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CATEGORIES OF SKETCHED STRUCTURES

by Andrée BASTIANI and Charles EHRESMANN

INTRODUCTION.

In the last decade algebraic structures have been defined on the objects of a category V:

1° A multiplication on an object e of V is a morphism k from a product $e \times e$ to e; monoids on e, groups on e, ... are obtained if further axioms are imposed on k by way of commuting diagrams ([Go], [EH]).

(The product may also be replaced by a «tensor product», but this point of view will not be considered here.)

2° The theory of fibre spaces and local structures led to p-structured categories (such as topological categories, differentiable categories, ordered categories, double categories) [E6] relative to a faithful functor p from V to the category of mappings (*), and more generally to categories in V (or category-objects in V).

Algebraic theories of Lawvere [Lw] (see also [B]) give an axiomatic way to define universal algebras; but they do not cover structures defined by partial laws, such as categories. However, all these structures may be defined by «sketches». Other examples of sketched structures are: categories equipped with a partial or a total choice of limits [Br], «discretely structured» categories [Bu], adjoint functors [L2], and also «less algebraic» structures, such as topologies [Br].

More precisely, let σ be a cone-bearing category, i.e. a category (or even a neocategory) Σ, equipped with a set of cones. A σ-structure in V is [E3] a functor from Σ to V, applying the distinguished cones on

(*) A category will be considered as the category of its morphisms and not, as usually, as the category of its objects.

limit-cones; a σ-morphism in V is a natural transformation between σ-structures in V. We denote by V^{σ} the category of σ-morphisms in V.

There are many cone-bearing categories σ' such that V^{σ} is equivalent to $V^{\sigma'}$; among them, we associate «universally» to σ:

- a limit-bearing category $\bar{\sigma}$ (i. e. the distinguished cones are limit-cones),
- a presketch $\bar{\bar{\sigma}}$ (i. e. a functor is at most the base of one distinguished cone),
- a prototype π (i. e. a presketch which is a limit-bearing category),

and, if $\mathcal{J}$ is a set of categories containing the indexing-category of each distinguished cone of σ,

- a loose $\mathcal{J}$-type τ' (i. e. a limit-bearing category in which each functor indexed by an element of $\mathcal{J}$ is the base of at least one distinguished cone); for a universal algebra, τ' «is» its algebraic theory;
- a $\mathcal{J}$-type τ (i. e. a loose $\mathcal{J}$-type which is a presketch).

Moreover:

- $\bar{\sigma}$, $\bar{\bar{\sigma}}$, π and τ are defined up to an isomorphism,
- τ' is defined up to an equivalence,
- if σ is a sketch (i. e. if it is injectively immersed in π), then π and $\bar{\sigma}$ are isomorphic, while τ and τ' are equivalent.

The existence of π and τ was proved in [E4] and [E5] under the stronger assumption that σ were a presketch; this was necessary, the proof using the existence theorem for free structures whose hypotheses are not satisfied in the case of general cone-bearing categories. But subsequent works, in particular [Bu] and the (yet unpublished) paper of Lair on tensor products of sketches [L], showed that cone-bearing categories are often more convenient, and so they convinced us of the importance of immersing them in «universal» loose types.

We achieve this here by giving an explicite construction (by transfinite induction) of $\bar{\sigma}$, π, τ and τ'. These constructions are suggested by the explicite construction of the free $\mathcal{J}$-projective completion of a category in [E]. When applied to a prototype, the constructions of τ and τ' generalize theorems of [E] on completions of categories.

These results are proved in Part I in the case where the distinguished cones are projective, in Part II when there are both projective and inductive distinguished cones. They may also be expressed as adjonctions between the category $\mathcal{S}''$ of morphisms between cone-bearing categories, and some of its full subcategories. In fact, $\mathcal{S}''$ is the category of 1-morphisms of a representable and corepresentable 2-category, and these adjonctions extend into 2-adjonctions.

Part III is devoted to the problem:

(P) If σ is a projective limit-bearing category on Σ and if V is underlying a symmetric monoidal closed category $\mathcal{V}$, does V^σ admit a symmetric monoidal closed structure?

We solve this problem in the case where σ is «cartesian», i.e. where the category $\mathcal{M}^\sigma$ of σ-morphisms in the category $\mathcal{M}$ of maps is cartesian closed (Proposition 20 is a characterisation of such a σ). More precisely, if σ is cartesian and if V admits «enough» limits, then V^σ is underlying a symmetric monoidal closed category as soon as either the tensor product of $\mathcal{V}$ commutes with the projective limits considered on σ, or the insertion functor from V^σ to V^Σ admits a left adjoint.

To prove this, we consider the symmetric monoidal closed category $\mathcal{V}^\Sigma$ defined by Day (Example 5-3 [D]) and we show that V^σ is closed for the closure functor (or Hom internal functor) of $\mathcal{V}^\Sigma$. The result is then deduced from a Proposition giving conditions under which a subcategory of a symmetric monoidal closed category admits such a structure (these conditions seem apparently slightly weaker than those we have just seen in a recent paper by Day [D1]). Notice that we use only a partial result of [D]; his general result is used in [FL] to get solutions of (P) under another kind of conditions (see Remark 2, page 82).

As an application, we deduce a symmetric monoidal closed structure on the category $\mathcal{F}(V)$ of functors in V (when σ is the prototype of categories), as was announced in [BE]. We finally show that the closure functor E on $\mathcal{F}(V)$ may also be constructed by a direct method (whose idea is to define the analogue of the «double category of quartets» generalizing the method used in a particular case in [BE]), which requires that V has only pullbacks and kernels (and not even finite sums, which have to be used in the first construction).

Sketched structures may be generalized in different ways: one of them (proposed two years ago by the first of the authors in a lecture) is to replace the cone-bearing categories by cone-bearing double categories (2-theories of [Du] and [G1] are examples of them). Another way consists in substituting «cylinders» to the cones, as is done in a just appeared paper by Freyd and Kelly [FK].

We use throughout the terminology of [E1], but we have tried to take lighter notations, nearer to those used in most papers on categorical Algebra. We stay in the frame of the Zermelo-Fraenkel set theory, with the supplementary axiom of universes: Any set belongs to a universe.

I. PROJECTIVE LIMIT-BEARING CATEGORIES

1. Neocategories and neofunctors.

Firstly, we recall the definition of a neocategory (or «graphe multiplicatif» [E1]). Graphs and also categories appear as «extreme» examples of neocategories.

A *neocategory* Σ is a couple formed by a set, denoted by $\underline{\Sigma}$, and a «partial law of composition» κ on $\underline{\Sigma}$ satisfying the following axioms:

1° κ is a mapping from a subset of $\underline{\Sigma}\times\underline{\Sigma}$ (denoted by $\Sigma * \Sigma$ and called *the set of composable couples*) into $\underline{\Sigma}$; instead of $\kappa(y, x)$, we write $y.x$ (or $y \circ x$, or yx, ...) and we call $y.x$ the *composite of* (y, x).

2° There exists a graph $(\underline{\Sigma}, \beta, \alpha)$ (i. e. α and β are retractions from $\underline{\Sigma}$ onto a subset of $\underline{\Sigma}$, denoted by Σ_0), such that:

a) For each element x of $\underline{\Sigma}$, the composites $x.\alpha(x)$ and $\beta(x).x$ are defined, and we have:

$$x.\alpha(x) = x = \beta(x).x \,;$$

b) If the composite $y.x$ is defined, then:

$$\alpha(y) = \beta(x), \quad \alpha(y.x) = \alpha(x), \quad \beta(y.x) = \beta(y).$$

From the condition 2, it follows that the graph (Σ, β, α) is uniquely defined; moreover the set Σ_0 of its vertices (called *objects of* Σ) is the set of unit elements (i. e. identities) of Σ. We say that $\alpha(x)$ is the *source* of x, and that $\beta(x)$ is the *target* of x. The elements of $\underline{\Sigma}$ are called *morphisms of* Σ. We write

$$x \in \Sigma \quad \text{or} \quad x: e \to e' \text{ in } \Sigma$$

instead of: x is a morphism of Σ, with source e and target e'. If e and e' are two objects of Σ, the set of morphisms $f: e \to e'$ in Σ will be denoted by $e'.\Sigma.e$ or by $\Sigma(e', e)$ (and not $\Sigma(e, e')$ as usually).

EXAMPLES. 1° A graph $(\underline{\Sigma}, \beta, \alpha)$ may be identified with the neocategory Σ admitting $\underline{\Sigma}$ as its set of morphisms and in which the only composites are $x.\alpha(x)$ and $\beta(x).x$, for every element x of $\underline{\Sigma}$.

2° A category is a neocategory in which all the couples (y, x) where $\alpha(y) = \beta(x)$ are composable (so that $\Sigma * \Sigma$ is the pullback of (α, β)), the law of composition being furthermore associative.

Let Σ and Σ' be two neocategories. A *neofunctor* ϕ from Σ toward Σ' is a triple $(\Sigma', \underline{\phi}, \Sigma)$, where $\underline{\phi}$ is a mapping from $\underline{\Sigma}$ into $\underline{\Sigma}'$ such that $\phi(e) \in \Sigma'_0$ for each $e \in \Sigma_0$ and that:

If $y.x$ is defined in Σ, then $\underline{\phi}(y).\underline{\phi}(x)$ is defined in Σ', and

$$\underline{\phi}(y).\underline{\phi}(x) = \underline{\phi}(y.x).$$

We say also that $\phi: \Sigma \to \Sigma'$ is a neofunctor; we write $\phi(x)$ instead of $\underline{\phi}(x)$ and ϕ_0 denotes the restriction $\phi_0: \Sigma_0 \to \Sigma'_0$ of ϕ.

If $\phi: \Sigma \to \Sigma'$ and $\phi': \Sigma' \to \Sigma''$ are two neofunctors, we denote by $\phi' \cdot \phi$, or by $\phi' \phi$, the neofunctor from Σ to Σ'' assigning

$$\phi'(\phi(x)) \quad \text{to } x \text{ in } \Sigma.$$

Neofunctors between graphs reduce to morphisms between graphs (in the usual meaning [E1]) and neofunctors between categories are ordinary *functors.*

Let Σ and Σ' be neocategories. If ϕ and ψ are two neofunctors from Σ to Σ', a *natural transformation* τ *from* ϕ *to* ψ is defined as a triple (ψ, τ_0, ϕ), where τ_0 is a mapping associating to each object e of Σ a morphism $\tau_0(e): \phi(e) \to \psi(e)$ of Σ' (also denoted by $\tau(e)$), such that the composites $\psi(x).\tau(e)$ and $\tau(e').\phi(x)$ be defined and that

$$\psi(x).\tau(e) = \tau(e').\phi(x),$$

for each $x: e \to e'$ in Σ. We say also that $\tau: \phi \to \psi$ is a natural transformation (defined by τ_0).

EXAMPLES. 1° If u is an object of Σ', the constant mapping assigning u to each morphism x in Σ defines a neofunctor $u^\wedge: \Sigma \to \Sigma'$. If $z: u \to u'$ is a morphism in Σ', we denote by $z^\wedge$ the natural transformation (said constant on z) from $u^\wedge$ to $u'^\wedge$ such that $z^\wedge(e) = z$ for each $e \in \Sigma_0$.

2° A natural transformation from a constant neofunctor, i.e. a natural transformation $\gamma: u^\wedge \to \psi$, is called a *projective cone in* Σ', *indexed by*

Σ, *with base* $\psi : \Sigma \to \Sigma'$ *and vertex* u. Similarly, a natural transformation $\gamma' : \phi \to u^\wedge$ is called an *inductive cone.*

3° Let $\tau : \phi \to \psi$ be a natural transformation with $\phi : \Sigma \to \Sigma'$. If $\phi' : \Sigma' \to \Sigma''$ is a neofunctor, the mapping $\underline{\phi}' \tau_0$ defines a natural transformation denoted by $\phi' \tau : \phi' \phi \to \phi' \psi$; if τ is a projective (resp. inductive) cone, $\phi' \tau$ is also one. If $\phi'' : \Sigma'' \to \Sigma$ is a neofunctor, the mapping $\tau_0 \phi''_0$ defines the natural transformation $\tau\phi'' : \phi\phi'' \to \psi\phi''$.

Let Σ be a neocategory and Σ' a category. If

$$\tau : \phi \to \psi \quad \text{and} \quad \tau' : \psi \to \psi'$$

are natural transformations, the mapping $\tau''_0 : \Sigma_0 \to \underline{\Sigma}'$ such that

$$\tau''_0(e) = \tau'(e).\tau(e) \quad \text{for each} \quad e \in \Sigma_0$$

defines a natural transformation $\tau'' : \phi \to \psi'$, denoted by $\tau' \boxminus \tau$. (This is not true if Σ' is only a neocategory.) With this law of composition, the set of natural transformations between neofunctors from Σ to Σ' becomes a category, denoted by $\mathfrak{N}(\Sigma', \Sigma)$ or by Σ'^{Σ}.

EXAMPLES. 1° Let $z : u' \to u$ be a morphism of Σ'. If $\gamma : u^\wedge \to \psi$ is a projective cone in Σ', with vertex u, we denote by γz the projective cone $\gamma \boxminus z^\wedge : u'^\wedge \to \psi$. If $\gamma' : \phi \to u'^\wedge$ is an inductive cone, we define $z\gamma'$ as the inductive cone $z^\wedge \boxminus \gamma'$.

2° Suppose that Σ is the category **2**, with only two objets 0 and 1, and one morphism $a = (0, 1)$ from 0 to 1. A functor $\phi : \mathbf{2} \to \Sigma'$ may be identified with the morphism $\phi(a)$ of the category Σ'; a natural transformation $\tau : \phi \to \phi'$ may be identified with the *quartet* (*commutative square*) $(\phi'(a), \tau(1), \tau(0), \phi(a))$. Then the category $\Sigma'^{\mathbf{2}}$ reduces to the *longitudinal category of quartets of* Σ' (often called category of pairs), denoted by $\boxminus\Sigma'$. By assigning (y', x', x, y) to the quartet (x', y', y, x), we define an isomorphism from $\boxminus\Sigma'$ onto a category $\boxminus\Sigma'$, called the *lateral category of quartets of* Σ'. The pair $(\boxminus\Sigma', \boxminus\Sigma')$ is a double category [E6], written $\square\Sigma'$.

A projective cone $\gamma : u^\wedge \to \phi$ in the category Σ' is called a *projective limit-cone* («limite projective naturalisée» in [E]) if, for any projective cone $\gamma' : u'^{\wedge} \to \phi$ in Σ admitting the same base than γ, there exists one and only one $z : u' \to u$ in Σ' such that $\gamma z = \gamma'$; in that case, z will be called *the factor of* γ' *through* γ, and denoted by $lim_{\gamma} \gamma'$. Dually, we define the notion of an *inductive limit-cone.*

If $\mathcal{J}$ is a set of categories, we say that thecategory Σ *admits* $\mathcal{J}$*-projective* (resp. $\mathcal{J}$*-inductive*) *limits* if each functor $\phi : K \to \Sigma$, where $K \in \mathcal{J}$, admits a projective (resp. an inductive) limit.

REMARK. Since we will essentially be concerned with projective cones or projective limits, we call them briefly cones or limits; but the dual notions will always be called explicitly inductive cone or inductive limit.

2. Cone-bearing neocategories.

By definition, a *cone-bearing neocategory* σ is a pair (Σ, Γ), where Σ is a neocategory and Γ a set of (projective) cones in Σ, said the *distinguished cones of* σ, indexed by categories. The set of the indexing-categories of all the distinguished cones is called *the set of indexing-categories of* σ.

If σ' is another cone-bearing neocategory (Σ', Γ'), a *morphism from* σ *to* σ' is defined as a triple $\overline{\psi} = (\sigma', \psi, \sigma)$, where $\psi : \Sigma \to \Sigma'$ is a neofunctor such that:

$$\psi\gamma \in \Gamma' \quad \text{for any} \quad \gamma \in \Gamma .$$

We say also that $\overline{\psi} : \sigma \to \sigma'$ is a morphism defined by ψ; we write:

$$\overline{\psi}(x) = \psi(x) \;\text{ if }\; x \in \Sigma , \quad \overline{\psi}\gamma = \psi\gamma \;\text{ if }\; \gamma \in \Gamma$$

or, more generally, if γ is a cone in Σ. Notice that the set of indexing-categories of σ must then be included in that of σ'.

If $\overline{\psi}' = (\sigma'', \psi', \sigma')$ is also a morphism from σ' to the cone-bearing neocategory σ'', then $\psi'\psi$ defines a morphism, denoted by

$$\overline{\psi}' \cdot \overline{\psi} : \sigma \to \sigma'' .$$

If ψ is an isomorphism and if its inverse defines also a morphism from σ' to σ, we say that $\overline{\psi}$ is an *isomorphism.*

Two cone-bearing neocategories σ and σ' are said *equivalent* if there exist morphisms

$$\overline{\psi} = (\sigma', \psi, \sigma) \quad \text{and} \quad \overline{\psi}' = (\sigma, \psi', \sigma')$$

such that $\psi\,\psi'$ and $\psi'\psi$ be equivalent to identities (which implies the equivalence of the underlying neocategories).

REMARK. Cone-bearing neocategories are sketches in the sense of [E3] (but the notions of a sketch considered in [E4] and [E5] are more strict, and here the word sketch will have the same meaning as in [E5]). They are used in [Bu] under the name «esquisse multiforme». Lair needs them in [L] to define tensor products of sketches. Morphisms between cone-bearing neocategories are called homomorphisms between sketches in [E3].

DEFINITION. A (projective) cone-bearing neocategory (Σ, Γ) is called a *limit-bearing category* if Σ is a category and if each distinguished cone $\gamma \in \Gamma$ is a (projective) limit-cone.

EXAMPLES. 1° Let Σ be a category and $\mathfrak{I}$ a set of categories. Let Γ be the set of all limit-cones in Σ with indexing-categories in $\mathfrak{I}$. Then (Σ, Γ) is a limit-bearing category, called the *full* $\mathfrak{I}$*-limit bearing category on* Σ.

2° Let σ be a limit-bearing category (Σ, Γ) and K a category. Consider the category of natural transformations Σ^K; for each object i of K, denote by $\pi_i : \Sigma^K \to \Sigma$ the functor associating $\tau(i)$ to the natural transformation τ. Let $\overline{\Gamma}$ be the set of cones γ in Σ^K such that:

$$\pi_i \gamma \in \Gamma \quad \text{for any} \quad i \in K_0 .$$

Then $(\Sigma^K, \overline{\Gamma})$ is a limit-bearing category [E3], denoted by σ^K. In particular, if K is the category $\mathbf{2}$ and if $\Sigma^{\mathbf{2}}$ is identified with the longitudinal category $\square\!\square\,\Sigma$ of quartets of Σ (see example 2-1), we get the *longitudinal limit-bearing category of quartets of* σ, denoted by $\square\!\square\,\sigma$. The canonical isomorphism from $\square\!\square\,\Sigma$ to $\boxminus\Sigma$ defines an isomorphism from $\square\!\square\,\sigma$ to the *lateral limit-bearing category of quartets of* σ, written $\boxminus\sigma$.

Let $\mathfrak{U}$ be a universe [AB]; an element of $\mathfrak{U}$ is called a $\mathfrak{U}$*-set* (or a small set). We denote by:

- $\mathcal{F}'_0$ (resp. $\mathcal{F}_0$) the set of neocategories (resp. of categories) Σ whose sets of morphisms are $\mathfrak{U}$-sets.

- $\mathcal{S}''_0$ the set of cone-bearing neocategories (Σ, Γ), where Γ and $\underline{\Sigma}$ are $\mathfrak{U}$-sets as well as $\underline{K}$, for each indexing-category K of $\sigma = (\Sigma, \Gamma)$.
- $\mathcal{P}'_0$ the set of limit-bearing categories belonging to $\mathcal{S}''_0$.
- $\mathfrak{M}$ the category of all mappings between $\mathfrak{U}$-sets (following our convention to name a category according to its morphisms).
- $\mathcal{F}'$ the category of all neofunctors $\phi: \Sigma \to \Sigma'$, where Σ and Σ' belong to $\mathcal{F}'_0$ (this category is denoted by $\mathfrak{N}'$ in [E1]).
- $p_{\mathcal{F}'}: \mathcal{F}' \to \mathfrak{M}$ the faithful functor which assigns the map

$$\underline{\phi}: \underline{\Sigma} \to \underline{\Sigma}' \quad \text{to} \quad \phi: \Sigma \to \Sigma'$$

and by $p'_{\mathcal{F}'}: \mathcal{F}' \to \mathfrak{M}$ the not-faithful functor associating

$$\phi_0: \Sigma_0 \to \Sigma'_0 \quad \text{to} \quad \phi: \Sigma \to \Sigma'.$$

- $\mathcal{F}$ the full subcategory of $\mathcal{F}'$ formed by the functors and by $p_{\mathcal{F}}: \mathcal{F} \to \mathfrak{M}$ the faithful functor restriction of $p_{\mathcal{F}'}$.

The morphisms $\overline{\phi}: (\Sigma, \Gamma) \to (\Sigma', \Gamma')$ between cone-bearing neocategories (resp. between limit-bearing categories) belonging to $\mathcal{S}''_0$ form a category $\mathcal{S}''$ (resp. $\mathcal{P}'$). Assigning $\phi: \Sigma \to \Sigma'$ to $\overline{\phi}$, we define a faithful functor

$$q_{\mathcal{S}''}: \mathcal{S}'' \to \mathcal{F}' \quad (\text{resp. } q_{\mathcal{P}'}: \mathcal{P}' \to \mathcal{F}).$$

Let $p_{\mathcal{S}''}$ and $p_{\mathcal{P}'}$ be the composite functors:

$$p_{\mathcal{S}''} = p_{\mathcal{F}'}\, q_{\mathcal{S}''}: \mathcal{S}'' \to \mathfrak{M}, \quad p_{\mathcal{P}'} = p_{\mathcal{F}}\, q_{\mathcal{P}'}: \mathcal{P}' \to \mathfrak{M}.$$

The following elementary proposition will be used later on.

PROPOSITION 1. *$\mathcal{S}''$ admits $\mathcal{F}_0$-projective limits and $\mathcal{F}_0$-inductive limits; $q_{\mathcal{S}''}$ commutes with projective and inductive limits; $p_{\mathcal{S}''}$ commutes with projective limits and filtered inductive limits; $\mathcal{P}'$ is closed in $\mathcal{S}''$ for projective limits.* (See also [E4] and [L1].)

Δ. The proof is straightforward. Let $F: K \to \mathcal{S}''$ be a functor, where $\underline{K}$ is a $\mathfrak{U}$-set, and write

$$F(i) = (\Sigma_i, \Gamma_i) \quad \text{for any} \quad i \in K_0.$$

1° Let Σ be a projective limit of the functor $q_{\mathcal{S}''}F$; then $\underline{\Sigma}$ is a projective limit of $p_{\mathcal{S}''}F$; denote by $\pi_i: \Sigma \to \Sigma_i$ the canonical projection and by Γ the set of cones γ in Σ such that

$$\pi_i \gamma \in \Gamma_i \text{ for any } i \in K_0 .$$

Then (Σ, Γ) is a projective limit of F. If moreover F takes its values in $\mathcal{P}'$, we have also $(\Sigma, \Gamma) \in \mathcal{P}'_0$.

2° $q_{\mathcal{S}''} F$ admits [E1] an inductive limit Σ', with canonical injections $\nu_i : \Sigma_i \to \Sigma'$. Let Γ' be the set of all cones

$$\nu_i \gamma_i , \text{ where } i \in K_0 \text{ and } \gamma_i \in \Gamma_i .$$

Then (Σ', Γ') is an inductive limit σ' of F. If K is filtered, $\underline{\Sigma}'$ is [E1] an inductive limit of $p_{\mathcal{S}''} F$. ∇

Let σ be a cone-bearing neocategory (Σ, Γ) and $\mathcal{J}$ its set of indexing-categories.

DEFINITION. If σ' is a limit-bearing category (Σ', Γ'), we define a *σ-structure in σ'* as a neofunctor $\psi : \Sigma \to \Sigma'$ defining a morphism $\overline{\psi} : \sigma \to \sigma'$ (we also say [E5] that ψ is a realization of σ in σ'). If Σ' is a category and if σ' is the full $\mathcal{J}$-limit-bearing category on Σ' (example 1), a σ-structure in σ' is called a *σ-structure in Σ'*.

The set $\mathcal{S}(\sigma', \sigma)_0$ of σ-structures in the limit-bearing category $\sigma' = (\Sigma', \Gamma')$ is the set of objects of a full subcategory of Σ'^{Σ} denoted by $\mathcal{S}(\sigma', \sigma)$, and called *the category of morphisms between σ-structures in σ'*, or *category of σ-morphisms in σ'*.

If Σ' is a category, a σ-structure in Σ' is just a neofunctor ψ from Σ to Σ' such that $\psi\gamma$ is a limit-cone, for any $\gamma \in \Gamma$. We will denote by $\mathcal{S}(\Sigma', \sigma)$, or by Σ'^{σ}, the full subcategory of Σ'^{Σ} whose objects are the σ-structures in Σ'. Remark that Σ'^{σ} admits $\mathcal{S}(\sigma', \sigma)$ as a full subcategory, for any limit-bearing category σ' on Σ'.

PROPOSITION 2. *Let σ be a cone-bearing neocategory, σ' a limit-bearing category and $\boxminus\sigma'$ the lateral limit-bearing category of quartets of σ' (example 2). Then there exists a canonical bijection from the set of morphisms of $\mathcal{S}(\sigma', \sigma)$ to the set $\mathcal{S}(\boxminus\sigma', \sigma)_0$ of σ-structures in $\boxminus\sigma'$.*

Δ. To a natural transformation $\tau : \psi \to \psi'$, where $\psi : \Sigma \to \Sigma'$, there corresponds the neofunctor $T : \Sigma \to \boxminus\Sigma'$ which assigns the quartet

$$(\psi'(z), \tau(e'), \tau(e), \psi(z)) \text{ to } z : e \to e' \text{ in } \Sigma.$$

The map f associating T to τ is a bijection from $\underline{\Sigma'^{\Sigma}}$ to the set of neofunctors from Σ to $\boxminus\Sigma'$. (If we identify τ with a functor from $\mathbf{2}$ to Σ'^{Σ}, this bijection f is deduced from the canonical isomorphism:

$$(\Sigma'^{\Sigma})^{\mathbf{2}} \approx (\Sigma'^{\mathbf{2}})^{\Sigma} \approx (\boxminus\Sigma')^{\Sigma}.)$$

The natural transformation τ is a morphism between σ-structures iff(*) T is a σ-structure in $\boxminus\sigma'$. Therefore f induces a bijection

$$f' : \underline{\mathcal{S}}(\sigma', \sigma) \to \mathcal{S}(\boxminus\sigma', \sigma)_0 . \quad \nabla$$

3. Limit-bearing category generated by a cone-bearing neocategory.

The study of the category $\mathcal{S}(\sigma', \sigma)$ of morphisms between σ-structures in the limit-bearing category σ' will be much easier when the cone-bearing neocategory σ is itself a limit-bearing category. Hence the question: Does there exist a limit-bearing category $\bar{\sigma}$ such that $\mathcal{S}(\sigma', \sigma)$ and $\mathcal{S}(\sigma', \bar{\sigma})$ are isomorphic? The following proposition not only answers affirmatively this question, but it gives an explicit construction of a smallest $\bar{\sigma}$ of this kind.

PROPOSITION 3. *Let σ be a cone-bearing neocategory (Σ, Γ). There exists a limit-bearing category $\bar{\sigma} = (\bar{\Sigma}, \bar{\Gamma})$ and a morphism $\bar{\delta} : \sigma \to \bar{\sigma}$ satisfying the following conditions:*

1° $\bar{\Gamma} = \{ \bar{\delta}\gamma \mid \gamma \in \Gamma \}$.

2° If $\mathfrak{U}$ is a universe such that $\sigma \in \mathcal{S}_0''$, then $\bar{\sigma} \in \mathcal{P}_0'$.

3° $\bar{\sigma}$ is characterized, up to an isomorphism, by the universal property: If σ' is a limit-bearing category and $\bar{\psi} : \sigma \to \sigma'$ a morphism, then there exists one and only one morphism $\bar{\psi}' : \bar{\sigma} \to \sigma'$ such that $\bar{\psi}' \cdot \bar{\delta} = \bar{\psi}$.

Δ. By transfinite induction, we shall construct a «tower» of cone-bearing neocategories σ_ξ such that σ_0 be σ and that $\sigma_{\xi+1}$ be deduced from σ_ξ by adding to σ_ξ «formal factors» through a distinguished cone γ for cones with the same base as γ. We will show that this tower ends for

(*) iff means if and only if.

some sufficiently big ordinal μ, and that σ_μ is the limit-bearing category $\bar{\sigma}$.

1° Let us decribe the step from σ_ξ to $\sigma_{\xi+1}$. Let σ_ξ be any cone-bearing neocategory (Σ_ξ, Γ_ξ).

a) If $\gamma \in \Gamma_\xi$ and if γ' is a cone in Σ_ξ with the same base as γ we consider the pair (γ, γ') (called the «formal factor» of γ' through γ). Let Ω be the set of all these pairs; let U be the sum («disjoint union») of $\underline{\Sigma}_\xi$ and Ω, with injections

$$v : \underline{\Sigma}_\xi \to U \quad \text{and} \quad v' : \Omega \to U.$$

We define a graph (U, β, α) in the following way:

- If $x : u \to u'$ in Σ_ξ, then

$$\alpha(v(x)) = v(u), \quad \beta(v(x)) = v(u').$$

- If $(\gamma, \gamma') \in \Omega$, where $\gamma : u^\wedge \to \phi$ and $\gamma' : u'^\wedge \to \phi$, then

$$\alpha(v'(\gamma, \gamma')) = v(u'), \quad \beta(v'(\gamma, \gamma')) = v(u).$$

Let L be the free category generated by (U, β, α); it is [E1] the «category of paths» on (U, β, α) and U is identified with paths of length 1. Consider the smallest equivalence relation r on $\underline{L}$ such that:

$$(P)\begin{cases} (v(x'), v(x)) \sim v(x'.x), \text{ if } x'.x \text{ is defined in } \Sigma_\xi, \\ (v(\gamma(i)), v'(\gamma, \gamma')) \sim v(\gamma'(i)), \text{ if } (\gamma, \gamma') \in \Omega \text{ and } i \in K_0, \\ v(z) \sim v'(\gamma, \gamma'), \text{ if } z \in \Sigma_\xi, \text{ if } (\gamma, \gamma') \in \Omega \text{ and if} \\ \qquad \gamma'(i) = \gamma(i).z \text{ for any } i \in K_0, \\ \text{where } K \text{ is the indexing-category of } \gamma. \end{cases}$$

There exists a quasi-quotient category [E1] of L by r, denoted by $\bar{\Sigma}_\xi$; since r identifies no objects, $\bar{\Sigma}_\xi$ is in fact the quotient category of L by the smallest equivalence relation compatible with the law of composition of L and containing r. Let ρ: $L \to \bar{\Sigma}_\xi$ be the canonical functor corresponding to r. The map $\underline{\rho}\, v$ defines a neofunctor $\delta_\xi : \Sigma_\xi \to \bar{\Sigma}_\xi$ by the first condition imposed on r. Denote by $\bar{\Gamma}_\xi$ the set of all the cones $\delta_\xi \gamma$, where $\gamma \in \Gamma_\xi$. Then $(\bar{\Sigma}_\xi, \bar{\Gamma}_\xi)$ is a cone-bearing neocategory $\bar{\sigma}_\xi$ and δ_ξ defines a morphism $\bar{\delta}_\xi : \sigma_\xi \to \bar{\sigma}_\xi$. Moreover, for each formal factor $(\gamma, \gamma') \in \Omega$, we have

$$(\delta_\xi \gamma) z = \gamma_\xi \gamma', \quad \text{where} \quad z = \underline{\rho}(v'(\gamma, \gamma')).$$

b) *Suppose that* σ' *is a limit-bearing category* (Σ', Γ') *and that* ψ *is a neofunctor defining a morphism* $\overline{\psi} : \sigma_\xi \to \sigma'$. *Then there exists a unique morphism*

$$\overline{\psi}' : \overline{\sigma}_\xi \to \sigma' \quad \text{such that} \quad \overline{\psi}' \cdot \overline{\delta}_\xi = \overline{\psi}.$$

Indeed, if $(\gamma, \gamma') \in \Omega$, where $\gamma : u^\wedge \to \not{c}$, the cone $\psi\gamma$ is a limit-cone with the same base as the cone $\psi\gamma'$; so there exists a unique y such that $(\psi\gamma) y = \psi\gamma'$, namely the factor of $\psi\gamma'$ through $\psi\gamma$. By assigning y to the formal factor (γ, γ'), we get a mapping $f : \Omega \to \underline{\Sigma}'$. The unique map $f' : U \to \underline{\Sigma}'$ such that

$$f' v = \underline{\psi} \quad \text{and} \quad f' v' = f$$

defines a neofunctor from (U, β, α) (considered as a neocategory) to Σ'. This neofunctor extends into a functor $F' : L \to \Sigma'$. Since σ' is a limit-bearing category, this F' is compatible with r, so that there exists one and only one functor

$$\psi' : \overline{\overline{\Sigma}}_\xi \to \Sigma' \quad \text{such that} \quad \psi' \rho = F'.$$

This functor defines the unique morphism

$$\overline{\psi}' : \overline{\sigma}_\xi \to \sigma' \quad \text{such that} \quad \overline{\psi}' \cdot \overline{\delta}_\xi = \overline{\psi}.$$

c) If $\mathfrak{U}$ is a universe such that $\sigma_\xi \in \mathcal{S}_0''$, then $\underline{K}$, for each indexing-category K of σ_ξ and Γ_ξ are $\mathfrak{U}$-sets; it results that the set Γ_K of cones in Σ_ξ indexed by K is also a $\mathfrak{U}$-set, as well as the set $\bigcup_{K \in \mathfrak{I}} \Gamma_K$, where $\mathfrak{I}$ is the set of indexing-categories of σ_ξ. From this we deduce succesively that Ω, U, $\underline{L}$ and $\underline{\overline{\Sigma}}_\xi$ are $\mathfrak{U}$-sets, and that $\overline{\sigma}_\xi$ belongs to $\mathcal{S}_0''$.

2° We are now ready to construct the tower. Let $\mathfrak{I}$ be the set of indexing-categories of σ. If $K \in \mathfrak{I}$, we denote by $\overline{\overline{K}}$ the cardinal of $\underline{K}$. (An ordinal number ζ is considered as the set of ordinals ξ such that $\xi < \zeta$; the cardinal of a set E is identified with the initial ordinal equipotent to E). Let λ be the ordinal which is the upper bound of the ordinals $\overline{\overline{K}}$, where $K \in \mathfrak{I}$, and let μ be the least regular ordinal satisfying $\lambda < \mu$.

Accepting the «axiom of universes», there exists a universe $\mathfrak{U}$ to which belongs $\Gamma \cup \underline{\Sigma} \cup \bigcup_{K \in \mathcal{J}} \underline{K}$, i. e. such that σ is an object of the category $\mathcal{S}''$ corresponding to $\mathfrak{U}$. As $\underline{K}$ is a $\mathfrak{U}$-set, $\bar{\bar{K}}$ belongs to $\mathfrak{U}$ and, $\mathcal{J}$ being equipotent to a subset of the $\mathfrak{U}$-set Γ, the ordinal λ belongs also to $\mathfrak{U}$, as well as μ. (Here we use the fact that the upper bound of the ordinals which belong to a universe $\mathfrak{U}$ is an inaccessible ordinal [AB]).

