\mathbf{V}J$ come from the conditions that are needed for J_A, J_f to form a right lax transformation $\bar{J}$: $\mathbf{V}J \to \mathbf{V}$; one uses the universal property of right liftings.

(b) Suppose $R: \mathbf{W} \to \mathbf{V}$ is such that $\overrightarrow{P}(R) = J$; that is, $R_A = J_A$ for all $A \in \mathbf{A}$. For $f: A \to B$ in $\mathbf{A}$, $R_f: J_B.\mathbf{W}f \to \mathbf{V}f.J_A$ uniquely determines a natural transformation $\phi_f: \mathbf{W}f \to (\mathbf{V}J)f$ such that $R_f = J_f.J_B\phi_f$. Then the following diagrams commute.





The universal property of liftings allow us to deduce that $\phi : \mathbf{W} \to \mathbf{V}J$ is a morphism of lax functors at $\mathbf{X}$; moreover, it is the unique one such that $J_\phi = R$.

Putting together Theorem 5, Theorem 6 and the definition of $E\mathbf{V}J$, we obtain:

COROLLARY. *If $\mathbf{V} : \mathbf{A} \to \mathbf{Cat}$ is a lax functor and, for each $A \in \mathbf{A}$, there is an adjunction ε_A, $\eta_A : J_A \dashv E_A : (\mathbf{V}A, \mathbf{X}_A)$, then $\mathbf{V}J = E\mathbf{V}J$ (where the liftings are those coming from the counits ε_A, $A \in \mathbf{A}$), and $(\mathbf{V}, J)$ is tractable.*

Let $\overrightarrow{Tract}_{\mathbf{A}}(\mathbf{X})$ denote the subcategory of $(\mathbf{X}, \overrightarrow{P})$ consisting of those objects $(\mathbf{V}, J)$ such that $\mathbf{V} : \mathbf{A} \to \mathbf{Cat}$ is a genuine functor and $(\mathbf{V}, J)$ is tractable, and those arrows $N : (\mathbf{V}, J) \to (\mathbf{V}', J')$ such that $N : \mathbf{V} \to \mathbf{V}'$ is a natural transformation. The «*opsemantics*» functor

$$\overrightarrow{Sem} : Fib_{\mathbf{A}}(\mathbf{X}) \longrightarrow \overrightarrow{Tract}_{\mathbf{A}}(\mathbf{X})$$

is defined by:

for a lax functor $\mathbf{W}$ at $\mathbf{X}$, $\overrightarrow{Sem}\,(\mathbf{W}) = (\widetilde{\mathbf{W}}, \widetilde{\mathbf{J}})$;

for a morphism $\phi : \mathbf{W} \to \mathbf{W}'$ of lax functors at $\mathbf{X}$, $\overrightarrow{Sem}\,(\phi)$ is the unique natural transformation such that

$$\begin{array}{ccc} \mathbf{W} & \xrightarrow{\;\widetilde{J}\;} & \widetilde{\mathbf{W}} \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle Sem\,(\phi)} \\ \mathbf{W}' & \xrightarrow[\;\widetilde{J}\;]{} & \widetilde{\mathbf{W}}' \end{array}$$

commutes.

Since $\widetilde{J}$ has a right adjoint the above Corollary implies that $\overrightarrow{Sem}\,(\mathbf{W})$ is tractable.

THEOREM 7. *The opsemantics functor*

$$\overrightarrow{Sem} : Fib_{\mathbf{A}}(\mathbf{X}) \longrightarrow \overrightarrow{Tract}_{\mathbf{A}}(\mathbf{X})$$

has a right adjoint called the «opstructure» functor

$$\overrightarrow{Str} : \overrightarrow{Tract}_{\mathbf{A}}(\mathbf{X}) \longrightarrow \overrightarrow{Fib}_{\mathbf{A}}(\mathbf{X})$$

and the unit of this adjunction is an isomorphism.

PROOF. Suppose $(\mathbf{V}, J) \in \overrightarrow{Tract}_{\mathbf{A}}(\mathbf{X})$, and choose a cartesian arrow $\overline{J} : J_* \mathbf{V} \to \mathbf{V}$ over J. Then, for each $\mathbf{W} \in Fib_{\mathbf{A}}(\mathbf{X})$, the correspondence $\phi \longleftrightarrow N$ set up by commutativity of the diagram

$$\begin{array}{ccccc} & & J_*\mathbf{V} & & \\ & {\scriptstyle \phi}\nearrow & & \searrow{\scriptstyle \overline{J}} & \\ \mathbf{W} & & & & \mathbf{V} \\ & {\scriptstyle \widetilde{J}}\searrow & & \nearrow{\scriptstyle N} & \\ & & \widetilde{\mathbf{W}} & & \end{array}$$

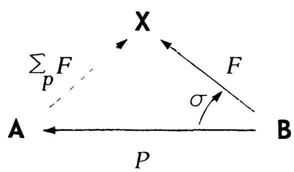
gives a bijection

$$Fib_{\mathbf{A}}(\mathbf{X})(\mathbf{W}, J_*\mathbf{V}) \cong Tract_{\mathbf{A}}(\mathbf{X})((\widetilde{\mathbf{W}}, \widetilde{J}), (\mathbf{V}, J))$$

(using the cartesian property of $\overline{J}$ and the reflection property of $\widetilde{J}$: see Theorem 3). The bijection is clearly natural in $\mathbf{W}$. The fact that the unit is an isomorphism follows from the fact that $\mathbf{W} = \widetilde{\mathbf{W}}\widetilde{J}$ and so $\widetilde{J} : \mathbf{W} \to \widetilde{\mathbf{W}}$ is cartesian (Theorem 6).

All the preceding work of this section can be dualized in **Cat**. The definitions make sense in any 2-category, so, instead of making them in **Cat**, we make them now in $\mathbf{Cat}^{op}$ and express them in terms of the data of **Cat**.

The dual of right lifting is *right extension* (usually called *right Kan extension*). The data for a right extension of $F: \mathbf{B} \to \mathbf{X}$ along $P: \mathbf{B} \to \mathbf{A}$ is contained in a diagram



THEOREM 5^{op}. *If* $P: \mathbf{B} \to \mathbf{A}$ *is a functor with a left adjoint* $\check{P}: \mathbf{A} \to \mathbf{B}$ *and* $\varepsilon: \check{P} P \to 1$ *is the counit of the adjunction, then any functor* $F: \mathbf{B} \to \mathbf{X}$ *has a right extension along* P *given by the functor* $F\check{P}: \mathbf{A} \to \mathbf{X}$ *and the natural transformation* $F\varepsilon: (F\check{P})P \to F$.

The functor

$$\overleftarrow{P}: \overleftarrow{Lax}\,[\mathbf{A}, \mathbf{Cat}] \longrightarrow Gen\,[\,|\mathbf{A}|, \mathbf{Cat}\,]$$

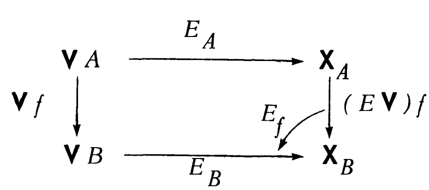
is given by

$$\overleftarrow{P}(\mathbf{W}) = (\mathbf{W}A)_{A \in |\mathbf{A}|}, \quad \overleftarrow{P}(L) = (L_A)_{A \in |\mathbf{A}|}.$$

However, nothing essentially new arises for the fibre category $\overleftarrow{P}^{-1}(\mathbf{X})$; it is just $Fib_{\mathbf{A}}(\mathbf{X})^{op}$.

An object $(\mathbf{V}, E)$ of $(\overleftarrow{P}, \mathbf{X})$ is said to be *tractable* when there exists a cocartesian arrow (with respect to the functor $\overleftarrow{P}$) over $E: \overleftarrow{P}\mathbf{V} \to X$ which has domain $\mathbf{V}$.

THEOREM 6^{op}. *Suppose* $(\mathbf{V}, E)$ *is an object of* $(\overleftarrow{P}, \mathbf{X})$. *Suppose that, for each* $f: A \to B$ *in* $\mathbf{A}$, *a diagram*



is given, in which the functor $(E\mathbf{V})f : \mathbf{X}_A \to \mathbf{X}_B$ *and the natural transformation* $E_f : (E\mathbf{V})f \,.\, E_A \to E_B \,.\, \mathbf{V}f$ *form a right extension of* $E_B \,.\, \mathbf{V}f$ *along* E_A. *Then:*

(*a*) *the data* $(E\mathbf{V})_A = \mathbf{X}_A$, $(E\mathbf{V})f$ *can be uniquely enriched to a lax functor* $E\mathbf{V} : \mathbf{A} \to \mathbf{Cat}$ *such that the data* E_A, E_f *form a left lax transformation* $E : \mathbf{V} \to E\mathbf{V}$;

(*b*) *the left lax transformation* $\bar{E} : \mathbf{V} \to E\mathbf{V}$ *is cocartesian over* $\acute{E} : \overrightarrow{P}\mathbf{V} \to \mathbf{X}$, *so* $(\mathbf{V}, E)$ *is tractable.*

COROLLARY. *If* $\mathbf{V} : \mathbf{A} \to \mathbf{Cat}$ *is a lax functor, and, for each* $A \in \mathbf{A}$, *there is an adjunction*

$$\varepsilon_A,\ \eta_A : J_A \dashv E_A : (\mathbf{V}A, \mathbf{X}_A),$$

then $E\mathbf{V} = E\mathbf{V}J$ (*where the extensions are those coming from the counits* ε_A, $A \in \mathbf{A}$), *and* $(\mathbf{V}, E)$ *is tractable.*

Let $\overleftarrow{Tract}_{\mathbf{A}}(\mathbf{X})$ denote the subcategory of $(\overleftarrow{P}, \mathbf{X})$ consisting of those objects $(\mathbf{V}, E)$ which are tractable and are such that $\mathbf{V} : \mathbf{A} \to \mathbf{Cat}$ is a genuine functor, and those arrows $N : (\mathbf{V}, E) \to (\mathbf{V}', E')$ such that $N : \mathbf{V} \to \mathbf{V}'$ is a natural transformation. The «semantics» functor

$$\overleftarrow{Sem} : Fib_{\mathbf{A}}(\mathbf{X})^{op} \longrightarrow \overleftarrow{Tract}_{\mathbf{A}}(\mathbf{X})$$

is defined by:

for a lax functor $\mathbf{W}$ at $\mathbf{X}$, $\overleftarrow{Sem}(\mathbf{W}) = (\hat{\mathbf{W}}, \hat{J})$;

for a morphism $\phi : \mathbf{W}' \to \mathbf{W}$ of lax functors at $\mathbf{X}$, $Sem(\phi)$ is the unique natural transformation such that

$$\begin{array}{ccc} \hat{\mathbf{W}} & \xrightarrow{\ \hat{E}\ } & \mathbf{W} \\ {\scriptstyle Sem(\phi)}\downarrow & & \downarrow{\scriptstyle \phi^l} \\ \hat{\mathbf{W}}' & \xrightarrow[\ \hat{E}\ ]{} & \mathbf{W}' \end{array}$$

commutes (where $\phi^l : \mathbf{W} \to \mathbf{W}'$ is $\phi : \mathbf{W}' \to \mathbf{W}$ regarded as a left lax transformation which is the identity on objects).

THEOREM 7^{op}. *The semantics functor*

$$\overleftarrow{Sem} : Fib_{\mathbf{A}}(\mathbf{X})^{op} \longrightarrow \overleftarrow{Tract}_{\mathbf{A}}(\mathbf{X})$$

has a left adjoint called the « structure » functor

$$\overleftarrow{Str} : \overleftarrow{Tract}_{\mathbf{A}}(\mathbf{X}) \longrightarrow Fib_{\mathbf{A}}(\mathbf{X})^{op}$$

and the counit of this adjunction is an isomorphism.

5. Distinguishing the second basic construction

The aim of this section is to examine properties of the second basic construction and to find necessary and sufficient conditions under which a given generator should be isomorphic to it.

THEOREM 8. *Suppose the lax functor* $\mathbf{W} : \mathbf{A} \to \mathbf{Cat}$ *satisfies the following condition :*

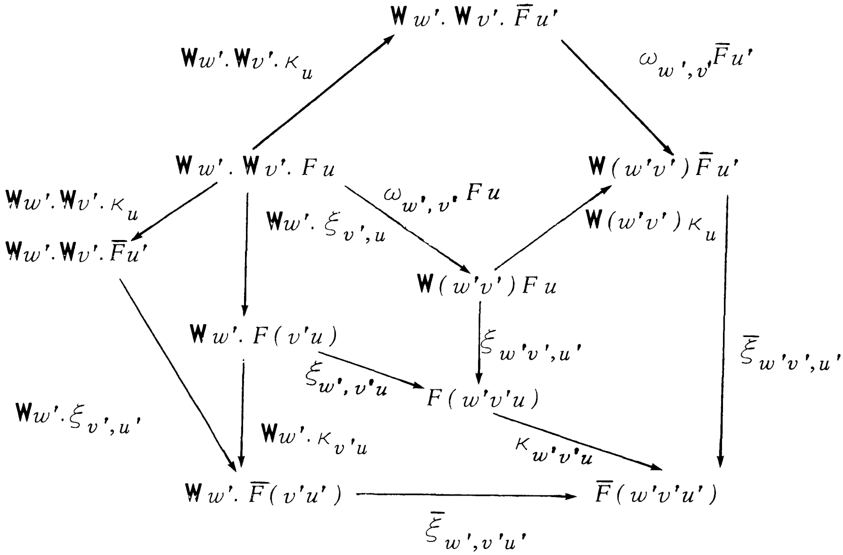
for each $A \in \mathbf{A}$, *the category* $\mathbf{W}A$ *has a coproduct for each family of objects indexed by any subset of any hom set of* $\mathbf{A}$, *and, for each* $u : A \to B$ *in* $\mathbf{A}$, *the functor* $\mathbf{W}u : \mathbf{W}A \to \mathbf{W}B$ *preserves these coproducts.*

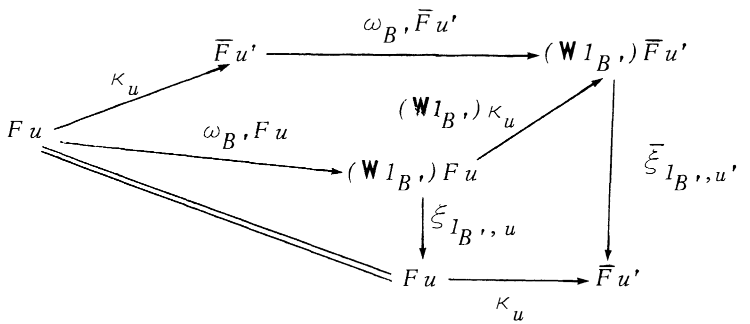
Then, for each $f : A' \to A$ *in* $\mathbf{A}$ *the functor* $\hat{\mathbf{W}}f : \hat{\mathbf{W}}A' \to \hat{\mathbf{W}}A$ *has a left adjoint.*