For each ordinal ξ, let $<\xi>$ be the category defining the canonical order on ξ; its set of objects is ξ. (In particular, $\mathbf{2} = <2>$). By transfinite induction, we define a functor $\omega : <\mu+1> \to \mathcal{S}''$:

- First, $\omega(0) = \sigma$.
- Let ζ be an ordinal, $\zeta \leqslant \mu$; suppose we have defined a functor

$$\omega_\zeta : <\zeta> \to \mathcal{S}'' \quad \text{such that} \quad \omega_\zeta(0) = \sigma,$$

and write

$$\omega_\zeta(\xi) = \sigma_\xi = (\Sigma_\xi, \Gamma_\xi) \quad \text{for any} \quad \xi < \zeta.$$

We extend ω_ζ into a functor $\omega_{\zeta+1} : <\zeta+1> \to \mathcal{S}''$ in the following way:

If ζ is a limit ordinal, $\omega_{\zeta+1}(\zeta)$ is the canonical inductive limit, denoted by $\sigma_\zeta = (\Sigma_\zeta, \Gamma_\zeta)$, of the functor ω_ζ (which exists, Proposition 1) and $\omega_{\zeta+1}(\zeta, \xi) : \sigma_\xi \to \sigma_\zeta$ is the canonical injection, for any $\xi < \zeta$. We recall (Proposition 1 and [E1]) that $\underline{\Sigma}_\zeta$ is the canonical inductive limit of the functor $p_{\mathcal{S}''} \omega_\zeta$ from $<\zeta>$ to $\mathfrak{M}$ and that each composite $\overline{y'} . \overline{y}$ in Σ_ζ is of the form $\omega_{\zeta+1}(\zeta, \xi)(y' . y)$, for some $\xi < \zeta$, where $y' . y$ is a composite in Σ_ξ, and $\overline{y} = \omega_{\zeta+1}(\zeta, \xi)(y)$, $\overline{y'} = \omega_{\zeta+1}(\zeta, \xi)(y')$.

If ζ is the successor of ξ (that is: $\zeta = \xi + 1$), then $\omega_{\zeta+1}(\zeta)$ will be the cone-bearing neocategory $(\overline{\Sigma}_\xi, \overline{\Gamma}_\xi)$ associated to σ_ξ in Part 1, and $\omega_{\zeta+1}(\xi+1, \xi) : \sigma_\xi \to \sigma_\zeta$ will be the morphism $\overline{\delta}_\xi$ constructed in Part 1. The induction hypothesis $\sigma_\xi \in \mathcal{S}''_0$ implies $\sigma_\zeta \in \mathcal{S}''_0$ (Part 1).

- Finally, we put

$$\omega = \omega_{\mu+1}, \quad \overline{\sigma} = (\overline{\Sigma}, \overline{\Gamma}) = \sigma_\mu, \quad \overline{\delta} = \omega(\mu, 0) = (\overline{\sigma}, \delta, \sigma).$$

By construction, $\overline{\sigma}$ is an object of $\mathcal{S}''$.

3° a) By transfinite induction, we prove that $\overline{\Sigma}$ is a category. Indeed, suppose that ζ is an ordinal, $\zeta \leqslant \mu$, and that Σ_ξ is a category for any

ξ such that $0 \neq \xi < \zeta$. If $\zeta = \xi + 1$, then $\Sigma_\zeta = \overline{\Sigma}_\xi$ is a category, by construction. If ζ is a limit ordinal, Σ_ζ is the inductive limit of the functor $q_{\mathcal{S}''} \omega_\zeta$; since $<\zeta>$ is a filtered category and since the Σ_ξ, for $\xi < \zeta$ and $0 \neq \xi$, are categories, the neocategory Σ_ζ is a category. Hence Σ_μ is a category $\overline{\Sigma}$.

b) We are going to prove, by transfinite induction, that each cone $\overline{\gamma}$ of $\overline{\Gamma}$ is of the form $\delta\gamma$, for some $\gamma \in \Gamma$. Indeed, we have $\Gamma_0 = \Gamma$. Let ζ be an ordinal, $\zeta \leqslant \mu$, and suppose that, for any $\xi < \zeta$, we have:

$$\Gamma_\xi = \{ \omega(\xi, 0)\gamma \mid \gamma \in \Gamma \}.$$

- If ζ is a limit ordinal, Proposition 1 asserts that

$$\Gamma_\zeta = \{ \omega(\zeta, \xi)\gamma_\xi \mid \gamma_\xi \in \Gamma_\xi, \quad \xi < \zeta \}$$

and the induction hypothesis implies that

$$\gamma_\xi = \omega(\xi, 0)\gamma, \quad \text{for some} \quad \gamma \in \Gamma;$$

hence

$$\omega(\zeta, \xi)\gamma_\xi = \omega(\zeta, \xi) \cdot \omega(\xi, 0)\gamma = \omega(\zeta, 0)\gamma.$$

It follows that:

$$\Gamma_\zeta = \{ \omega(\zeta, 0)\gamma \mid \gamma \in \Gamma \}.$$

- If $\zeta = \xi + 1$, by Part 1-a, we have:

$$\Gamma_\zeta = \overline{\Gamma}_\xi = \{ \overline{\delta}_\xi \gamma_\xi \mid \gamma_\xi \in \Gamma_\xi \},$$

where

$$\overline{\delta}_\xi \gamma_\xi = \omega(\xi + 1, \xi)\gamma_\xi = \omega(\xi + 1, \xi) \cdot \omega(\xi, 0)\gamma = \omega(\zeta, 0)\gamma,$$

since $\gamma_\xi = \omega(\xi, 0)\gamma$, for some $\gamma \in \Gamma$. Therefore, in this case also,

$$\Gamma_\zeta = \{ \omega(\zeta, 0)\gamma \mid \gamma \in \Gamma \}.$$

c) The category $\overline{\Sigma}$ is determined independently of the universe $\mathcal{U}$. For, let $\hat{\mathcal{U}}$ be another universe such that

$$(\Gamma \cup \underline{\Sigma} \cup \bigcup_{K \in \mathcal{J}} \underline{K}) \in \hat{\mathcal{U}},$$

and let $\hat{\mathcal{S}}''$ be the category of morphisms between cone-bearing neocategories corresponding to $\hat{\mathcal{U}}$. If $F: C \to \mathcal{S}''$ and $\hat{F}: C \to \hat{\mathcal{S}}''$ are two functors taking the same values (i.e. $F(z) = \hat{F}(z)$ for any $z \in C$), then they have the same canonical inductive limit, according to the construction of this limit as a quotient of a sum. So the inductive limit σ_ζ of ω_ζ, for a limit ordinal ζ, does not depend on the choice of $\mathfrak{U}$. In particular, $\bar{\sigma}$ is independent of $\mathfrak{U}$.

d) $\bar{\delta}$ satisfies the condition 3 of the Proposition. Indeed, let $\bar{\psi}$ be a morphism (σ', ψ, σ) from σ to a limit-bearing category $\sigma' = (\Sigma', \Gamma')$. Part c above shows that we may suppose $\sigma' \in \mathcal{S}_0''$. Using the universal property of the inductive limit σ_ζ of ω_ζ and that of $\bar{\delta}_\xi : \sigma_\xi \to \bar{\sigma}_\xi = \sigma_{\xi+1}$ (Part 1-b), we construct by transfinite induction a sequence of morphisms $\bar{\psi}_\zeta : \sigma_\zeta \to \sigma'$, where $\zeta \leqslant \mu$, such that $\bar{\psi}_0 = \bar{\psi}$ and

$$\bar{\psi}_\zeta \cdot \omega(\zeta, \xi) = \bar{\psi}_\xi \quad \text{for any} \quad \xi < \zeta .$$

Then $\bar{\psi}_\mu$ is the unique morphism $\bar{\psi}' : \bar{\sigma} \to \sigma'$ satisfying $\bar{\psi}' \cdot \bar{\delta} = \bar{\psi}$.

(Notice that, up to now, we have not used the fact that μ is a given regular ordinal.)

4° To complete the proof, we have yet to show that $\bar{\sigma}$ is a limit-bearing category, i.e. that each cone $\bar{\gamma} \in \bar{\Gamma}$ is a limit-cone. This will imply that the tower ends with $\bar{\sigma}$ (this means that $\sigma_{\mu+1}$ is isomorphic to $\bar{\sigma}$). Suppose that $\bar{\gamma}$ is a distinguished cone of $\bar{\sigma}$; then there exists some cone $\gamma \in \Gamma$ such that $\bar{\gamma} = \delta\gamma$ (Part 3-b). Denote by ϕ the base of γ, by K its indexing-category, by γ_ξ the cone $\omega(\xi, 0)\gamma \in \Gamma_\xi$, for each $\xi < \mu$. Let $\bar{\gamma}' : u'^{\wedge} \to \phi$ be a cone in $\bar{\Sigma}$ with the same base as $\bar{\gamma}$.

a) We are going to prove the existence of an ordinal $\xi < \mu$ and of a cone γ' with the same base as the distinguished cone γ_ξ such that we have $\bar{\gamma}' = \omega(\mu, \xi)\gamma'$. Then the «formal factor» (γ_ξ, γ') determines a morphism z of $\bar{\Sigma}_\xi = \Sigma_{\xi+1}$ satisfying the equalities:

$$(\omega(\xi+1, \xi)\gamma_\xi) z = \omega(\xi+1, \xi)\gamma' ,$$

which gives, after transformation by $\omega(\mu, \xi+1)$:

$$(\delta\gamma)\bar{z} = \bar{\gamma}' , \quad \text{where} \quad \bar{z} = \omega(\mu, \xi+1)(z).$$

Indeed, since $\bar{\Sigma}$ is the inductive limit of $p_{\mathcal{S}''}\omega_\mu : \langle\mu\rangle \to \mathfrak{M}$, for each object i of K there exists an ordinal $\xi_i < \mu$ and a $x_i \in \Sigma_{\xi_i}$ such that

$$\overline{\gamma}'(i) = \omega(\mu, \xi_i)(x_i).$$

Let $k: i \to i'$ be a morphism in K. By construction of the inductive limit $\overline{\sigma}$, the equality $\overline{\gamma}'(i') = \delta\phi(k).\overline{\gamma}'(i)$ means that there exists an ordinal ξ_k such that $\xi_i < \xi_k \lesssim \mu$, $\xi_{i'} < \xi_k < \mu$ and

(1) $$\omega(\xi_k, \xi_{i'})(x_{i'}) = \omega(\xi_k, 0)(\phi(k)).\omega(\xi_k, \xi_i)(x_i).$$

μ being a regular ordinal such that

$$\overline{\overline{K}} < \mu \quad \text{and} \quad \xi_k < \mu \quad \text{for any } k \in K,$$

the upper bound ξ of the ξ_k, where $k \in K$, verifies $\xi < \mu$. For this ordinal ξ and for each $k: i \to i'$ in K, we get from (1):

$$\begin{aligned}\omega(\xi, \xi_{i'})(x_{i'}) &= \omega(\xi, \xi_k)\omega(\xi_k, \xi_{i'})(x_{i'}) \\ &= \omega(\xi, \xi_k)(\omega(\xi_k, 0)(\phi(k)).\omega(\xi_k, \xi_i)(x_i)) \\ &= \omega(\xi, 0)(\phi(k)).\omega(\xi, \xi_i)(x_i).\end{aligned}$$

This shows that the map

$$\gamma'_0 : K_0 \to \underline{\Sigma}_\xi \quad \text{such that} \quad \gamma'_0(i) = \omega(\xi, \xi_i)(x_i)$$

defines a cone γ' in Σ_ξ with the same base as $\gamma_\xi = \omega(\xi, 0)\gamma$. Moreover $\omega(\mu, \xi)\gamma' = \overline{\gamma}'$, since, for each object i of K, we have:

$$\omega(\mu, \xi)\gamma'(i) = \omega(\mu, \xi_i)(x_i) = \overline{\gamma}'(i).$$

b) We have found a $\overline{z}$ such that

$$\overline{\gamma}\,\overline{z} = \overline{\gamma}', \qquad \text{namely} \qquad \overline{z} = \omega(\mu, \xi + 1)(z).$$

Suppose that $\overline{z}'$ is another morphism of $\overline{\Sigma}$ satisfying $\overline{\gamma}\,\overline{z}' = \overline{\gamma}'$; we show that $\overline{z} = \overline{z}'$. Indeed, there exists an ordinal $\zeta < \mu$ and a morphism z' in Σ_ζ with $\omega(\mu, \zeta)(z') = \overline{z}'$. We may suppose $\xi < \zeta$. For each $i \in K_0$, the equality $\overline{\gamma}'(i) = \overline{\gamma}(i).\overline{z}'$, which may also be written

$$\omega(\mu, \xi)\gamma'(i) = \omega(\mu, 0)\gamma(i).\omega(\mu, \zeta)(z')$$

implies the existence of an ordinal ζ_i such that $\zeta < \zeta_i < \mu$ and

$$\omega(\zeta_i, \xi)\gamma'(i) = \omega(\zeta_i, 0)\gamma(i).\omega(\zeta_i, \zeta)(z').$$

If ζ' is the upper bound of the ζ_i, for $i \in K_0$, we get as above $\zeta' < \mu$ and

$$\omega(\zeta', \xi)\gamma' = (\omega(\zeta', 0)\gamma)\omega(\zeta', \zeta)(z') = \gamma_{\zeta'}\hat{z}',$$

where $\hat{z}' = \omega(\zeta', \zeta)(z')$. From the equality

$$\omega(\xi+1,\xi)\gamma' = (\omega(\xi+1,\xi)\gamma_\xi)z\ ,$$

it follows, by applying $\omega(\zeta', \xi+1)$:

$$\omega(\zeta',\xi)\gamma' = (\omega(\zeta',\xi)\gamma_\xi)\hat{z} = \gamma_{\zeta'}\hat{z}\ ,$$

where $\hat{z} = \omega(\zeta', \xi+1)(z)$. Hence $\hat{z}'$ and $\hat{z}$ are two morphisms such that $\gamma_{\zeta'}\hat{z} = \gamma_{\zeta'}\hat{z}'$, which implies

$$\omega(\zeta'+1, \zeta')(\hat{z}) = \omega(\zeta'+1,\zeta')(\hat{z}'),$$

by construction of $\Sigma_{\zeta'+1} = \overline{\Sigma}_{\zeta'}$ (Part 1). Finally, applying $\omega(\mu, \zeta'+1)$, we get $\overline{z} = \overline{z}'$. ∇

DEFINITION. With the hypotheses of Proposition 3, we call $\overline{\sigma}$ the *limit-bearing category generated by* σ.

COROLLARY 1. *The insertion functor* $I: \mathcal{P}' \to \mathcal{S}''$ *admits a (left) adjoint.* $\mathcal{P}'$ *admits* $\mathcal{F}_0$ *-inductive limits, and there exist quasi-quotient limit-bearing categories.*

Δ. The first statement results from Proposition 3.

If $F: C \to \mathcal{P}'$ is a functor, where $\underline{C}$ is a $\mathfrak{U}$-set, then $IF: C \to \mathcal{S}''$ admits an inductive limit σ (Proposition 1), and the limit-bearing category $\overline{\sigma}$ generated by σ is an inductive limit of F.

Let σ' be a cone-bearing neocategory (Σ', Γ') and ρ an equivalence relation on $\underline{\Sigma}'$. There exists a quasi-quotient cone-bearing neocategory σ of σ' by ρ (i. e. a quasi-quotient $p_{\mathcal{S}''}$-structure [E1]); namely, $\sigma = (\Sigma, \Gamma)$, where Σ is the neocategory quotient of Σ' by the smallest compatible equivalence relation on Σ' containing ρ and where

$$\Gamma = \{\ \hat{\rho}\gamma' \mid \gamma' \in \Gamma'\ \},\ \text{ if }\ \hat{\rho}: \Sigma' \to \Sigma$$

is the neofunctor corresponding to ρ. Hence the limit-bearing category $\overline{\sigma}$ generated by σ is the quasi-quotient limit-bearing category of σ' by ρ. If $\sigma' \in \mathcal{P}'_0$, then $\overline{\sigma}$ is a quasi-quotient $p_{\mathcal{P}'}$-structure of σ' by ρ. ∇

COROLLARY 2. *Let* σ *be a cone-bearing neocategory and* $\overline{\sigma}$ *the limit-bearing category generated by* σ. *If* σ' *is a limit-bearing category, then the* *categories* $\mathcal{S}(\sigma', \sigma)$ *and* $\mathcal{S}(\sigma', \overline{\sigma})$ *are isomorphic. In particular,* Σ'^{σ} *and* $\Sigma'^{\overline{\sigma}}$ *are isomorphic, for every category* Σ'.

Δ. Let $\boxminus\sigma'$ be the lateral limit-bearing category of quartets of σ' (Example 2-2). We have constructed, in Proposition 2, bijections

$$g : \underline{\mathcal{S}}(\sigma', \sigma) \to \mathcal{S}(\boxminus\sigma', \sigma)_0 \quad \text{and} \quad h : \underline{\mathcal{S}}(\sigma', \overline{\sigma}) \to \mathcal{S}(\boxminus\sigma', \overline{\sigma})_0 .$$

By Proposition 3, there is a canonical bijection

$$d : \mathcal{S}(\boxminus\sigma', \overline{\sigma})_0 \to \mathcal{S}(\boxminus\sigma', \sigma)_0 ,$$

assigning $\psi'\delta$ to the $\overline{\sigma}$-structure ψ', where $\overline{\delta} = (\overline{\sigma}, \delta, \sigma)$ is the canonical morphism. The bijection $g^{-1}dh$ defines the isomorphism from the category $\mathcal{S}(\sigma', \overline{\sigma})$ to $\mathcal{S}(\sigma', \sigma)$ assigning $\tau\delta$ to τ. ∇

REMARKS. 1° $\overline{\sigma}$ is «universal» relative to all σ-structures, and not only to those which are «small enough». The universe $\mathcal{U}$ is used as a tool in in the proof of Proposition 3, and it does not appear in the conclusion (as we have shown in Part 3-c). We could have omitted $\mathcal{U}$ by considering «the category of morphisms between all cone-bearing neocategories» (i. e. by admitting a theory of sets and classes).

2° In [L], Corollary 1 of Proposition 3 is deduced from the general existence theorem for free structures of [E], the proof being identical with the argument used in [E5] to prove the existence of the prototype of σ. Above, we have not only shown the existence of $\overline{\sigma}$, but we have also given an explicit construction of it, from which many properties of $\overline{\sigma}$ may be deduced. This construction is suggested by the explicit construction of a free $\mathcal{I}$-projective completion of a category (Theorem 7 of [E]); the main difference, apart from adding «no objects», lies in the fact that the hypotheses of Theorem 7 of [E] (after adding «all formal cones») implied the injectivity of the functor $\delta_\xi : \Sigma_\xi \to \Sigma_{\xi+1}$, for any ordinal (which was difficult to prove and required a detailed description of the morphisms of $\Sigma_{\xi+1}$ as «reduced paths»); so, the category Σ_ζ, for a limit ordinal ζ, was just the union of the categories Σ_ξ, for $\xi < \zeta$. This is no more true here, and we have to define Σ_ζ, for a limit ordinal ζ, as the inductive limit of the functor $q_{\mathcal{S}''}\omega_\zeta : \langle \zeta \rangle \to \mathcal{F}'$.

3° Proposition 3 may also be expressed as follows: Let σ be a cone-bearing neocategory. *There exists a limit-bearing category* $\overline{\sigma} = (\overline{\Sigma}, \overline{\Gamma})$, *characterized up to an isomorphism by the property:*

If $\mathfrak{U}$ *is a universe such that* $\sigma \in \mathcal{S}''_0$, *then* $\overline{\sigma}$ *is a free structure generated by* σ *relative to the insertion functor from* $\mathcal{P}'$ *to* $\mathcal{S}''$.

Intuitively, if $\sigma = (\Sigma, \Gamma)$, the set $\underline{\overline{\Sigma}}$ belongs to the smallest universe to which belongs $\underline{\Sigma}$, while $\overline{\sigma}$ solves the universal problem for *any* universe to which belongs $\underline{\Sigma}$.

4. Loose types.

Let σ be a cone-bearing neocategory and σ' a limit-bearing category. We have seen that there exist limit-bearing categories $\overline{\sigma}$ such that the category $\mathcal{S}(\sigma', \sigma)$ is isomorphic with $\mathcal{S}(\sigma', \overline{\sigma})$. In fact, we have constructed a $\overline{\sigma}$ which is minimal. Now the question arises: If any functor is the base of a distinguished cone in σ', does there exist a $\overline{\sigma}$ with the same property? We are going to solve this problem relative to a given set of categories.

We denote by $\mathcal{J}$ a set of categories.

DEFINITION. If σ is a cone-bearing neocategory (resp. a limit-bearing category) whose set of indexing-categories $\mathcal{J}_\sigma$ is a subset of $\mathcal{J}$, we also say that σ is a $\mathcal{J}$*-cone-bearing neocategory* (resp. a $\mathcal{J}$*-limit-bearing category).*

In particular, σ is a $\mathcal{J}_\sigma$-cone-bearing neocategory.

DEFINITION. Let σ be a limit-bearing category (Σ, Γ) and $\mathcal{J}$ its set of indexing-categories. We say that σ is *a loose type* (or, more precisely, a loose $\mathcal{J}$-type) if each functor $\phi : K \to \Sigma$, where $K \in \mathcal{J}$, is the base of at least one distinguished limit-cone $\gamma \in \Gamma$.

This condition implies that Σ admits $\mathcal{J}$-projective limits.

If $\mathfrak{U}$ is a universe such that $\mathcal{J}$ is a $\mathfrak{U}$-set, we denote by $\mathcal{S}''^{\mathcal{J}}$ (resp. $\mathcal{P}'^{\mathcal{J}}$, resp. $\mathcal{L}^{\mathcal{J}}$) the full subcategory of $\mathcal{S}''$ whose objects are the $\mathcal{J}$-cone-bearing neocategories (resp. the $\mathcal{J}$-limit-bearing categories, resp. the loose

$\mathcal{J}$-types) σ belonging to $\mathcal{S}_0''$.

PROPOSITION 4. *Let σ be a $\mathcal{J}$-cone-bearing neocategory (Σ, Γ). There exists a loose $\mathcal{J}$-type $\bar{\sigma}$ (unique up to an equivalence) and a morphism $\bar{\delta} = (\bar{\sigma}, \delta, \sigma)$ satisfying the following condition where σ' is a loose $\mathcal{J}'$-type for a set $\mathcal{J}'$ of categories containing $\mathcal{J}$:*

1° If $\bar{\psi} : \sigma \to \sigma'$ is a morphism, there exist morphisms $\bar{\psi}' : \bar{\sigma} \to \sigma'$ such that $\bar{\psi}' \cdot \bar{\delta} = \bar{\psi}$, and two such morphisms are equivalent.

2° If $\bar{\psi}' = (\sigma', \psi', \bar{\sigma})$ and $\bar{\psi}'' = (\sigma', \psi'', \bar{\sigma})$ are morphisms and if $\tau : \psi' \delta \to \psi'' \delta$ is a natural transformation (resp. an equivalence), there exists one and only one natural transformation (resp. equivalence)

$$\tau' : \psi' \to \psi'' \quad \text{such that} \quad \tau' \delta = \tau .$$

Moreover, if $\mathcal{U}$ is a universe such that $\mathcal{J}$ is a $\mathcal{U}$-set and $\sigma \in \mathcal{S}_0''$, then we have $\bar{\sigma} \in \mathcal{L}_0^{\mathcal{J}}$.

Δ. We will again construct, by transfinite induction, a tower of cone-bearing-neocategories which stops («up to an equivalence») at the first regular ordinal μ greater than all the ordinals $\bar{\bar{K}}$, where $K \in \mathcal{J}$. The method is similar to that used in Proposition 3, but, in the «non-limit step» from σ_ξ to $\sigma_{\xi+1}$, we will add also «formal cones» for each neofunctor indexed by an element of $\mathcal{J}$.

1° Let us first describe this non-limit step. We suppose that σ_ξ is a cone-bearing neocategory (Σ_ξ, Γ_ξ).

a) Let us consider:

- the set Ω of pairs (γ, γ') (or «formal factors»), where $\gamma \in \Gamma_\xi$ and γ' is a cone in Σ_ξ with the same base as γ,
- the set M of neofunctors $\phi : K \to \Sigma_\xi$, where $K \in \mathcal{J}$, which are not the base of any distinguished cone $\gamma \in \Gamma_\xi$,
- the set M' of pairs (i, ϕ), where $\phi \in M$ and where i is an object of the indexing-category of ϕ,
- the sum («disjoint union») U of $\underline{\Sigma}_\xi$, Ω, M and M', with injections:

$$v : \underline{\Sigma}_\xi \to U, \quad v' : \Omega \to U, \quad w : M \to U, \quad w' : M' \to U .$$

We describe a graph (U, β, α) in the following way:

- if $x : u \to u'$ in Σ_ξ, then

$$\alpha(v(x)) = v(u), \quad \beta(v(x)) = v(u'),$$

- if $(\gamma, \gamma') \in \Omega$, with $\gamma : u^\wedge \to \phi$ and $\gamma' : u'^\wedge \to \phi$,

$$\alpha(v'(\gamma, \gamma')) = v(u'), \quad \beta(v'(\gamma, \gamma')) = v(u),$$

- $w(\phi)$ is a vertex, for each $\phi \in M$,
- if $(i, \phi) \in M'$, we have:

$$\alpha(w'(i, \phi)) = w(\phi), \quad \beta(w'(i, \phi)) = v(\phi(i)).$$

We denote by L the free category generated by (U, β, α) and by r the equivalence relation on $\underline{L}$ satisfying both the condition (P) of Part 1, Proposition 3 and the condition

$$\text{(P')} \begin{cases} (v(\phi(k)), w'(i, \phi)) \sim w'(i', \phi) \\ \text{if} \quad \phi \in M, \quad \phi : K \to \Sigma_\xi, \quad k : i \to i' \text{ in } K. \end{cases}$$

There exists a quotient category $\overline{\Sigma}_\xi$ of L by the smallest equivalence relation compatible on L and containing r. Let $\rho : L \to \overline{\Sigma}_\xi$ be the canonical functor corresponding to r; the map $\underline{\rho}\, v$ defines a neofunctor δ_ξ from Σ_ξ to $\overline{\Sigma}_\xi$.

If $\phi : K \to \Sigma_\xi$ belongs to M, let γ_ϕ be the cone in $\overline{\Sigma}_\xi$ with vertex $\rho(w(\phi))$ and base $\delta_\xi \phi$ such that

$$\gamma_\phi(i) = \rho(w'(i, \phi)) \text{ for any } i \in K_0$$

(it will be called «the formal cone associated to ϕ»). Put:

$$\overline{\Gamma}_\xi = \{ \delta_\xi \gamma \mid \gamma \in \Gamma_\xi \} \cup \{ \gamma_\phi \mid \phi \in M \}.$$

Then $(\overline{\Sigma}_\xi, \overline{\Gamma}_\xi)$ is a cone-bearing (neo)category $\overline{\sigma}_\xi$ and δ_ξ defines a morphism $\overline{\delta}_\xi : \sigma_\xi \to \overline{\sigma}_\xi$.

When $\mathfrak{U}$ is a universe such that $\mathcal{J}$ is a $\mathfrak{U}$-set and $\sigma_\xi \in \mathcal{S}_0''$, the set of neofunctors $\phi : K \to \Sigma_\xi$, where $K \in \mathcal{J}$, is a $\mathfrak{U}$-set, as well as the set of cones in Σ_ξ indexed by elements of $\mathcal{J}$. It follows that M, M' and Ω are $\mathfrak{U}$-sets. Hence $\overline{\sigma}_\xi \in \mathcal{S}_0''$.

b) Let $\mathcal{J}'$ be a set of categories containing $\mathcal{J}$ and σ' a loose $\mathcal{J}'$-type (Σ', Γ'). If $\overline{\psi} = (\sigma', \psi, \sigma_\xi)$ is a morphism, there exists at least one morphism $\overline{\psi}': \overline{\sigma}_\xi \to \sigma'$ such that $\overline{\psi}' \cdot \overline{\delta}_\xi = \overline{\psi}$.

If $\overline{\psi}'$ exists and if $\phi \in M$, then ψ' will transfer the formal cone γ_ϕ into a distinguished cone γ' of σ' admitting $\psi\phi$ as base; since there may be several cones of this kind, ψ' will be defined «up to a choice» of cones γ'.

Hence, for each element $\phi: K \to \Sigma_\xi$ of M, we choose a distinguished cone $\eta_\phi: e_\phi{}^\wedge \to \psi\phi$ of Γ' and we define mappings

- $g: M \to \underline{\Sigma}'$ by $g(\phi) = e_\phi$,
- $g': M' \to \underline{\Sigma}'$ by $g'(i, \phi) = \eta_\phi(i)$,
- $f: \Omega \to \underline{\Sigma}'$ by $f(\gamma, \gamma') = y$, where y is the unique morphism such that $(\psi\gamma)y = \psi\gamma'$.

As in Part 1, Proposition 3, there exists a unique functor $F': L \to \Sigma'$ «extending» ψ, f, g, g', and a unique functor:

$$\psi': \overline{\Sigma}_\xi \to \Sigma' \quad \text{such that} \quad \psi'\rho = F'.$$

Moreover the equivalence relation r is such that ψ' is a $\overline{\sigma}_\xi$-structure in σ'. By construction, ψ' defines the unique morphism $\overline{\psi}': \overline{\sigma}_\xi \to \sigma'$ satisfying the conditions:

$$\overline{\psi}' \cdot \overline{\delta}_\xi = \overline{\psi} \quad \text{and} \quad \overline{\psi}'\gamma_\phi = \eta_\phi \quad \text{for any} \quad \phi \in M.$$

c) If σ' is a loose type, if $\overline{\psi}' = (\sigma', \psi', \overline{\sigma}_\xi)$ and $\overline{\psi}'' = (\sigma', \psi'', \overline{\sigma}_\xi)$ are morphisms and if $\tau: \psi'\delta_\xi \to \psi''\delta_\xi$ is a natural transformation, there exists a unique natural transformation $\tau': \psi' \to \psi''$ such that $\tau'\delta_\xi = \tau$.

Indeed, let us consider the lateral limit-bearing category $\boxminus\sigma' = (\boxminus\Sigma', \hat{\Gamma})$ of quartets of σ'. We identify the objects of $\boxminus\Sigma'$ with the morphisms of Σ'. Since Σ' is a loose type, $\boxminus\sigma'$ is also one. Proposition 2 canonically associates to τ a neofunctor $T: \Sigma_\xi \to \boxminus\Sigma'$ defining a morphism $\overline{T}: \sigma_\xi \to \boxminus\sigma'$. If $\phi: K \to \Sigma_\xi$ belongs to M, the cones $\psi'\gamma_\phi$ and $\psi''\gamma_\phi$ are two distinguished limit-cones with bases $\psi'\delta_\xi\phi$ and $\psi''\delta_\xi\phi$. Since $\tau\phi: \psi'\delta_\xi\phi \to \psi''\delta_\xi\phi$ is a natural transformation, there exists a unique morphism x_ϕ in Σ' such that:

$$(\psi''\gamma_\phi)x_\phi = \tau\phi \boxminus (\psi'\gamma_\phi).$$

By assigning to an object i of K the quartet

$$\hat{\eta}_\phi(i) = (\psi''\gamma_\phi(i), \tau\phi(i), x_\phi, \psi'\gamma_\phi(i)),$$

we define a cone $\hat{\eta}_\phi : x_\phi\hat{} \to T\phi$ in $\boxminus\Sigma'$ which belongs to $\hat{\Gamma}$ (by the definition of $\boxminus\sigma'$). Part b asserts the existence of a unique morphism

$$\overline{T}' = (\boxminus\sigma', T', \overline{\sigma}_\xi) \text{ such that } T'\delta_\xi = T \text{ and } T'\gamma_\phi = \hat{\eta}_\phi$$
$$\text{for any } \phi \in M.$$

Let $\tau' : \theta \to \theta'$ be the natural transformation to which T' is associated; the equality $T'\delta_\xi = T$ implies

$$\tau'\delta_\xi = \tau, \quad \theta\delta_\xi = \psi'\delta_\xi, \quad \theta'\delta_\xi = \psi''\delta_\xi$$

and, for each $\phi \in M$, from $T'\gamma_\phi = \hat{\eta}_\phi$, we deduce

$$\tau'(\rho(w(\phi))) = x_\phi, \quad \theta\gamma_\phi = \psi'\gamma_\phi, \quad \theta'\gamma_\phi = \psi''\gamma_\phi.$$

θ and θ' define morphisms from $\overline{\sigma}_\xi$ to σ'. Hence, using Part b, we get

$$\theta = \psi' \quad \text{and} \quad \theta' = \psi''.$$

Since $\overline{T}'$ and x_ϕ are determined in a unique way, τ' is the unique natural transformation from ψ' to ψ'' satisfying $\tau'\delta_\xi = \tau$. Moreover, if τ is an equivalence, x_ϕ is invertible for every $\phi \in M$, so that τ' is also an equivalence.

2° a) Let λ be the upper bound of the ordinals $\overline{\overline{K}}$, where $K \in \mathcal{J}$, and μ the least regular ordinal greater than λ. We can choose a universe $\mathcal{U}$, such that

$$(\mathcal{J} \cup \underline{\Sigma} \cup \Gamma \cup \bigcup_{K \in \mathcal{J}} \underline{K}) \in \mathcal{U};$$

then $\sigma \in \mathcal{S}''_0$. As in Part 2, Proposition 3, we see that μ is a $\mathcal{U}$-set and we define by transfinite induction a functor $\omega : \langle \mu + 1 \rangle \to \mathcal{S}''$ (whose values are independent of $\mathcal{U}$) satisfying the following conditions, where

$$\omega(\xi) = \sigma_\xi = (\Sigma_\xi, \Gamma_\xi) \quad \text{for any} \quad \xi \leq \mu:$$

- $\omega(0) = \sigma$;
- for each limit ordinal ζ, with $\zeta \leq \mu$, we take for σ_ζ the canonical inductive limit of the functor $\omega_\zeta : \langle \zeta \rangle \to \mathcal{S}''$ restriction of ω, and for $\omega(\zeta, \xi)$ the injection from σ_ξ to σ_ζ, if $\xi < \zeta$.
- If $\zeta = \xi + 1$, where $\xi < \mu$, then σ_ζ is the cone-bearing (neo)cate-

gory $\overline{\sigma_\xi}$ associated to σ_ξ in Part 1 and $\omega(\zeta,\xi)$ is the morphism $\overline{\delta}_\xi$. We write

$$\begin{cases} \overline{\sigma} = \sigma_\mu, \quad \overline{\delta} = (\overline{\sigma}, \delta, \sigma) = \omega(\mu, 0) \\ \omega(\zeta,\xi) = (\sigma_\zeta, \omega_{\zeta\xi}, \sigma_\xi) \text{ when } \xi < \zeta \leqslant \mu . \end{cases}$$

b) Let σ' be a loose $\mathcal{J}'$-type, where $\mathcal{J}'$ contains $\mathcal{J}$, and $\overline{\psi}: \sigma \to \sigma'$ a morphism. Using Part 1-b, we construct by transfinite induction morphisms $\overline{\psi}_\zeta : \sigma_\zeta \to \sigma'$ for each $\zeta \leqslant \mu$, such that

$$\overline{\psi}_0 = \overline{\psi}, \quad \overline{\psi}_\zeta \cdot \omega(\zeta,\xi) = \overline{\psi}_\xi \quad \text{if } \xi < \zeta .$$

In particular $\overline{\psi}_\mu$ is a morphism $\overline{\psi}' : \overline{\sigma} \to \sigma'$ for which $\overline{\psi}' \cdot \overline{\delta} = \overline{\psi}$.

Now, let ψ' and ψ'' be two $\overline{\sigma}$-structures in σ' and $\tau: \psi'\delta \to \psi''\delta$ a natural transformation. Suppose that ζ is an ordinal, $\zeta \leqslant \mu$, and that, for each $\xi < \zeta$, there exists a natural transformation

$$\tau_\xi : \psi'\omega_{\mu\xi} \to \psi''\omega_{\mu\xi} \text{ such that } \tau_\xi \omega_{\xi 0} = \tau \text{ for any } \xi < \zeta .$$

- If $\zeta = \xi + 1$, Part 1-c shows the existence and the unicity of a natural transformation $\tau_\zeta : \psi'\omega_{\mu\zeta} \to \psi''\omega_{\mu\zeta}$ such that $\tau_\zeta\omega_{\zeta\xi} = \tau_\xi$, and so

$$\tau_\zeta \omega_{\zeta 0} = \tau_\xi \omega_{\xi 0} = \tau .$$

- If ζ is a limit ordinal and if $T_\xi : \Sigma_\xi \to \boxminus\Sigma'$ is the neofunctor associated to τ_ξ, for any $\xi < \zeta$, there exists a unique neofunctor

$$T_\zeta : \Sigma_\zeta \to \boxminus\Sigma' \text{ such that } T_\zeta\omega_{\zeta\xi} = T_\xi \text{ for any } \xi < \zeta ,$$

since Σ_ζ is the inductive limit of ω_ζ. Hence the natural transformation corresponding to T_ζ is the unique natural transformation

$$\tau_\zeta : \psi'\omega_{\mu\zeta} \to \psi''\omega_{\mu\zeta} \text{ satisfying } \tau_\zeta\omega_{\zeta\xi} = \tau_\xi \text{ for any } \xi < \zeta .$$

- By transfinite induction, we so define a natural transformation τ_μ, which is the unique natural transformation

$$\tau' : \psi' \to \psi'' \text{ such that } \tau'\delta = \tau .$$

3° We have yet to prove that $\overline{\sigma} = (\overline{\Sigma}, \overline{\Gamma})$ is a loose $\mathcal{J}$-type.

a) We see that $\overline{\Sigma}$ is a category as in Part 3-a Proposition 3. Suppose that $\overline{\gamma}$ is a distinguished cone of $\overline{\Gamma}$. By a method similar to that used in Part 4-a, Proposition 3, we get an ordinal $\zeta < \mu$ and a cone $\gamma \in \Gamma_\zeta$ such that $\overline{\gamma} = \omega_{\mu\zeta}\gamma$, and we still deduce similarly that $\overline{\gamma}$ is a limit-cone.

b) Let $\phi' : K \to \overline{\Sigma}$ be a functor, where $K \in \mathcal{J}$. There exists a cone $\overline{\gamma} \in \overline{\Gamma}$ with base ϕ'.