PROOF. Take $(F, \xi) \in \hat{\mathbf{W}}A$. For $u' : A' \to B'$, define $\bar{F}u' = \coprod_{u' = uf} Fu$; that is, $\bar{F}u'$ is the coproduct of the family of objects Fu indexed by the subset of $\mathbf{A}(A, B')$ consisting of those arrows $u : A \to B'$ such that $u' = uf$. Let $\kappa_u : Fu \to \bar{F}u'$ be the injection corresponding to the u-component Fu of the coproduct. By this condition of the theorem, for each $v' : B' \to C'$, the arrows $(\mathbf{W}v')\kappa_u : (\mathbf{W}v')Fu \to (\mathbf{W}v')\bar{F}u'$ have the properties of injections into a coproduct. So an arrow $\bar{\xi}_{v',u'} : (\mathbf{W}v')\bar{F}u' \to \bar{F}(v'u')$ is defined uniquely by commutativity of the diagram

$$\begin{array}{ccc} (\mathbf{W}v')Fu & \xrightarrow{(\mathbf{W}v')\kappa_u} & (\mathbf{W}v')\bar{F}u' \\ \xi_{v',u}\downarrow & & \downarrow \bar{\xi}_{v',u'} \\ F(v'u) & \xrightarrow[\kappa_{v'u}]{} & \bar{F}(v'u') \end{array}$$

where $u' = uf$; then the following diagrams commute



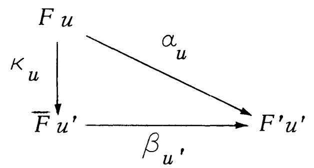


The arrows $\mathbf{W}w'.\mathbf{W}v'.\kappa_u$ are injections into a coproduct by the condition of the theorem. So $(F, \xi) \in \hat{\mathbf{W}}A'$. From the definition of $\bar{\xi}$ it then follows that the arrows κ_u are the components of an arrow

$$\kappa : (F, \xi) \longrightarrow (\hat{\mathbf{W}}f)(\bar{F}, \bar{\xi})$$

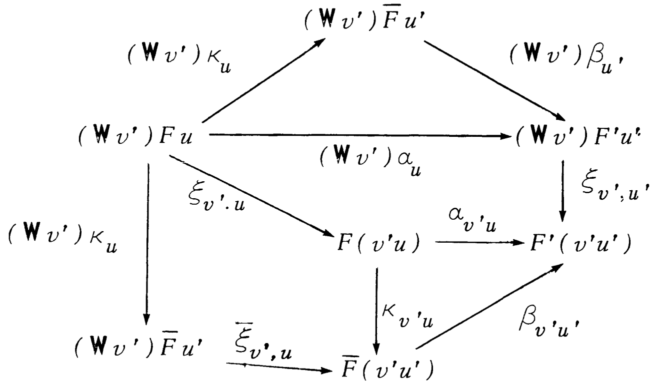
of $\hat{\mathbf{W}}A$.

Suppose $(F', \xi') \in \mathbf{W}A'$ and $\alpha : (F, \xi) \to (\hat{\mathbf{W}}f)(F', \xi')$ is an arrow of $\hat{\mathbf{W}}A$. For $u' : A' \to B'$ define $\beta_{u'} : \bar{F}u' \to F'u'$ by

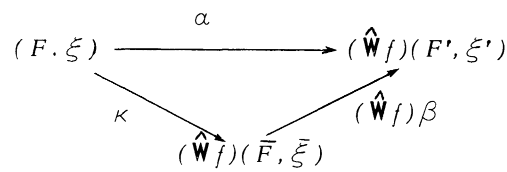


$u' = uf$

These are the components of an arrow $\beta : (\bar{F}, \bar{\xi}) \to (F', \xi')$ in $\hat{\mathbf{W}} A'$ since, for $u' = uf$, the following commutes.

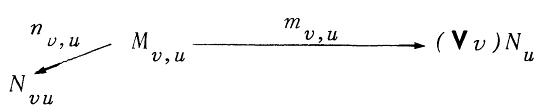


From the definition of β it follows that β is unique with the property that



commutes. If follows that $\hat{\mathbf{W}}f$ has a left adjoint and κ is the unit of the adjunction.

Suppose $\mathbf{V} : \mathbf{A} \to \mathbf{Cat}$ is a functor and A is an object of $\mathbf{A}$. An *A-centred centipede in* $\mathbf{V}$ is a quadruple (M, N, m, n) which assigns to each pair of arrows $u : A \to B$, $v : B \to C$ of $\mathbf{A}$ a diagram



in $\mathbf{V}C$. A *reflection of the centipede* (M, N, m, n) is a pair (H, h), where H is an oject of $\mathbf{V}A$ and h is a family of arrows $h_u : N_u \to (\mathbf{V}u)H$ indexed by the arrows $u : A \to B$ in $\mathbf{A}$ out of A, such that the diagram

$$\begin{array}{ccc} M_{v,u} & \xrightarrow{\;m_{v,u}\;} & (\mathbf{V}v)N_u \\ {\scriptstyle n_{v,u}}\downarrow & & \downarrow{\scriptstyle (\mathbf{V}v)h_u} \\ N_{vu} & \xrightarrow[\;h_{vu}\;]{} & \mathbf{V}(vu)H \end{array}$$

commutes. If (H, h) has the property that, for any reflection (H', h') of (M, N, m, n), there exists a unique arrow $k : H \to H'$ of $\mathbf{V}A$ such that

$$\begin{array}{ccc} N_u & \xrightarrow{\;h_u\;} & (\mathbf{V}u)H \\ & {\scriptstyle h'_u}\searrow & \downarrow{\scriptstyle (\mathbf{V}u)k} \\ & & (\mathbf{V}u)H' \end{array}$$

commutes, then (H, h) is called a *universal reflection of the centipede.*

The category $\mathbf{A}[A]$ will be defined. The objects are either of the form $[u]$ where $u : A \to B$ is an arrow of $\mathbf{A}$, or of the form $[v, u]$ where $u : A \to B$, $v : B \to C$ are arrows of $\mathbf{A}$. For each pair of arrows $u : A \to B$, $v : B \to C$ of $\mathbf{A}$ there is exactly one arrow $[v, u] \to [u]$ and exactly one arrow $[v, u] \to [vu]$ in $\mathbf{A}[A]$ (in the case $vu = u$, there are exactly two arrows $[v, u] \to [u]$), and the only other arrows of $\mathbf{A}[A]$ are the identities.

THEOREM 9. *Suppose* $\mathbf{V} : \mathbf{A} \to \mathbf{Cat}$ *is a functor and* $A \in \mathbf{A}$, *and suppose:*

- *for each arrow* $u : A \to B$ *in* $\mathbf{A}$, *the functor* $\mathbf{V}u : \mathbf{V}A \to \mathbf{V}B$ *has a left adjoint,*
- *every functor from* $\mathbf{A}[A]$ *into* $\mathbf{V}A$ *has a colimit.*

Then every A-centred centipede in $\mathbf{V}$ *has a universal reflection.*

PROOF. For each $u : A \to B$ in $\mathbf{A}$, let $\check{\mathbf{V}}u : \mathbf{V}B \to \mathbf{V}A$ be the left adjoint of $\mathbf{V}u$ with $\lambda_u : (\check{\mathbf{V}}u)(\mathbf{V}u) \to 1$, $\kappa_u : 1 \to (\mathbf{V}u)(\check{\mathbf{V}}u)$ as counit and unit. Suppose (M, N, m, n) is a centipede in $\mathbf{V}$ centred at A. Excluding the dotted arrows from the following diagram, we note that a functor from

$\mathbf{A}\,[\,A\,]$ to $\mathbf{V}A$ is determined.