Indeed, for each $k \in K$, there exists an ordinal $\xi_k < \mu$ and a morphism x_k of Σ_{ξ_k} such that $\phi'(k) = \omega(\mu, \xi_k)(x_k)$. If the composite $k'.k$ is defined in K, the equality $\phi'(k').\phi'(k) = \phi'(k'.k)$ implies the existence of an ordinal $\xi_{k',k}$ greater than ξ_k, $\xi_{k'}$ and $\xi_{k'.k}$ such that $\xi_{k',k} < \mu$ and (1):

$$\omega(\xi_{k',k}, \xi_{k'})(x_{k'}).\omega(\xi_{k',k}, \xi_k)(x_k) = \omega(\xi_{k',k}, \xi_{k'.k})(x_{k'.k}).$$

We denote by ξ the ordinal upper bound of the family of the $\xi_{k',k}$, where (k', k) belongs to the set $K*K$ of composable couples. Since $\overline{\overline{K}} < \mu$, the cardinal of $K*K$ is strictly less than the regular ordinal μ, so that $\xi < \mu$. Put

$$\phi(k) = \omega(\xi, \xi_k)(x_k) \quad \text{for any} \quad k \in K;$$

if the composite $k'.k$ is defined, we get

$$\phi(k').\phi(k) = \phi(k'.k)$$

(by applying $\omega(\xi, \xi_{k',k})$ to (1)); so, we have defined a functor

$$\phi : K \to \Sigma_\xi \quad \text{such that} \quad \omega_{\mu\xi}\phi = \phi'.$$

By construction of $\Sigma_{\xi+1} = \overline{\Sigma}_\xi$ (Part 1), there exists a distinguished cone $\gamma_\phi \in \Gamma_{\xi+1}$ with base $\omega_{\xi+1\,\xi}\phi$. Hence $\omega_{\mu\,\xi+1}\gamma_\phi$ is a cone of $\overline{\Gamma}$, admitting $\omega_{\mu\,\xi}\phi = \phi'$ as its base. ∇

DEFINITION. If $\overline{\sigma} = (\overline{\Sigma}, \overline{\Gamma})$ is a loose type satisfying the conditions of Proposition 4, we call $\overline{\sigma}$ a *loose $\mathcal{J}$-type generated by* σ (or of σ) and $\overline{\Sigma}$ *a loose $\mathcal{J}$-projective completion of* σ.

COROLLARY 1. *Let* σ *be a $\mathcal{J}$-cone-bearing neocategory and* $\overline{\sigma}$ *a loose $\mathcal{J}$-type generated by* σ. *If* σ' *is a loose $\mathcal{J}'$-type, where $\mathcal{J}'$ contains $\mathcal{J}$, the categories* $\mathcal{S}(\sigma', \sigma)$ *and* $\mathcal{S}(\sigma', \overline{\sigma})$ *are equivalent.*

Δ. Let $\overline{\delta} = (\overline{\sigma}, \delta, \sigma)$ be the canonical morphism. We have a functor $F : \mathcal{S}(\sigma', \overline{\sigma}) \to \mathcal{S}(\sigma', \sigma)$ assigning $\tau'\delta$ to the natural transformation τ' between $\overline{\sigma}$-structures in σ'. This functor defines an equivalence. Indeed, for each σ-structure ψ in σ', Proposition 4 asserts the existence of $\bar{\sigma}$-structures ψ' in σ' for which $\psi'\delta = \psi$; choosing one of them, we denote it by $G(\psi)$. If $\tau: \psi \to \theta$ is an element of $\mathcal{S}(\sigma', \sigma)$, there exists a unique natural transformation

$$G(\tau): G(\psi) \to G(\theta) \text{ such that } G(\tau)\delta = \tau$$

(Proposition 4). From the unicity of $G(\tau)$, it results that we define in this way a functor $G: \mathcal{S}(\sigma', \sigma) \to \mathcal{S}(\sigma', \bar{\sigma})$. The equalities

$$FG(\tau) = G(\tau)\delta = \tau, \quad \text{for any } \tau,$$

mean that FG is an identity.

On the other hand, for each $\bar{\sigma}$-structure ψ' in σ', we have

$$GF(\psi')\delta = F(\psi') = \psi'\delta,$$

so that there exists a unique equivalence $\eta(\psi'): \psi' \to GF(\psi')$ for which $\eta(\psi')\delta$ is an identity. If $\tau': \psi' \to \theta'$ is an element of $\mathcal{S}(\sigma', \bar{\sigma})$, we get

$$\eta(\theta') \boxminus \tau' = GF(\tau') \boxminus \eta(\psi'),$$

since

$$(\eta(\theta') \boxminus \tau')\delta = \eta(\theta')\delta \boxminus \tau'\delta = \tau'\delta = F(\tau')$$

and

$$(GF(\tau') \boxminus \eta(\psi'))\delta = GF(\tau')\delta \boxminus \eta(\psi')\delta =$$
$$= GF(\tau')\delta = F(\tau').$$

Hence we have defined an equivalence $\eta: Id_{\mathcal{S}(\sigma', \bar{\sigma})} \to GF$. ∇

COROLLARY 2. *Let σ be a $\mathcal{J}$-cone-bearing neocategory (Σ, Γ) and Σ' a category admitting $\mathcal{J}$-projective limits. Then the category Σ'^{σ} is equivalent to the full subcategory of $\Sigma'^{\bar{\Sigma}}$ whose objects are the functors from $\bar{\Sigma}$ to Σ' which commute with $\mathcal{J}$-projective limits, $\bar{\Sigma}$ denoting a loose $\mathcal{J}$-projective completion of σ.*

Δ. Let us denote by $\bar{\sigma} = (\bar{\Sigma}, \bar{\Gamma})$ a loose $\mathcal{J}$-type generated by σ and let σ' be the full $\mathcal{J}$-limit-bearing category on Σ'; since σ' is a loose type, the categories

$$\Sigma'^{\sigma} = \mathcal{S}(\sigma', \sigma) \quad \text{and} \quad \Sigma'^{\bar{\sigma}} = \mathcal{S}(\sigma', \bar{\sigma})$$

are equivalent, by Corollary 1. If $\psi': \bar{\Sigma} \to \Sigma'$ is a functor, it commutes with

$\mathcal{J}$-projective limits iff each functor from $K \in \mathcal{J}$ to $\overline{\Sigma}$ is the base of a limit-cone γ, where $\psi'\gamma$ is a limit-cone. Hence ψ' is a $\overline{\sigma}$-structure in Σ' iff ψ' commutes with $\mathcal{J}$-projective limits. This means that $\Sigma'^{\overline{\sigma}}$ is the full subcategory of $\Sigma'^{\overline{\Sigma}}$ whose objects are functors commuting with $\mathcal{J}$-projective limits. ∇

COROLLARY 3. *A loose $\mathcal{J}$-projective completion $\overline{\Sigma}$ of a $\mathcal{J}$-cone-bearing neocategory σ is characterized up to an equivalence by the conditions:*

1° $\overline{\Sigma}$ admits $\mathcal{J}$-projective limits.

2° There exists a σ-structure δ in $\overline{\Sigma}$ satisfying the universal property: If Σ' is a category admitting $\mathcal{J}$-projective limits and if ψ is a σ-structure in Σ', there exists a functor ψ', unique up to an equivalence, such that ψ' commutes with $\mathcal{J}$-projective limits and $\psi'\delta = \psi$.

Δ. Condition 2 results from Proposition 4, applied to the full $\mathcal{J}$-limit-bearing category σ' on Σ'. ∇

REMARKS. 1° The construction of the loose $\mathcal{J}$-type $\overline{\sigma} = (\overline{\Sigma}, \overline{\Gamma})$, generated by $\sigma \doteq (\Sigma, \Gamma)$, is yet suggested by the explicit construction of a free $\mathcal{J}$-projective completion of a category (Theorem 7 [E], in which Σ is a category and Γ is void); the difference is that we do not require that there exists only one cone of $\overline{\Gamma}$ with a given base (this problem will be studied in Paragraph 5). Notice that the general Proposition 4 and Corollaries cannot be immediatly deduced from the general existence theorem of free structures. Indeed, if σ' is a loose $\mathcal{J}$-type (Σ', Γ') and if A is a subset of $\underline{\Sigma}'$, there does not exist a «smallest» loose $\mathcal{J}$-type extracted from σ' and containing A.

2° The loose $\mathcal{J}$-type $\overline{\sigma}$ is defined up to an equivalence, and not up to an isomorphism (as the limit-bearing category generated by σ); so, Proposition 4 does not imply the existence of an adjoint for the insertion functor from $\mathcal{L}^{\mathcal{J}}$ to $\mathcal{S}''^{\mathcal{J}}$. In fact, we have proved the following result:

Let $\mathcal{L}^{\mathcal{J}}_{\sim}$ (resp. $\mathcal{S}''^{\mathcal{J}}_{\sim}$) be the quotient category of $\mathcal{L}^{\mathcal{J}}$ (resp. of $\mathcal{S}''^{\mathcal{J}}$) by the equivalence (generated by): $\overline{\psi}$ and $\overline{\psi}'$ are equivalent iff there exists an equivalence between the neofunctors defining them. This category has the same objects as $\mathcal{L}^{\mathcal{J}}$ (resp. as $\mathcal{S}''^{\mathcal{J}}$). From Proposition 4, it results:

COROLLARY 4. *Let σ be a $\mathcal{J}$-cone-bearing neocategory; there exists a loose $\mathcal{J}$-type $\bar{\sigma}$ satisfying the following condition:*

If $\mathcal{U}$ is a universe such that $\mathcal{J}$ is a $\mathcal{U}$-set and $\sigma \in \mathcal{S}_0''$, then $\bar{\sigma}$ is a free structure generated by σ relative to the insertion functor $\mathcal{L}^{\mathcal{J}}_{\sim} \hookrightarrow \mathcal{S}''^{\mathcal{J}}_{\sim}$.

5. Presketches. Prototypes. Sketches.

These are special cone-bearing neocategories and limit-bearing categories. We are going to show that a cone-bearing neocategory generates a presketch and a prototype π. If σ is mapped injectively into π (we then call σ a sketch), the limit-bearing category generated by σ is itself a prototype, isomorphic with π.

DEFINITION. A cone-bearing neocategory (Σ, Γ) is called a (projective) *presketch* if there exists at most one distinguished cone $\gamma \in \Gamma$ with base a given neofunctor ϕ. A limit-bearing category which is a presketch is called a *prototype*.

The cone-bearing neocategory (Σ, Γ) is a presketch iff Γ is the image of a mapping assigning to some neofunctors $\phi: K \to \Sigma$ a cone in Σ with base ϕ. So, the notion of a presketch is equivalent to that used in [E5]. In particular, as in [E5] a prototype «is» a category equipped with a partial choice of projective limit-cones.

$\mathcal{U}$ being a universe, we denote by $\mathcal{S}'$ (resp. by $\mathcal{P}$) the full subcategory of $\mathcal{S}''$ whose objects are the presketches (resp. the prototypes) belonging to $\mathcal{S}''$. It results from [E5] that $\mathcal{S}'$ and $\mathcal{P}$ are closed in $\mathcal{S}''$ for projective limits.

PROPOSITION 5. *Let σ be a cone-bearing neocategory (Σ, Γ). There exists a presketch $\bar{\sigma} = (\bar{\Sigma}, \bar{\Gamma})$, determined up to an isomorphism by the following condition:*

If $\mathcal{U}$ is a universe such that $\sigma \in \mathcal{S}_0''$, then $\bar{\sigma}$ is a free structure generated by σ relative to the insertion functor from $\mathcal{S}'$ to $\mathcal{S}''$.

Δ. We shall construct $\bar{\sigma}$ by transfinite induction, the idea being at each step to «identify» distinguished cones with the same base.

1° Let σ_ξ be a cone-bearing neocategory (Σ_ξ, Γ_ξ). We consider the smallest equivalence relation r on $\underline{\Sigma}_\xi$ such that:

$$(\mathrm{P''})\begin{cases} \gamma(i) \sim \gamma'(i), \text{ for each } i \in K_0, \text{ if } \gamma \text{ and } \gamma' \text{ are two cones of } \Gamma_\xi \\ \qquad\qquad\qquad \text{with the same base, indexed by } K. \end{cases}$$

There exists a canonical quasi-quotient neocategory $\overline{\Sigma}_\xi$ of Σ_ξ by r (it is [E1] the quotient neocategory of Σ_ξ by the smallest equivalence relation containing r and compatible with the law of composition and with the maps source and target of Σ_ξ). Let $\delta_\xi : \Sigma_\xi \to \overline{\Sigma}_\xi$ be the canonical neofunctor and put:

$$\overline{\Gamma}_\xi = \{ \delta_\xi \gamma \mid \gamma \in \Gamma_\xi \}, \quad \overline{\sigma}_\xi = (\overline{\Sigma}_\xi, \overline{\Gamma}_\xi).$$

Then $\overline{\sigma}_\xi$ is a cone-bearing neocategory and δ_ξ defines a morphism $\overline{\delta}_\xi$ from σ_ξ to $\overline{\sigma}_\xi$.

If $\mathfrak{U}$ is a universe, the quotient of a $\mathfrak{U}$-set is a $\mathfrak{U}$-set, so that $\underline{\overline{\Sigma}}_\xi$ is a $\mathfrak{U}$-set when $\underline{\Sigma}_\xi$ is a $\mathfrak{U}$-set; if Γ_ξ is also a $\mathfrak{U}$-set, $\overline{\Gamma}_\xi$ is a $\mathfrak{U}$-set.

If σ' is a presketch (Σ', Γ') and if $\overline{\psi} = (\sigma', \psi, \sigma)$ is a morphism, ψ is compatible with r, and the unique neofunctor

$$\psi' : \overline{\Sigma}_\xi \to \Sigma' \quad \text{such that } \psi' \delta_\xi = \psi$$

defines the unique morphism

$$\overline{\psi}' : \overline{\sigma}_\xi \to \sigma' \quad \text{such that } \overline{\psi}' \cdot \overline{\delta}_\xi = \overline{\psi}.$$

2° Let μ be the smallest regular ordinal such that $\bar{\bar{K}} < \mu$ for each indexing-category of σ. As in Proposition 3, by transfinite induction we construct a functor $\omega : \langle \mu \rangle \to \mathcal{S}''$ satisfying the following properties, where

$$\sigma_\xi = \omega(\xi) = (\Sigma_\xi, \Gamma_\xi) \quad \text{for any } \xi \leqslant \mu :$$

- $\omega(0) = \sigma$;
- $\omega(\zeta)$, for any limit-ordinal $\zeta \leqslant \mu$, is the canonical inductive limit of the functor $\omega_\zeta : \langle \zeta \rangle \to \mathcal{S}''$, restriction of ω, and $\omega(\zeta, \xi) : \sigma_\xi \to \sigma_\zeta$ is the canonical injection;
- σ_ζ, for an ordinal $\zeta = \xi + 1 < \mu$, is the cone-bearing neocategory $\overline{\sigma}_\xi$ associated to σ_ξ in Part 1, and $\omega(\zeta, \xi)$ is the morphism $\overline{\delta}_\xi$ of Part 1.

We denote by $\overline{\sigma} = (\overline{\Sigma}, \overline{\Gamma})$ the neocategory σ_μ thus obtained, and by $\overline{\delta} = (\overline{\sigma}, \delta, \sigma)$ the morphism $\omega(\mu, 0)$. As in Part 3-b, Proposition

3, we see that

$$\Gamma_\xi = \{ \omega(\xi, 0)\gamma \mid \gamma \in \Gamma \} \text{ for any } \xi \leqslant \mu .$$

Let σ' be a presketch and $\overline{\psi}: \sigma \to \sigma'$ a morphism; the universal properties of the inductive limit and of δ_ξ (Part 1) permit to define by transfinite induction a unique sequence of morphisms $\overline{\psi}_\xi : \sigma_\xi \to \sigma'$, $\xi \leqslant \mu$, such that $\overline{\psi}_0 = \overline{\psi}$ and

$$\overline{\psi}_\zeta \cdot \omega(\zeta, \xi) = \overline{\psi}_\xi \quad \text{for} \quad \xi < \zeta \leqslant \mu .$$

3° It remains to prove that $\overline{\sigma}$ is a presketch. We suppose that $\overline{\gamma}$ and $\overline{\gamma}'$ are two distinguished cones of $\overline{\Gamma}$ with the same base. Then there exist cones $\gamma: u^\wedge \to \phi$ and $\gamma': u'^\wedge \to \phi'$ of Γ such that

$$\overline{\gamma} = \delta\gamma \quad \text{and} \quad \overline{\gamma}' = \delta\gamma'.$$

Let K be the indexing-category of γ (and of γ'). For each morphism k of K, the equality $\delta\phi(k) = \delta\phi'(k)$ implies the existence of an ordinal ξ_k such that $\xi_k < \mu$ and

$$\omega(\xi_k, 0)(\phi(k)) = \omega(\xi_k, 0)(\phi'(k)) .$$

If ξ is the ordinal upper bound of the family of the ξ_k, for $k \in K$, we have $\xi < \mu$ (since μ is regular and $\bar{\bar{K}} < \mu$). By construction the cones

$$\omega(\xi, 0)\gamma \quad \text{and} \quad \omega(\xi, 0)\gamma'$$

are distinguished cones of σ_ξ with the same base. Hence they are identified in $\sigma_{\xi+1}$, i. e. we get

$$\omega(\xi+1, 0)\gamma = \omega(\xi+1, 0)\gamma' .$$

Applying $\omega(\mu, \xi+1)$, it follows $\overline{\gamma} = \overline{\gamma}'$. ∇

COROLLARY 1. *The insertion functor from $\mathcal{S}'$ to $\mathcal{S}''$ admits a left adjoint.*

DEFINITION. A presketch $\overline{\sigma}$ satisfying Proposition 5 is called a *presketch generated by σ.*

COROLLARY 2. *Let σ be a cone-bearing neocategory, $\overline{\sigma}$ a presketch generated by σ and σ' a prototype. Then the category $\mathcal{S}(\sigma', \sigma)$ is isomorphic with $\mathcal{S}(\sigma', \overline{\sigma})$.*

Δ. The Proof is similar to that of Corollary 2, Proposition 3. ∇

REMARK. In [F] Proposition 5 is proved more generally for $\mathbf{V}$ -categories, where $\mathbf{V}$ is a monoidal closed category.

PROPOSITION 6. *Let σ be a cone-bearing neocategory (Σ, Γ). There exists a prototype $\bar{\sigma}$ defined up to an isomorphism by the condition:*

If $\mathfrak{U}$ is a universe such that $\sigma \in \mathcal{S}''_0$, then $\bar{\sigma}$ is a free structure generated by σ relative to the insertion functor from $\mathcal{P}$ to $\mathcal{S}''$.

Δ. The prototype $\bar{\sigma}$ will be constructed by transfinite induction, as end of a «tower». The method is similar to that used in Proposition 3, the only difference being that the non-limit step has to be slightly modified in the following way:

Let us suppose that σ_ξ is a cone-bearing neocategory (Σ_ξ, Γ_ξ). As in Part 1, Proposition 3, we consider the set Ω of the «formal factors» (γ, γ'), where $\gamma \in \Gamma_\xi$ and γ' is a cone in Σ_ξ with the same base as γ, the same graph (U, β, α), on the sum U of $\underline{\Sigma}_\xi$ and Ω, and the free category L generated by it. Let r' be the smallest equivalence relation on $\underline{L}$ satisfying the condition $(\hat{\mathrm{P}})$ formed by the condition (P) of Part 1, Proposition 3 and the condition

$$(v\mathrm{P}'')\begin{cases} v(\gamma(i)) \sim v(\gamma'(i)) & \text{for any object } i \text{ of the indexing-category of } \gamma \\ & \text{when } (\gamma, \gamma') \in \Omega \text{ and } \gamma' \in \Gamma_\xi \end{cases}$$

(deduced from the condition (P") of Proposition 5), where $v: \underline{\Sigma}_\xi \to U$ still denotes the canonical injection.

Then $\bar{\Sigma}_\xi$ is the canonical quasi-quotient category of L by r' (we recall [E1] that $\bar{\Sigma}_\xi$ is defined as follows: let L' be the quotient neocategory of L by the smallest compatible equivalence relation containing r' and the free category L'' generated by the graph underlying L'; the category $\bar{\Sigma}_\xi$ is the quotient category of L'' by the smallest compatible equivalence relation such that

$$(x', x) \sim x'.x \quad \text{if} \quad x'.x \text{ is defined in } L'\text{)}.$$

If $\underline{\Sigma}_\xi$ is a $\mathfrak{U}$-set, $\underline{\bar{\Sigma}}_\xi$ is also one.

Apart from this modification (i. e. r' satisfies both (P) and $(v\mathrm{P}'')$, not only (P)), the construction of $\bar{\sigma}$ and of the canonical morphism $\bar{\delta}$ from

σ to $\bar{\sigma}$ is essentially the same as done in Proposition 3; the proof of Proposition 3 may also be copied to prove that each morphism ψ from σ to a prototype is of the form $\overline{\psi'} \cdot \bar{\delta}$. Finally an argument similar to that of Proposition 3 shows that the distinguished cones of $\bar{\sigma}$ are limit-cones, and we prove as in Part 3, Proposition 5, that two distinguished cones admitting the same base are identical. Hence $\bar{\sigma}$ is a prototype. ∇

COROLLARY 1. *The insertion functors from $\mathcal{P}$ to $\mathcal{S}''$ and from $\mathcal{P}$ to $\mathcal{S}'$ admit (left) adjoints. The categories $\mathcal{S}'$ and $\mathcal{P}$ admit $\mathcal{F}_0$-inductive limits. If (Σ, Γ) is a cone-bearing neocategory, there exists a quasi-quotient prototype of it, by an equivalence relation on $\underline{\Sigma}$.*

Δ. The proof is similar to that of Corollary 1, Proposition 3. ∇

DEFINITION. A prototype $\bar{\sigma}$ satisfying Proposition 6 will be called a *prototype generated by* σ. If the canonical morphism $\bar{\delta} : \sigma \to \bar{\sigma}$ is injective, we say that σ is a *sketch*.

COROLLARY 2. *If σ is a cone-bearing neocategory and $\bar{\sigma}$ a prototype generated by σ, for every prototype σ', the category $\mathcal{S}(\sigma', \sigma)$ is isomorphic with $\mathcal{S}(\sigma', \bar{\sigma})$.*

Δ. The proof is similar to that of Corollary 2, Proposition 3. ∇

REMARK. The existence of an adjoint for the insertion functor from $\mathcal{P}$ to $\mathcal{S}'$ is deduced in [E5] from the general existence theorem for free structures. This fact is generalized in [F] for **V**-categories, where **V** is a monoidal closed category. Sketches are introduced in [E5]. Naturally each prototype is also a sketch, and every sketch σ generates a prototype of which σ is a subsketch.

PROPOSITION 7. *Let σ be a sketch, $\bar{\sigma}$ a limit-bearing category generated by σ and π a prototype generated by σ. Then $\bar{\sigma}$ and π are isomorphic.*

Δ. Let us denote by

$$\bar{\delta} = (\bar{\sigma}, \delta, \sigma) \quad \text{and} \quad \bar{\Pi} = (\pi, \Pi, \sigma)$$

the canonical morphisms. Since π is a fortiori a limit-bearing category, it exists a unique morphism

$$\overline{\Pi}' = (\pi, \Pi', \overline{\sigma}) \text{ such that } \overline{\Pi}' \cdot \overline{\delta} = \overline{\Pi}$$

(this is valid even is σ is not a sketch). If $\overline{\sigma}$ is also a prototype, then there exists a unique morphism

$$\overline{\delta}' : \pi \to \overline{\sigma} \text{ such that } \overline{\delta}' \cdot \overline{\Pi} = \overline{\delta},$$

and, from the equalities

$$\overline{\delta}' \cdot \overline{\Pi}' \cdot \overline{\delta} = \overline{\delta} \text{ and } \overline{\Pi}' \cdot \overline{\delta}' \cdot \overline{\Pi} = \overline{\Pi},$$

we deduce that $\overline{\delta}'$ is an isomorphism, whose inverse is $\overline{\Pi}'$. Hence Proposition 7 will be proved if we show that $\overline{\sigma} = (\overline{\Sigma}, \overline{\Gamma})$ is a prototype, when σ is a sketch.

Indeed, let $\overline{\gamma}$ and $\overline{\gamma}'$ be two distinguished cones of $\overline{\Gamma}$ with the same base ϕ'. Since

$$\overline{\Gamma} = \{ \delta\gamma \mid \gamma \in \Gamma \}$$

(Proposition 3), there exist cones γ and γ' of Γ such that $\overline{\gamma} = \delta\gamma$ and $\overline{\gamma}' = \delta\gamma'$. The cones $\Pi\gamma$ and $\Pi\gamma'$ are distinguished cones of the prototype π; as $\overline{\Pi} = \overline{\Pi}' \cdot \overline{\delta}$, we get

$$\Pi\gamma = \Pi'\overline{\gamma} \text{ and } \Pi\gamma' = \Pi'\overline{\gamma}',$$

so that $\Pi\gamma$ and $\Pi\gamma'$ have the same base $\Pi'\phi'$. Hence $\Pi\gamma = \Pi\gamma'$. The injectivity of Π implies $\gamma = \gamma'$ and, therefore, $\overline{\gamma} = \overline{\gamma}'$. ∇

We denote by $\mathcal{S}$ the full subcategory of $\mathcal{S}'$ whose objects are the sketches $\sigma \in \mathcal{S}''_0$.

PROPOSITION 8. *Let σ be a cone-bearing neocategory (Σ, Γ) and let $\overline{\Pi} = (\pi, \Pi, \sigma)$ be the canonical morphism from σ to a prototype $\pi = (\overline{\Sigma}, \overline{\Gamma})$ generated by σ. The presketch $\overline{\overline{\sigma}}$ image of σ by Π is a sketch, characterized up to an isomorphism by the condition:*

If $\mathcal{U}$ is a universe such that $\sigma \in \mathcal{S}''_0$, then $\overline{\overline{\sigma}}$ is a free structure generated by σ relative to the insertion functor from $\mathcal{S}$ to $\mathcal{S}''$.

Δ. We denote by $\overline{\overline{\Sigma}}$ the sub-neocategory of $\overline{\Sigma}$ defined by the set $\Pi(\underline{\Sigma})$ and by $\eta : \overline{\overline{\Sigma}} \to \overline{\Sigma}$ the insertion neofunctor. Let $\Pi' : \Sigma \to \overline{\overline{\Sigma}}$ be the neofunctor restriction of Π and $\overline{\overline{\Gamma}}$ the set of cones $\Pi'\gamma$, where $\gamma \in \Gamma$. Then, $\overline{\overline{\sigma}} = (\overline{\overline{\Sigma}}, \overline{\overline{\Gamma}})$ is a cone-bearing neocategory, Π' defines a morphism $\overline{\Pi}'$

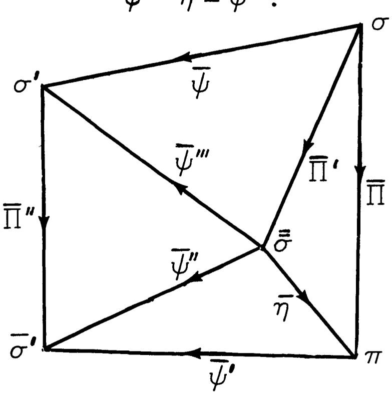
from σ to $\bar{\bar{\sigma}}$ and η defines a morphism $\bar{\eta}: \bar{\bar{\sigma}} \to \pi$. Moreover $\bar{\eta} \cdot \bar{\Pi}' = \bar{\Pi}$.

2° We are going to prove that π is also a prototype generated by $\bar{\bar{\sigma}}$, the canonical morphism being $\bar{\eta}$; it will follow that $\bar{\bar{\sigma}}$ is a sketch η being injective. Indeed, let $\bar{\sigma}'$ be a prototype and $\bar{\psi}'': \bar{\bar{\sigma}} \to \bar{\sigma}'$ a morphism. By definition of π, there exists a unique morphism

$$\bar{\psi}' : \pi \to \bar{\sigma}' \quad \text{such that} \quad \bar{\psi}' \cdot \bar{\Pi} = \bar{\psi}'' \cdot \bar{\Pi}' ;$$

this equality may also be written $\bar{\psi}' \cdot \bar{\eta} \cdot \bar{\Pi}' = \bar{\psi}'' \cdot \bar{\Pi}'$ and, $\bar{\Pi}'$ being surjective, it follows that $\bar{\psi}'$ is also the unique morphism satisfying

$$\bar{\psi}' \cdot \bar{\eta} = \bar{\psi}'' .$$



3° Let σ' be a sketch (Σ', Γ') and $\bar{\psi}: \sigma \to \sigma'$ a morphism. It remains to exhibit a morphism

$$\bar{\psi}''': \bar{\bar{\sigma}} \to \sigma' \quad \text{such that} \quad \bar{\psi}''' \cdot \bar{\Pi}' = \bar{\psi};$$

the surjectivity of $\bar{\Pi}'$ will imply the unicity of such a morphism. Indeed, the canonical morphism $\bar{\Pi}'' = (\bar{\sigma}', \Pi'', \sigma')$ from σ' to a prototype $(\bar{\Sigma}', \bar{\Gamma}')$ generated by σ' is injective, σ' being a sketch. As π is a prototype generated by σ, there exists a unique morphism $\bar{\psi}' = (\bar{\sigma}', \psi', \pi)$ such that

$$\bar{\Pi}'' \cdot \bar{\psi} = \bar{\psi}' \cdot \bar{\Pi} = \bar{\psi}' \cdot \bar{\eta} \cdot \bar{\Pi}' .$$

As $\bar{\psi}' \bar{\eta}$ maps $\bar{\bar{\Sigma}} = \Pi'(\underline{\Sigma})$ into $\Pi''(\underline{\Sigma}')$ and as Π'' is injective, there is a unique neofunctor

$$\psi''': \bar{\bar{\Sigma}} \to \Sigma' \quad \text{such that} \quad \Pi'' \psi''' = \psi' \eta ;$$

it satisfies $\psi''' \Pi' = \psi$, since Π'' is injective and

$$\Pi'' \psi''' \Pi' = \psi' \eta \Pi' = \psi' \Pi = \Pi'' \psi .$$

If $\bar{\bar{\gamma}} \in \bar{\bar{\Gamma}}$, we have $\bar{\bar{\gamma}} = \Pi' \gamma$ for some $\gamma \in \Gamma$; from the equality

$$\Pi'' \psi''' \bar{\bar{\gamma}} = \Pi'' \psi''' \Pi' \gamma = \Pi'' \psi \gamma ,$$

we deduce $\Pi'' \psi''' \bar{\bar{\gamma}} \in \bar{\Gamma}'$, the neofunctor $\Pi'' \psi$ defining a morphism. Now, Π'' is injective and $\bar{\Gamma}'$ is formed by the cones $\Pi'' \gamma'$, where $\gamma' \in \Gamma'$. Hence, $\psi''' \bar{\bar{\gamma}} \in \Gamma'$.

So, ψ''' defines the unique morphism

$$\bar{\psi}''' : \bar{\bar{\sigma}} \to \sigma' \quad \text{such that} \quad \bar{\psi}''' \cdot \bar{\Pi}' = \bar{\psi} . \quad \nabla$$

COROLLARY 1. *The insertion functors from* $\mathcal{S}$ *to* $\mathcal{S}'$ *and* $\mathcal{S}''$ *admit left adjoints;* $\mathcal{S}$ *admits* $\mathcal{F}_0$*-inductive limits.* ∇

DEFINITION. A sketch $\bar{\bar{\sigma}}$ satisfying the condition of Proposition 8 is called a *sketch generated by* σ.

COROLLARY 2. *Let* σ *be a cone-bearing neocategory,* $\bar{\bar{\sigma}}$ *a sketch generated by* σ *and* σ' *a prototype. The categories* $\mathcal{S}(\sigma', \sigma)$ *and* $\mathcal{S}(\sigma', \bar{\bar{\sigma}})$ *are isomorphic.*

Δ. The proofs of these corollaries are similar to that of Corollaries 1 and 2, Proposition 3. ∇

6. Types.

A loose type which is a presketch will be called a type. We are going to show that each $\mathcal{I}$-cone-bearing neocategory σ generates a $\mathcal{I}$-type τ which is defined up to an isomorphism (and not only up to an equivalence, as the loose $\mathcal{I}$-type $\bar{\sigma}$ generated by σ). Moreover τ is equivalent to $\bar{\sigma}$, when σ is a sketch.

We still denote by $\mathcal{I}$ a given set of categories.

DEFINITION. A $\mathcal{I}$-cone-bearing neocategory which is a presketch (resp. a sketch, or a prototype) will be called a $\mathcal{I}$*-presketch* (resp. a $\mathcal{I}$*-sketch* or a $\mathcal{I}$*-prototype*). A loose $\mathcal{I}$-type which is a presketch is called a $\mathcal{I}$*-type*.

A $\mathcal{I}$-type $\sigma = (\Sigma, \Gamma)$ may be identified with a category Σ admitting $\mathcal{J}$-projective limits, equipped with a choice of a limit-cone with base ϕ for each functor $\phi: K \to \Sigma$, where $K \in \mathcal{J}$ (i.e. with a $\mathcal{J}$-type as defined in [E5]). If $\mathcal{J}$ is a $\mathfrak{U}$-set, denote by $\mathcal{S}'^{\mathcal{J}}$, $\mathcal{S}^{\mathcal{J}}$, $\mathcal{P}^{\mathcal{J}}$ and $\mathcal{F}^{\mathcal{J}}$ the full sub-categories of $\mathcal{S}''^{\mathcal{J}}$ whose objects are respectively the $\mathcal{J}$-presketches, the $\mathcal{J}$-sketches, the $\mathcal{J}$-prototypes and the $\mathcal{J}$-types belonging to $\mathcal{S}''_0$.

PROPOSITION 9. *Let σ be a $\mathcal{J}$-cone-bearing neocategory. There exists a $\mathcal{J}$-type $\overline{\sigma}$ characterized up to an isomorphism by the condition:*

If $\mathfrak{U}$ is a universe such that $\mathcal{J}$ is a $\mathfrak{U}$-set and $\sigma \in \mathcal{S}''_0$, then $\overline{\sigma}$ is a free structure generated by σ relative to the insertion functor $\mathcal{F}^{\mathcal{J}} \hookrightarrow \mathcal{S}''^{\mathcal{J}}$.

Δ. The construction of $\overline{\sigma}$ is obtained by modifying the construction of the loose $\mathcal{J}$-type generated by σ (Proposition 4) in a way similar to that used to deduce in Proposition 6 the construction of the prototype from Proposition 3. In fact, we have only to modify the transition from σ_ξ to $\sigma_{\xi+1}$ by also identifying two distinguished cones with the same base. More precisely:

1° If σ_ξ is a cone-bearing neocategory (Σ_ξ, Γ_ξ), we define as in Part 1, Proposition 4, the graph (U, β, α) and the free category L it generates. But now we denote by $\overline{\Sigma}_\xi$ the canonical quasi-quotient category of L by the equivalence relation satisfying not only conditions (P) and (P') as in Proposition 4, but also the condition (νP") of Proposition 6. After this modification,

$$\overline{\Gamma}_\xi \,,\; \overline{\sigma}_\xi \text{ and } \overline{\delta}_\xi = (\overline{\sigma}_\xi, \delta_\xi, \sigma_\xi)$$

are defined formally as in Part 1, Proposition 4.