$$\begin{array}{ccccc}
\check{\mathbf V}(vu)M_{v,u} & \xrightarrow{\check{\mathbf V}(vu)m_{v,u}} & \mathbf V(vu)(\mathbf V v)N_u & \xrightarrow{\check{\mathbf V}(vu)(\mathbf V v)\kappa_u N_u} & \check{\mathbf V}(vu)\mathbf V(vu)(\check{\mathbf V}u)N_u \\
\downarrow{\scriptstyle \check{\mathbf V}(vu)n_{v,u}} & & & & \downarrow{\scriptstyle \lambda_{vu}(\check{\mathbf V}u)N_u} \\
\check{\mathbf V}(vu)N_{\bar v\bar u} & \overset{\bar h_{vu}}{\dashrightarrow} & H & \overset{\bar h_u}{\dashleftarrow} & (\check{\mathbf V}u)N_u
\end{array}$$

Let (H, h) be an upper bound of this functor as illustrated by the dotted arrows. This is equivalent to (H, h) a reflection of the centipede, where h, $\bar h$ are related by

$$\begin{array}{ccc}
N_u & \xrightarrow{h_u} & (\mathbf V u)H \\
{\scriptstyle \kappa_u N_u}\searrow & & \nearrow{\scriptstyle (\mathbf V u)\bar h_u} \\
& (\mathbf V u)(\check{\mathbf V}u)N_u &
\end{array}$$

The diagram

$$\begin{array}{ccc}
N_u & \xrightarrow{h_u} & (\mathbf V u)H \\
& {\scriptstyle h_{u'}}\searrow & \downarrow{\scriptstyle (\mathbf V u)k} \\
& & (\mathbf V u)H'
\end{array}$$

commutes if and only if the diagram

$$\begin{array}{ccc}
(\mathbf V u)N_u & \xrightarrow{\bar h_u} & H \\
& {\scriptstyle \bar h'_u}\searrow & \downarrow{\scriptstyle k} \\
& & H'
\end{array}$$

commutes. So $(H, \bar h)$ is a colimit of the functor if and only if (H, h) is a universal reflection of the centipede.

REMARK. Professor Mac Lane has made the following observations on centipedes. The category $\mathbf{A}\,[\,A\,]$ is exactly the Kan subdivision category (see Mac Lane's forthcoming book) of the category $A/\mathbf{A}$ of objects under A. For any functor $\mathbf{V}: \mathbf{A} \to \mathbf{Cat}$, define the *join* $J(\mathbf{V})$ of $\mathbf{V}$ to be the category $\tilde{\mathbf{V}}*$, where $*$ is the terminal object of $\mathbf{A}$ (added if $\mathbf{A}$ does not already have one).

Given a functor $\mathbf{V}: \mathbf{A} \to \mathbf{Cat}$, define a functor $\mathbf{V}^{\#}: \mathbf{A}[A]^{op} \to \mathbf{Cat}$ as indicated by the diagram

$$\begin{array}{ccccc} [u] & \longleftarrow & [v,u] & \longrightarrow & [vu] \\ & & \Big\downarrow{\scriptstyle \mathbf{V}^{\#}} & & \\ \mathbf{V}B & \xrightarrow[\mathbf{V}v]{} & \mathbf{V}C & \xleftarrow[1]{} & \mathbf{V}C \end{array}$$

Then we have the category $J(\mathbf{V}^{\#})$ and the projection $P: J(\mathbf{V}^{\#}) \to \mathbf{A}[A]$. An A-centred centipede in $\mathbf{V}$ is precisely a functor $Q: \mathbf{A}[A] \to J(\mathbf{V}^{\#})$ with $PQ = 1$; that is a *section* of P.

We further observe that, if $Cpd_{\mathbf{A}}(\mathbf{V})$ denote the full subcategory of the category of functors from $\mathbf{A}[A]$ to $J(\mathbf{V}^{\#})$ consisting of the sections of P, then there is an inclusion functor $\mathbf{V}A \to Cpd_{\mathbf{A}}(\mathbf{V})$ given by $H \mapsto \overline{H}$, where $\overline{H}[u] = ([u], (\mathbf{V}u)H)$, $\overline{H}[v,u] = ([v,u], \mathbf{V}(vu)H)$. The reflection of a centipede Q *is* its reflection in $\mathbf{V}A$ with respect to this inclusion.

Suppose $\mathbf{X}$ is an object of $Gen[|A|, \mathbf{Cat}]$, and suppose $(\mathbf{V}, E)$ is an object of $(\overleftarrow{P}, E)$ where $\mathbf{V}: \mathbf{A} \to \mathbf{Cat}$ is a functor. The family E of functors $E_B: \mathbf{V}B \to \mathbf{X}_B$, $B \in \mathbf{A}$, is said to *split the centipede* (M, N, m, n) *in* $\mathbf{V}$ *centred at* A when there exist, for arrows $u: A \to B$, $v: B \to C$ in $\mathbf{A}$,

objects X_u of $\mathbf{X}_B$,
arrows $p_{v,u}: E_C(\mathbf{V}v)N_u \to X_{vu}$ of $\mathbf{X}_C$,
arrows $s_u: X_u \to E_B N_u$ of $\mathbf{X}_B$,
arrows $t_{v,u}: E_C(\mathbf{V}v)N_u \to E_C M_{v,u}$ of $\mathbf{X}_C$,

such that the following diagrams commute:

$$\begin{array}{ccc} E_D(\mathbf{V}w)M_{v,u} & \xrightarrow{E_D(\mathbf{V}w)m_{v,u}} & E_D\mathbf{V}(wv)N_u \\ {\scriptstyle E_D(\mathbf{V}w)n_{v,u}}\Big\downarrow & & \Big\downarrow{\scriptstyle p_{wv,u}} \\ E_D(\mathbf{V}w)N_{vu} & \xrightarrow{p_{w,vu}} & X_{wvu} \end{array}$$

$$\begin{array}{ccc} X_u & \xrightarrow{s_u} & E_B N_u \\ & \searrow^{=} & \downarrow p_{1,u} \\ & & X_u \end{array} \qquad \begin{array}{ccc} E_C(\mathbf{V}v)N_u & \xrightarrow{t_{v,u}} & E_C M_{v,u} \\ & \searrow^{=} & \downarrow E_C m_{v,u} \\ & & E_C(\mathbf{V}v)N_u \end{array}$$

$$\begin{array}{ccc} E_C(\mathbf{V}v)N_u & \xrightarrow{t_{v,u}} & E_C M_{v,u} \\ p_{v,u}\downarrow & & \downarrow E_C n_{v,u} \\ X_{vu} & \xrightarrow[s_{vu}]{} & E_C N_{vu} \end{array}$$

The family E of functors is said to *create universal reflections of A-centred centipedes which it splits* when it has the following property: given an A-centred centipede (M, N, m, n) in $\mathbf{V}$ which the family of functors splits via X_u, $p_{v,u}$, s_u, $t_{v,u}$ as in the definition, then there exists a unique reflection (H, h) of the centipede such that $X_u = E_B(\mathbf{V}u)H$ and $p_{v,u} = (\mathbf{V}v)h_u$; moreover, this reflection is universal.