Now, let $\mathcal{J}'$ be a set of categories containing $\mathcal{J}$, let σ' be a $\mathcal{J}'$-type and $\overline{\psi} = (\sigma', \psi, \sigma_\xi)$ be a morphism. For each functor $\phi: K \to \Sigma_\xi$, where $K \in \mathcal{J}$, there exists one and only one cone $\eta_\phi \in \Gamma'$ with base $\psi\phi$. Hence, by the method of Part 1-b, Proposition 4, we get one and only one morphism

$$\overline{\psi}': \overline{\sigma}_\xi \to \sigma' \text{ such that } \overline{\psi}' \cdot \overline{\delta}_\xi = \overline{\psi}$$

(while in Proposition 4 the morphism $\overline{\psi}'$ was only defined up to an equivalence, the choice of η_ϕ being not unique).

2° By transfinite induction, exactly as in Proposition 4:

a) we construct a functor $\omega : < \mu + 1 > \rightarrow \mathcal{S}''$, where μ is yet the least regular ordinal such that $\bar{\bar{K}} < \mu$, for each $K \in \mathcal{J}$;

b) putting $\bar{\sigma} = \omega(\mu)$ and $\bar{\delta} = \omega(\mu, 0)$, we prove that $\bar{\sigma}$ is a loose $\mathcal{J}$-type;

c) using the last statement of Part 1, we show that, if $\bar{\psi} : \sigma \rightarrow \sigma'$ is a morphism from σ to a $\mathcal{J}'$-type, where $\mathcal{J}'$ contains $\mathcal{J}$, there exists a unique morphism

$$\bar{\psi}' : \bar{\sigma} \rightarrow \sigma' \quad \text{satisfying} \quad \bar{\psi}' \cdot \bar{\delta} = \bar{\psi} .$$

Finally, we see that $\bar{\sigma}$ is also a presketch (and therefore a $\mathcal{J}$-type), by an argument similar to that used in Part 3, Proposition 5. ∇

COROLLARY 1. *Let $\mathcal{J}$ be a $\mathfrak{U}$-set; the insertion functors from $\mathcal{F}^{\mathcal{J}}$ to $\mathcal{S}'^{\mathcal{J}}$, to $\mathcal{S}''^{\mathcal{J}}$, to $\mathcal{S}^{\mathcal{J}}$, to $\mathcal{P}^{\mathcal{J}}$ and to $\mathcal{L}^{\mathcal{J}}$ admit left adjoints. $\mathcal{F}^{\mathcal{J}}$ admits $\mathcal{F}_0$-inductive limits. There exists a quasi-quotient $\mathcal{J}$-type of a $\mathcal{J}$-cone-bearing neocategory (Σ, Γ) by an equivalence relation on $\underline{\Sigma}$.*

Δ. The proof is similar to that of Corollary 1, Proposition 3. ∇

DEFINITION. A $\mathcal{J}$-type $\bar{\sigma}$ satisfying the condition of Proposition 8 is called a *$\mathcal{J}$-type generated by σ*.

COROLLARY 2. *Let σ be a $\mathcal{J}$-cone-bearing neocategory and $\bar{\sigma}$ a $\mathcal{J}$-type generated by σ. If σ' is $\mathcal{J}'$-type, where $\mathcal{J}'$ contains $\mathcal{J}$, the categories $\mathcal{S}(\sigma', \sigma)$ and $\mathcal{S}(\sigma', \bar{\sigma})$ are isomorphic.*

Δ. The proof is similar to that of Corollary 2, Proposition 3. ∇

REMARKS. In [E5] Proposition 9 is deduced from the existence theorem for free structures. The explicit construction of $\bar{\sigma}$ given here generalizes that of Theorem 7 [E] (where Γ is supposed void). Proposition 9 may be extended for **V**-categories (see [F]).

PROPOSITION 10. *Let σ be a $\mathcal{J}$-presketch (Σ, Γ),*

$$\bar{\delta} = (\bar{\sigma}, \delta, \sigma) \quad \text{and} \quad \bar{\theta} = (\tau, \theta, \sigma)$$

the canonical morphisms from σ to a loose $\mathcal{J}$-type $\bar{\sigma} = (\bar{\Sigma}, \bar{\Gamma})$ and to a $\mathcal{J}$-type τ generated by σ. Then the following conditions are equivalent:

1° σ is a sketch;

2° δ is injective;

3° θ is injective.

If they are satisfied, τ and $\overline{\sigma}$ are equivalent.

Δ. 1° Notice that, to prove the injectivity of δ, it is sufficient to exhibit an injective σ-structure ψ in a loose $\mathfrak{I}$-type σ'; indeed, there exists then a neofunctor ψ' defining a morphism from $\overline{\sigma}$ to σ' which satisfies $\psi'\delta = \psi$; this equality implies the injectivity of δ when ψ is injective. In particular, if θ is injective, δ is also injective, for θ is a σ-structure in the (loose) $\mathfrak{I}$-type τ. Similarly, θ will be injective as soon as there exists an injective σ-structure in a $\mathfrak{I}$-type.

We denote by π a prototype $(\hat{\Sigma}, \hat{\Gamma})$ generated by σ and by $\overline{\overline{\Pi}}$ the canonical morphism from σ to π.

a) If θ is injective, then σ is a sketch. Indeed, since τ is also a prototype, there exists a unique morphism

$$\overline{\overline{\Pi}}': \pi \to \tau \text{ such that } \overline{\overline{\Pi}}' \cdot \overline{\overline{\Pi}} = \overline{\theta}.$$

$\overline{\theta}$ being injective, $\overline{\overline{\Pi}}$ is injective, i. e. σ is a sketch.

b) Supposing σ is a sketch, we now prove the injectivity of δ. Let $\mathfrak{U}$ be a universe to which belong $\underline{K}$, for any $K \in \mathfrak{I}$, and $u'.\hat{\Sigma}.u$, for any pair (u', u) of objects of $\hat{\Sigma}$. The category $\mathfrak{M}$ of maps between $\mathfrak{U}$-sets admits then $\mathfrak{I}$-projective limits, so that the category $\Sigma' = \mathfrak{M}^{\hat{\Sigma}^*}$ of natural transformations, where $\hat{\Sigma}^*$ is the dual of $\hat{\Sigma}$, admits $\mathfrak{I}$-projective limits. Hence the full $\mathfrak{I}$-limit-bearing category σ' on Σ' is a loose $\mathfrak{I}$-type. If we consider the Yoneda immersion Y from $\hat{\Sigma}$ to Σ', it is injective and it commutes with projective limits; so Y defines a morphism $\overline{Y}: \pi \to \sigma'$. A fortiori, $\overline{Y} \cdot \overline{\overline{\Pi}}: \sigma \to \sigma'$ is a morphism from σ to a loose $\mathfrak{I}$-type and, $\overline{\overline{\Pi}}$ being injective by definition of a sketch, $\overline{Y} \cdot \overline{\overline{\Pi}}$ is injective. From the initial remark, we deduce that δ is also injective.

2° We have yet to show that, if δ is injective, θ is injective and τ is equivalent to $\overline{\sigma}$. For this, we will use the following result:

a) *Let σ' be a loose $\mathfrak{I}$-type (Σ', Γ') and ψ an injective σ-structure* *in* σ'. *Then there exists a subset* Γ'' *of* Γ' *such that* (Σ'', Γ'') *is a* $\mathcal{G}$-*type* σ'' *and that* ψ *defines also a morphism* $\overline{\psi}: \sigma \to \sigma''$.

Indeed, let $\phi' : K \to \Sigma'$ be a functor, where $K \in \mathcal{G}$. Since ψ is injective, there is at most one neofunctor

$$\phi: K \to \Sigma \quad \text{such that} \quad \psi\,\phi = \phi'$$

and, σ being a presketch, there exists at most one distinguished cone $\gamma \in \Gamma$ with ϕ as its base; hence there is at most one cone $\gamma \in \Gamma$ such that ϕ' is the base of $\psi\gamma \in \Gamma'$. If such a cone γ exists, we denote the cone $\psi\gamma$ by $\gamma_{\phi'}$; otherwise, we choose one cone $\gamma' \in \Gamma'$ with ϕ' as its base, and we denote it by $\gamma_{\phi'}$. The set Γ'' of cones

$$\gamma_{\phi'}\ , \ \text{where} \ \ \phi': K \to \Sigma' \ \ \text{and} \ \ K \in \mathcal{G},$$

is a subset of Γ', and (Σ', Γ'') is a $\mathcal{G}$-type σ''; by construction, ψ defines a morphism from σ to σ''.

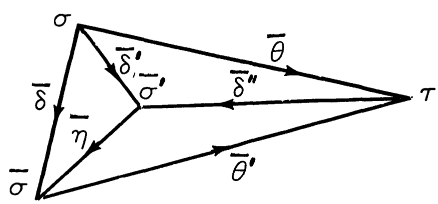
b) We suppose now that δ is injective. Part a applied to $\overline{\delta}: \sigma \to \overline{\sigma}$ asserts the existence of a $\mathcal{G}$-type $\overline{\sigma}' = (\overline{\Sigma}, \overline{\Gamma}')$ such that $\overline{\Gamma}'$ is a subset of $\overline{\Gamma}$ and that δ defines a morphism $\overline{\delta}': \sigma \to \overline{\sigma}'$. By definition of the $\mathcal{G}$-type generated by σ, there exists a unique morphism $\overline{\delta}'' = (\overline{\sigma}', \delta'', \tau)$ satisfying $\overline{\delta}'' \cdot \overline{\theta} = \overline{\delta}'$. This implies the injectivity of θ.

The identity of $\overline{\Sigma}$ defines a morphism $\overline{\eta}: \overline{\sigma}' \to \overline{\sigma}$ and we have: $\overline{\eta} \cdot \overline{\delta}' = \overline{\delta}$. There exists a morphism $\overline{\theta}' = (\tau, \theta', \overline{\sigma})$ such that $\overline{\theta}' \cdot \overline{\delta} = \overline{\theta}$. From the equalities

$$\overline{\eta} \cdot \overline{\delta}'' \cdot \overline{\theta}' \cdot \overline{\delta} = \overline{\eta} \cdot \overline{\delta}'' \cdot \overline{\theta} = \overline{\eta} \cdot \overline{\delta}' = \overline{\delta}\,,$$

it follows (Proposition 4, condition 2) that the functor $\delta''\theta'$ which defines the morphism $\overline{\eta} \cdot \overline{\delta}'' \cdot \overline{\theta}' : \overline{\sigma} \to \overline{\sigma}$ is equivalent to the identity of $\overline{\Sigma}$. On the other hand, the equalities

$$\overline{\theta}' \cdot \overline{\eta} \cdot \overline{\delta}'' \cdot \overline{\theta} = \overline{\theta}' \cdot \overline{\eta} \cdot \overline{\delta}' = \overline{\theta}' \cdot \overline{\delta} = \overline{\theta}$$



imply that the functor $\theta' \delta''$ defining the morphism $\overline{\theta}' \cdot \overline{\eta} \cdot \overline{\delta}'': \tau \to \tau$ is an identity. Hence, $\overline{\theta}'$ defines an equivalence from $\overline{\sigma}$ to τ. ∇

COROLLARY. *Let σ be a $\mathcal{I}$-prototype (Σ, Γ). The canonical morphism $\overline{\theta}: \sigma \to \tau$ from σ to a $\mathcal{I}$-type $\tau = (\overline{\Sigma}, \overline{\Gamma})$ generated by σ is injective. Moreover, $\overline{\Sigma}$ is a loose $\mathcal{I}$-projective completion of σ.* ∇

REMARK. The injectivity of $\overline{\theta}$ was shown in Theorem 6 of [E] .

II. MIXED LIMIT-BEARING CATEGORIES

7. Mixed sketches and mixed types.

Up to now, we have always considered neocategories equipped with projective cones. Dually, we could deduce similar results for neocategories Σ equipped with a set of inductive cones (since this is equivalent with equipping the dual of Σ with projective cones). In this paragraph, we will generalize all the preceding results to the case where the neocategory is equipped with both projective cones and inductive cones.

We denote by $\mathcal{I}$ and $\mathcal{J}$ two sets of categories.

DEFINITIONS. 1° *A mixed cone-bearing neocategory* (resp. *category*) is a triple (Σ, Γ, ∇), where Σ is a neocategory (resp. a category), Γ a set of projective cones in Σ indexed by categories and ∇ a set of inductive cones in Σ indexed by categories. We say more precisely that (Σ, Γ, ∇) is a $(\mathcal{I}, \mathcal{J})$-*cone-bearing neocategory* if the indexing-category of each γ of Γ belongs to $\mathcal{I}$ and that of each $\kappa \in \nabla$ belongs to $\mathcal{J}$.

2° If moreover Σ is a category, if Γ is a set of projective limit-cones and ∇ a set of inductive limit-cones, then (Σ, Γ, ∇) is called a *mixed limit-bearing category* (or, more precisely, a $(\mathcal{I}, \mathcal{J})$-*limit-bearing category).*

3° A $(\mathcal{I}, \mathcal{J})$-limit-bearing category (Σ, Γ, ∇) is called a (mixed) *loose* $(\mathcal{I}, \mathcal{J})$-*type* if each functor $\phi: K \to \Sigma$, where $K \in \mathcal{I}$ (resp. where $K \in \mathcal{J}$) is the base of at least one cone $\gamma \in \Gamma$ (resp. of at least one cone $\kappa \in \nabla$).

4° A $(\mathcal{I}, \mathcal{J})$-cone-bearing neocategory $(\Sigma, \Gamma . \nabla)$ is called a (mixed)

$(\mathcal{I},\mathcal{J})$*-presketch* if two different cones of Γ (resp. of ∇) have different bases. A mixed presketch which is a mixed limit-bearing category (resp. a loose $(\mathcal{I},\mathcal{J})$-type) is called a *mixed prototype* (resp. a $(\mathcal{I},\mathcal{J})$*-type*).

5° A *morphism between mixed cone-bearing neocategories* is a triple (σ',ψ,σ), where

$$\sigma = (\Sigma,\Gamma,\nabla) \quad \text{and} \quad \sigma' = (\Sigma',\Gamma',\nabla')$$

are mixed cone-bearing neocategories and $\psi: \Sigma \to \Sigma'$ is a neofunctor such that

$$\psi\gamma \in \Gamma' \text{ for any } \gamma \in \Gamma, \quad \psi\kappa \in \nabla' \text{ for any } \kappa \in \nabla .$$

6° Let σ be a mixed cone-bearing neocategory and $\sigma' = (\Sigma',\Gamma',\nabla')$ a mixed cone-bearing category. A neofunctor ψ defining a morphism $\overline{\psi}$ from σ to σ' (still denoted by $\overline{\psi}: \sigma \to \sigma'$) is called a σ*-structure in* σ'. We denote by $\mathcal{S}(\sigma',\sigma)$ the full subcategory of Σ'^{Σ} formed by the natural transformations between σ-structures in σ'.

EXAMPLES. 1° Let Σ' be a category. The *full* $(\mathcal{I},\mathcal{J})$*-limit-bearing category on* Σ' is the triple $(\Sigma',\Gamma',\nabla') = \sigma'$, where Γ' is the set of all the projective limit-cones in Σ' indexed by a category $K \in \mathcal{I}$ and ∇' the set of all the inductive limit-cones in Σ' indexed by a $K \in \mathcal{J}$. If σ is a mixed cone-bearing neocategory, a σ-structure ψ in σ' is called a σ*-structure in* Σ', and $\mathcal{S}(\sigma',\sigma)$ is then denoted by $\mathcal{S}(\Sigma',\sigma)$, or by Σ'^{σ}.

2° Let K be a category and $\sigma = (\Sigma,\Gamma,\nabla)$ a mixed cone-bearing category. We denote by σ^K the mixed cone-bearing category $(\Sigma^K,\overline{\Gamma},\overline{\nabla})$, where $\overline{\Gamma}$ is defined as in Example 2-2 and $\overline{\nabla}$ is defined dually from ∇. When σ is a mixed limit-bearing category, such is σ^K. If K is the category $\mathbf{2}$, as in Example 2-2, we deduce from $\sigma^{\mathbf{2}}$ the *longitudinal mixed cone-bearing category* $\square\square\sigma$ *of quartets of* σ and the *lateral mixed cone-bearing category* $\boxminus\sigma$ *of quartets of* σ (they are mixed limit-bearing categories when such is σ).

PROPOSITION 11. *Let σ be a mixed cone-bearing neocategory and σ' a mixed cone-bearing category. There is a canonical bijection from the set of morphisms of the category $\mathcal{S}(\sigma',\sigma)$ onto $\mathcal{S}(\boxminus\sigma',\sigma)_0$.*

Δ. The proof is similar to that of Proposition 2. ∇

Let $\mathcal{U}$ be a universe. We denote by:

- $\mathcal{S}m''_0$ the set of mixed cone-bearing neocategories (Σ, Γ, ∇) such that $\underline{\Sigma}$, Γ, ∇ are $\mathcal{U}$-sets, as well as $\underline{K}$, for any indexing category K of a cone γ of Γ or ∇.
- $\mathcal{S}m''$ the category of morphisms between elements of $\mathcal{S}m''_0$.
- $q_{\mathcal{S}m''} : \mathcal{S}m'' \to \mathcal{F}'$ the functor associating ψ to (σ', ψ, σ).
- $\mathcal{P}m'$, $\mathcal{S}m'$, $\mathcal{P}m$ the full subcategories of $\mathcal{S}m''$ whose objects are those $\sigma \in \mathcal{S}m''_0$ which are respectively mixed limit-bearing categories, mixed pre-sketches and mixed prototypes.
- $\mathcal{S}''^{\mathcal{I}\mathcal{J}}$, $\mathcal{L}^{\mathcal{I}\mathcal{J}}$ and $\mathcal{F}^{\mathcal{I}\mathcal{J}}$, if $\mathcal{I}$ and $\mathcal{J}$ are $\mathcal{U}$-sets, the full subcategories of $\mathcal{S}m''$ whose objects are those $\sigma \in \mathcal{S}m''_0$ which are respectively $(\mathcal{I}, \mathcal{J})$-cone-bearing neocategories, loose $(\mathcal{I}, \mathcal{J})$-types and $(\mathcal{I}, \mathcal{J})$-types.
- $\mathcal{S}''^{\mathcal{I}\mathcal{J}}_{\sim}$ and $\mathcal{L}^{\mathcal{I}\mathcal{J}}_{\sim}$ the quotient categories of $\mathcal{S}''^{\mathcal{I}\mathcal{J}}$ and $\mathcal{L}^{\mathcal{I}\mathcal{J}}$ by the equivalence relation generated by:

$$(\sigma', \psi, \sigma) \sim (\sigma', \psi', \sigma) \text{ iff there exists an equivalence } \eta : \psi \to \psi'.$$

The category $\mathcal{S}''$ may be identified with the full subcategory of $\mathcal{S}m''$, whose objects are those $(\Sigma, \Gamma, \nabla) \in \mathcal{S}m''_0$ such that ∇ is void; similarly $\mathcal{P}'$, $\mathcal{S}'$ and $\mathcal{P}$ may be identified with subcategories of $\mathcal{P}m'$, $\mathcal{S}m'$ and $\mathcal{P}m$. The categories $\mathcal{L}^{\mathcal{I}}$ and $\mathcal{F}^{\mathcal{I}}$ will be identified with $\mathcal{L}^{\mathcal{I}\mathcal{J}}$ and $\mathcal{F}^{\mathcal{I}\mathcal{J}}$ corresponding to the case where the set $\mathcal{J}$ is void.

We also obtain the analogous categories of morphisms between inductive cone-bearing neocategories as subcategories of $\mathcal{S}m''$.

PROPOSITION 12. *$\mathcal{S}m''$ admits $\mathcal{F}_0$-projective limits and $\mathcal{F}_0$-inductive limits; $q_{\mathcal{S}m''}$ commutes with projective limits and with inductive limits. The categories $\mathcal{P}m'$, $\mathcal{S}m'$, $\mathcal{P}m$ are closed for projective limits in $\mathcal{S}m''$, as well as $\mathcal{F}^{\mathcal{I}\mathcal{J}}$, when $\mathcal{I}$ and $\mathcal{J}$ are $\mathcal{U}$-sets.*

Δ. The proof is similar to that of Proposition 1. The distinguished projective cones on the limit are defined as in Proposition 1, while the distinguished inductive cones are defined dually. ∇

PROPOSITION 13. *Let σ be a mixed cone-bearing neocategory. There exist:*

a mixed limit-bearing category $\overline{\sigma}$,

a mixed presketch π' ,

a mixed prototype π,

characterized up to an isomorphism by the condition:

If $\mathcal{U}$ *is a universe such that* σ *belongs to* $\mathcal{S}m''_0$, *then* $\overline{\sigma}$, π' *and* π *are free structures generated by* σ *relative to the insertion functors toward* $\mathcal{S}m''$ *from respectively* $\mathcal{P}_{m}'$, $\mathcal{S}m'$ *and* $\mathcal{P}_m$.

Δ. Let μ be the least regular ordinal greater than $\overline{\overline{K}}$, for any category K indexing either a cone of Γ or a cone of ∇. We construct $\overline{\sigma}$ (resp. π', resp. π) by transfinite induction, as the end of a tower of mixed cone-bearing neocategories σ_ξ, for $\xi < \mu$, as in Proposition 3 (resp. 5, resp. 6), the only difference being in the non-limit step which we now describe.

We suppose for this that σ_ξ is any mixed cone-bearing neocategory $(\Sigma_\xi, \Gamma_\xi, \nabla_\xi)$.

1° In the construction of $\overline{\sigma}$, we associate to σ_ξ the mixed cone-bearing neocategory $\overline{\sigma}_\xi = (\overline{\Sigma}_\xi, \overline{\Gamma}_\xi, \overline{\nabla}_\xi)$ defined as follows. We denote by:

- Ω the set of pairs (γ, γ') (or «formal factors»), where $\gamma \in \Gamma_\xi$ and γ' is a projective cone in Σ_ξ with the same base as γ.
- $\hat{\Omega}$ the set of pairs (κ', κ) (or «formal cofactors»), where $\kappa \in \nabla_\xi$ and κ' is an inductive cone in Σ' with the same base as κ.
- U the sum of $\underline{\Sigma}_\xi$, Ω and $\hat{\Omega}$, with injections:

$$v: \underline{\Sigma}_\xi \to U, \quad v': \Omega \to U, \quad \hat{v}': \hat{\Omega} \to U.$$

- (U, β, α) the graph such that:

$$(G)\quad \begin{cases} v(x): v(u) \to v(u') & \text{if } x: u \to u' \text{ is in } \Sigma_\xi . \\ v'(\gamma, \gamma'): v(u') \to v(u) & \text{if } (\gamma, \gamma') \in \Omega \text{ and if } \gamma \text{ and } \gamma' \text{ have } u \\ & \text{and } u' \text{ as vertices}. \\ \hat{v}'(\kappa', \kappa): v(u) \to v(u') & \text{if } (\kappa', \kappa) \in \hat{\Omega} \text{ and if } \kappa \text{ and } \kappa' \text{ have } u \\ & \text{and } u' \text{ as vertices}. \end{cases}$$

- L the free category generated by (U, β, α) and r the smallest equivalence relation on $\underline{L}$ satisfying the condition (Pm) obtained by adding

Condition (P), Part 1, Proposition 3, and

$$(\mathrm{Pi}) \begin{cases} (\hat{v}'(\kappa',\kappa), v(\kappa(i))) \sim v(\kappa'(i)), \text{ if } i \in K_0, \\ \hat{v}'(\kappa',\kappa) \sim v(z), \text{ if } z \in \Sigma_\xi \text{ and } z.\kappa(i) = \kappa'(i) \text{ for any } i \in K_0, \\ \text{where } (\kappa',\kappa) \in \hat{\Omega} \text{ and } K \text{ is the indexing-category of } \kappa. \end{cases}$$

- $\overline{\overline{\Sigma}}_\xi$ the quasi-quotient category of L by r and $\rho: L \to \overline{\overline{\Sigma}}_\xi$ the canonical functor.

The map $\rho . v$ defines a neofunctor $\delta_\xi : \Sigma_\xi \to \overline{\overline{\Sigma}}_\xi$. The triple $\overline{\overline{\sigma}}_\xi = (\overline{\overline{\Sigma}}_\xi, \overline{\overline{\Gamma}}_\xi, \overline{\overline{\nabla}}_\xi)$, where

$$\overline{\overline{\Gamma}}_\xi = \{ \delta_\xi \gamma \mid \gamma \in \Gamma_\xi \} \text{ and } \overline{\overline{\nabla}}_\xi = \{ \delta_\xi \kappa \mid \kappa \in \nabla_\xi \},$$

is a mixed cone-bearing neocategory and δ_ξ defines a morphism

$$\overline{\overline{\delta}}_\xi : \sigma_\xi \to \overline{\overline{\sigma}}_\xi .$$

2° In the construction of π', we associate to σ_ξ the following mixed cone-bearing neocategory $\overline{\sigma}_\xi$: Let r be the smallest equivalence relation on $\underline{\Sigma}_\xi$ satisfying the condition (P" m) obtained by adding to the condition (P") of Proposition 5 the condition:

$$(\mathrm{P''i}) \begin{cases} \kappa(i) \sim \kappa'(i), \text{ for any } i \in K_0, \text{ if } \kappa \text{ and } \kappa' \text{ are two cones of } \nabla_\xi \\ \text{with the same base, indexed by } K. \end{cases}$$

We denote by $\overline{\Sigma}_\xi$ the quasi-quotient category of Σ_ξ by r and we define the canonical neofunctor $\delta_\xi : \Sigma_\xi \to \overline{\Sigma}_\xi$ and the sets $\overline{\Gamma}_\xi$ and $\overline{\nabla}_\xi$ formally as in Part 1. Then $\overline{\sigma}_\xi = (\overline{\Sigma}_\xi, \overline{\Gamma}_\xi, \overline{\nabla}_\xi)$ and $\overline{\delta}_\xi : \sigma_\xi \to \overline{\sigma}_\xi$ is defined by δ_ξ.

3° In order to get π, we construct $\overline{\sigma}_\xi$ as in Part 1, replacing only the condition (Pm) by the condition ($\hat{\mathrm{P}}$m) deduced from the conditions (Pm) and (P"m) (as ($\hat{\mathrm{P}}$) was deduced from (P) and (P") in Proposition 6).

4° To prove that $\overline{\sigma}$ (resp. π', resp. π) has the properties indicated in Proposition 13, we use the same arguments as in Proposition 3 (resp. 5, resp. 6) for the distinguished projective cones, and dual arguments for the distinguished inductive cones. (This is possible, since the parts of the constructions involving inductive cones are just deduced by duality from those involving projective cones.) ∇

DEFINITION. With the hypotheses of Proposition 13, we call $\overline{\sigma}$ (resp. π', resp. π) a *mixed limit-bearing category* (resp. a *mixed presketch*, resp. a *mixed prototype)* generated by σ. We say that σ is a *mixed sketch* if the canonical morphism from σ to π is injective.

We denote by $\mathcal{S}m$ the full subcategory of $\mathcal{S}m'$ whose objects are the mixed sketches $\sigma \in \mathcal{S}m'_0$.

PROPOSITION 14. *Let σ be a mixed cone-bearing neocategory. There exists a mixed sketch $\bar{\bar{\sigma}}$ defined up to an isomorphism by the condition:*

If $\mathcal{U}$ is a universe such that $\sigma \in \mathcal{S}m''_0$, then $\bar{\bar{\sigma}}$ is a free structure generated by σ relative to the insertion functor from $\mathcal{S}m$ to $\mathcal{S}m''$.

Δ. Let (π, Π, σ) be the canonical morphism from σ to a prototype generated by σ and $\bar{\bar{\sigma}}$ the mixed presketch image of σ by Π. Then a proof similar to that of Proposition 8 shows that $\bar{\bar{\sigma}}$ is a sketch satisfying the condition of Proposition 14. ∇

PROPOSITION 15. *Let σ be a $(\mathcal{I}, \mathcal{J})$-cone-bearing neocategory. There exist*

- *a loose $(\mathcal{I}, \mathcal{J})$-type $\bar{\sigma}$, defined up to an equivalence,*
- *a $(\mathcal{I}, \mathcal{J})$-type τ, defined up to an isomorphism,*

satisfying the condition:

Let $\mathcal{U}$ be a universe such that $\mathcal{I}$ and $\mathcal{J}$ are $\mathcal{U}$-sets and $\sigma \in \mathcal{S}m''_0$. Then $\bar{\sigma}$ and τ are free structures generated by σ relative to the insertion functors respectively form $\underset{\sim}{\mathcal{L}}^{\mathcal{I}\mathcal{J}}$ to $\mathcal{S}''^{\mathcal{I}\mathcal{J}}_{\sim}$ and from $\mathcal{F}^{\mathcal{I}\mathcal{J}}$ to $\mathcal{S}''^{\mathcal{I}\mathcal{J}}$.

Δ. The construction of $\bar{\sigma}$ (resp. of τ) is done by transfinite induction by a method similar to that used in Proposition 4 (resp. 9), the only modification occuring in the non-limit step, which we now describe.

Let σ_ξ be a $(\mathcal{I}, \mathcal{J})$-cone-bearing neocategory. We consider the sets

- $\underline{\Sigma}_\xi$, Ω, M and M', defined as in Part 1, Proposition 4,
- $\hat{\Omega}$, $\hat{M}$ and $\hat{M}'$ defined dually as follows:

$\hat{\Omega}$ is the set of pairs of cones (κ', κ), where $\kappa \in \nabla_\xi$ and κ' is an inductive cone in Σ_ξ with the same base as κ,
$\hat{M}$ is the set of neofunctors $\phi: K \to \Sigma_\xi$, where $K \in \mathcal{J}$, which are not the base of any inductive cone $\kappa \in \nabla_\xi$,
$\hat{M}'$ is the set of pairs (ϕ, i), where $\phi \in \hat{M}$ and $i \in K_0$.

We denote by U the sum of these seven sets, by

$$v,\ v',\ w,\ w',\ \hat{v}',\ \hat{w},\ \hat{w}'$$

the canonical injections into U. We get a graph (U, β, α) by imposing

$$\begin{cases} \text{condition (G) of Proposition 13 and} \\ w'(i,\phi): w(\phi) \to v(\phi(i)) \quad \text{if } (i,\phi) \in M', \\ \hat{w}'(\phi,i): v(\phi(i)) \to \hat{w}(\phi) \quad \text{if } (\phi,i) \in \hat{M}'. \end{cases}$$

Let L be the free category generated by this graph and r the smallest equivalence relation on $\underline{L}$ satisfying the condition (Pm) of Part 1 (resp. ($\hat{\text{P}}$m), of Part 3), Proposition 13, the condition (P') of Proposition 4 and

$$\text{(P'i)} \begin{cases} (\hat{w}'(\phi,i), v(\phi(k))) \sim \hat{w}'(\phi,i'), \ \text{if } (\phi,i) \in \hat{M}',\ \phi: K \to \Sigma_\xi, \\ \qquad k: i' \to i \text{ in } K. \end{cases}$$

There exists a quasi-quotient category $\overline{\Sigma}_\xi$ of L by r and, if ρ is the canonical functor from L to $\overline{\Sigma}_\xi$, then $\underline{\rho}\, v$ defines a neofunctor δ_ξ, from Σ_ξ to $\overline{\Sigma}_\xi$.

Let $\phi: K \to \Sigma_\xi$ be a functor. If $\phi \in M$, we define a projective cone $\gamma_\phi: \rho(w(\phi))^\wedge \to \delta_\xi \phi$, the «formal projective cone associated to ϕ», by

$$\gamma_\phi(i) = \rho(w'(i,\phi)), \text{ for any } i \in K_0.$$

If $\phi \in \hat{M}$, we define an inductive cone $\kappa_\phi: \delta_\xi \phi \to \rho(\hat{w}(\phi))^\wedge$, the «formal inductive cone associated to ϕ», by:

$$\kappa_\phi(i) = \rho(\hat{w}'(\phi,i)), \text{ for any } i \in K_0.$$

We denote by

- $\overline{\Gamma}_\xi$ the set of cones $\delta_\xi \gamma$ where $\gamma \in \Gamma_\xi$, and γ_ϕ where $\phi \in M$,
- $\overline{\nabla}_\xi$ the set of cones $\delta_\xi \kappa$ where $\kappa \in \nabla_\xi$, and κ_ϕ where $\phi \in \hat{M}$,
- $\overline{\sigma}_\xi$ the mixed cone-bearing category $(\overline{\Sigma}_\xi, \overline{\Gamma}_\xi, \overline{\nabla}_\xi)$,
- $\overline{\delta}_\xi: \sigma_\xi \to \overline{\sigma}_\xi$ the morphism defined by δ_ξ.

If $\overline{\psi} = (\sigma', \psi, \overline{\sigma}_\xi)$ is a morphism from $\overline{\sigma}_\xi$ to a loose $(\mathcal{I}, \mathcal{J})$-type σ', we can choose one (resp. to a $(\mathcal{I}, \mathcal{J})$-type σ', there exists one unique) distinguished projective cone η_ϕ in σ' with $\psi\phi$ as its base, for each $\phi \in M$, and one distinguished inductive cone $\hat{\eta}_{\phi'}$ with $\psi\phi'$ as its base, for each $\phi' \in \hat{M}$. As in Part 1 Proposition 4, we see there is a unique morphism $\overline{\psi}': \overline{\sigma}_\xi \to \sigma'$ such that

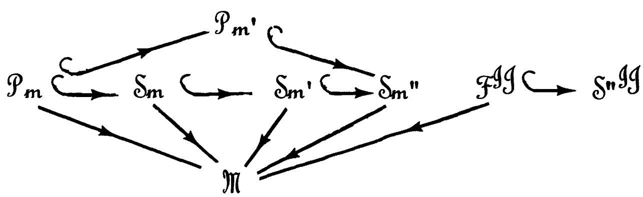
$$\overline{\psi}' \cdot \overline{\delta}_\xi = \overline{\psi}, \quad \overline{\psi}' \gamma_\phi = \eta_\phi \text{ if } \phi \in M,$$
$$\overline{\psi}' \kappa_{\phi'} = \hat{\eta}_{\phi'} \text{ if } \phi' \in \hat{M}.$$

The construction of $\overline{\sigma}$ (resp. of τ) is done just as in Proposition 4 (resp. 9), but with this modified definition of $\overline{\sigma}_\xi$. The proof of Proposition 15 is completed similarly. ∇

DEFINITION. With the hypotheses of Proposition 15, we call $\overline{\sigma}$ a *loose* $(\mathcal{I}, \mathcal{J})$*-type generated by* σ and τ a $(\mathcal{I}, \mathcal{J})$*-type generated by* σ. The category underlying $\overline{\sigma}$ is called a *loose* $(\mathcal{I}, \mathcal{J})$*-completion of* σ.

The preceding Propositions admit the following corollaries:

COROLLARY 1. *In the diagram*



the insertion functors admit left adjoints, all the categories admit $\mathcal{F}_0$*-inductive limits and the functors toward* $\mathfrak{M}$ *admit quasi-quotient structures.*

COROLLARY 2. *The corollaries of Propositions 3, 4, 5, 6, 8 and 9 are still valid when (projective) cone-bearing neocategories are replaced by mixed cone-bearing neocategories.*

Let σ be a $(\mathcal{I}, \mathcal{J})$-cone-bearing neocategory. We will denote by:

- $\overline{\sigma}$ a mixed limit-bearing category,
- π a mixed prototype,
- τ' a loose $(\mathcal{I}, \mathcal{J})$-type,
- τ a $(\mathcal{I}, \mathcal{J})$-type,

generated by σ. From Corollary 2, we deduce:

COROLLARY 3. *1° If* σ' *is a* $(\mathcal{I}, \mathcal{J})$*-type, the categories*

$$\mathcal{S}(\sigma', \sigma), \quad \mathcal{S}(\sigma', \pi), \quad \mathcal{S}(\sigma', \overline{\sigma}) \text{ and } \mathcal{S}(\sigma', \tau)$$

are isomorphic, and they are equivalent to $\mathcal{S}(\sigma', \tau')$.

2° If Σ' is a category admitting $\mathcal{I}$-projective limits and $\mathcal{J}$-inductive limits, the categories Σ'^{σ} and $\Sigma'^{\overline{\sigma}}$ are isomorphic, and they are equivalent to the category $\Sigma'^{\tau'}$.

PROPOSITION 16. *Let σ be a $(\mathcal{I}, \mathcal{J})$-cone-bearing neocategory. The following conditions are equivalent:*

1° σ is a mixed sketch.

2° The canonical morphism $\overline{\delta}: \sigma \to \tau'$ is injective.

3° The canonical morphism $\overline{\theta}: \sigma \to \tau$ is injective.

If they are satified, then

$$\begin{cases} \pi \text{ is isomorphic with } \overline{\sigma}, \\ \tau \text{ is equivalent to } \tau'. \end{cases}$$

Δ. The proof is just similar to that of Propositions 7 and 10, except that Part 1-b of Proposition 10 must be modified as follows.

We suppose that σ is a mixed sketch (Σ, Γ, ∇); we want to exhibit an injective σ-structure in a loose $(\mathcal{I}, \mathcal{J})$-type. As in Proposition 10, we consider the canonical morphism $\overline{\Pi} = (\pi, \Pi, \sigma)$ from σ to a prototype $\pi = (\hat{\Sigma}, \hat{\Gamma}, \hat{\nabla})$ generated by σ, a universe $\mathcal{U}$ such that $\underline{K}$, for any category K belonging to $\mathcal{I}$ or $\mathcal{J}$, and $u'.\hat{\Sigma}.u$, for any pair (u', u) of objects of $\hat{\Sigma}$, are $\mathcal{U}$-sets, and the Yoneda immersion Y from $\hat{\Sigma}$ to $\mathcal{M}^{\hat{\Sigma}^*}$. But Y does not commute with inductive limits. So we take the full subcategory Σ'' of $\Sigma' = \mathcal{M}^{\hat{\Sigma}^*}$ whose objects are functors $F: \hat{\Sigma}^* \to \mathcal{M}$ commuting with $\mathcal{J}$-projective limits. It is known (see, for example, [J]) that Σ'' admits $\mathcal{F}_0$-projective and inductive limits. (In fact, Σ'' is closed for projective limits in Σ' and the insertion functor from Σ'' to Σ' admits a left adjoint). Moreover, there exists [Lb] a restriction

$$Y': \hat{\Sigma} \to \Sigma'' \quad \text{of} \quad Y: \hat{\Sigma} \to \Sigma',$$

which commutes with projective limits and with $\mathcal{J}$-inductive limits. It follows that the full $(\mathcal{I}, \mathcal{J})$-limit-bearing category on Σ'' is a loose $(\mathcal{I}, \mathcal{J})$-type σ'', and that Y' defines an injective morphism $\overline{Y'}: \pi \to \sigma''$. Hence $Y'\Pi$ is an injective σ-structure in the loose $(\mathcal{I}, \mathcal{J})$-type σ''. ∇

REMARK. If σ is a mixed limit-bearing category, the «type part» of Proposition 15 and the injectivity of $\overline{\theta}$ are stated in Theorem 15 [E]. The explicit constructions of the generated loose $(\mathcal{I},\mathcal{J})$-type and $(\mathcal{I},\mathcal{J})$-type τ are yet suggested by that of Theorem 8 [E] (the construction of the type τ has also be done for **V**-categories [F]). Proposition 16 generalizes Theorem 14 of [E] (which corresponds to the case $\Gamma = \emptyset = \nabla$).

DEFINITION. Let σ be a mixed $(\mathcal{I},\mathcal{J})$-cone-bearing neocategory. If σ' is a mixed limit-bearing category $(\Sigma', \Gamma', \nabla')$, we say that σ is *σ'-regular* if each σ-structure in Σ' is equivalent to a σ-structure in σ'. If σ is σ'-regular for each $(\mathcal{I},\mathcal{J})$-type σ', we say that σ is *regular.*

This definition means that the insertion functor from $\mathcal{S}(\sigma',\sigma)$ to Σ'^{σ} defines an equivalence between these two categories.

COROLLARY. *Let σ be a $(\mathcal{I},\mathcal{J})$-sketch and σ' a mixed prototype (resp. a $(\mathcal{I},\mathcal{J})$-type) $(\Sigma', \Gamma', \nabla')$. Then σ is σ'-regular iff a prototype (resp. a $(\mathcal{I},\mathcal{J})$-type) $\hat{\sigma}$ generated by σ is σ'-regular.*

Δ. We denote by $(\hat{\sigma}, \delta, \sigma)$ the canonical morphism, by F the functor from $\Sigma'^{\hat{\sigma}}$ to Σ'^{σ} assigning $\theta'\delta$ to θ'. By Proposition 16, $\hat{\sigma}$ is also a limit-bearing category (resp. a loose $(\mathcal{I},\mathcal{J})$-type) generated by σ. So, according to the proof of Corollary 2, Proposition 15 via Corollary 2, Proposition 3 (resp. via Corollary 1, Proposition 4), there exists a functor G from Σ'^{σ} to $\Sigma'^{\hat{\sigma}}$ such that G is an inverse of F (resp. such that FG is an identity and GF is equivalent to an identity).

1° If $\hat{\sigma}$ is σ'-regular and if μ is a σ-structure in Σ', there exists an equivalence η' from the $\hat{\sigma}$-structure $G(\mu)$ in Σ' to a $\hat{\sigma}$-structure ψ' in σ', and $\eta'\delta : G(\mu)\delta \to \psi'\delta$ is an equivalence from μ to the σ-structure $\psi'\delta$ in σ', since $G(\mu)\delta = FG(\mu) = \mu$. So σ is σ'-regular.

2° We suppose that σ is σ'-regular. Let ν be a $\hat{\sigma}$-structure in Σ'; there exists an equivalence ξ from $\nu\delta$ to a σ-structure ψ in σ', and $G(\xi)$ is an equivalence from $G(\nu\delta)$ to $G(\psi)$. By definition of $\hat{\sigma}$, there exists a $\hat{\sigma}$-structure ψ' in σ' satisfying $F(\psi') = \psi'\delta = \psi$. As ψ' is equivalent to $GF(\psi') = G(\psi)$ and ν to $G(\nu\delta) = GF(\nu)$, the functors ν and ψ' are equivalent. Hence, $\hat{\sigma}$ is σ'-regular. ∇

REMARK. Most usual sketches are regular. More generally, we say that

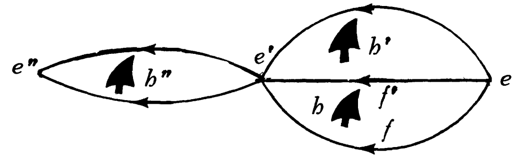
σ is loosely σ'-regular if the categories Σ'^{σ} and $\mathcal{S}(\sigma', \sigma)$ are equivalent. From Corollary 2, Proposition 15, we deduce at once that σ is loosely σ'-regular, where σ' is a mixed prototype (resp. a loose $(\mathcal{I}, \mathcal{J})$-type, resp. a $(\mathcal{I}, \mathcal{J})$-type) iff so is a prototype (resp. a loose $(\mathcal{I}, \mathcal{J})$-type, resp. a $(\mathcal{I}, \mathcal{J})$-type) generated by σ. In several papers, regular means loosely regular. In particular in [L], each mixed cone-bearing neocategory is universally immersed into a loosely regular one.

8. Corresponding 2-categories of bimorphisms.

In this paragraph, we give a reformulation of the preceding results in terms of 2-categories. The categories $\mathcal{P}_{m'}$, $\mathcal{P}_m$,... appear as the categories of 1-morphisms of representable and corepresentable 2-categories and the adjoint functors constructed above extend into 2-adjoints.

2-categories will be considered as those special double categories (or category-objects in $\mathcal{F}$) $(C^{\cdot}, C^{\perp})$ for which the objects of the category $C^{\cdot}$ are also objects of the category $C^{\perp}$ (they are often considered as $\mathcal{F}$-categories, relative to the closed cartesian category $\mathcal{F}$).

Let $\mathcal{C}$ be a 2-category $(C^{\cdot}, C^{\perp})$. The categories $C^{\cdot}$ and $C^{\perp}$ have the same set of morphisms, denoted by $\underline{\mathcal{C}}$, and whose elements are called *bimorphisms* (or 2-cells) of $\mathcal{C}$. The category $C^{\cdot}$ will be called the *category of bimorphisms of* $\mathcal{C}$ (or «strong category» [G]), and written $\mathcal{C}^{\cdot}$, while $C^{\perp}$, also denoted by $\mathcal{C}^{\perp}$, is called the *transverse category* (or «weak» category) of $\mathcal{C}$. We say that an object of $\mathcal{C}^{\cdot}$ is a *vertex of* $\mathcal{C}$, and that an object of $\mathcal{C}^{\perp}$ is a *1-morphism* (or 1-cell) of $\mathcal{C}$. The set of 1-morphisms defines a subcategory of $\mathcal{C}^{\cdot}$, denoted by $|\mathcal{C}|$. If h is an element of $\underline{\mathcal{C}}$, it is both a morphism $h: f \to f'$ in $\mathcal{C}^{\perp}$ and a morphism in $\mathcal{C}^{\cdot}$, with source the source e of the 1-morphism f (or f') in $|\mathcal{C}|$ and with target the target e'



of f in $|\mathcal{C}|$; to «visualize» the two laws, we will write:

$$h : e \rightrightarrows e' \text{ in } \mathcal{C}$$

or, more precisely:

$$h : f \to f' : e \rightrightarrows e' \text{ in } \mathcal{C}.$$

The 2-category $\mathcal{C}$ is said *representable* (resp. *corepresentable*) [G1] if the insertion functor I from $|\mathcal{C}|$ to the category of bimorphisms $\mathcal{C}^{\cdot}$ admits a coadjoint (resp. a left adjoint) $\square$. A cofree (resp. free) structure generated by a vertex e of $\mathcal{C}$ is called a *representation* (resp. a *corepresentation*) *of* e. Hence $\square e$ is a representation of e iff there exists a bimorphism $\partial e : \square e \rightrightarrows e$ such that, for each bimorphism $h : e' \rightrightarrows e$, there is a unique 1-morphism

$$h' : e' \to \square e \quad \text{satisfying} \quad \partial e . h' = h .$$

If $\mathcal{C}$ is a representable 2-category, the triple on $|\mathcal{C}|$ associated to the pair $(I, \square)$ of adjoint functors admits the category of bimorphisms of $\mathcal{C}$ as its Kleisli category.

We still denote by $\mathfrak{N}$ the 2-category of natural transformations associated to the universe $\mathcal{U}$ (we call a 2-category by its bimorphisms, and not by its vertices, as usual). Its category of 1-morphisms $|\mathfrak{N}|$ is the category $\mathcal{F}$ of functors associated to $\mathcal{U}$. Its transverse category is the sum of the categories Σ'^{Σ}, where Σ and Σ' are categories whose sets of morphisms belong to $\mathcal{U}$. The law of its category of bimorphisms is the *lateral composition* of natural transformations: If

$$\tau : \phi \to \phi' : \Sigma \rightrightarrows \Sigma' \quad \text{and} \quad \tau' : \nu \to \nu' : \Sigma' \rightrightarrows \Sigma''$$

are natural transformations, their lateral composite, denoted by $\tau' \cdot \tau$ or by $\tau' \tau$ is the natural transformation:

$$\tau' \phi' \boxminus \nu \tau : \nu \phi \to \nu' \phi' : \Sigma \rightrightarrows \Sigma'' .$$

$\mathfrak{N}$ is representable and corepresentable, a representation of the category Σ being the lateral category $\boxminus \Sigma$ of quartets of Σ and a corepresentation of Σ being the product category $\Sigma \times \mathbf{2}$ (see [G1]).

Using $\mathfrak{N}$, we are going to define a representable and corepresentable 2-category, whose category of 1-morphisms is the category of morphisms between mixed cone-bearing categories.

DEFINITION. A *bimorphism between mixed cone-bearing categories* is defined as a triple $\overline{\tau} = (\overline{\psi}', \tau, \overline{\psi})$, where

$$\overline{\psi} = (\sigma', \psi, \sigma) \quad \text{and} \quad \overline{\psi}' = (\sigma', \psi', \sigma)$$

are morphisms between mixed cone-bearing categories and $\tau : \psi \to \psi'$ is a natural transformation between the underlying functors.

We also say that $\overline{\tau}$ is a bimorphism from $\overline{\psi}$ to $\overline{\psi}'$ defined by τ, denoted by one of the following formulas:

$$\overline{\tau} : \overline{\psi} \to \overline{\psi}', \quad \overline{\tau} : \sigma \rightrightarrows \sigma', \quad \overline{\tau} : \overline{\psi} \to \overline{\psi}' : \sigma \rightrightarrows \sigma' .$$

Let σ and σ' be two mixed cone-bearing categories. We define the longitudinal category $\overline{\mathcal{S}}(\sigma', \sigma)$ of bimorphisms between σ and σ' as the set of bimorphisms $\overline{\tau} : \sigma \rightrightarrows \sigma'$ equipped with the *longitudinal composition:* The longitudinal composite of $(\overline{\tau}', \overline{\tau})$ exists iff

$$\overline{\tau} = (\overline{\psi}', \tau, \overline{\psi}) \quad \text{and} \quad \overline{\tau}' = (\overline{\psi}'', \tau', \overline{\psi}'),$$

and it is then equal to the bimorphism, denoted by $\overline{\tau}' \boxminus \overline{\tau}$,

$$\overline{\tau}' \boxminus \overline{\tau} : \overline{\psi} \to \overline{\psi}'' : \sigma \rightrightarrows \sigma' ,$$

defined by the natural transformation $\tau' \boxminus \tau$.

The category $\overline{\mathcal{S}}(\sigma', \sigma)$ is trivially isomorphic with $\mathcal{S}(\sigma', \sigma)$.

If $\overline{\tau} : \sigma \rightrightarrows \sigma'$ and $\overline{\tau}'' : \sigma' \to \sigma''$ are bimorphisms, where

$$\overline{\tau} = (\overline{\psi}', \tau, \overline{\psi}) \quad \text{and} \quad \overline{\tau}'' = (\overline{\nu}', \tau'', \overline{\nu}) ,$$

the natural transformation $\tau'' \tau$ defines a bimorphism

$$\overline{\theta} : \overline{\nu}\overline{\psi} \to \overline{\nu}' \overline{\psi}' : \sigma \rightrightarrows \sigma'' ;$$

we call $\overline{\theta}$ the *lateral composite of* $(\overline{\tau}'', \overline{\tau})$ and we denote it by $\overline{\tau}'' \cdot \overline{\tau}$.

We consider still the set $\mathcal{S}m_0''$ of mixed cone-bearing neocategories associated to the universe $\mathcal{U}$ and the corresponding category of morphisms $\mathcal{S}m''$. We denote by:

- $\mathcal{F}m_0''$ the subset of $\mathcal{S}m_0''$ formed by those σ whose underlying neocategory is a category,

- $\mathcal{F}m''$ the full subcategory of $\mathcal{S}m''$ of morphisms between mixed cone-bearing categories belonging to $\mathcal{F}m''_0$,

- $\mathfrak{N}\mathcal{F}m''$ the 2-category of bimorphisms associated to $\mathfrak{U}$: its category of bimorphisms is formed by the bimorphisms $\bar{\tau}: \sigma \rightrightarrows \sigma'$ such that σ and σ' belong to $\mathcal{F}m''_0$, the law of composition being the lateral composition; the law of its transverse category is the longitudinal composition (category sum of the categories $\bar{\mathcal{S}}(\sigma', \sigma)$). In particular, the category of 1-morphisms is $\mathcal{F}m''$.

- $\mathfrak{N}\mathcal{F}m'$ and $\mathfrak{N}\mathcal{F}m$ the 2-categories of bimorphisms between mixed pre-sketches and sketches on a category, i.e. the full sub-2-category of $\mathfrak{N}\mathcal{F}m''$ whose sets of vertices are respectively

$$\mathcal{F}m'_0 = \mathcal{F}m''_0 \cap \mathcal{S}m'_0 \quad \text{and} \quad \mathcal{F}m_0 = \mathcal{F}m''_0 \cap \mathcal{S}m_0 .$$

- $\mathfrak{N}\mathcal{P}m'$, $\mathfrak{N}\mathcal{P}m$, $\mathfrak{N}\mathcal{L}^{\mathcal{I}\mathcal{J}}$ and $\mathfrak{N}\mathcal{F}^{\mathcal{I}\mathcal{J}}$, where $\mathcal{I}$ and $\mathcal{J}$ are $\mathfrak{U}$-sets of categories, the 2-categories of bimorphisms between mixed limit-bearing categories, prototypes, loose $(\mathcal{I}, \mathcal{J})$-types and $(\mathcal{I}, \mathcal{J})$-types, i.e. the full sub-2-categories of $\mathfrak{N}\mathcal{F}m''$ whose sets of vertices are respectively $\mathcal{P}m'_0$, $\mathcal{P}m_0$, $\mathcal{L}_0^{\mathcal{I}\mathcal{J}}$ and $\mathcal{F}_0^{\mathcal{I}\mathcal{J}}$.

All these 2-categories are canonically equipped with a faithful 2-functor toward $\mathfrak{N}$.

PROPOSITION 17. *The 2-category $\mathfrak{N}\mathcal{F}m''$ is representable and corepresentable.*

Δ. Let σ be a mixed cone-bearing category (Σ, Γ, ∇).

1° σ admits as a representation the lateral mixed cone-bearing category $\boxminus\sigma$ of quartets of σ, for any universe $\mathfrak{U}$ such that $\sigma \in \mathcal{F}m''_0$.

Indeed, let a and b be the functors from $\boxminus\Sigma$ to Σ defined by the mappings source and target of the longitudinal category $\square\square\Sigma$. By definition (Example 2-7), $\boxminus\sigma$ is the category $\boxminus\Sigma$ equipped with the sets

- $\bar{\Gamma}$ of projective cones $\bar{\gamma}$ such that $a\bar{\gamma} \in \Gamma$ and $b\bar{\gamma} \in \Gamma$,
- $\bar{\nabla}$ of inductive cones $\bar{\kappa}$ such that $a\bar{\kappa} \in \nabla$ and $b\bar{\kappa} \in \nabla$.

In particular, a and b define morphisms

$$\bar{a}: \boxminus\sigma \to \sigma \quad \text{and} \quad \bar{b}: \boxminus\sigma \to \sigma .$$

To the identical morphism of $\boxminus\sigma$, Proposition 11 associates a bimorphism

$$\partial\,\sigma = (\overline{b}, j, \overline{a}) : \boxminus\sigma \rightrightarrows \sigma$$

(where j is the natural transformation from a to b assigning the morphism x of Σ to the object x of $\boxminus\Sigma$).

Let σ' be a mixed cone-bearing category $(\Sigma', \Gamma', \nabla')$ and

$$\overline{\tau} = (\overline{\psi}', \tau, \overline{\psi}) : \sigma' \rightrightarrows \sigma$$

a bimorphism. The unique functor $T : \Sigma' \to \boxminus\Sigma$ such that $jT = \tau$ defines a morphism from σ' to $\boxminus\sigma$ (Proposition 11), which is the unique morphism

$$\overline{T} : \sigma' \to \boxminus\sigma \quad \text{such that} \quad \partial\sigma \cdot \overline{T} = \overline{\tau}.$$

Hence $\mathfrak{N}\mathfrak{F}m''$ is representable, $\boxminus\sigma$ being a representation of σ.

2° We denote by:

- $\hat{\Sigma}$ the category $\Sigma \times \mathbf{2}$,
- ν and ν' the functors from Σ to $\hat{\Sigma}$ associating respectively $(x, 0)$ and $(x, 1)$ to the morphism x of Σ,
- $\hat{\Gamma}$ the set of cones $\nu\gamma$ and $\nu'\gamma$, where $\gamma \in \Gamma$,
- $\hat{\nabla}$ the set of cones $\nu\kappa$ and $\nu'\kappa$, where $\kappa \in \nabla$.

Then $(\hat{\Sigma}, \hat{\Gamma}, \hat{\nabla})$ is a cone-bearing category $\hat{\sigma}$ and ν and ν' define morphisms $\overline{\nu}$ and $\overline{\nu}'$ from σ to $\hat{\sigma}$. By assigning $(e, (1, 0))$ to an object e of Σ, we get a natural transformation $\theta : \nu \to \nu'$, and therefore a bimorphism $\overline{\theta} = (\overline{\nu}', \theta, \overline{\nu}) : \sigma \rightrightarrows \hat{\sigma}$.

$\hat{\sigma}$ is a corepresentation of σ in $\mathfrak{N}\mathfrak{F}m''$ for any universe $\mathfrak{U}$ such that $\sigma \in \mathfrak{F}m''_0$. Indeed, let

$$\overline{\tau}' = (\overline{\psi}', \tau', \overline{\psi}) : \sigma \rightrightarrows \sigma'$$

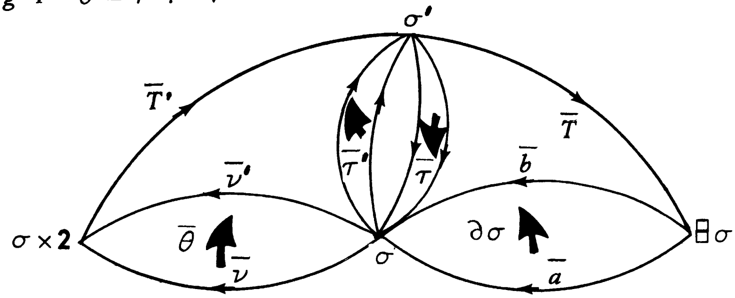
be a bimorphism, where σ' is a mixed cone-bearing category $(\Sigma', \Gamma', \nabla')$. As $\hat{\Sigma}$ is a corepresentation of Σ in $\mathfrak{N}$, there exists a unique functor

$$T' : \hat{\Sigma} \to \Sigma' \quad \text{such that} \quad T'\theta = \tau'.$$

This functor defines a morphism $\overline{T}' : \hat{\sigma} \to \sigma'$, since

$$T'\nu\gamma = \overline{\psi}\gamma \quad \text{and} \quad T'\nu'\gamma = \overline{\psi}'\gamma,$$

for any distinguished cone γ in σ. Then $\overline{T}'$ is the unique morphism satisfying $\overline{T}' \cdot \overline{\theta} = \overline{\tau}'$. ∇



The cone-bearing category $\hat{\sigma}$ considered here above will be denoted by $\sigma \times \mathbf{2}^{\bullet}$.

REMARK. $\sigma \times \mathbf{2}$ is not defined as the product of two cone-bearing categories. However, if $\mathcal{I}$ and $\mathcal{J}$ are given sets of categories, we can define a prototype $\hat{\mathbf{2}}$ by equipping $\mathbf{2}$ with the set of all «constant» projective cones in $\mathbf{2}$ indexed by a category $K \in \mathcal{I}$, and with the set of all constant inductive cones indexed by a category of $\mathcal{J}$. Then, for each $(\mathcal{I}, \mathcal{J})$-cone-bearing category σ, the product $\sigma \times \hat{\mathbf{2}}$ in $\mathcal{F}m''$ is identical with $\sigma \times \mathbf{2}$.

PROPOSITION 18. *Let* $\mathcal{X}$ *denote anyone of the symbols*

$$\mathcal{F}m'',\ \mathcal{F}m',\ \mathcal{F}m,\ \mathcal{P}m',\ \mathcal{P}m,\ \mathcal{L}^{\mathcal{I}\mathcal{J}},\ \mathcal{F}^{\mathcal{I}\mathcal{J}},$$

where $\mathcal{I}$ *and* $\mathcal{J}$ *are* $\mathcal{U}$*-sets of categories.*

1° If $\mathcal{X} \neq \mathcal{F}m'$, *then* $\mathcal{N}\mathcal{X}$ *is representable, a representation of a vertex* σ *being* $\boxminus\sigma$.

2° If $\mathcal{X} \neq \mathcal{L}^{\mathcal{I}\mathcal{J}}$ *and* $\mathcal{X} \neq \mathcal{F}^{\mathcal{I}\mathcal{J}}$, *then* $\mathcal{N}\mathcal{X}$ *is corepresentable, a corepresentation of* σ *being* $\sigma \times \mathbf{2}$.

3° $\mathcal{N}\mathcal{F}^{\mathcal{I}\mathcal{J}}$ *is corepresentable, a corepresentation of* σ *being a* $(\mathcal{I}, \mathcal{J})$*-type generated by* $\sigma \times \mathbf{2}$.

Δ. 1° A full sub-2-category $\mathcal{N}\mathcal{X}$ of the representable (resp. corepresentable) 2-category $\mathcal{N}\mathcal{F}m''$ to which belongs a representation (resp. a corepresentation) of each vertex σ of $\mathcal{N}\mathcal{X}$ is representable (resp. corepresentable). So assertions 1 and 2 result from the following facts.

a) If σ is a mixed limit-bearing category, so is $\boxminus\sigma$. Since a constant functor toward **2** admits its unique value both as a projective limit and as an inductive limit, $\sigma \times \mathbf{2}$ is also a limit-bearing category. Hence $\mathfrak{N}\mathcal{P}_m'$ is representable and corepresentable.

b) If σ is a mixed presketch (resp. prototype), $\sigma \times \mathbf{2}$ is also one, so that $\mathfrak{N}\mathcal{F}_m'$ and $\mathfrak{N}\mathcal{P}_m$ are corepresentable.

c) Let σ be a mixed prototype (Σ, Γ, ∇). Then the mixed limit-bearing category $\boxminus\sigma$ is also a presketch, i.e. a prototype. Indeed, let $\bar{\gamma}$ be a distinguished projective cone in $\boxminus\sigma$, with base T and vertex x. Objects of $\boxminus\Sigma$ are identified with morphisms of Σ and we denote yet by a and b the functors from $\boxminus\Sigma$ to Σ determined by the mappings source and target of $\square\Sigma$. By construction, $a\bar{\gamma}$ and $b\bar{\gamma}$ belong to Γ, so that $a\bar{\gamma}$ and $b\bar{\gamma}$ are the only cones of Γ with bases aT and bT (for σ is a presketch). Moreover, $b\bar{\gamma}$ being a projective limit-cone, x is the unique morphism of Σ such that

$$(b\bar{\gamma})x = \theta \,\square\, a\bar{\gamma}, \quad \text{where} \quad \theta : aT \to bT$$

is the natural transformation canonically associated to the functor T toward $\boxminus\Sigma$. Hence $\bar{\gamma}$ is the unique distinguished projective cone in $\boxminus\sigma$, with base T. Similarly, there is at most one distinguished inductive cone of $\boxminus\sigma$ with a given base. This proves that $\boxminus\sigma$ is a mixed prototype. A fortiori, $\mathfrak{N}\mathcal{P}_m$ is representable.

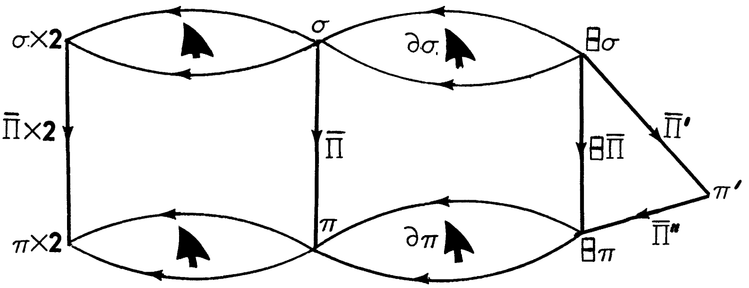
d) If σ is a $(\mathcal{I}, \mathcal{J})$-type (resp. a loose $(\mathcal{I}, \mathcal{J})$-type), so is $\boxminus\sigma$, which implies that $\mathfrak{N}\mathcal{F}^{\mathcal{I}\mathcal{J}}$ and $\mathfrak{N}\mathcal{L}^{\mathcal{I}\mathcal{J}}$ are representable.

e) Let σ be a mixed sketch (Σ, Γ, ∇), where Σ is a category; let $\bar{\Pi}$ be the canonical morphism (π, Π, σ) from σ to a prototype π generated by σ.

$\boxminus\sigma$ is a mixed sketch. Indeed, let $\bar{\Pi}'$ be the canonical morphism from $\boxminus\sigma$ to a mixed prototype it generates. From Part c, it follows that $\boxminus\pi$ is a prototype. The functor $\boxminus\Pi$ (assigning

$$(\Pi(y'), \Pi(x'), \Pi(y), \Pi(x)) \quad \text{to} \quad (y', x', x, y))$$

defines a morphism $\boxminus\bar{\Pi} : \boxminus\sigma \to \boxminus\pi$. So there exists a unique morphism $\bar{\Pi}''$ such that $\boxminus\bar{\Pi} = \bar{\Pi}'' \cdot \bar{\Pi}'$. Since Π is injective, $\boxminus\Pi$ is also injective



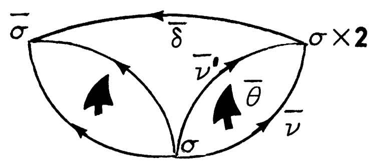
and the preceding equality implies the injectivity of $\bar{\Pi}'$. Therefore, $\boxminus\sigma$ is a mixed sketch, and $\mathcal{N}\mathcal{F}_m$ is representable.

Similarly, $\sigma \times 2$ is a mixed sketch, because the functor $\Pi \times 2$ defines an injective morphism $\bar{\Pi} \times 2$ from $\sigma \times 2$ to the prototype $\pi \times 2$ (Part b). So $\mathcal{N}\mathcal{F}_m$ is corepresentable.

2° Let σ be a $(\mathcal{I}, \mathcal{J})$-type. Then $\sigma \times 2$ is not a $(\mathcal{I}, \mathcal{J})$-type, but it generates a $(\mathcal{I}, \mathcal{J})$-type $\bar{\sigma}$. By transitivity of free structures, $\bar{\sigma}$ is a free structure generated by σ relative to the composite insertion functor

$$\mathcal{F}^{\mathcal{I}\mathcal{J}} \hookrightarrow \mathcal{F}_m" \hookrightarrow (\mathcal{N}\mathcal{F}_m")^{\cdot} .$$

A fortiori $\bar{\sigma}$ is a corepresentation of σ in the full sub-2-category $\mathcal{N}\mathcal{F}^{\mathcal{I}\mathcal{J}}$ of $\mathcal{N}\mathcal{F}_m"$. ∇



COROLLARY. *$\mathcal{N}\mathcal{F}_m'$ is not representable.*

Δ . Let σ be a mixed presketch (Σ, Γ, ∇), where Σ is a category.

1° $\boxminus\sigma$ may not be a presketch. Indeed, we still denote by a and b the functors from $\boxminus\Sigma$ to Σ determined by the mappings source and target of $\square\square\Sigma$. Let $T: K \to \boxminus\Sigma$ be a functor and $\tau: K \rightrightarrows \Sigma$ the corresponding natural transformation. If

$$\gamma : e^{\wedge} \to aT \quad \text{and} \quad \gamma' : e'^{\wedge} \to bT$$

are cones of Γ , for any morphism $x : e \to e'$ in Σ such that $\gamma' x = \tau \square\square \gamma$, there exists a cone $\overline{\gamma}_x : x^\wedge \to T$ in $\boxminus\Sigma$ such that

$$a\overline{\gamma}_x = \gamma \quad \text{and} \quad b\overline{\gamma}_x = \gamma' ,$$

and this cone is distinguished in $\boxminus\sigma$. As γ' is not necessarily a limit-cone, there may exist another

$$y \in \Sigma \quad \text{such that} \quad \gamma' y = \tau \boxbox \gamma ,$$

and so another distinguished cone $\overline{\gamma}_y$ in $\boxminus\sigma$ with base T. Then $\boxminus\sigma$ is not a presketch.

2° Let us suppose there exists a representation $\hat{\sigma}$ of σ in $\mathcal{NF}_m$' and denote by $\overline{\eta} = (\sigma, \eta, \hat{\sigma})$ the canonical bimorphism. Let

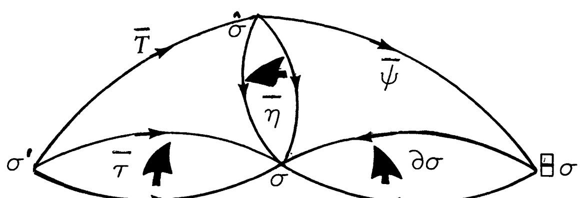
$$\partial\sigma = (\overline{b}, j, \overline{a}) : \boxminus\sigma \rightrightarrows \sigma$$

be the canonical bimorphism defining $\boxminus\sigma$ as a representation of σ in the 2-category $\mathcal{NF}_m$'' (Proposition 17). There exists a unique morphism

$$\overline{\psi} = (\boxminus\sigma, \psi, \hat{\sigma}) \quad \text{such that} \quad \partial\sigma \cdot \overline{\psi} = \overline{\eta} .$$

We are going to show that $\overline{\psi}$ is an isomorphism, which is impossible in the case where $\boxminus\sigma$ is not a presketch.

a) ψ is an isomorphism. Indeed, let $\tau : \Sigma' \rightrightarrows \Sigma$ be any natural transformation. It defines a bimorphism $\overline{\tau} : \sigma' \rightrightarrows \sigma$, where σ' is the mixed presketch on Σ' without any distinguished cone. There exists a unique morphism $\overline{T} = (\hat{\sigma}, T, \sigma')$ such that $\overline{\eta} \cdot \overline{T} = \overline{\tau}$; this means that T is the unique functor satisfying $\eta T = \tau$. Hence η defines the underlying category of $\hat{\sigma}$ as a representation of Σ in $\mathcal{N}$. As j defines $\boxminus\Sigma$ as a representation of Σ in $\mathcal{N}$, the functor ψ such that $j\psi = \eta$ is an isomorphism.



b) The inverse ψ^{-1} of ψ defines a morphism from $\boxminus\sigma$ to $\hat{\sigma}$. Indeed, let $\overline{\gamma}$ be a distinguished cone of $\boxminus\sigma$. We get a mixed presketch $\bar{\bar{\sigma}}$ by equipping $\boxminus\Sigma$ with $\overline{\gamma}$ as its only distinguished cone; j defines a morphism $\bar{\bar{j}} : \bar{\bar{\sigma}} \to \sigma$. So there exists a unique morphism

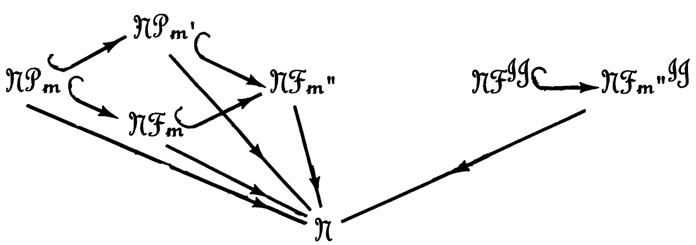
$$\overline{\psi}' = (\hat{\sigma}, \psi', \bar{\bar{\sigma}}) \quad \text{satisfying} \quad \overline{\eta} \cdot \overline{\psi}' = \bar{\bar{j}} \ ;$$

in other words, ψ' is a functor such that $\psi'\overline{\gamma}$ is a distinguished cone of $\hat{\sigma}$ and $\eta\psi' = j$. It follows $j\psi\psi' = j$, which implies that $\psi\psi'$ is an identity functor. ψ being an isomorphism, we have $\psi' = \psi^{-1}$. Hence, $(\boxminus\sigma, \psi^{-1}, \hat{\sigma})$ is a morphism, inverse of $\overline{\psi}$. ∇

REMARK. The 2-category $\mathfrak{N}\mathfrak{L}^{\mathfrak{I}\mathfrak{J}}$ is not corepresentable, but it is weakly corepresentable [G1], a vertex σ admitting as a weak corepresentation a loose $(\mathfrak{I},\mathfrak{J})$-type $\overline{\sigma}$ generated by $\sigma\times\mathbf{2}$. More precisely, let $\overline{\theta}$ be the canonical bimorphism defining $\sigma\times\mathbf{2}$ as a corepresentation of σ in $\mathfrak{N}\mathfrak{F}_m''$ (Proposition 17, Part 2) and $\overline{\delta} : \sigma\times\mathbf{2} \to \overline{\sigma}$ the canonical morphism. If σ' is a loose $(\mathfrak{I},\mathfrak{J})$-type, $\overline{\tau}' : \sigma \rightrightarrows \sigma'$ a bimorphism, there exists a morphism $\overline{T}'$, defined up to an *equivalence*, such that $\overline{T}' \cdot (\overline{\delta} \cdot \overline{\theta}) = \overline{\tau}'$.