THEOREM 10. *For any lax functor $\mathbf{W}: \mathbf{A} \to \mathbf{Cat}$ the family $\hat{E}$ of functors $\hat{E}_A : \hat{\mathbf{W}}A \to \mathbf{W}A$, $A \in \mathbf{A}$ creates universal reflections of all centipedes in $\hat{\mathbf{W}}$ which it splits.*

PROOF. Let (M, N, m, n) be a centipede at A in $\hat{\mathbf{W}}$ which is split by the family of functors in the theorem. Put

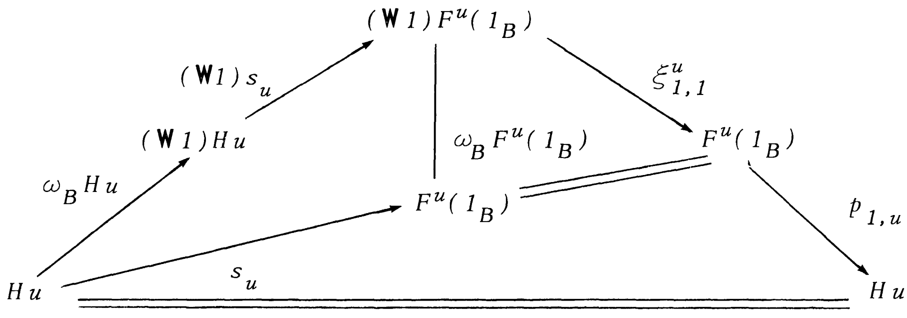
$$M_{v,u} = (G^{v,u}, \gamma^{v,u}) \in \hat{\mathbf{W}}C, \quad N_u = (F^u, \xi^u) \in \hat{\mathbf{W}}B.$$

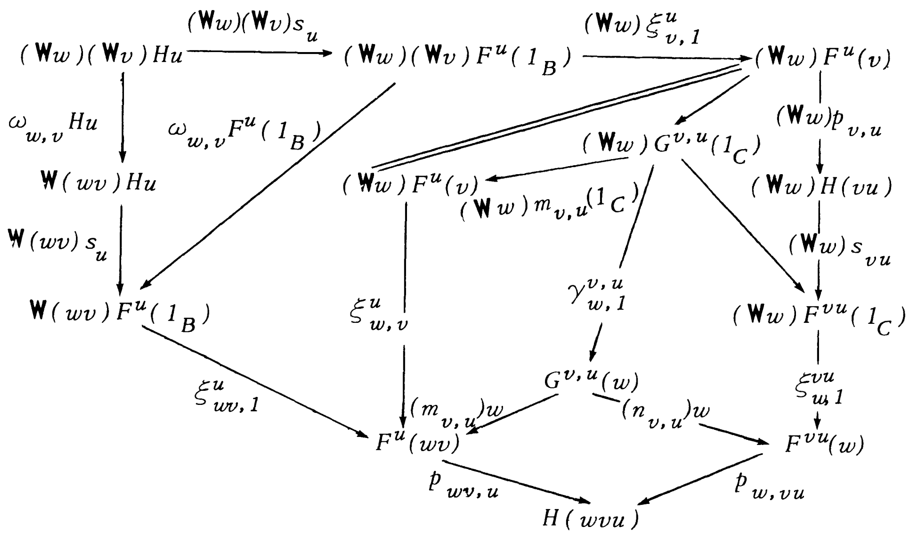
We have X_u, $p_{v,u}$, s_u, $t_{v,u}$ as in the definitions. Define $Hu = X_u$ and define $\tau_{v,u} : (\mathbf{W}v)Hu \to H(vu)$ to be the composite

$$(\mathbf{W}v)Hu = (\mathbf{W}v)X_u \xrightarrow{(\mathbf{W}v)s_u} (\mathbf{W}v)\hat{E}_B(F^u, \xi^u) = (\mathbf{W}v)F^u(1_B) \xrightarrow{\xi^u_{v,1}} F^u(v)$$

$$= \hat{E}_C(\hat{\mathbf{W}}v)(F^u, \xi^u) \xrightarrow{p_{v,u}} H(vu).$$

The following diagrams show that $(H, \tau) \in \hat{\mathbf{W}}A$.

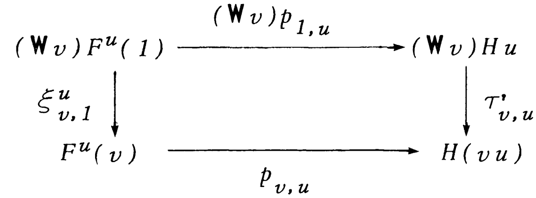




For $u: A \to B$, $v: B \to C$ define $h_u(v): F^u(v) \to H(vu)$ to be just $p_{v,u}$. The right side of the last diagram shows that $h_u(v)$ are the components of an arrow $h_u: (F^u, \xi^u) \to (\hat{\mathbf{W}} u)(H, \tau)$ in $\hat{\mathbf{W}} B$. Then $((H, \tau), h)$ is a reflection of the centipede (M, N, m, n), and

$$X_u = Hu = \hat{E}_B(\hat{\mathbf{W}} u)(H, \tau), \quad p_{v,u} = (\hat{\mathbf{W}} v) h_u.$$

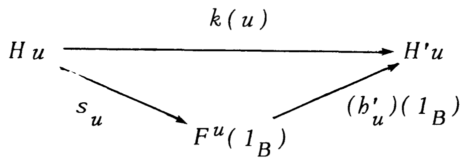
We must show that $((H, \tau), h)$ is unique with these properties. Suppose $((H, \tau'), h)$ is a reflection of the centipede. Then



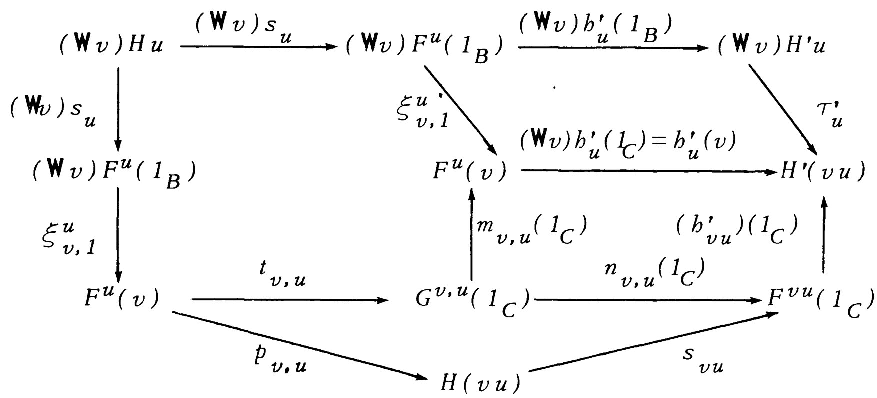
commutes. So

$$\tau'_{v,u} = \tau'_{v,u} \cdot (\mathbf{W} v) p_{1,u} \cdot (\mathbf{W} v) s_u = p_{v,u} \cdot \xi^u_{v,1} \cdot (\mathbf{W} v) s_u = \tau_{v,u}.$$

It remains to prove that $((H, \tau), h)$ is a universal reflection. Suppose $((H', \tau'), h')$ is another reflection of the same centipede. Define $k(u)$ by the commutative diagram



These arrows are the components of an arrow $k : (H, \tau) \to (H', \tau')$ in $\hat{\mathbf{W}} A$ as the following diagram shows:



The following diagram commutes:

$$\begin{array}{ccccc}
F^u(v) & \xrightarrow{p_{v,u}} & H(vu) & \xrightarrow{s_u} & F^{vu}(1_C) \\
 & \searrow^{t_{v,u}} & & \nearrow_{n_{v,u}(1_C)} & \Big\downarrow (h'_{vu})(1_C) \\
\Big\| & & G^{v,u}(1_C) & & H'(vu) \\
 & & \Big\downarrow m_{v,u}(1_C) & \nearrow & \\
 & & F^u(v) & & (\hat{\mathbf{W}} v) h'_u(1_C) = h'_u(v)
\end{array}$$

So the diagram

$$\begin{array}{ccc}
(F^u, \xi^u) & \xrightarrow{h_u} & (\hat{\mathbf{W}} u)(H, \tau) \\
 & \searrow^{h'_u} & \Big\downarrow (\hat{\mathbf{W}} u) k \\
 & & (\hat{\mathbf{W}} u)(H', \tau')
\end{array}$$

commutes. Moreover, k is unique with this property. For if k' also makes this triangle commute, then

$$k'(u) = k'(u) . p_{1,u} . s_u = k'(u) . h_u(1) . s_u = h'_u(1) . s_u = k(u) .$$