If $\mathfrak{I}$ and $\mathfrak{J}$ are $\mathfrak{U}$-sets of categories, we denote by $\mathfrak{N}\mathfrak{F}_m''^{\mathfrak{I}\mathfrak{J}}$ the full sub-2-category of $\mathfrak{N}\mathfrak{F}_m''$ whose vertices are those $(\mathfrak{I},\mathfrak{J})$-cone-bearing categories belonging to $\mathfrak{S}_0''^{\mathfrak{I}\mathfrak{J}}$.

PROPOSITION 19. *In the following diagram of 2-functors,*



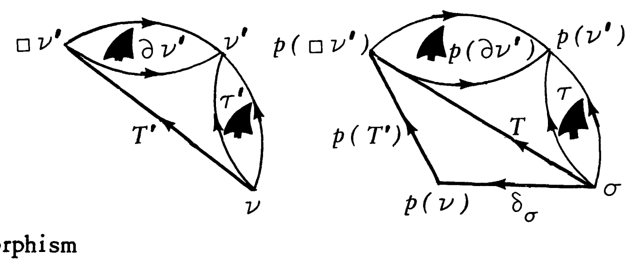
where the 2-functors toward $\mathfrak{N}$ assign to a bimorphism $(\sigma', \theta, \sigma)$ the natural transformation θ, all the 2-functors admit 2-adjoints.

Δ. All the 2-functors of the diagram are 2-functors between representable 2-categories, which commute with the representations by Proposition 18. Moreover their restrictions to the categories of 1-morphisms admit left adjoints. Indeed, this results from Corollary 1, Proposition 15, for the insertion 2-functors. Now let $p_{\mathfrak{X}} : \mathfrak{N}\mathfrak{X} \to \mathfrak{N}$ be one of the 2-functors toward $\mathfrak{N}$. Assigning to a category Σ the trivial mixed prototype $\bar{\bar{\Sigma}}$ on Σ (without any distinguished cone) and to a natural transformation $\tau : \Sigma \rightrightarrows \Sigma'$ the bimorphism $\overline{\tau}: \overline{\Sigma} \rightrightarrows \overline{\Sigma}'$ defined by τ, we get a 2-functor $J_{\mathfrak{X}}: \mathfrak{N} \to \mathfrak{N}\mathfrak{X}$; its restriction $|J_{\mathfrak{X}}|: \mathfrak{F} \to \mathfrak{X}$ is an adjoint of the restriction $|p_{\mathfrak{X}}|: \mathfrak{X} \to \mathfrak{F}$ of $p_{\mathfrak{X}}$. So Proposition 19 follows from the lemma:

2° LEMMA. *Let* $\mathcal{H}$ *be a representable 2-category and* $p: \mathcal{H} \to \mathcal{C}$ *a 2-functor satisfying the following conditions:*

- *For each vertex* ν *of* $\mathcal{H}$, *let* $\partial\nu: \square\nu \to \nu$ *be a bimorphism which defines* $\square\nu$ *as a representation of* ν; *then* $p(\partial\nu)$ *defines* $p(\square\nu)$ *as a representation of* $p(\nu)$ *in* $\mathcal{C}$.
- *The functor* $|p|: |\mathcal{H}| \to |\mathcal{C}|$ *restriction of* p *admits an adjoint* q. *Then* q *extends into a 2-adjoint of* p.

a) The functor $p^{\cdot}: \mathcal{H}^{\cdot} \to \mathcal{C}^{\cdot}$ underlying p admits an adjoint $Q^{\cdot}$ extending q. More precisely, for each vertex σ of $\mathcal{C}$, the canonical morphism $\delta_\sigma: \sigma \to p(q(\sigma))$ corresponding to the pair of adjoint functors $(|p|, q)$ defines also $q(\sigma)$ as a free structure generated by σ relative to $p^{\cdot}$. Indeed, let σ be a vertex of $\mathcal{C}$ and $\nu = q(\sigma)$. If ν' is a vertex of $\mathcal{H}$ and $\tau: \sigma \rightrightarrows p(\nu')$ a bimorphism in $\mathcal{C}$, there exists a unique 1-morphism $T: \sigma \to p(\square\nu')$ such that $p(\partial\nu') \cdot T = \tau$, since $p(\partial\nu')$ defines $p(\square\nu')$ as a representation of $p(\nu')$. To T is associated a unique 1-morphism



$$T': \nu \to \square\nu' \quad \text{such that} \quad p(T') \cdot \delta_\sigma = T.$$

From the equalities

$$p(\partial\nu' \cdot T') \cdot \delta_\sigma = p(\partial\nu') \cdot p(T') \cdot \delta_\sigma = p(\partial\nu') \cdot T = \tau,$$

it follows that $\partial\nu' \cdot T'$ is the unique bimorphism

$$\tau': \nu \rightrightarrows \nu' \quad \text{such that} \quad p(\tau') \cdot \delta_\sigma = \tau.$$

Hence, ν is a free structure generated by σ relative to $p^{\cdot}$.

b) The map $\underset{\sim}{Q}$ underlying the adjoint functor $Q^{\cdot}$ of $p^{\cdot}$ also defines a 2-functor $Q: \mathcal{C} \to \mathcal{H}$, so that Q is a 2-adjoint [G] of p. Indeed, we denote by:

- $\mathcal{C}(\sigma', \sigma)^{\cdot}$ the subcategory of the transverse category $\mathcal{C}^{\perp}$ of $\mathcal{C}$ formed by the bimorphisms $\theta: \sigma \rightrightarrows \sigma'$.

- $\mathcal{C}(\lambda', \lambda)$, if $\lambda: \rho \to \sigma$ and $\lambda': \sigma' \to \rho'$ are 1-morphisms of $\mathcal{C}$, the functor from $\mathcal{C}(\sigma', \sigma)$ to $\mathcal{C}(\rho', \rho)$ assigning the composite $\lambda' \cdot \theta \cdot \lambda$ to $\theta: \sigma \rightrightarrows \sigma'$.

Let σ and σ' be vertices of $\mathcal{C}$; we write

$$\nu = q(\sigma) \text{ and } \nu' = q(\sigma').$$

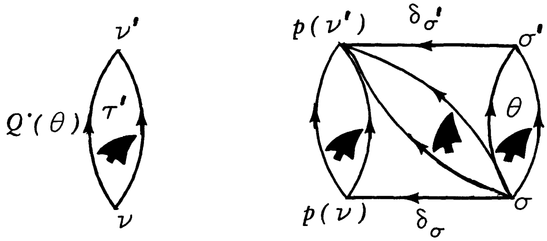
As p is a 2-functor, there exists a functor

$$p_{\nu'\nu}: \mathcal{H}(\nu', \nu) \to \mathcal{C}(p(\nu'), p(\nu))$$

defined by a restriction of p. The map g_σ assigning $p(\tau') \cdot \delta_\sigma$ to the bimorphism $\tau': \nu \rightrightarrows \nu'$ defines the functor

$$\hat{g}_\sigma = \mathcal{C}(p(\nu', \delta_\sigma) p_{\nu'\nu}: \mathcal{H}(\nu', \nu) \to \mathcal{C}(p(\nu'), \sigma).$$

Part a proves that g_σ is a bijection; it follows that $\hat{g}_\sigma$ is an isomorphism.



The functor

$$\hat{g}_\sigma^{-1} \mathcal{C}(\delta_{\sigma'}, \sigma): \mathcal{C}(\sigma', \sigma) \to \mathcal{H}(\nu', \nu)$$

associates $Q^{\cdot}(\theta)$ to $\theta: \sigma \rightrightarrows \sigma'$, since $Q^{\cdot}(\theta)$ is the unique bimorphism

$$\theta': \nu \rightrightarrows \nu' \quad \text{such that} \quad p(\theta') \cdot \delta_\sigma = \delta_{\sigma'} \cdot \theta.$$

The category $\mathcal{C}^{\perp}$ being a sum of the categories $\mathcal{C}(\sigma', \sigma)$, we deduce that $\underset{\sim}{Q}$ defines a functor from $\mathcal{C}^{\perp}$ to $\mathcal{H}^{\perp}$, and also a 2-functor Q. ∇

REMARK. Proposition 19 gives a more axiomatic proof of Corollary 2, Propositions 3 or 15.

III. MONOIDAL CLOSED CATEGORIES OF SKETCHED MORPHISMS

Let σ be a projective limit-bearing category and $\mathcal{V}$ a symmetric monoidal closed category [EK]. Under some conditions on σ, the category V^σ of σ-morphisms in the underlying category V of $\mathcal{V}$ admits a symmetric monoidal closed structure. This is applied to the category of functors (or «category of category objects») in V.

10. Cartesian closed structures on $\mathfrak{M}^\sigma$.

After some notations, we give conditions on σ, insuring that $\mathfrak{M}^\sigma$ admits a cartesian closed structure.

If $\theta: p \to p': L \rightrightarrows K$ is a natural transformation, for any $y: e \to e'$ in L, we denote by $\theta(y)$ the morphism $\theta(e').p(y) = p'(y).\theta(e)$.

Let $P: K' \times K \to C$ be a functor (of «two variables»). If $p: L \to K$ is a functor, we denote by $P(s, p\text{-})$ the functor from L to C assigning $P(s, p(y))$ to $y \in L$, for each object s of K; we denote by $P(x, p\text{-})$ (or by $P(x, p)$, if this does not lead to any confusion) the natural transformation from $P(s, p\text{-})$ to $P(s', p\text{-})$ such that $P(x, p\text{-})(y) = P(x, p(y))$, for any $y \in L$, if $x: s \to s'$ is a morphism in K'.

If $p = Id_K$, we write $P(x, \text{-})$ instead of $P(x, p\text{-})$. If p is the dual q^* of a functor q, we write also $P(x, q\text{-})$ instead of $P(x, q^*\text{-})$.

Similar notations are used relative to the other «variable», and for functors of «more than two variables».

Let K be a category. The functor $Hom_K: K \times K^* \to \mathfrak{M}$ will often be denoted by $K(\text{-},\text{-})$, so that the set of morphisms $x: e \to e'$ in K is written $K(e', e)$ (and not $K(e, e')$ as usual).

We say that K *admits a cartesian closed structure* if there exists a cartesian closed category $\mathbf{K}$, whose underlying category is K. This means that K admits finite products and that, for each object e of K, the partial product functor $\text{-}\times e: K \to K$ (corresponding to a choice of finite products on K) admits a right adjoint. Then we call *closure functor* on K a functor $D: K \times K^* \to K$ such that $D(\text{-}, e)$ is a right adjoint of $\text{-}\times e$, for any object e of K (such a functor is the internal Hom-functor for a closed category [EK] underlying a cartesian closed structure on K). The product functor $\times$ and D are defined up to an equivalence, so that **K** is determined up to an isomorphism of cartesian closed categories.

From now on, we denote by:

- σ a projective limit-bearing category (Σ, Γ),
- $\mathcal{I}$ the set of indexing categories of σ,
- σ^* the « dual of σ», which is the inductive limit-bearing category (Σ^*, Γ^*), whose distinguished inductive cones correspond by duality to projective cones $\gamma \in \Gamma$,
- $\mathfrak{U}$ a universe, to which belong $\underline{\Sigma}$ and $\underline{I}$, for each $I \in \mathcal{I}$,
- $\mathfrak{M}$ the category of maps between $\mathfrak{U}$-sets.

The functor $\Sigma(-, u): \Sigma \to \mathfrak{M}$ commuting with projective limits, it is a σ-structure in $\mathfrak{M}$ for each object u of Σ. Hence the Yoneda immersion $\hat{Y}$ from Σ^* to $\mathfrak{M}^\Sigma$ takes its values in the category $\mathfrak{M}^\sigma$ of σ-morphisms (i. e. of morphisms between σ-structures) in $\mathfrak{M}$. We denote by Y the functor from Σ^* to $\mathfrak{M}^\sigma$, restriction of $\hat{Y}$.

This functor Y is in fact a σ^*-structure in $\mathfrak{M}^\sigma$, called the *Yoneda σ^*-structure*. (Indeed, this will result from Proposition 3-1 [Lb], if $\mathfrak{M}^\sigma(F, Y-)$ is a σ-structure in $\mathfrak{M}$ for each object F of $\mathfrak{M}^\sigma$, i. e. for each σ-structure F; this holds since, by Yoneda Lemma, we get

$$\mathfrak{M}^\sigma(F, Y-) = \mathfrak{M}^\sigma(F, -)Y^* = \mathfrak{M}^\Sigma(F, -)\hat{Y}^* \approx F).$$

Let V be a category. The category V^σ of σ-morphisms in V is a full subcategory of V^Σ, closed for equivalences (i. e. a functor equivalent to a σ-structure in V is also one). If V admits projective limits indexed by a category K, the category V^σ admits also projective limits indexed by K, and the insertion functor from V^σ to V^Σ commutes with these limits [E4] (since in V^Σ these limits are computed evaluationwise and projective limit functors commute with projective limits of any kind). In other words, V^σ is closed in V^Σ for projective limits indexed by K.

Since $\mathfrak{M}$ admits $\mathcal{F}_0$-projective limits, where $\mathcal{F}_0$ is the set of all the categories whose sets of morphisms are $\mathfrak{U}$-sets, $\mathfrak{M}^\sigma$ admits also $\mathcal{F}_0$-projective limits. In particular, $\mathfrak{M}^\sigma$ admits finite products. From the canonical product functor on $\mathfrak{M}$, we deduce the product functors $\hat{\times}$ and $\times$ on $\mathfrak{M}^\Sigma$ and on $\mathfrak{M}^\sigma$. For each σ-structure F in $\mathfrak{M}$, we denote by $-\times F$ the canonical partial product functor from $\mathfrak{M}^\sigma$ to $\mathfrak{M}^\sigma$, which assigns to a σ-structure F' in $\mathfrak{M}$ the σ-structure $F'\times F$ in $\mathfrak{M}$ such that

$$(F'\times F)(x) = F'(x)\times F(x)$$

for each morphism $x: u \to u'$ in Σ. It is a restriction of $-\hat{\times} F: \mathfrak{M}^\Sigma \to \mathfrak{M}^\Sigma$.

In particular, for any object u of Σ, we have the partial product functor $-\times Y(u): \mathfrak{M}^\sigma \to \mathfrak{M}^\sigma$.

PROPOSITION 20. *$\mathfrak{M}^\sigma$ admits a cartesian closed structure iff the functor $-\times Y(u)$ commutes with $\mathcal{J}$-inductive limits, for each object u of Σ. In this case:*

1° $\mathfrak{M}^\sigma$ admits a closure functor $\overline{M}$ assigning to the pair (F', F) of σ-structures in $\mathfrak{M}$ the functor $\mathfrak{M}^\sigma(F', F\times Y-) = \mathfrak{M}^\sigma(F', -)(F\times -)^* Y^*$.

2° For each σ-structure F in $\mathfrak{M}$, the functor $\overline{M}(F, Y-): \Sigma \to \mathfrak{M}^\sigma$ is a σ-structure in $\mathfrak{M}^\sigma$, and $\overline{M}(F', F) = \mathfrak{M}^\sigma(\overline{M}(F', Y-), F)$.

Δ. If $\mathfrak{M}^\sigma$ admits a cartesian closed structure, the partial product functor $-\times Y(u)$ admits a right adjoint, so that it commutes with inductive limits, for any object u of Σ.

We suppose now that $-\times Y(u)$ commutes with $\mathcal{J}$-inductive limits, for each object u of Σ.

$\mathfrak{M}^\Sigma$ admits a cartesian closed structure whose closure functor $\hat{M}$ associates $\mathfrak{M}^\Sigma(\theta', -)(\theta \hat{\times} \hat{Y}-)^*$ to each pair (θ', θ) of morphisms of $\mathfrak{M}^\Sigma$ (see for example [GZ], Chapter 2-1). To show that $\mathfrak{M}^\sigma$ admits a cartesian closed structure, it is sufficient to prove that $\hat{M}(F', F)$ is a σ-structure when F and F' are σ-structures, for this implies the existence of a functor $\overline{M}: \mathfrak{M}^\sigma \times (\mathfrak{M}^\sigma)^* \to \mathfrak{M}^\sigma$ restriction of $\hat{M}$, and $\overline{M}$ is a closure functor on $\mathfrak{M}^\sigma$. The proof will go in three steps.

1° Let F be a σ-structure in $\mathfrak{M}$ and u an object of Σ. Then, the functor $\hat{M}(F, Y(u))$ is a σ-structure in $\mathfrak{M}$. Indeed, by definition,

$$\hat{M}(F, Y(u)) = \mathfrak{M}^\Sigma(F, Y(u)\hat{\times}\hat{Y}-): \Sigma \to \mathfrak{M}.$$

As F and $Y(u)\times Y(u')$, for each object u' of Σ, are objects of the full subcategory $\mathfrak{M}^\sigma$ of $\mathfrak{M}^\Sigma$, we also have

$$\hat{M}(F, \text{v}(u)) = \mathfrak{M}^\sigma(F, Y(u) \times Y\text{-}),$$

so that this functor is the dual of the composite functor G:

$$\Sigma^* \xrightarrow{Y} \mathfrak{M}^\sigma \xrightarrow{Y(u)\times -} \mathfrak{M}^\sigma \xrightarrow{\mathfrak{M}^\sigma(F, -)^*} \mathfrak{M}^*$$

where

- Y is a σ^*-structure,

- the functor $Y(u)\times -$ commutes with $\mathcal{J}$-inductive limits, since it is equivalent to the functor $-\times Y(u)$ (the product functor being symmetrical) which commutes with $\mathcal{J}$-inductive limits according to the hypothesis,

- $\mathfrak{M}^\sigma(F, -)^*$ commutes with inductive limits

Hence G is a σ^*-structure in $\mathfrak{M}^*$ and its dual $\hat{M}(F, Y(u))$ is a σ-structure in $\mathfrak{M}$.

2° Let F be a σ-structure in $\mathfrak{M}$. From Part 1, it follows that the functor $\hat{M}(F, -)\hat{Y}^*: \Sigma \to \mathfrak{M}^\Sigma$ takes its values in $\mathfrak{M}^\sigma$. So it admits as a restriction a functor $L: \Sigma \to \mathfrak{M}^\sigma$. This functor L is a σ-structure in $\mathfrak{M}^\sigma$. Indeed let us denote by π_u for each object u of Σ the «projection functor» from $\mathfrak{M}^\sigma$ to $\mathfrak{M}$, which assigns $\theta(u)$ to the σ-morphism θ. Projective limits being computed evaluationwise in $\mathfrak{M}^\sigma$ (since the insertion functor from $\mathfrak{M}^\sigma$ to $\mathfrak{M}^\Sigma$ commutes with projective limits), L is a σ-structure in $\mathfrak{M}^\sigma$ iff $\pi_u L$ is a σ-structure in $\mathfrak{M}$ for each object u of Σ. As

$$\pi_u L(x) = \hat{M}(F, \hat{Y}(x))(u) = \mathfrak{M}^\sigma(F, Y(x) \times Y(u))$$

for each $x \in \Sigma$, we get

$$\pi_u L = \mathfrak{M}^\sigma(F, (Y\text{-}) \times Y(u)).$$

The product being symmetrical, the functor $(Y\text{-}) \times Y(u)$ is equivalent to $Y(u) \times (Y\text{-})$; a fortiori $\pi_u L$ is equivalent to $\mathfrak{M}^\sigma(F, Y(u) \times Y\text{-})$, which is identical to the σ-structure $\hat{M}(F, Y(u))$. So $\pi_u L$ is a σ-structure in $\mathfrak{M}$ for each u, and L is a σ-structure in $\mathfrak{M}^\sigma$ denoted by $\overline{M}(F, Y\text{-})$.

3° Let F and F' be σ-structures in $\mathfrak{M}$. Then $\hat{M}(F', F)$ is a σ-structure in $\mathfrak{M}$. Indeed, we have

$$\hat{M}(F', F) = \mathfrak{M}^\Sigma(F', F \hat{\times} \hat{Y}\text{-}).$$

As $\hat{M}$ is a closure functor on $\mathfrak{M}^{\Sigma}$, the functor $\mathfrak{M}^{\Sigma}(F', F \hat{\times} -)$ is equivalent to $\mathfrak{M}^{\Sigma}(\hat{M}(F', -), F) = \mathfrak{M}^{\Sigma}(-, F)\hat{M}(F', -)$. It follows that the functor

$$\hat{M}(F', F) = \mathfrak{M}^{\Sigma}(F', F \hat{\times} \hat{Y}-) = \mathfrak{M}^{\Sigma}(F', F \hat{\times} -)\hat{Y}^*$$

is equivalent to the functor

$$\mathfrak{M}^{\Sigma}(\hat{M}(F', -), F)\hat{Y}^* = \mathfrak{M}^{\Sigma}(\hat{M}(F', \hat{Y}-), F) = \mathfrak{M}^{\sigma}(\bar{M}(F', Y-), F).$$

This last functor is a σ-structure in $\mathfrak{M}$, since it is the composite of the σ-structure $\bar{M}(F', Y-)$ in $\mathfrak{M}^{\sigma}$ with the functor $\mathfrak{M}^{\sigma}(-, F)$ which commutes with projective limits. Hence $\hat{M}(F', F)$ is a σ-structure in $\mathfrak{M}$, and there exists a functor $\bar{M} : \mathfrak{M}^{\sigma} \times (\mathfrak{M}^{\sigma})^* \to \mathfrak{M}^{\sigma}$ restriction of $\hat{M}$. ∇

DEFINITION. With the hypothesis of Proposition 20, for each σ-structure F in $\mathfrak{M}$ we call $\bar{M}(F, Y-)$ ($= \bar{M}(F, -)Y^*$) the *σ-structure in $\mathfrak{M}^{\sigma}$ associated to F.*

COROLLARY. *If the insertion functor I from $\mathfrak{M}^{\sigma}$ to $\mathfrak{M}^{\Sigma}$ commutes with $\mathfrak{I}$-inductive limits, then $\mathfrak{M}^{\sigma}$ admits a cartesian closed structure.*

Δ. Let u be an object of Σ. The partial product functor $- \hat{\times} Y(u)$ from $\mathfrak{M}^{\Sigma}$ to $\mathfrak{M}^{\Sigma}$ commutes with $\mathfrak{I}$-inductive limits, since it admits a right adjoint $\hat{M}(-, Y(u))$. It follows that the functor

$$P = (- \hat{\times} Y(u))I : \mathfrak{M}^{\sigma} \to \mathfrak{M}^{\Sigma}$$

also commutes with $\mathfrak{I}$-inductive limits. As P takes its values in the full subcategory $\mathfrak{M}^{\sigma}$ of $\mathfrak{M}^{\Sigma}$, there exists a functor P' from $\mathfrak{M}^{\sigma}$ to $\mathfrak{M}^{\sigma}$ restriction of P, and P' commutes with $\mathfrak{I}$-inductive limits. P' being the partial product functor $- \times Y(u)$ on $\mathfrak{M}^{\sigma}$, the hypothesis of Proposition 20 is satisfied. So the Corollary results from this Proposition. ∇

REMARK. The insertion functor I from $\mathfrak{M}^{\sigma}$ to $\mathfrak{M}^{\Sigma}$ always admits a left adjoint and $\mathfrak{M}^{\sigma}$ admits $\mathcal{F}_0$-inductive limits ([J] or [Br]). If I commutes with $\mathcal{F}_0$-inductive limits, it admits a right adjoint (Theorem 2-1 [GZ]). So the Corollary may then be deduced from the following result:

If V is a category admitting a cartesian closed structure and if V' is a full subcategory of V such that the insertion functor from V' to V admits both a left adjoint and a right adjoint, then V' admits a cartesian closed structure.

This last result proves also that, *if* σ' *is a mixed limit-bearing category* (Σ, Γ, ∇) *and if the insertion functor from* $\mathcal{M}^{\sigma'}$ *to* $\mathcal{M}^{\Sigma}$ *admits both a left adjoint and a right adjoint, then* $\mathcal{M}^{\sigma'}$ *admits a cartesian closed structure.* However this condition on σ' is very restrictive.

10. Monoidal closed categories.

A) The monoidal closed category $\mathcal{V}^{\Sigma}$.

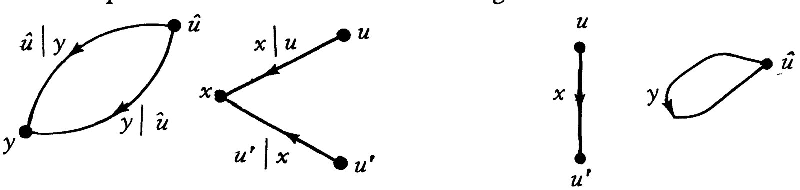
Let Σ be a category. We recall here the definition and some properties of the symmetric monoidal closed category $\mathcal{V}^{\Sigma}$ constructed by Day [D], where $\mathcal{V}$ is a symmetric monoidal closed category.

We denote by $\therefore\Sigma$ the *subdivision category of* Σ:

- its objects are the morphisms of Σ,
- for each morphism $x: u \to u'$ of Σ which does not belong to Σ_0, there are in $\therefore\Sigma$ two morphisms

$$x \,|\, u : u \to x \quad \text{and} \quad u' \,|\, x : u' \to x,$$

- there are no other morphisms in $\therefore\Sigma$, and the only composites are those of a morphism with its source and its target.



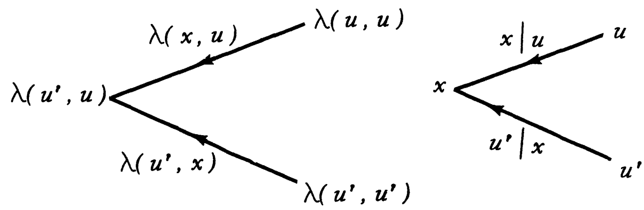
(Intuitively, x is replaced by «an abstract triangle» with vertex x). Naturally, $\therefore\Sigma$ depends on the graph underlying the category Σ and not on the law of composition of Σ.

Let V be a category. We define as follows a functor $\therefore$ from the category $V^{\Sigma\times\Sigma^*}$ to the category $V^{\therefore\Sigma}$:

If $\lambda: \Sigma\times\Sigma^* \to V$ is a functor, $\therefore(\lambda) : \therefore\Sigma \to V$ is the functor assigning $\lambda(u,u)$ to $u \in \Sigma_0$ and

$$\lambda(x,u) \;\text{ to }\; x\,|\,u, \quad \lambda(u',x) \;\text{ to }\; u'\,|\,x, \quad \lambda(u',u) \;\text{ to }\; x,$$

for each morphism $x: u \to u'$ in Σ.



- If $\theta : \lambda \rightarrow \lambda' : \Sigma \times \Sigma^* \rightrightarrows V$ is a natural transformation, $\therefore(\theta)$ is the natural transformation from $\therefore(\lambda)$ to $\therefore(\lambda')$ assigning $\theta(u', u)$ to the morphism $x : u \rightarrow u'$ in Σ.

If the functor $\therefore(\lambda)$ admits a projective limit s, then s is called *an end of* $\lambda : \Sigma \times \Sigma^* \rightarrow V$.

We say that V *admits* Σ*-ends* if V admits $\therefore\Sigma$-projective limits, i.e. if each functor $\lambda : \Sigma \times \Sigma^* \rightarrow V$ admits an end. In that case, if a choice of $\therefore\Sigma$-projective limits is done in V and if $L : V^{\therefore\Sigma} \rightarrow V$ is the corresponding canonical projective limit functor, we denote by $\int\theta$ the morphism $L(\therefore(\theta))$, for each $\theta \in V^{\Sigma \times \Sigma^*}$. We write also

$$\int_{x'x} \theta(x', x) \quad \text{instead of} \quad \int\theta$$

(the usual notation, which does not seem explicite enough, is $\int_u \theta(u,u)$).

EXAMPLE. $\mathfrak{M}$ admits Σ-ends, when $\Sigma \in \mathfrak{F}_0$. Let ψ and ψ' be two functors from Σ to $\Sigma' \in \mathfrak{F}_0$ and consider the functor

$$\Sigma'(\psi'-, \psi-) : \Sigma \times \Sigma^* \rightarrow \mathfrak{M},$$

which assigns $\Sigma'(\psi'(x'), \psi(x))$ to the pair (x', x) of morphisms of Σ. The canonical end of this functor is the set $\Sigma'^{\Sigma}(\psi', \psi)$ of natural transformations from ψ to ψ'.

From now on, we denote by $\mathcal{V}$ a symmetric monoidal closed category $(V, \tau, i, a, b, c, m, D)$. In this notation:

- V is the underlying category,
- $\tau : V \times V \rightarrow V$ is the «tensor product functor» and we write

$$g \tau f \quad \text{instead of} \quad \tau(g, f),$$

- i is the «unit» (up to an equivalence) of τ,
- the equivalences defining i as a unit of τ are

$$a: Id_V \to -\tau i \quad \text{and} \quad b: Id_V \to i\tau - ,$$

- the equivalence defining the «associativity» of τ is

$$c: -\tau(-\tau-) \to (-\tau-)\tau- : V\times V\times V \rightrightarrows V,$$

- the equivalence defining the «symmetry» of τ is

$$m: (-\tau-) \to (-\tau-)\mu : V\times V \rightrightarrows V,$$

where μ is the symmetry functor from $V\times V$ to $V\times V$, assigning

$$(g, f) \text{ to } (f, g) \in V\times V,$$

- $D: V\times V^* \to V$ is the closure functor, so that $D(-, s)$ is a right adjoint of $-\tau s$, for each object s of V.

We suppose that $V(s, i)$ belongs to the universe $\mathfrak{U}$ for each object s of V, and that V admits sums indexed by $\mathfrak{U}$-sets. Then the functor $V(-, i): V \to \mathfrak{M}$ admits a left adjoint, which we denote by q. If E is a $\mathfrak{U}$-set, $q(E)$ is a sum $\coprod_E i$ in V (of the family

$$(s_z)_{z\in E} \quad \text{where} \quad s_z = i \quad \text{for each} \quad z \in E \text{).}$$

In fact, q defines [K] a monoidal closed functor from the canonical cartesian closed category over $\mathfrak{M}$ to $\mathcal{V}$, so that the functors

$$q(-\times-) \quad \text{and} \quad \tau(q-, q-): \mathfrak{M}\times\mathfrak{M} \to V$$

are canonically equivalent.

Let Σ be a category such that V admits Σ-ends. Then Day ([D], example 5-3) has defined a symmetric monoidal closed category

$$\mathcal{V}^\Sigma = (V^\Sigma, \hat{\tau}, i\hat{}, \hat{a}, \hat{b}, \hat{c}, \hat{m}, \hat{D})$$

as follows:

- If G and G' are functors from Σ to V, the functor $G'\hat{\tau}G : \Sigma \to V$ is the functor $\tau[G', G]$ which assigns $G'(x)\tau G(x)$ to $x \in \Sigma$. If

$$\theta: G \to F : \Sigma \rightrightarrows V \quad \text{and} \quad \theta': G' \to F'$$

are natural transformations, the natural transformation

$$\theta'\hat{\tau}\theta : G'\hat{\tau}G \to F'\hat{\tau}F : \Sigma \rightrightarrows V$$

assigns $\theta'(u)\tau\theta(u)$ to the object u of Σ.

- $i^\wedge$ is the constant functor from Σ to V, whose value is the unit i,
- the natural equivalences $\hat{a}$ and $\hat{b}$ assign the natural transformations aG and bG to the functor $G : \Sigma \to V$,
- the natural equivalence $\hat{c}$ assigns

$$c\,[\,G'', G', G\,] : \Sigma \rightrightarrows V \quad \text{to} \quad (G'', G', G),$$

where G'', G' and G are functors from Σ to V,

- the equivalence $\hat{m}$ assigns $m\,[\,G', G\,] : \Sigma \rightrightarrows V$ to the pair (G', G) of functors from Σ to V,
- if G' and G are functors from Σ to V, then $\hat{D}(G', G)$ is an end of the functor from $\Sigma \times \Sigma^*$ to V^Σ assigning the natural transformation

$$D(G'(x'), -)((G(x)\,\tau -)\,q\Sigma(x, -))^* : \Sigma \rightrightarrows V$$

to the pair (x', x) of morphisms of Σ. We will write:

$$\hat{D}(G', G) = \int_{x'x} D(G'(x'), G(x)\,\tau\, q\Sigma(x, -)).$$

In fact, Day proves a stronger result: $\mathcal{V}^\Sigma$ is a symmetric monoidal closed category over $\mathcal{V}$, which means that the functors and natural transformations in the construction above underly $\mathcal{V}$-functors or $\mathcal{V}$-natural transformations. From this, we will use only that, G and G' being functors from Σ to V, the functors

$$\int_{x'x} D(-, G(x))\,\hat{D}(G', -)(x') \quad \text{and} \quad \int_{x'x} D(G'(x'), -)(G\,\hat{\tau} -)(x)^*$$

from $(V^\Sigma)^*$ to V are equivalent. (This may be proved directly, using Fubini Theorem on ends [ML] and the $\mathcal{V}$-Yoneda Lemma [K] .)

B) Subcategories of a symmetric monoidal closed category.

We suppose here that $\mathcal{V}$ is a symmetric monoidal closed category

$$\mathcal{V} = (V, \tau, i, a, b, c, m, D),$$

and V' a full subcategory of V which is closed for D, i.e. such that it exists a functor $D' : V' \times V'^* \to V'$ restriction of D. Then, under some conditions, V' underlies a symmetric monoidal closed category having D' as its closure functor. This will be applied in the next Section to the subcategory V^σ of V^Σ.

If V' is also closed for τ, i.e. if it exists a functor

$$\tau' : V' \times V' \to V' \quad \text{restriction of } \tau,$$

and if i is an object of V', then the natural equivalences a, b, c and m admit restrictions a', b', c' and m' such that

$$(V', \tau', i, a', b', c', m', D')$$

is a symmetric monoidal closed subcategory of $\mathcal{V}$. More generally:

PROPOSITION 21. *We suppose that V' is a full subcategory of V, such that:*

1° there exists a functor $D' : V' \times V'^* \to V'$ restriction of D,

2° the insertion functor I from V' to V admits a left adjoint J.

Then there exists a symmetric monoidal closed category

$$(V', \tau', J(i), a', b', c', m', D'), \text{ where } f'\tau' f = J(f'\tau f).$$

Δ. We denote by $\delta : Id_V \to IJ$ the natural transformation defining J as an adjoint of I, by i' the object $J(i)$ of V' and by τ' the composite functor $J\,\tau(I-, I-)$:

$$V' \times V' \xrightarrow{I \times I} V \times V \xrightarrow{\tau} V \xrightarrow{J} V'$$

which assigns $J(f'\tau f)$ to the pair (f', f) of morphisms of V'.

1° Let s' be an object of V'. The functor $-\tau' s' : V' \to V'$ admits $D'(-, s')$ as a right adjoint. Indeed, as $D(-, s')$ is a right adjoint of $-\tau s'$, the functor $D(-, s')I$ is a right adjoint of $J(-\tau s')$. As V' is a full subcategory of V in which $D(-, s')I$ takes its values, the restriction

$$D'(-, s') : V' \to V' \quad \text{of} \quad D(-, s')I$$

is also a right adjoint of the functor from V' to V' restriction of $J(-\tau s')$, i. e. of the functor $-\tau' s'$.

If τ' is a tensor-product functor on V' whose unit is i', Proposition 21 will result from Theorem II-5-8 of [EK].

2° We will establish some facts to be used afterwards.

a) Let s' and s'' be objects of V' and e of V. Then the maps

$$V(s'', \delta(e)\tau s') \quad \text{and} \quad V(s'', s'\tau\delta(e))$$

are bijections. Indeed, we denote by:

- $p(s'', s') : D(s'', s')\tau s' \to s''$ the morphism defining $D(s'', s')$

as a cofree structure generated by s'' relative to the functor $-\tau s'$,

- $P(s'', s', e): V(D(s'', s'), e) \to V(s'', e\tau s')$ the bijection assigning $p(s'', s').(g\tau s')$ to $g: e \to D(s'', s')$.