THEOREM 11. *Suppose* $(\mathbf{V}, E, J)$ *is a generator of the lax functor* $\mathbf{W} : \mathbf{A} \to \mathbf{Cat}$ *with* $\mathbf{V} : \mathbf{A} \to \mathbf{Cat}$ *a genuine functor, and suppose* $N : \mathbf{V} \to \hat{\mathbf{W}}$ *is the unique natural transformation such that* $\hat{E} N = E$. *If the family* E *of functors* $E_A : \mathbf{V} A \to \mathbf{W} A$, $A \in \mathbf{A}$, *creates universal reflections of* A*-centred centipedes in* $\mathbf{V}$ *which it splits, then the functor* $N_A : \mathbf{V} A \to \hat{\mathbf{W}} A$ *is an isomorphism.*

PROOF. For each $A \in \mathbf{A}$ we define a functor $\check{N}_A : \hat{\mathbf{W}} A \to \mathbf{V} A$. Take (F, ξ) in $\hat{\mathbf{W}} A$. This gives rise to the following centipede in $\mathbf{V}$ centred at A:

$$\begin{array}{ccc}
J_C E_C(\mathbf{V} v) J_B F u & \xrightarrow{\varepsilon_C(\mathbf{V} v) J_B F u} & (\mathbf{V} v) J_B F u \\
J_C \xi_{v,u} \Big\downarrow & & \\
J_C F(vu) & &
\end{array}$$

The family E of functors splits this centipede; the splitting is given by the data:

$$F u, \quad E_C(\mathbf{V} v) J_B F u \xrightarrow{\xi_{v,u}} F(vu), \quad F u \xrightarrow{\eta_B F u} E_B J_B F u,$$

$$E_C(\mathbf{V}v)J_B Fu \xrightarrow{E_C\eta_C(\mathbf{V}v)J_B Fu} E_C J_C E_C(\mathbf{V}v)J_B Fu.$$

Let (H, h) be the unique reflection of this centipede with the property $Fu = E_B(\mathbf{V}u)H$ and $\xi_{v,u} = E_C(\mathbf{V}v)h_u$. Define $\check{N}_A(F,\xi) = H$.

Let (H', h') be the corresponding reflection for (F', ξ') and suppose $\alpha : (F,\xi) \to (F',\xi')$ is an arrow of $\hat{\mathbf{W}}A$. The following diagram commutes:

$$\begin{array}{ccccc}
J_C F(vu) & \xleftarrow{J_C\xi_{v,u}} & J_C E_C(\mathbf{V}v)J_B Fu & \xrightarrow{\varepsilon_C(\mathbf{V}v)J_B Fu} & (\mathbf{V}v)J_B Fu \\
{\scriptstyle J_C\alpha_{vu}}\downarrow & & \downarrow{\scriptstyle J_C(\mathbf{W}v)\alpha_u} & & \downarrow{\scriptstyle (\mathbf{V}v)J_B\alpha_u} \\
J_C F'(vu) & \xleftarrow[J_C\xi'_{v,u}]{} & J_C E_C(\mathbf{V}v)J_B F'u & \xrightarrow[\varepsilon_C(\mathbf{V}v)J_B F'u]{} & (\mathbf{V}v)J_B F'u
\end{array}$$

It follows that there exists a unique $k : H \to H'$ such that

$$\begin{array}{ccc}
J_B Fu & \xrightarrow{h_u} & (\mathbf{V}u)H \\
{\scriptstyle J_B\alpha_u}\downarrow & & \downarrow{\scriptstyle (\mathbf{V}u)k} \\
J_B F'u & \xrightarrow[h'_u]{} & (\mathbf{V}u)H'
\end{array}$$

commutes. Define $\check{N}_A\alpha = k$. Then $\check{N}_A : \mathbf{W}A \to \mathbf{V}A$ is a functor.

Next we show that $\check{N}_A N_A = 1$. Take $K \in \mathbf{V}A$. Then $N_A K = (F, \xi)$ where

$$Fu = E_B(\mathbf{V}u)K \quad\text{and}\quad \xi_{v,u} = E_C(\mathbf{V}v)\,\varepsilon_B(\mathbf{V}u)K.$$

The following diagram commutes:

$$\begin{array}{ccc}
J_C E_C(\mathbf{V}v)J_B E_B(\mathbf{V}u)K & \xrightarrow{\varepsilon_C(\mathbf{V}v)J_B E_B(\mathbf{V}u)K} & (\mathbf{V}v)J_B E_B(\mathbf{V}u)K \\
\downarrow{\scriptstyle J_C E_C(\mathbf{V}v)\,\varepsilon_B(\mathbf{V}u)K} & & {\scriptstyle (\mathbf{V}v)\,\varepsilon_B(\mathbf{V}u)K}\downarrow \\
J_C E_C\mathbf{V}(vu)K & \xrightarrow[\varepsilon_C\mathbf{V}(vu)K]{} & \mathbf{V}(vu)
\end{array}$$

So $(K, \varepsilon_B\mathbf{V}(-)K)$ is a reflection of the centipede used in the construction of $\check{N}_A N_A K$; moreover,

$$E_B(\mathbf{V}u)K = Fu \text{ and } E_C(\mathbf{V}v)\,\varepsilon_B(\mathbf{V}u)K = \xi_{v,u}\ ,$$

while $(\check{N}_A(F,\xi), h)$ was the unique reflection with this property. So

$$\check{N}_A N_A K = K \text{ and } h = \varepsilon_B \mathbf{V}(-)K.$$

Let $l : K \to K'$ be an arrow of $\mathbf{V}A$. Then $(N_A l)_u = E_B(\mathbf{V}u)l$, so $\check{N}_A N_A l = k$ is the unique arrow such that

$$\begin{array}{ccc} J_B E_B(\mathbf{V}u)K & \xrightarrow{\varepsilon_B(\mathbf{V}u)K} & (\mathbf{V}u)K \\ {\scriptstyle J_B E_B(\mathbf{V}u)l}\downarrow & & \downarrow{\scriptstyle (\mathbf{V}u)k} \\ J_B E_B(\mathbf{V}u)K' & \xrightarrow[\varepsilon_B(\mathbf{V}u)K']{} & (\mathbf{V}u)K' \end{array}$$

commutes. But by naturality of ε_B, l does this. So $\check{N}_A N_A l = l$.

Finally we show that $\check{N}_A N_A = 1$. Take $(F,\xi) \in \hat{\mathbf{W}}A$ and let (H, h) be the reflection of the centipede used in the construction of $\check{N}_A(F,\xi)$. Then $N_A H = (\bar{F}, \bar{\xi})$ where

$$\bar{F}u = E_B(\mathbf{V}u)H = Fu \text{ and } \bar{\xi}_{v,u} = E_C(\mathbf{V}v)\,\varepsilon_B(\mathbf{V}u)H.$$

Now

$$\begin{aligned} \xi_{v,u}\cdot(\mathbf{W}v)\xi_{1,u} = \xi_{v,u}\cdot\omega_{v,1}Fu = \xi_{v,u}\cdot E_C(\mathbf{V}v)\,\varepsilon_B J_B E_B(\mathbf{V}u)H \\ = E_C(\mathbf{V}v)\,\varepsilon_B(\mathbf{V}u)H\,.\,(\mathbf{W}v)\xi_{1,u} = \bar{\xi}_{v,u}\cdot(\mathbf{W}v)\xi_{1,u}, \end{aligned}$$

and $(\mathbf{W}v)\omega_B Fu$ is a right inverse for $(\mathbf{W}v)\xi_{1,u}$; so $\xi_{v,u} = \bar{\xi}_{v,u}$. So

$$N_A \check{N}_A(F,\xi) = (F,\xi).$$

Take $\alpha : (F,\xi) \to (F',\xi')$ and put $k = \check{N}_A\alpha$. Then

$$\begin{aligned} (N_A k)_u \xi_{1,u} = (E_B(\mathbf{V}u)k)(E_B h_u) = E_B((\mathbf{V}u)k\,.\,h_u) \\ = E_B(h'_u\cdot J_B\alpha_u) = \xi'_{1,u}\cdot(\mathbf{W}1_B)\alpha_u = \alpha_u\cdot\xi_{1,u}, \end{aligned}$$

and $\omega_B Fu$ is a right inverse for $\xi_{1,u}$. So $N_A k = \alpha$.