Since V' is closed for D, the object $D(s'', s')$ belongs to V', and, from the adjonction between I and J, we deduce that

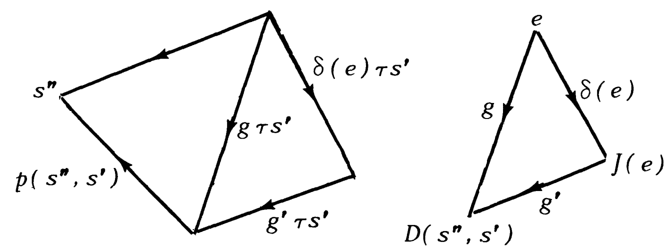
$$V(D(s'', s'), \delta(e)): V(D(s'', s'), J(e)) \to V(D(s'', s'), e)$$

is a bijection. The composite bijection

$$P(s'', s', e)\, V(D(s'', s'), \delta(e))\, P(s'', s', J(e))^{-1}$$

assigns $f'.(\delta(e)\tau s')$ to $f': J(e)\tau s' \to s''$; so it is the map

$$V(s'', \delta(e)\tau s'): V(s'', J(e)\tau s') \to V(s'', e\tau s').$$



The map $V(s'', s'\tau\delta(e))$ is also a bijection, the equality

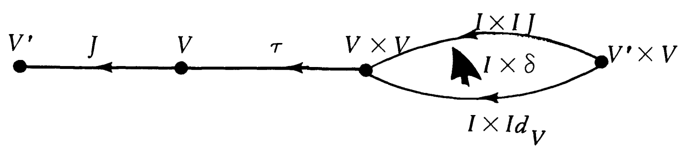
$$s'\tau\delta(e) = m(J(e), s').(\delta(e)\tau s').m(s', e)^{-1}.$$

implying that $V(s'', s'\tau\delta(e))$ is the composite bijection

$$V(s'', m(s', e)^{-1})\, V(s'', \delta(e)\tau s')\, V(s'', m(J(e), s')).$$

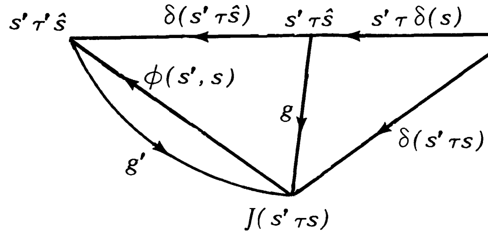
b) ϕ is the natural transformation

$$J\tau(I\times\delta): J\tau(I-,-) \to \tau'(-, J-): V'\times V \rightrightarrows V'.$$



It is an equivalence, i.e. $J(s'\tau J(s))$ is a free structure generated by $s'\tau s$ relative to I. Indeed. let s be an object of V and s' of V'; we write $\hat{s} = J(s)$; then $\phi(s', s) = J(s'\tau\delta(s))$ is the unique f such that

$$f.\delta(s'\tau s) = \delta(s'\tau\hat{s}).(s'\tau\delta(s)).$$



From Part a, it follows that $V(J(s' \tau s), s'\tau\delta(s))$ is a bijection, so that there exists a unique morphism

$$g : s' \tau \hat{s} \to J(s' \tau s) \quad \text{satisfying} \quad g.(s'\tau\delta(s)) = \delta(s' \tau s).$$

There also exists a unique morphism g' such that $g'. \delta(s' \tau \hat{s}) = g$. From the equalities

$$g'.f. \delta(s' \tau s) = g'. \delta(s' \tau \hat{s}).(s'\tau\delta(s)) = g.(s'\tau\delta(s)) = \delta(s' \tau s),$$

we get $g'.f = J(s' \tau s)$ and, from the equalities

$$f. g'. \delta(s' \tau \hat{s}).(s' \tau\delta(s)) = f. \delta(s' \tau s) = \delta(s' \tau \hat{s}).(s' \tau\delta(s)),$$

we deduce successively

$$f. g'. \delta(s' \tau \hat{s}) = \delta(s' \tau \hat{s}),$$

since $V(s'\tau' \hat{s}, s' \tau\, \delta(s))$ is a bijection and $f. g' = J(s' \tau \hat{s})$.

This proves that $\phi(s', s)$ admits g' as an inverse, and ϕ is an equivalence.

c) Similarly,

$$\phi' = J\, \tau (\delta \times I) : V \times V' \rightrightarrows V'$$

is an equivalence.

3° We are going to show that τ' is a tensor-product functor whose unit is $i' = J(i)$.

a) If s is an object of V', we denote by $a'(s)$ the morphism

$$\delta(s\, \tau i').(s\, \tau\, \delta(i)).a(s) : s \to s\, \tau' i'.$$

We so define a natural transformation $a' : Id_{V'} \to - \tau' i'$ such that $I a'$ is the natural transformation

$$(\delta(- \tau i') \square\square (- \tau\, \delta(i)) \square\square a) I.$$

The morphism $a(s)$ being invertible and $\delta(s\tau i').(s\tau\delta(i))$ defining $s\tau' i'$ as a free structure generated by $s\tau i$ (Part 2-b), the morphism $a'(s)$ defines $s\tau' i'$ as a free structure generated by the object s of the full subcategory V' relative to the insertion functor I. Hence $a'(s)$ is invertible. So a' is an equivalence.

We define similarly the equivalence $b': Id_{V'} \to i'\tau'-$, which assigns $\delta(i'\tau s).(\delta(i)\tau s).b(s)$ to the object s of V'.

b) $\mu: V\times V \to V\times V$ and $\mu': V'\times V' \to V'\times V'$ being the «symmetry functors», we have $\mu(I-, I-) = (I\times I)\mu'$. The equivalence $m: \tau \to \tau\mu$ defining the symmetry of τ gives rise to the equivalence

$$m' = J\,m(I-, I-): \tau' \to J\,\tau\mu(I-, I-),$$

which assigns the invertible morphism $J(m(s', s))$ to the pair (s', s) of objects of V'. As

$$J\,\tau\mu(I-, I-) = J\,\tau(I-, I-)\mu' = \tau'\mu',$$

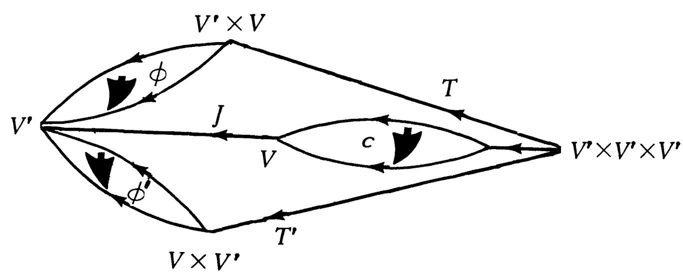
the equivalence m' is a symmetry of τ'.

c) We consider the functors

$$T: V'\times V'\times V' \to V'\times V \quad\text{and}\quad T': V'\times V'\times V' \to V\times V'$$

assigning to (x'', x', x) respectively $(x'', x'\tau x)$ and $(x''\tau x', x)$. With the notations of Part 2, let c' be the natural transformation

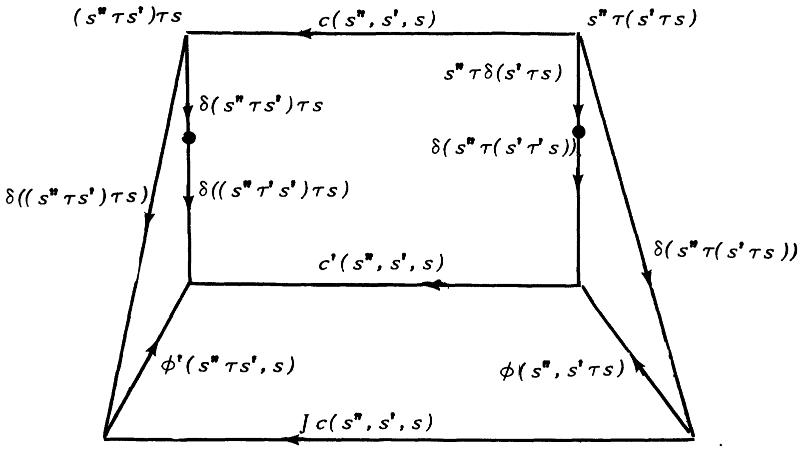
$$\phi' T' \square\square\, J\,c(I\times I\times I) \square\square (\phi T)^{-1}: -\tau'(-\tau'-) \to (-\tau'-)\tau'-,$$



which assigns

$$c'(s'', s', s) = \phi'(s''\tau s', s).\,J\,c(s'', s', s).\,\phi(s'', s'\tau s)^{-1}$$

to (s'', s', s), where s'', s' and s are objects of V'.



Since ϕ, Jc and ϕ' are equivalences, c' is also an equivalence. To prove that

$$(V', \tau', i', a', b', c', m', D')$$

is a symmetric monoidal closed category, we have yet to show that the three coherence axioms are satisfied.

d) The coherence axiom on units asserts that, if s and s' are objects of V', then

$$c'(s', i', s).(s' \tau' b'(s)) = a'(s') \tau' s.$$

Indeed, we have the following diagram, where

① is a quartet, c being a natural transformation,

② is a quartet, by definition of c',

③ is commutative, since we have

$$a'(s') = \delta(s' \tau i').(s' \tau \delta(i)).a(s'),$$

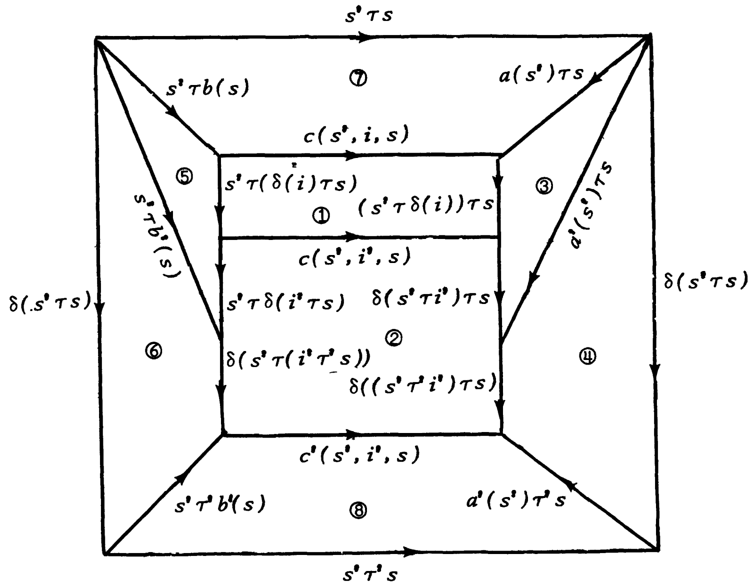
④ is commutative, as a consequence of the equality

$$a'(s') \tau' s = J(a'(s') \tau s),$$

⑤ is commutative, by definition of b',

⑥ is commutative, by definition of τ' (similarly to ④),

⑦ is commutative, the first coherence axiom being satisfied in the monoidal category (V, τ, i, a, b, c).



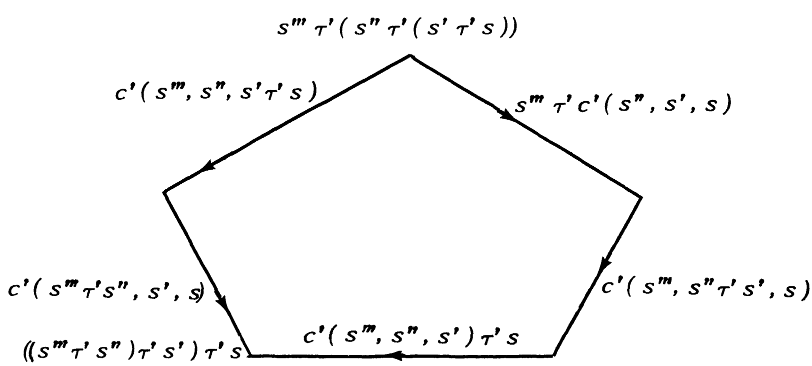
From this diagram we deduce

$$c'(s', i', s).(s' \tau' b'(s)).\ \delta(s' \tau s) = (a'(s') \tau' s).\delta(s' \tau s),$$

which implies, $\delta(s' \tau s)$ defining $s' \tau' s$ as a free structure generated by $s' \tau s$ relative to I,

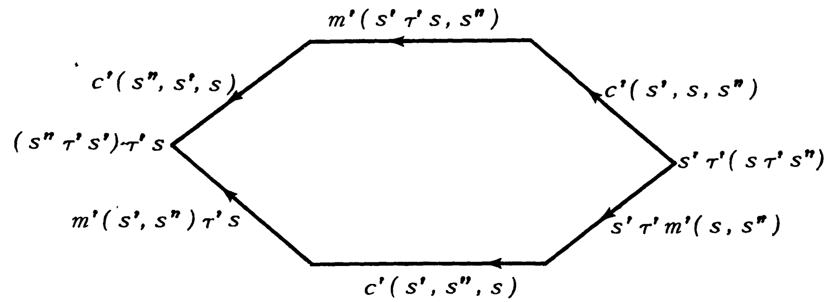
$$c'(s', i', s).(s' \tau' b'(s)) = a'(s') \tau' s.$$

e) We consider the second coherence axiom (on associativity), called axiom MC3 [EK], which says that, if s, s', s'' and s''' are objects of V', the following diagram commutes.



This may be proved directly from the axiom MC3 satisfied by τ and from the definition of c'. But we can also use Proposition II-2-1 [EK], since D' and τ' are in the «basic situation» of Chapter II-4 [EK]. So τ' satisfies the axiom MC3 iff D' satisfies the axiom CC3 of [EK] (associativity coherence axiom for closed categories). As D is a closure functor on V, it satisfies CC3 and, V' being a full subcategory of V, the restriction D' of D also verifies CC3 (which is independent of i and τ). Hence τ' is a tensor-product functor on V'.

f) The coherence axiom on symmetry asserts that, if s, s' and s'' are objects of V', the following diagram commutes:



This diagram is the exterior border of the following diagram, where:

① , ④ and ⑦ commute, by definition of c',

② commutes, m being a natural transformation,

③ , ⑤ and ⑧ commute, since $m' = J\,m(I-, I-)$,

⑥ and ⑨ commute, as $\tau' = J\,\tau(I-, I-)$,

⑩ commutes, (V, τ, i, a, b, c, m) being a symmetric monoidal category, the mapping

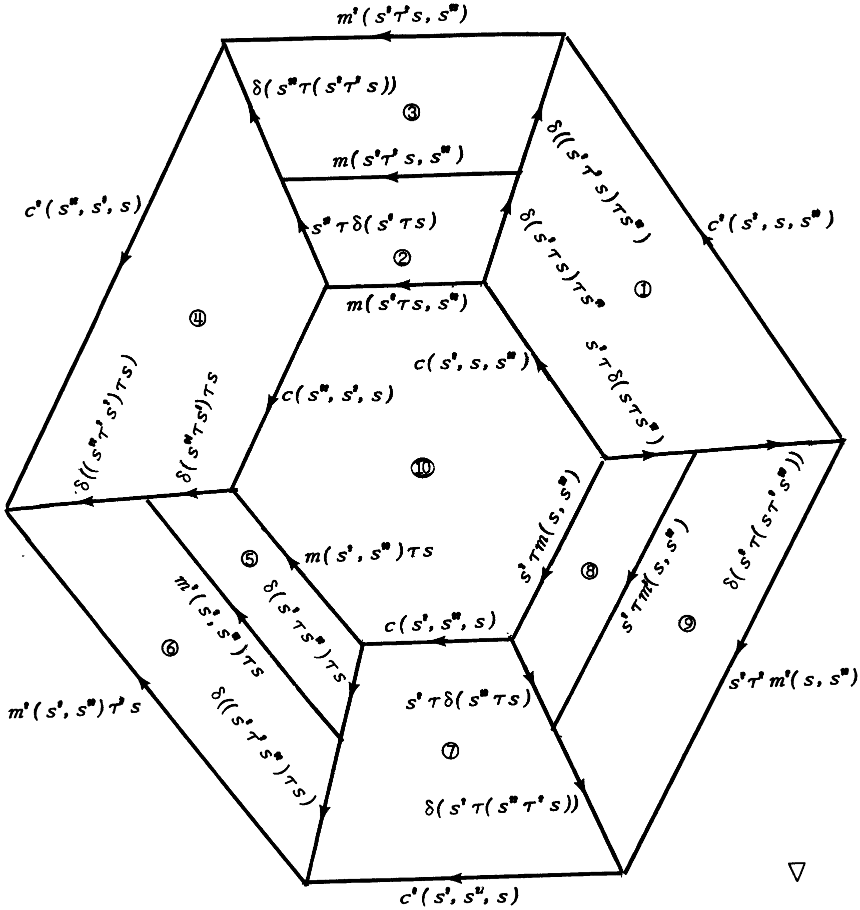
$$V((s''\tau' s')\tau' s,\ \delta(s'\tau(s\tau' s'')).(s'\tau\delta(s\tau s'')))$$

is a bijection (Part 2-b).

From all these properties, we deduce that the exterior border of this diagram commutes. Hence all the coherence axioms are satisfied, so that

$$(V', \tau', i', a', b', c', m', D')$$

is a symmetric monoidal closed category.



▽

COROLLARY. *With the hypotheses of Section A on* $\mathcal{V}$, *let* σ' *be a mixed limit-bearing category* (Σ, Γ, ∇) *such that:*

1° the insertion functor from $V^{\sigma'}$ *to* V^{Σ} *admits a left adjoint,*

2° $V^{\sigma'}$ *is closed for the closure functor* $\hat{D}$ *of the symmetric monoidal closed category* $\mathcal{V}^{\Sigma}$.

Then $V^{\sigma'}$ *is underlying a symmetric monoidal closed category whose closure functor is a restriction of* $\hat{D}$.

Δ. This results from Proposition 21 applied to $\mathcal{V}^{\Sigma}$ and $V^{\sigma'}$. ▽

PROPOSITION 22. *Let* V' *be a full subcategory of* V *such that:*

1° $i \in V'$ *and* τ *admits a restriction* $\tau' : V' \times V' \to V'$, *2° the insertion functor I from V' to V admits a right adjoint I'. Then there exists a symmetric monoidal closed category*

$$(V', \tau', i, a', b', c', m', D'), \quad \text{where} \quad D' = I'D(I-, I-).$$

Δ. The first condition implies that V' defines a symmetric monoidal subcategory $(V', \tau', i, a', b', c', m')$ of (V, τ, i, a, b, c, m).

Proposition 22 will result from Theorem II-5-8 [EK] if we prove that, for each object s' of V', the functor $-\tau' s' : V' \to V'$ admits $D'(-, s')$ as a right adjoint. Indeed, $I'D(-, s')$ is a right adjoint of $(-\tau s')I$. As V' is a full subcategory of V, it follows that the functor $D'(-, s'): V' \to V'$, restriction of $I'D(-, s')$, is a right adjoint of the functor $-\tau' s'$, restriction of $(-\tau s')I$. ∇

COROLLARY 1. *With the hypotheses of Section A on $\mathcal{V}$, let σ' be a mixed limit-bearing category (Σ, Γ, ∇) such that:*

1° the insertion functor from $V^{\sigma'}$ to V^{Σ} admits a right adjoint,

2° $V^{\sigma'}$ is closed for the tensor product $\hat{\tau}$ of $\mathcal{V}^{\Sigma}$, and $i^{\wedge}$ is a σ'-structure in V.

Then $V^{\sigma'}$ is underlying a symmetric monoidal closed category whose tensor product is a restriction of $\hat{\tau}$.

COROLLARY 2. *Let σ be a $\mathcal{J}$-limit-bearing category (Σ, Γ) and V a category admitting $\mathcal{J}$-projective limits, sums indexed by $\mathcal{U}$-sets and Σ-ends. If V admits a cartesian closed structure, and if the insertion functor from V^{σ} to V^{Σ} admits a right adjoint, then V^{σ} admits a cartesian closed structure (deduced from that of V^{Σ}).*

Δ. V^{σ} being closed for finite products in V^{Σ}, this results from Corollary 1, applied to a symmetric cartesian closed category $\mathcal{V}$ over V. ∇

11. Symmetric monoidal closed category $\mathcal{V}^{\sigma}$.

If σ is «cartesian», V^{σ} is closed for the closure functor of $\mathcal{V}^{\Sigma}$ (section 10-A), so that the preceding corollaries give symmetric monoidal closed structures on V^{σ}.

As in the sections 10 and 11, we still denote by σ a projective limit-bearing category (Σ, Γ) whose set of morphisms is a $\mathcal{U}$-set, by $\mathcal{J}$ its set of indexing-categories, by Y the Yoneda σ^*-structure in $\mathfrak{M}^\sigma$.

DEFINITION. We say that σ is *cartesian* if the functor $- \times Y(u) : \mathfrak{M}^\sigma \to \mathfrak{M}^\sigma$ commutes with $\mathcal{J}$-inductive limits, for each object u of Σ.

Proposition 20 says that σ is cartesian iff $\mathfrak{M}^\sigma$ admits a cartesian closed structure.

In all this Section, we will denote by V a category satisfying the following condition

(L)
- $V(s', s)$ is a $\mathfrak{U}$-set, for any pair (s', s) of objects of V.
- V admits Σ-ends.
- V admits sums indexed by $\mathfrak{U}$-sets (this property may be replaced by a weaker one, as is shown in Remark 2, after Proposition 23).

Finally, $\mathcal{V}$ denotes a symmetric monoidal closed category

$$(V, \tau, i, a, b, c, m, D),$$

$q : \mathfrak{M} \to V$ an adjoint of $V(-, i)$ and $\mathcal{V}^\Sigma$ the corresponding symmetric monoidal closed category constructed by Day (Section 10-A):

$$\mathcal{V}^\Sigma = (V^\Sigma, \hat{\tau}, i\hat{}, \hat{a}, \hat{b}, \hat{c}, \hat{m}, \hat{D}).$$

PROPOSITION 23. *We suppose σ is cartesian. Then: there exists a functor $\hat{D}' : V^\sigma \times (V^\sigma)^* \to V^\sigma$ restriction of $\hat{D}$; for each σ-structure G in V, the functor $\hat{D}'(G, qY-)$ assigning $\hat{D}(G, qY(x))$ to $x \in \Sigma$ is a σ-structure in V^σ. Finally, if G and G' are σ-structures in V, we have*

$$\hat{D}'(G', G) \approx \int_{x'x} D(-, G(x)) \hat{D}'(G', qY(x')).$$

Δ. 1° Let u be an object of Σ and G a σ-structure in V. For each object s of V, let G_s be the functor $V(\hat{D}(G, qY(u)), s)$. We are going to prove that the functors G_s are σ-structures in $\mathfrak{M}$. This will imply [Lb] that $\hat{D}(G, qY(u))$ is a σ-structure in V, so that the functor $\hat{D}'(G, qY-)$ from Σ to V^σ exists. Indeed, as $V(-, s)$ commutes with projective limits and

$$\hat{D}(G, qY(u)) = \int_{x'x} D(G(x'), qY(u)(x) \tau q\Sigma(x, -)),$$

the functor G_s is an end of the functor $F : \Sigma \times \Sigma^* \to \mathfrak{M}^\Sigma$ assigning

$$V(-, s) D(G(x'), -)(qY(u)(x) \tau q\Sigma(x, -))^* \text{ to } (x', x).$$

The functors $V(D(G-, -), s)$ and $V(D(G-, s), -)$, from $\Sigma \times V^*$ to $\mathfrak{M}$

are equivalent, $\mathcal{V}$ being a symmetric monoidal closed category, as well as the functors

$$(q-)\tau(q-) \text{ and } q(-\times-): \mathfrak{M}\times\mathfrak{M} \to V.$$

So F is equivalent to the functor F' assigning

$$V(D(G(x'),s),q-)(Y(u)(x)\times\Sigma(x,-))^* \text{ to } (x',x)$$

where x and x' are morphisms of Σ. Since q is adjoint to $V(-,i)$ and, by definition of a symmetric monoidal closed category, $V(D-,i)$ is equivalent to $V(-,-)$, the functor F' is equivalent to the functor F'' from $\Sigma\times\Sigma^*$ to $\mathfrak{M}^\Sigma$ assigning to (x',x):

$$\mathfrak{M}(V(G(x'),s),-)(Y(u)\times Y(-))(x)^*: \Sigma \rightrightarrows \mathfrak{M}.$$

It follows that G_s is also an end of F''.

As G is a σ-structure in V and $V(-,s)$ commutes with projective limits, the functor $\hat{G} = V(G-,s)$ is a σ-structure in $\mathfrak{M}$. We consider the σ-structure $\bar{M}(\hat{G},Y-)$ in $\mathfrak{M}^\sigma$ associated to $\hat{G}$ (Proposition 20); we have

$$\bar{M}(\hat{G},Y(u)) = \mathfrak{M}^\sigma(\hat{G},Y(u)\times Y-) \approx \int F'',$$

by definition of the set of natural transformations between two functors as an end. So G_s is equivalent to the σ-structure $\bar{M}(\hat{G},Y(u))$ in $\mathfrak{M}$. A fortiori, G_s is a σ-structure in $\mathfrak{M}$, for any object s of V.

2° Let G be a σ-structure in V. Then $\hat{D}'(G,qY-)$ is a σ-structure $\bar{G}$ in V^σ, equivalent to $\bar{\bar{G}}: \Sigma \to V^\sigma$, where $\bar{\bar{G}}(x'): \Sigma \rightrightarrows V$ is defined by $\bar{\bar{G}}(x')(y) = \bar{G}(y)(x)$, for $y \in \Sigma$. The proof is similar to Part 2, Prop. 20.

3° If G and G' are σ-structures in V, then $\hat{D}(G',G)$ is a σ-structure in V. Indeed, $\hat{D}(G',G)$ is an end of the functor $H: \Sigma\times\Sigma^* \to V^\Sigma$, assigning

$$D(G'(x'),-)(G(x)\tau q-)^*\Sigma(x,-)^* = D(G'(x'),-)(G\hat{\tau}qY(-))(x)^*$$

to (x',x). The functors

$$\int_{x'x} D(-,G(x))\hat{D}(G',-)(x') \text{ and } \int_{x'x} D(G'(x'),-)(G\hat{\tau}-)(x)^*$$

being equivalent (section 10-A), $\hat{D}(G',G)$ is also an end of the functor H' from $\Sigma\times\Sigma^*$ to V^Σ assigning

$$D(-,G(x))\hat{D}'(G',qY(x')) \text{ to } (x',x).$$

If u and u' are objects of Σ, the functor $H'(u', u)$ is a σ-structure in V, for it is the composite functor

$$\Sigma \xrightarrow{\hat{D}(G', qY(u'))} V \xrightarrow{D(-, G(u))} V$$

where $\hat{D}(G', qY(u'))$ is a σ-structure in V (Part 1) and $D(-, G(u))$ commutes with projective limits. Hence, H' takes its values in V^σ. As V^σ is closed for Σ-ends in V^Σ (the category V admitting Σ-ends), it follows that the end $\hat{D}(G', G)$ of H' is a σ-structure in V.

This proves the existence of a functor $\hat{D}': V^\sigma \times (V^\sigma)^* \to V^\sigma$, restriction of $\hat{D}$. ∇

COROLLARY 1. *If σ is cartesian and if the insertion functor $\hat{I}$ from V^σ to V^Σ admits a left adjoint $\hat{J}$, it exists a symmetric monoidal closed category*

$$\mathcal{V}^\sigma = (V^\sigma, \hat{\tau}', \hat{i}', \hat{a}', \hat{b}', \hat{c}', \hat{m}', \hat{D}'),$$

where $\hat{D}'$ is a restriction of $\hat{D}$ and $\hat{\tau}'$ a restriction of $\hat{J}\,\hat{\tau}$.

Δ. By Proposition 23, V^σ is closed for $\hat{D}$. So this corollary results from the corollary of Proposition 21. ∇

COROLLARY 2. *Under the following conditions, V^σ defines a symmetric monoidal closed subcategory $\mathcal{V}^\sigma$ of $\mathcal{V}^\Sigma$:*

1° σ is cartesian,

2° τ commutes with $\mathcal{G}$-projective limits,

3° $i^\wedge$ is a σ-structure in V (for example, if all the indexing-categories of σ are connected or if i is a final object of V).

Δ. Proposition 23 asserts that V^σ is closed for $\hat{D}$.

If G and G' are σ-structures in V, the functor

$$G' \hat{\tau} G = \tau [G', G] : \Sigma \to V$$

is a σ-structure in V, since $[G', G]$ is a σ-structure in $V \times V$ and τ is commuting with $\dot{\mathcal{G}}$-projective limits. Hence V^σ is also closed for $\hat{\tau}$. Since $i^\wedge$ belongs to V^σ (condition 3), V^σ defines a symmetric monoidal closed subcategory of $\mathcal{V}^\Sigma$. ∇

COROLLARY 3. *We suppose σ is cartesian and V is a category admitting* *a cartesian closed structure. If V satisfies condition* (L), *then V^σ admits a cartesian closed structure.*

Δ. Let $\mathcal{V}$ be any symmetric cartesian closed category whose underlying category is V (it is defined up to an isomorphism). Its tensor-product functor τ is in fact a product functor, so that it commutes with $\mathcal{J}$-projective limits. As i is then a final object of V, the constant functor $i^\wedge$ commutes with projective limits; a fortiori, it is a σ-structure in V. Hence Corollary 2 asserts that V^σ defines a symmetric monoidal closed subcategory $\mathcal{V}^\sigma$ of $\mathcal{V}^\Sigma$. Since $\mathcal{V}^\Sigma$ is cartesian, $\hat{\tau}$ being a product functor, so is $\mathcal{V}^\sigma$. ∇

EXAMPLE. Let V be a category admitting $\mathcal{F}_0$-projective limits and $\mathcal{F}_0$-inductive limits; so it satisfies (L). If V satisfies also the condition:

(L') There exists an $\mathcal{U}$-ordinal ξ such that $\mathcal{J}$-projective limits commute with inductive limits indexed by $<\xi>$, in V,

the insertion functor from V^σ to V^Σ admits a left adjoint [F1]. Then, by Corollary 1, V^σ underlies a symmetric monoidal closed category as soon as σ is cartesian. The condition (L') is verified, for instance, when V is locally ξ-presentable [GU], or when V is a fibred category over a category satisfying (L') (see [W]).

REMARKS. 1° The third property of Condition (L) may be replaced everywhere by the less restrictive condition:

V admits sums indexed by the sets $\Sigma(u', u)$, where u and u' are objects of Σ.

Indeed, in this case, let $\mathcal{M}'$ be the full subcategory of $\mathcal{M}$ whose objects are the $\mathcal{U}$-sets E such that there exists in V a sum $\coprod_E i$ (of E exemplars of i), where i is still the unit of the symmetric monoidal closed category $\mathcal{V}$. Choosing such a sum $q'(E)$ for each object E of $\mathcal{M}'$, we get a functor $q' : \mathcal{M}' \to V$, which is a «partial adjoint» of $V(\cdot, i)$. The sets $\Sigma(u', u)$ are objects of $\mathcal{M}'$. If E and E' are objects of $\mathcal{M}'$, then $q'(E')\tau q'(E)$ is a sum of $E' \times E$ exemplars of i, since $\cdot\, \tau q'(E)$, being a left adjoint, commutes with sums. Hence $\mathcal{M}'$ is closed for finite products, and the functors

$$q'(\cdot \times \cdot) : \mathcal{M}' \times \mathcal{M}' \to V \quad \text{and} \quad \tau(q'\cdot, q'\cdot)$$

are equivalent. The proofs of Proposition 23 and of its Corollaries using only the values of q on sets $E' \times E$, where E and E' are of the form $\Sigma(u', u)$, they are also valid if we replace q by q'.

2° We have not used the general result of Day [D], but a very special case of it (Example 5-3 of [D]). In fact, Day associates to any «premonoidal symmetric structure» $P : \Sigma^* \times \Sigma^* \times \Sigma \to V$ a symmetric monoidal closed category P^{Σ} whose underlying category is V^{Σ}. In a forthcoming paper [FL] Foltz and Lair prove that V^{σ} is also closed for the closure functor of P^{Σ} when P defines a double σ-costructure in V, i.e. when there exists a σ^*-structure p in $(V^{\sigma})^{\sigma^*}$ such that

$$p(y)(x')(x) = P(y, x', x), \text{ when } x, x' \text{ and } y \text{ belong to } \underline{\Sigma}.$$

So, in this case and if the insertion functor from V^{σ} to V^{Σ} admits a left adjoint, V^{σ} is underlying a symmetric monoidal closed category P^{σ}. Notice that Proposition 23 and its corollaries cannot be deduced from this result of [FL]. Indeed, the category $\mathfrak{V}^{\Sigma}$ used here is the category P^{Σ} associated to the premonoidal structure P such that

$$P(y, x', x) = (qY(y) \hat{\tau}\, qY(x'))(x),$$

and P does not define a double σ-costructure, even if σ is cartesian.

Application.

We denote by:

- $\mathfrak{S}$ a sketch (definition p. 30) $(\hat{\Sigma}, \hat{\Gamma})$, where $\hat{\underline{\Sigma}}$ is a $\mathfrak{U}$-set, $\mathfrak{I}$ its set of indexing-categories and σ a prototype (Σ, Γ) generated by $\mathfrak{S}$,
- σ' a $\mathfrak{I}$-type (V, Γ'), where V is a category satisfying Condition (L),
- $\mathcal{S}(\sigma', \mathfrak{S})$ the category of $\mathfrak{S}$-morphisms (i.e. of morphisms between $\mathfrak{S}$-structures) in σ',
- $\mathcal{S}(\sigma_{\mathfrak{M}}, \mathfrak{S})$ the category of $\mathfrak{S}$-morphisms in the canonical $\mathfrak{I}$-type $\sigma_{\mathfrak{M}} = (\mathfrak{M}, \Gamma_{\mathfrak{M}})$ on $\mathfrak{M}$.

PROPOSITION 24. *We suppose that $\mathfrak{S}$ is σ'-regular and $\sigma_{\mathfrak{M}}$-regular (definition p. 47) and that $\mathcal{S}(\sigma_{\mathfrak{M}}, \mathfrak{S})$ admits a cartesian closed structure. Then:*

1° $\mathcal{S}(\sigma', \mathfrak{S})$ is underlying a symmetric monoidal closed category if V underlies a symmetric monoidal closed category $\mathfrak{V}$ and if one of the following *conditions is satisfied:*

a) There exists an ordinal ξ *in* $\mathfrak{U}$ *such that the* $\mathfrak{I}$*-projective limits in* V *commute with the inductive limits indexed by* $<\xi>$.

b) τ *commutes with* $\mathfrak{I}$*-projective limits and* $i\hat{}$ *is a* σ*-structure in* V.

2° $\mathcal{S}(\sigma', \hat{\sigma})$ *admits a cartesian closed structure if* V *admits one.*

Δ. 1° σ is cartesian. Indeed, $\hat{\sigma}$ being $\sigma_{\mathfrak{M}}$-regular, the categories $\mathfrak{M}^{\hat{\sigma}}$ and $\mathcal{S}(\sigma_{\mathfrak{M}}, \hat{\sigma})$ are equivalent. By Proposition 7 (page 30), the prototype σ generated by $\hat{\sigma}$ is also a limit-bearing category generated by $\hat{\sigma}$, so that $\mathfrak{M}^{\sigma}$ is isomorphic to $\mathfrak{M}^{\hat{\sigma}}$. Hence, $\mathfrak{M}^{\sigma}$ is equivalent to $\mathcal{S}(\sigma_{\mathfrak{M}}, \hat{\sigma})$ and, $\mathcal{S}(\sigma_{\mathfrak{M}}, \hat{\sigma})$ admitting a cartesian closed structure, $\mathfrak{M}^{\sigma}$ also admits one. It follows (Proposition 20) that σ is cartesian.

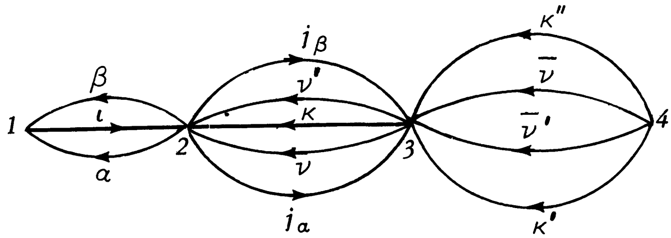
2° The categories $\mathcal{S}(\sigma', \hat{\sigma})$ and $\mathcal{S}(\sigma', \sigma)$ are isomorphic (Corollary 2, Proposition 6). As $\hat{\sigma}$ is σ'-regular, the Corollary of Proposition 16 asserts that the prototype σ generated by $\hat{\sigma}$ is also σ'-regular. This implies that the category V^{σ} is equivalent to $\mathcal{S}(\sigma', \sigma)$, and a fortiori to $\mathcal{S}(\sigma', \hat{\sigma})$. Hence $\mathcal{S}(\sigma', \hat{\sigma})$ underlies a symmetric monoidal closed category iff V^{σ} is underlying one; so the proposition results from the corollaries of Proposition 23 and from the Example. ∇

12. Application to categories of structured functors.

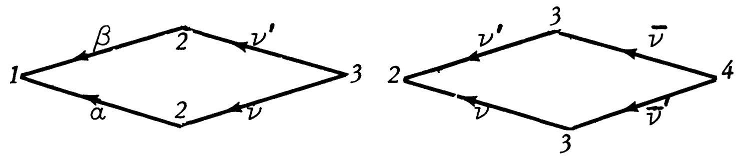
Applying the preceding results to the «sketch of categories», we deduce, from a monoidal closed structure on V, a similar one on the category of functors in V (or category of categories in V).