The usual variety of weaker assumptions than those of the last theorem lead to the usual variety of weaker conclusions as in the «triples» case. Among these is the following theorem, whose proof, after Theorem 11, we leave to the reader. See Theorem 9 for a simple test for the validity of the hypothesis of the next theorem.

THEOREM 12. *In the circumstances described in the first sentence of Theorem 11, if universal reflections centred at* $A \in \mathbf{A}$ *exist in* $\mathbf{V}$, *then the functor* $N_A : \mathbf{V}A \to \hat{\mathbf{W}}A$ *has a left adjoint.*

6. Appendix: Enrichment of results. Limits of lax functors into Cat.

Generalizations of the two basic constructions can be pursued in several directions. We do not wish to examine any of these in detail here, only some brief outlines.

1° Suppose $\mathbf{C}$ is a complete and cocomplete symmetric monoidal closed category. Let **C-Cat** denote the *2*-category of $\mathbf{C}$-categories, $\mathbf{C}$-functors and $\mathbf{C}$-natural transformations. Suppose $\mathbf{A}$ is a small category. A lax functor $\mathbf{W} : \mathbf{A} \to \mathbf{C}\text{-}\mathbf{Cat}$ is a morphism of bicategories in the sense of Benabou, so that each $\mathbf{W}A$ is a $\mathbf{C}$-category, each $\mathbf{W}f$ is a $\mathbf{C}$-functor and $\omega_{g,f}$, ω_A are $\mathbf{C}$-natural transformations.

For $A \in \mathbf{A}$, $\tilde{\mathbf{W}}A$ becomes a $\mathbf{C}$-category as follows. The objects (u, X) are as before. For $(u, X), (u', X') \in \tilde{\mathbf{W}}A$,

the object $\mathbf{W}A((u,X),(u',X'))$ of $\mathbf{C}$ is given by the coproduct

$$\coprod_{u' = uh} \mathbf{W}A'(X, (\mathbf{W}h)X').$$

Composition is given by the composite

$$\coprod_{u'' = u'h'} \mathbf{W}A''(X', (\mathbf{W}h')X'') \otimes \coprod_{u' = uh} \mathbf{W}A'(X, (\mathbf{W}h)X')$$

$$= \coprod_{\substack{u'' = u'h' \\ u' = uh}} (\mathbf{W}A''(X', (\mathbf{W}h')X'') \otimes \mathbf{W}A'(X, (\mathbf{W}h)X'))$$

$$\xrightarrow{\coprod((\mathbf{W}h)\otimes 1)} \coprod_{\substack{u'' = u'h' \\ u' = uh}} (\mathbf{W}A'((\mathbf{W}h)X', (\mathbf{W}h)(\mathbf{W}h')X'') \otimes \mathbf{W}A'(X, (\mathbf{W}h)X'))$$

$$\xrightarrow{\coprod(\text{comp in } \mathbf{W}A')} \coprod_{\substack{u'' = u'h' \\ u' = uh}} \mathbf{W}A'(X, (\mathbf{W}h)(\mathbf{W}h')X'')$$

$$\xrightarrow{\coprod(\mathbf{W}A'(1, \omega_{h,h'} X''))} \coprod_{\substack{u'' = u'h' \\ u' = uh}} \mathbf{W}A'(X, (\mathbf{W}(hh'))X'')$$

$$\xrightarrow{\text{injection}} \coprod_{u'' = uk} \mathbf{W}A'(X, (\mathbf{W}k)X'').$$

The identity of (u, X) is enriched to the composite

$$I \xrightarrow{id\ of\ \mathbf{W}A} \mathbf{W}A(X,X) \xrightarrow{\mathbf{W}A(1,\omega_A X)} \mathbf{W}A(X,(\mathbf{W}1_A)X) \xrightarrow{inj} \coprod_{u = uh} \mathbf{W}A(X,(\mathbf{W}h)X).$$

Now $\mathbf{C}$ itself may be regarded as a functor $\mathbf{C}: \mathbf{A} \to \mathbf{Cat}$ given by $\mathbf{C}A = \mathbf{C}$, $\mathbf{C}f = 1_C$. So the notion of cocentipede makes sense in $\mathbf{C}: \mathbf{A} \to \mathbf{Cat}$. Since all the functors $\mathbf{C}f = 1_C$ have right adjoints, $\mathbf{A}$ is small, and $\mathbf{C}$ is complete, all the cocentipedes in $\mathbf{C}: \mathbf{A} \to \mathbf{Cat}$ have universal coreflections (dual of Theorem 9). For $A \in \mathbf{A}$, $\hat{\mathbf{W}}A$ becomes a $\mathbf{C}$-category as follows. The objects (F, ξ) are as before. For (F, ξ), $(F', \xi') \in \hat{\mathbf{W}}A$, the object $\hat{\mathbf{W}}A((F,\xi),(F',\xi'))$ of $\mathbf{C}$ is the universal coreflection of the cocentipede

$$\begin{array}{ccc}
\hat{\mathbf{W}}A((F,\xi),(F',\xi')) & \dashrightarrow & \mathbf{W}A(Fu, F'u) \\
\vdots & & \downarrow{\scriptstyle \mathbf{W}v} \\
\vdots & & \mathbf{W}A((\mathbf{W}v)Fu, (\mathbf{W}v)F'u) \\
\downarrow & & \downarrow{\scriptstyle \mathbf{W}A(1,\xi'_{v,u})} \\
\mathbf{W}A(F(vu), F'(vu)) & \xrightarrow[\mathbf{W}A(\xi_{v,u}1)]{} & \mathbf{W}A((\mathbf{W}v)Fu, F'(vu))
\end{array}$$

(excluding the dotted arrows) in $\mathbf{C}$. Compositions and identities are readily supplied.

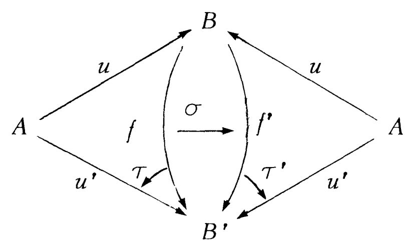
In fact, $\tilde{\mathbf{W}}$, $\hat{\mathbf{W}}$ become genuine functors from $\mathbf{A}$ to $\mathbf{C}$-$\mathbf{Cat}$, and the general theory of this work (excluding § 5) goes through with minor changes.

2° Another direction of generalization is to consider lax functors $\mathbf{W}: \mathbf{A} \to \mathbf{Cat}$ where $\mathbf{A}$ is a 2-category. Then $\tilde{\mathbf{W}}$ and $\hat{\mathbf{W}}$ may be defined suitably on 2-cells giving the procedures for creating 2-functors into $\mathbf{Cat}$ from lax functors into $\mathbf{Cat}$, each procedure with its appropriate universal property (Theorems 3 and 4).

Even if **A** is a bicategory, no new problems seem to arise other than book-keeping.

3° The generalization we wish to mention now seems to have more content. Here we would like to change the codomain of our lax functors to other 2-categories besides **Cat**.

For any 2-category **A**, and any object A of **A**, the 2-category $A/\mathbf{A}$ has objects pairs (B, u) where $u: A \to B$ is an arrow of **A**, has arrows $(f, \tau):(B, u) \to (B', u')$, pairs consisting of an arrow $f: B \to B'$ in **A** and a 2-cell $\tau: fu \to u'$, and has 2-cells $\sigma:(f, \tau) \to (f', \tau')$ just 2-cells $\sigma: f \to f'$ of **A** such that $\tau' . \sigma u = \tau$.



An alternative definition of $A/\mathbf{A}$ can be made as follows. Let

$$\mathbf{H}_A = \mathbf{A}(A, -)^{op} : {}^{op}\mathbf{A} \to \mathbf{Cat},$$

a hom 2-functor for the 2-category ${}^{op}\mathbf{A}$ obtained from **A** by reversing 2-cells. If **A** does not have a terminal object $\star$, one is easily added

$$(A(B, \star) = 1 \text{ for all } B \in A).$$

Then $(A/\mathbf{A})^{op} = \widetilde{\mathbf{H}}\,\star$. Let $\mathbf{Pr}: A/\mathbf{A} \to \mathbf{A}$ denote the projection 2-functor given by

$$(B, u) \to B, \ (f, \tau) \to f, \ \sigma \to \sigma.$$

Suppose $\mathbf{W}: \mathbf{A} \to \mathbf{C}$ is a lax functor between 2-categories **A** and **C** and suppose $A \in \mathbf{A}$, $C \in \mathbf{C}$. Then a 2-category $\hat{\mathbf{W}}(C, A)$ can be defined, for which we give the objects and arrows. The objects are lax functors $\mathbf{F}: A/\mathbf{A} \to C/\mathbf{C}$ such that the square

$$\begin{array}{ccc} A/\mathbf{A} & \xrightarrow{\quad \mathbf{F} \quad} & C/\mathbf{C} \\ {\scriptstyle \mathbf{Pr}}\downarrow & & \downarrow{\scriptstyle \mathbf{Pr}} \\ \mathbf{A} & \xrightarrow[\quad \mathbf{W} \quad]{} & \mathbf{C} \end{array}$$

commutes. The arrows $\alpha : \mathbf{F} \to \mathbf{F}'$ are left lax transformations which project to the identity of $\mathbf{W}$; right lax transformations with this projection property amount to the same thing.

If $\mathbf{C} = \mathbf{Cat}$ and $C = \mathbf{1}$, then the 2-category $C/\mathbf{C}$ might well be called $\mathbf{Obj}$; it is the 2-category of «all objects of all categories». If $\mathbf{A}$ is a category, then $A/\mathbf{A}$ is the category of objects under A.

When we presented the second basic construction (for a lax functor $\mathbf{W} : \mathbf{A} \to \mathbf{Cat}$ with $\mathbf{A}$ a category) to John Gray, he suggested the equality $\hat{\mathbf{W}} A = \hat{\mathbf{W}}(\mathbf{1}, A)$; this is indeed the case.

4° Finally, as promised in the introduction, we show how «limits and colimits» for lax functors into $\mathbf{Cat}$ may be obtained from the constructions.

There are two «diagonal» 2-functors

$$\overrightarrow{\Delta} : \mathbf{Cat} \longrightarrow \overrightarrow{Lax}\,[\mathbf{A}, \mathbf{Cat}], \quad \overleftarrow{\Delta} : \mathrm{Cat} \longrightarrow \overleftarrow{Lax}\,[\mathbf{A}, \mathbf{Cat}]$$

which take each category to the lax functor whose value all over $\mathbf{A}$ is that category.

THEOREM 13. *The 2-functor* $\overrightarrow{\Delta}$ *has a left adjoint*

$$\underleftarrow{lim} : \overrightarrow{Lax}\,[\mathbf{A}, \mathbf{Cat}] \longrightarrow \mathbf{Cat}$$

while the 2-functor $\overleftarrow{\Delta}$ *has a right adjoint*

$$\underrightarrow{lim} : \overleftarrow{Lax}\,[\mathbf{A}, \mathbf{Cat}] \longrightarrow \mathbf{Cat}.$$

PROOF. The diagonal functor $\Delta : \mathbf{Cat} \to Gen\,[\mathbf{A}, \mathbf{Cat}]$, induced by the functor $\mathbf{A} \to \mathbf{1}$, has both a left and a right 2-adjoint (2-Kan extension of the [DK] type). Composing with the inclusions

$$Gen\,[\mathbf{A}, \mathbf{Cat}] \longrightarrow \overrightarrow{Lax}\,[\mathbf{A}, \mathbf{Cat}], \qquad Gen\,[\mathbf{A}, \mathbf{Cat}] \longrightarrow \overleftarrow{Lax}\,[\mathbf{A}, \mathbf{Cat}],$$

we obtain the 2-functors $\overrightarrow{\Delta}, \overleftarrow{\Delta}$. The result follows from Theorems 3 and 4.

The following construction of $\underrightarrow{lim}\,\mathbf{W}$ for a lax functor $\mathbf{W} : \mathbf{A} \to \mathbf{Cat}$ may be of interest to formal category theorists.

From the codomain functor $\partial_1 : \mathbf{A}^2 \to \mathbf{A}$ and the projection $\mathbf{Pr}: \mathbf{Obj} \to \mathbf{Cat}$, form the pullback

$$\begin{array}{ccc}
\overleftarrow{Cnstrn}\,\mathbf{A} & \xrightarrow{\quad \mathbf{U} \quad} & \overleftarrow{Lax}\,[\,\mathbf{A}, \mathbf{Cat}\,] \\
{\scriptstyle \mathbf{V}}\big\downarrow & & \big\downarrow{\scriptstyle \overleftarrow{Lax}\,[\,\partial_1,\, 1\,]} \\
\overleftarrow{Lax}\,[\,\mathbf{A}^2, \mathbf{Obj}\,] & \xrightarrow[\overleftarrow{Lax}\,[\,1, \mathbf{Pr}\,]]{} & \overleftarrow{Lax}\,[\,\mathbf{A}^2, \mathbf{Cat}\,]\,.
\end{array}$$

Then $\underrightarrow{lim}\,\mathbf{W}$ is the fibre category $\mathbf{U}^{-1}(\mathbf{W})$ over $\mathbf{W}$ with respect to $\mathbf{U}$.

References

[B] BENABOU J., Introduction to bicategories. *Lecture Notes in Mathematics N°47*, Springer-Verlag (1967) 1-91.

[BR] BENABOU J. et ROUBAUD J., Monades et descente. *C.R.Acad. Sc. Paris*, t.270 (1970) 96-98.

[DK] DAY B. and KELLY G.M., Enriched functor categories. *Lecture Notes in Mathematics N°106*, Springer (1969) 178-191.

[Du] DUBUC E., Kan extensions in enriched category theory. *Lecture Notes in Mathematics N°145*, Springer (1970).

[EK] EILENBERG S. and KELLY G.M., Closed categories. *Proc. Conf. Categ. Alg. at La Jolla*, Springer, Berlin (1966) 421-562.

[EM] EILENBERG S. and MOORE J.C., Adjoint functors and triples. *Illinois J. Math. 9* (1965) 381-398.

[G1] GRAY J., Fibred and cofibred categories. *Proc. Conf. Categ. Alg. at La Jolla*, Springer, Berlin (1966) 21-83.

[G2] GRAY J., The categorical comprehension scheme. *Lecture Notes in Mathematics N°99*, Springer (1969) 242-312.

[Ke] KELLY G.M., Adjunction for enriched categories. *Lecture Notes in Mathematics N°106*, Springer (1969) 166-177.

[Kl] KLEISLI H., Every standard construction is induced by a pair of adjoint functors. *Proc. Amer. Math. Soc. 16* (1965) 544-546.

[Li] LINTON F., Relative functorial semantics: adjointness results. *Lecture Notes in Mathematics N°99*, Springer (1969) 166-177.

School of Mathematics and Physics
Macquarie University
North Ryde, N.S.W 2113
AUSTRALIE