An integer n is considered as being the set $\{0, 1, \dots, n-1\}$ (i.e. as a finite ordinal); we denote by $\mathbf{n}$ the category $<n>$ defining the usual order on n.

Let Δ be the simplicial category: its objects are the integers, its morphisms are the monotone maps between integers equipped with their usual order. We denote by Σ the dual of the full subcategory of Δ whose objects are *1, 2, 3* and *4*. A set of generators of Σ is formed by the morphisms drawn in the following diagram and by three other morphisms from *3* to *4*. The denomination of the morphisms will result from the following properties:



In Σ, we have the two pullbacks



and ι is a kernel of the pair $(2, \iota . \alpha)$.

We denote by:

- I the subdivision category of **2** and I' the category with two objects and two morphisms with the same source and the same target:

- γ and γ' the projective cones indexed by I defining respectively the pullbacks $((\alpha, \nu), (\beta, \nu'))$ and $((\nu, \bar{\nu}'), (\nu', \bar{\nu}))$, so that

$$\gamma(0) = \nu, \quad \gamma(1) = \nu', \quad \gamma'(0) = \bar{\nu}', \quad \gamma'(1) = \bar{\nu}.$$

- γ'' the projective cone indexed by I' and defining ι as a kernel of $(2, \iota . \alpha)$, so that $\gamma''(0) = \iota$.
- Γ the set $\{\gamma, \gamma'\}$ and $\bar{\Gamma}$ the set $\{\gamma, \gamma', \gamma''\}$.
- $\mathfrak{I}$ the singleton $\{I\}$ and $\bar{\mathfrak{I}}$ the set $\{I, I'\}$.
- σ and $\bar{\sigma}$ the pairs (Σ, Γ) and $(\Sigma, \bar{\Gamma})$.
- $\sigma_{\mathfrak{M}}$ the canonical $\bar{\mathfrak{I}}$-type $(\mathfrak{M}, \Gamma_{\mathfrak{M}})$ on the category $\mathfrak{M}$.

PROPOSITION 25. *σ and $\bar{\sigma}$ are regular prototypes, which are cartesian. The category $\mathcal{S}(\sigma_{\mathfrak{M}}, \bar{\sigma})$ is isomorphic to the category $\mathcal{F}$ of functors and $\mathfrak{M}^{\sigma}$ is equivalent to $\mathcal{F}$.*

Δ. 1° $\bar{\sigma}$ is a prototype, γ, γ' and γ'' being limit-cones. Let U be the subcategory of Σ generated by the set of morphisms drawn in the diagram above and $\sigma_{\mathcal{F}}$ the presketch obtained by equipping U with the cones (restrictions to U of) γ, γ' and γ''. Then $\sigma_{\mathcal{F}}$ is the sketch of categories considered in [E2], and $\mathcal{S}(\sigma_{\mathfrak{M}}, \sigma_{\mathcal{F}})$ is isomorphic to $\mathcal{F}$.

Each morphism y in Σ and not in U having the vertex 4 of γ' as its target, it is the factor of $\gamma' y$ through γ'; so the construction of the prototype generated by $\sigma_{\mathcal{F}}$ (Proposition 6) stops at the first step and gives $\bar{\sigma}$. Corollary 2, Proposition 6, asserts that $\mathcal{S}(\sigma_{\mathfrak{M}}, \bar{\sigma})$ is isomorphic to $\mathcal{S}(\sigma_{\mathfrak{M}}, \sigma_{\mathcal{F}})$. So there exists an isomorphism from $\mathcal{S}(\sigma_{\mathfrak{M}}, \bar{\sigma})$ to $\mathcal{F}$; it assigns to the $\bar{\sigma}$-structure F in $\sigma_{\mathfrak{M}}$ the category whose set of morphisms is $F(2)$, the law of composition being $F(\kappa)$, the maps source and target $F(\alpha)$ and $F(\beta)$.

As $\sigma_{\mathcal{F}}$ is a regular sketch (Propositions 4 and 5 of [E2]), its prototype $\bar{\sigma}$ is also regular (Corollary, Proposition 16). In particular, $\mathfrak{M}^{\bar{\sigma}}$ is equivalent to $\mathcal{F}$.

2° For each category V, the categories V^{σ} and $V^{\bar{\sigma}}$ are identical. Indeed, a $\bar{\sigma}$-structure in V is also a σ-structure in V. Now let F be a σ-structure in V. Since ι is a right inverse of α in Σ, the morphism $F(\iota)$ is a right inverse of $F(\alpha)$; this implies that $F(\iota)$ is a kernel of the pair $(F(2), F(\iota).F(\alpha))$ in V. Hence F is a $\bar{\sigma}$-structure in V.

It follows that $\mathfrak{M}^{\bar{\sigma}} = \mathfrak{M}^{\sigma}$ is equivalent to $\mathcal{F}$. Since $\mathcal{F}$ admits a cartesian closed structure, $\mathfrak{M}^{\sigma}$ also, i.e. σ and $\bar{\sigma}$ are cartesian (p. 78). ∇

REMARKS. 1° A σ-structure F in $\sigma_{\mathfrak{M}}$ corresponds to a category on $F(2)$ whose law of composition is $F(\kappa)$, equipped with an injection $F(\iota)$ defining $F(1)$ as a set of objects. So, $\mathcal{S}(\sigma_{\mathfrak{M}}, \sigma)$ is isomorphic to the category of functors between categories with a given set of objects.

2° In [E2] the sketch of categories was in fact defined as a «pointed sketch», i.e. the ι had to be mapped on a canonical injection. This condition is expressed here by asking $F(\iota)$ to be a «canonical» kernel, so that we have no need of pointed sketches.

DEFINITION. σ is called the *prototype of categories with objects* and $\bar{\sigma}$ the *prototype of categories*. If V is a category, we define a *category in* V as a σ-structure in V, a *functor in* V as a σ-morphism in V. If σ' is a

$\mathcal{T}$-type, the $\bar{\sigma}$-structures and σ-morphisms in σ' are called *categories in* σ' and *functors in* σ'.

The categories in V are called generalized structured categories in [E2], category-objects in V in most papers. We denote by:

- $\mathcal{F}(V)$ the category V^{σ} of functors in V: in particular $\mathcal{F}(\mathfrak{M}) = \mathfrak{M}^{\sigma}$.
- $\mathcal{F}(\sigma')$ the category $\mathcal{S}(\sigma', \bar{\sigma})$ of functors in a $\mathcal{T}$-type σ'.

PROPOSITION 26. *We suppose that V is a category which admits pullbacks, kernels and sums of pairs.*

1° If V admits a cartesian closed structure, $\mathcal{F}(V)$ admits also one.

2° Let $\mathcal{V} = (V, \tau, i, a, b, c, m, D)$ be a symmetric monoidal closed category.

a) If τ commutes with pullbacks, $\mathcal{F}(V)$ defines a symmetric monoidal closed subcategory $\mathcal{F}(\mathcal{V})$ of $\mathcal{V}^{\Sigma}$ (Section 10).

b) If the insertion functor from $\mathcal{F}(V)$ to V^{Σ} admits a left adjoint J, there exists a symmetric monoidal closed category $\mathcal{F}(\mathcal{V})$ whose underlying category is $\mathcal{F}(V)$ and whose tensor product assigns $J\tau[G', G]$ to the pair (G', G) of functors in V.

Δ. σ is cartesian (Proposition 25) and the only category I belonging to $\mathcal{I}$ is connected. So Proposition 26 will result from the Corollaries of Proposition 23, if we prove that V satisfies the condition (L) of page 78 (modified according to Remark 1, page 81).

We may choose a universe $\mathfrak{U}$ to which belong the sets $V(s', s)$, where s and s' are objects of V (since we suppose the axiom of universes satisfied). As $\Sigma(u', u)$, where u and u' are equal to $1, 2, 3$ or 4, is a non void finite set, V admits sums indexed by $\Sigma(u', u)$. Finally, the subdivision category $\therefore\Sigma$ of Σ is a finite connected category, so that the existence of Σ-ends in V follows from the

LEMMA. *If V is a category admitting pullbacks and kernels of pairs, it admits projective limits indexed by any category generated by a sub-neocategory which is finite and connected.*

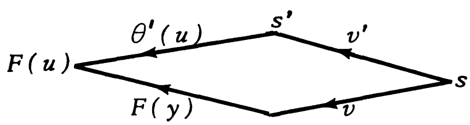
Δ. This (probably well-known) result is proved by induction on the number n of proper morphisms (i. e. different from an object) of the finite connected generating sub-neocategory. The assertion is evident if $n = 1$. We suppose it valid for $n = i$ and we take a functor $F: C \to V$, where C admits a generating sub-neocategory B which is finite, connected, and has $i+1$ proper morphisms. We can find a sub-neocategory B' of B which is connected and has i proper morphisms. We denote by:

- $y: e \to u$ the unique proper morphism of B not in B',
- C' the sub-category of C generated by B',
- $F': C' \to V$ the functor, restriction of F.

By the induction hypothesis, there exists a limit-cone $\theta': s'^{\wedge} \to F'$.

a) If u is not an object of B', then s' is also a projective limit of F.

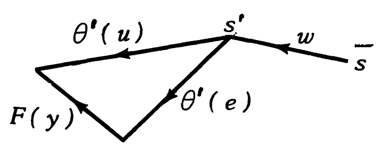
b) If u is an object of B' and e is not in B', there exists a pullback



and we get a limit-cone $\theta: s^{\wedge} \to F$ by defining

$$\theta(e) = v, \quad \theta(u') = \theta'(u').v' \text{ if } u' \in B'_0 .$$

c) If u and e are objects of B', let w be a kernel of the pair $(\theta'(u),$



$F(y).\theta'(e))$. Assigning $\theta(u') = \theta'(u').w$ to $u' \in B_0$ we define a limit-cone $\theta: \bar{s}^{\wedge} \to F$. This proves the Lemma by induction. ∇

COROLLARY. *Let σ' be a $\mathfrak{T}$-type (V, Γ'), where V is a category admitting sums of pairs. The properties 1 and 2 of Proposition 26 are also valid if we replace $\mathcal{F}(V)$ by $\mathcal{F}(\sigma')$ (resp. by $\mathcal{S}(\sigma', \sigma)$).*

Δ. This is deduced from Proposition 24 applied to $\bar{\sigma}$ (resp. to σ) by an argument similar to the proof of Proposition 26. ∇

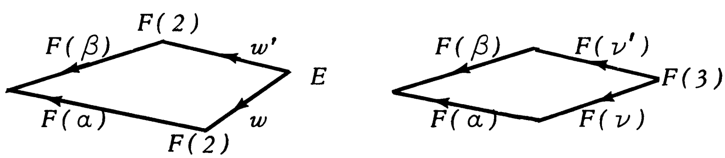
EXAMPLE. Let p be a saturated homomorphism functor [E1], i.e. p is a faithful functor from V to the category $\mathfrak{M}$ of maps and, if s is an object of V and f a bijection with source $p(s)$, there exists one and only one invertible morphism f' of V with source s satisfying $p(f') = f$. Let σ' be a $\hat{\mathcal{F}}$-type (V, Γ') such that $p\hat{\gamma}$ is a canonical limit-cone in $\mathcal{M}$ for any cone $\hat{\gamma}$ of Γ'. A category in σ' is called a *p-structured category* and $\mathcal{F}(\sigma')$ is identified with the category $\mathcal{F}(p)$ of p-structured functors [E2]. The Corollary gives conditions for a symmetric monoidal (resp. a cartesian) closed structure $\mathcal{V}$ on V to determine a similar structure on $\mathcal{F}(p)$. This statement generalizes Proposition 10 [BE], relative to the case where p is equivalent to the base functor $V(-, i)$ of $\mathcal{V}$ (this condition is very restrictive, since p is supposed faithful). It implies for instance, if p is the faithful functor $p_{\mathcal{F}}: \mathcal{F} \to \mathcal{M}$, that the category $\mathcal{F}(p_{\mathcal{F}})$ of double functors admits a cartesian closed structure, since $\mathcal{F}$ admits one (this does not result from [BE], the base functor $\mathcal{F}(-, 1)$ of $\mathcal{F}$ being equivalent to the not faithful functor $p'_{\mathcal{F}}$ which assigns to a functor ϕ its restriction ϕ_0).

13. Another construction of a closure functor on $\mathcal{F}(V)$.

We are going to give a direct construction of a closure functor of $\mathcal{F}(\mathcal{V})$; this construction proves that such a functor may be defined even if V admits pullbacks and kernels of pairs, but not sums of pairs.

A) Closure functor on $\mathcal{F}(\mathcal{M})$.

Let $F: \Sigma \to \mathcal{M}$ be an object of $\mathcal{F}(\mathcal{M})$. In $\mathcal{M}$, we have the pullbacks



where the first one is the canonical one, i. e. E is the set of pairs

$$(y, x) \in F(2) \times F(2) \text{ such that } F(\alpha)(y) = F(\beta)(x).$$

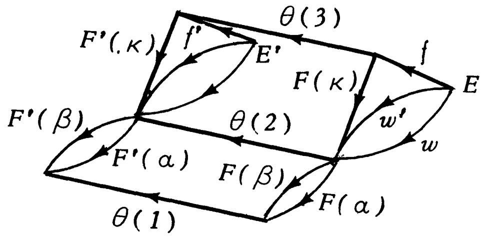
So there exists a unique bijection $f: E \to F(3)$ satisfying

$$F(\nu).f = w \text{ and } F(\nu').f = w'.$$

The map $F(\kappa).f$ is the law of composition of a category C whose set of morphisms is $F(2)$. We say that C is the category *determined by* F, and we denote it by $\eta(F)$.

We get an equivalence $\eta: \mathcal{F}(\mathcal{M}) \to \mathcal{F}$ by assigning to a morphism

$\theta : F \to F'$ of $\mathcal{F}(\mathfrak{M})$ the functor $\eta(\theta) : \eta(F) \to \eta(F')$ defined by the map $\theta(2)$.

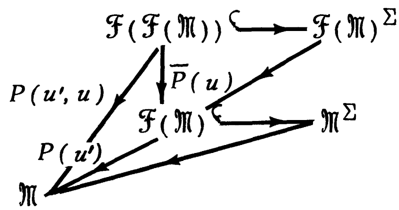


We consider the category $\mathcal{F}(\mathcal{F}(\mathfrak{M}))$ of functors in $\mathcal{F}(\mathfrak{M})$ and, for each object u of Σ, the «evaluation functors»

$$\overline{P}(u) : \mathcal{F}(\mathcal{F}(\mathfrak{M})) \to \mathcal{F}(\mathfrak{M}) \quad \text{and} \quad P(u) : \mathcal{F}(\mathfrak{M}) \to \mathfrak{M},$$

which assign $\theta(u)$ to θ. If u' is also an object of Σ, we write

$$P(u', u) = P(u')\overline{P}(u) : \mathcal{F}(\mathcal{F}(\mathfrak{M})) \to \mathfrak{M}.$$



There exists an equivalence $\overline{\eta}$ from $\mathcal{F}(\mathcal{F}(\mathfrak{M}))$ to the category $\mathcal{F}(p\mathcal{F})$ of double functors, described as follows:

- Let G be an object of $\mathcal{F}(\mathcal{F}(\mathfrak{M}))$. Then $G(2) = \overline{P}(2)(G) : \Sigma \to \mathfrak{M}$ determines a category $K^{\perp}$ and the functor

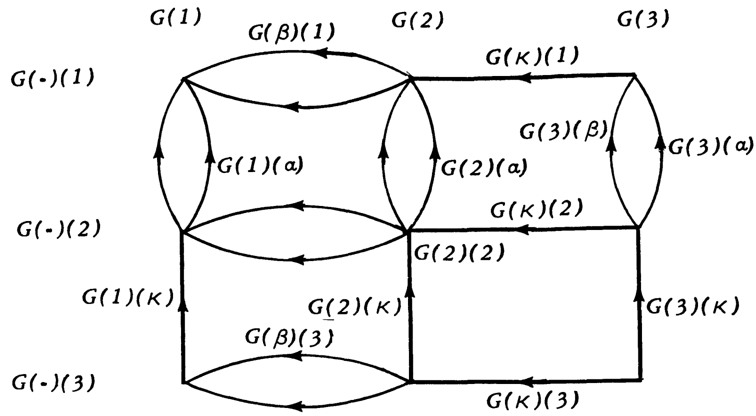
$$P(2)G = G(-)(2) : \Sigma \to \mathfrak{M} \quad \text{assigning} \quad G(x)(2) \quad \text{to } x \in \Sigma$$

determines a category $K^{\cdot}$, since $G(2)$ and $G(-)(2)$ are objects of $\mathcal{F}(\mathfrak{M})$. The categories $K^{\cdot}$ and $K^{\perp}$ have $G(2)(2) = P(2,2)(G)$ as their sets of morphisms, and their laws of composition are, respectively, $G(\kappa)(2)$ and $G(2)(\kappa)$. The pair $(K^{\cdot}, K^{\perp})$ is a double category, called *the double category determined by* G. We denote it by $\overline{\eta}(G)$.

- If $\overline{\theta} : G \to G'$ is a morphism of $\mathcal{F}(\mathcal{F}(\mathfrak{M}))$, the map

$$P(2,2)(\overline{\theta}) = \overline{\theta}(2)(2) : G(2)(2) \to G'(2)(2)$$

defines the double functor $\overline{\eta}(\overline{\theta})$ from $\overline{\eta}(G)$ to $\overline{\eta}(G')$.



PROPOSITION 27. *There exists a functor* $\partial : \mathfrak{F}(\mathfrak{M}) \to \mathfrak{F}(\mathfrak{F}(\mathfrak{M}))$ *satisfying the following conditions:*

1° If C is the category determined by an object F of $\mathfrak{F}(\mathfrak{M})$, *the double category determined by* $\partial(F)$ *is isomorphic to the double category of quartets of C.*

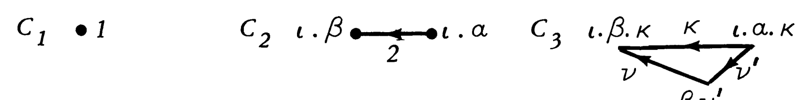
2° If u and u' are objects of Σ, *the functor* $P(u', u)\partial$ *is equivalent to* $P(u, u')\partial$, *and* $P(u, 1)\partial$ *is equivalent to* $P(u)$.

3° $\mathfrak{F}(\mathfrak{M})$ *admits a closure functor* $\overline{M}$ *such that*

$$\overline{M}(F', F)(x) = \mathfrak{F}(\mathfrak{M})(\partial(F')(x), F),$$

for a pair (F', F) *of objects of* $\mathfrak{F}(\mathfrak{M})$ *and a morphism x of* Σ.

Δ. We denote by Y the Yoneda σ^*-structure in $\mathfrak{F}(\mathfrak{M})$. For an object n of Σ, the category C_n determined by the object $Y(n)$ of $\mathfrak{F}(\mathfrak{M})$ is isomorphic to the category **n**; in particular:



The image of Y is isomorphic to the full subcategory of $\mathfrak{F}$ whose objects are the categories **1**, **2**, **3** and **4**. (It follows that a category K is isomorphic to the category determined by the object $\mathfrak{F}(K, \eta Y\text{-})$ of $\mathfrak{F}(\mathfrak{M})$.)

1° Let $\overline{M}$ be the closure functor on $\mathfrak{F}(\mathfrak{M})$ constructed in Proposition 20. For an object F of $\mathfrak{F}(\mathfrak{M})$, this proposition shows that $\overline{M}(F, Y\text{-})$ is the σ-structure in $\mathfrak{M}^{\sigma}$ (i.e. the category in $\mathfrak{F}(\mathfrak{M})$) assigning the natural transformation

$$\mathfrak{F}(\mathfrak{M})(F, Y(x) \times Y\text{-}) \quad \text{to} \quad x \in \Sigma .$$

Denoting $\overline{M}(F, Y\text{-})$ by $\partial(F)$, Proposition 20 also proves that $\overline{M}$ satisfies the third condition.

To the functor $\overline{M}(\text{-}, Y\text{-}) : \mathfrak{F}(\mathfrak{M}) \times \Sigma \to \mathfrak{F}(\mathfrak{M})$ is canonically associated the functor $M' : \mathfrak{F}(\mathfrak{M}) \to \mathfrak{F}(\mathfrak{M})^{\Sigma}$ such that $M'(\theta) = \overline{M}(\theta, Y\text{-})$ for any θ in $\mathfrak{F}(\mathfrak{M})$. This functor takes its values in the full subcategory $\mathfrak{F}(\mathfrak{F}(\mathfrak{M}))$ of $\mathfrak{F}(\mathfrak{M})^{\Sigma}$, since $M'(F) = \partial(F)$ for each object F of $\mathfrak{F}(\mathfrak{M})$. Hence M' admits as a restriction a functor $\partial : \mathfrak{F}(\mathfrak{M}) \to \mathfrak{F}(\mathfrak{F}(\mathfrak{M}))$.

2° As $Y(1)$ is a final object of $\mathfrak{F}(\mathfrak{M})$, for each object u of Σ the functor

$$P(u, 1)\partial = \mathfrak{F}(\mathfrak{M})(\text{-}, Y(1) \times Y(u))$$

is equivalent to $\mathfrak{F}(\mathfrak{M})(\text{-}, Y(u))$, and therefore (by Yoneda Lemma) to $P(u)$. Let u and u' be objects of Σ. We have:

$$P(u', u)\partial = \mathfrak{F}(\mathfrak{M})(\text{-}, Y(u) \times Y(u')).$$

If we consider the « symmetry equivalence »

$$\pi(u', u) : Y(u') \times Y(u) \to Y(u) \times Y(u')$$

(such that the isomorphism

$$\eta(\pi(u', u)) : C_{u'} \times C_u \to C_u \times C_{u'}$$

assigns (y, x) to (x, y)), we get the equivalence

$$\mathfrak{F}(\mathfrak{M})(\text{-}, \pi(u', u)) : P(u', u)\partial \to P(u, u')\partial .$$

3° Let F be an object of $\mathfrak{F}(\mathfrak{M})$. We denote by C the category determined by F.

a) $K^{\cdot}$ being the category determined by $\partial(F)(\text{-})(2)$, there exists an isomorphism $\phi(F)$ from $K^{\cdot}$ to the longitudinal category $\boxminus C$ of quartets of C. Indeed, the functor

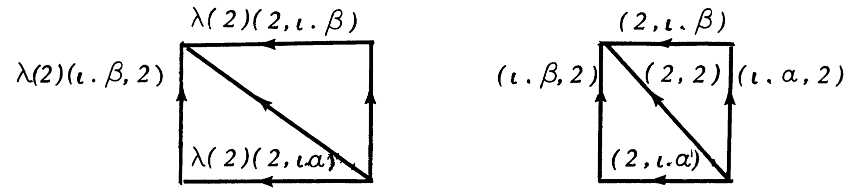
$$\partial(F)(\text{-})(2) = \mathfrak{F}(\mathfrak{M})(F, Y\text{-} \times Y(2))$$

is equivalent to the functor $\mathfrak{F}(C, \eta Y\text{-} \times C_2)$. So we get an isomorphism

$\phi(F): K^{\cdot} \to ⊏⊐ C$ assigning the quartet

$$(\lambda(2)(2,\iota.\beta), \lambda(2)(\iota.\beta,2), \lambda(2)(\iota.\alpha,2), \lambda(2)(2,\iota.\alpha))$$

to the natural transformation $\lambda: Y(2)\times Y(2) \to F$, element of K.

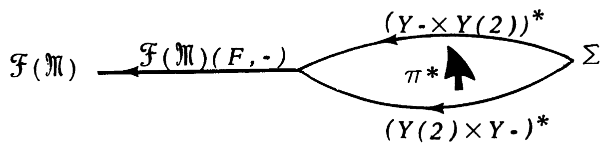


b) $\varphi(F)$ defines a double functor from the double category $\bar{\eta}(\partial(F)) = (K^{\cdot}, K^{\perp})$ determined by $\partial(F)$ to the double category $(⊏⊐ C, \boxminus C)$ of quartets. Indeed, $K^{\perp}$ is the category determined by

$$\partial(F)(2) = \mathfrak{F}(\mathfrak{M})(F, Y(2)\times Y\text{-}).$$

From the symmetry of the product on $\mathfrak{F}(\mathfrak{M})$ we deduce that the functors $-\times Y(2)$ and $Y(2)\times -$ from $\mathfrak{F}(\mathfrak{M})$ to $\mathfrak{F}(\mathfrak{M})$ are equivalent and that there exists an equivalence $\pi: Y\text{-}\times Y(2) \to Y(2)\times\text{-}$ where $\pi(u)$ is the equivalence $\pi(u, 2)$ considered in Part 2, for any object u of Σ. So, if π^* is the equivalence dual of π, we have the equivalence

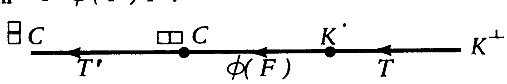
$$\Pi = \mathfrak{F}(\mathfrak{M})(F, -)\pi^* : \partial(F)(2) \to \partial(F)(-)(2),$$



and $\Pi(2)$ assigns $\lambda ⊏⊐ \pi(2,2)$ to $\lambda: Y(2)\times Y(2) \to F$. The isomorphism $T = \eta(\Pi): K^{\perp} \to K^{\cdot}$ associates to λ the natural transformation $T(\lambda)$ such that, if x and y are morphisms of Σ,

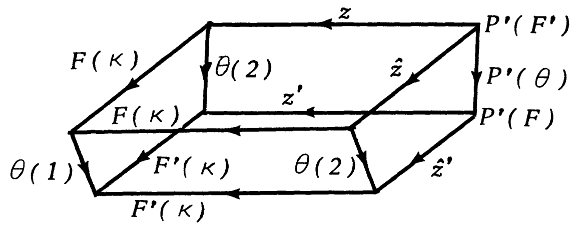
$$T(\lambda)(2)(y, x) = \lambda(2)(x, y).$$

If T' denotes the canonical isomorphism from $⊏⊐ C$ to $\boxminus C$, it follows that the isomorphism $T'\phi(F)T$:



is defined by the same map $\underline{\phi}(F)$ as the isomorphism $\phi(F)$. ∇

COROLLARY. *The functor* $P(2,2)\partial$ *is equivalent to the functor* P', *from* $\mathcal{F}(\mathfrak{M})$ *to* $\mathfrak{M}$ *assigning to* $\theta : F \to F'$ *the canonical pullback* $P'(\theta)$, *defined by the following diagram, whose bases are canonical pullbacks in* $\mathfrak{M}$:



Δ. Let F be an object of $\mathcal{F}(\mathfrak{M})$ and $\phi(F)$ the bijection considered in the preceding proof, from $K = P(2,2)\partial(F)$ to the set $\square C$ of quartets of the category C determined by F. There exists a bijection $\phi'(F)$ from $\square C$ to the canonical pullback $P'(F)$ of $(F(\kappa), F(\kappa))$, assigning

$$((y', x), (x', y)) \text{ to the quartet } (x', y', y, x).$$

If we associate to F the composite bijection

$$\psi(F) = \phi'(F)\,\phi(F) : K \to P'(F),$$

we get an equivalence $\psi : P(2,2)\partial \to P'$. ∇

B) Closure functors on $\mathcal{F}(V)$.

PROPOSITION 28. *Let* V *be a category admitting pullbacks. There exists a functor* $\overline{\partial} : \mathcal{F}(V) \to \mathcal{F}(\mathcal{F}(V))$ *such that, if* G *is a category in* V, *then* $\overline{\partial}(G)(2)$ *is equivalent to* $\overline{\partial}(G)(-)(2)$ *and, for* $s \in V_0$ *and* $x \in \Sigma$,

$$(A) \qquad V(-,s)\overline{\partial}(G)(x) \approx \partial(V(G-,s))(x).$$

Δ. We denote by $\mathfrak{L}$ the full subcategory of $\mathcal{F}(\mathfrak{M})^{V^*}$ whose objects are the functors H such that the functor $P(2)H = H(-)(2) : V^* \to \mathfrak{M}$ is representable.

1° There exists an equivalence $d : \mathcal{F}(V) \to \mathfrak{L}$. Indeed we have a functor $d' : V^* \times \mathcal{F}(V) \to \mathfrak{M}^{\Sigma}$ such that

$$d'(f, \theta) = V(-, f)\,\theta, \quad \text{for } f \in V, \ \theta \in \mathcal{F}(V).$$

As $V(-,s)$ commutes with pullbacks, $d'(s,G) = V(G-,s)$ is an object of $\mathcal{F}(\mathfrak{M})$ for each object (s,G); hence there exists a functor d'', from $V^* \times \mathcal{F}(V)$ to $\mathcal{F}(\mathfrak{M})$, restriction of d'. The functor $\hat{d}'' : \mathcal{F}(V) \to \mathcal{F}(\mathfrak{M})^{V^*}$

canonically associated to d'' is injective, and it takes its values in $\mathfrak{L}$ (since $P(u)\hat{d}''(G)$ is the representable functor $V(G(u), -)$ for any object u of Σ). So it admits as a restriction a functor d from $\mathfrak{F}(V)$ to $\mathfrak{L}$; if $\theta \in \mathfrak{F}(V)$, then $d(\theta): V^* \rightrightarrows \mathfrak{F}(\mathfrak{M})$ is the natural transformation such that $d(\theta)(f) = V(-, f)\,\theta$, for any $f \in V$. It is known (see [Go] and [E3]) that d is an equivalence; $d^{-1}: \mathfrak{L} \to \mathfrak{F}(V)$ will denote an equivalence.

2° We denote by:

- $Q'': V^* \times \mathfrak{F}(V) \to \mathfrak{F}(\mathfrak{M})^{\Sigma}$ the composite functor

$$V^* \times \mathfrak{F}(V) \xrightarrow{d''} \mathfrak{F}(\mathfrak{M}) \xrightarrow{\partial} \mathfrak{F}(\mathfrak{F}(\mathfrak{M})) \hookrightarrow \mathfrak{F}(\mathfrak{M})^{\Sigma}$$

which assigns $\partial(V(-, f)\,\theta)$ to (f, θ).

- Q' the functor from $\Sigma \times \mathfrak{F}(V)$ to $\mathfrak{F}(\mathfrak{M})^{V^*}$ associated to Q''.

If $x: u \to u'$ is in V and $\theta: G \to G'$ in $\mathfrak{F}(V)$, we have

$$Q'(u, G)(f) = \partial(V(G-, f))(u) = \overline{P}(u)\,\partial(V(G-, f)),$$

for any morphism f of V, and the natural transformation

$$Q'(x, \theta): Q'(u, G) \to Q'(u', G')$$

is such that, for any f in V, we have

$$Q'(x, \theta)(f) = \partial(V(-, f)\,\theta)(x) \in \mathfrak{F}(\mathfrak{M}).$$

Let G be a category in V. We are going to show that the functor $Q'(-, G)$ takes its values in $\mathfrak{L}$. This will imply that Q' takes also its values in the category $\mathfrak{L}$.

a) $Q'(-, G)$ is a category in $\mathfrak{F}(\mathfrak{M})^{V^*}$. Indeed, for each object s of V, the functor $Q'(-, G)(s): \Sigma \to \mathfrak{F}(\mathfrak{M})$ is the object $\partial(V(G-, s))$ of $\mathfrak{F}(\mathfrak{F}(\mathfrak{M}))$. It follows that the cone $Q'(-, G)\hat{\gamma}$, whose components in $\mathfrak{F}(\mathfrak{M})$ are the limit-cones $Q'(-, G)(s)\hat{\gamma}$, is a limit-cone in $\mathfrak{F}(\mathfrak{M})^{V^*}$, if $\hat{\gamma}$ is equal to γ or to γ'. Hence, $Q'(-, G)$ is a category in $\mathfrak{F}(\mathfrak{M})^{V^*}$.

b) We denote by R the functor from Σ to $\mathfrak{M}^{V^*}$ assigning

$$P(2)\,Q'(x, G): V^* \rightrightarrows \mathfrak{M} \quad \text{to any } x \in \Sigma.$$

The functor $Q'(-, G)$ will take its values in $\mathfrak{L}$ if we prove that $R(u)$ is representable for each object u of Σ. Indeed, for any f in V, we have

$$R(u)(f) = P(2)Q'(u,G)(f) = P(2)\overline{P}(u)\partial(V(G-,f)) =$$
$$= P(2,u)\partial(V(G-,f)).$$

- The functor $P(2,1)\partial$ being equivalent to $P(2)$ (Proposition 27), the functor $R(1)$ is equivalent to the functor assigning

$$P(2)(V(G-,f)) = V(G(2),f) \text{ to } f,$$

so that it is representable by $G(2)$.

- The functor $P(2,2)\partial$ being equivalent to the functor P' considered in the corollary of Proposition 27, the functor $R(2)$ is equivalent to the functor R' assigning $P'(V(G-,f))$ to $f \in V$. By definition of P' and pullbacks being computed evaluationwise in $\mathcal{F}(\mathfrak{M})^{V^*}$, the functor R' is a pullback in $\mathfrak{M}^{V^*}$ of $(V(G(\kappa),-), V(G(\kappa),-))$. Such a pullback is equivalent to $V(S,-)$, where S is a pullback of $(G(\kappa), G(\kappa))$ in V. So $R(2)$ is representable by S.

- $Q'(3,G)$ is a pullback of $(Q'(\alpha,G), Q'(\beta,G))$ (Part a) and $P(2)$ commutes with pullbacks, so that $R(3)$ is a pullback of $(R(\alpha), R(\beta))$ in $\mathfrak{M}^{V^*}$. We have just seen that $R(\alpha)$ and $R(\beta)$ are natural transformations between representable functors; hence $R(3)$ is representable.

- $Q'(4,G)$ being a pullback of $(Q'(\nu',G), Q'(\nu,G))$, we deduce similarly that $R(4)$ is representable, as a pullback of $(R(\nu'), R(\nu))$.

3° Q' taking its values in $\mathfrak{L}$, there exists a functor $Q: \Sigma \times \mathcal{F}(V) \to \mathfrak{L}$ restriction of Q'. We denote by $\overline{\partial}'$ the functor from $\mathcal{F}(V)$ to $\mathcal{F}(V)^{\Sigma}$ canonically associated to the composite functor $d^{-1}Q$:

$$\Sigma \times \mathcal{F}(V) \xrightarrow{\ Q\ } \mathfrak{L} \xrightarrow{\ d^{-1}\ } \mathcal{F}(V).$$

a) $\overline{\partial}'$ takes its values in $\mathcal{F}(\mathcal{F}(V))$. Indeed, if G is a category in V, we have $\overline{\partial}'(G) = d^{-1}Q(-,G)$. As $\mathfrak{L}$ is closed for pullbacks in $\mathcal{F}(\mathfrak{M})^{V^*}$ the functor $Q(-,G): \Sigma \to \mathfrak{L}$, restriction of the category $Q'(-,G)$ in the category $\mathcal{F}(\mathfrak{M})^{V^*}$, is also a category in $\mathfrak{L}$. The equivalence d^{-1} commuting with pullbacks, $\overline{\partial}'(G)$ is a category in $\mathcal{F}(V)$. It follows that there exists a functor $\overline{\partial}: \mathcal{F}(V) \to \mathcal{F}(\mathcal{F}(V))$, restriction of $\overline{\partial}'$. If G is a category in V, if f is a morphism of V and if $x \in \Sigma$, we get

$$V(-,f)\overline{\partial}(G)(x) = d(\overline{\partial}(G)(x))(f) \approx Q(x,G)(f) = \partial(V(G-,f))(x).$$

A fortiori $\overline{\partial}$ satisfies the condition (A) (we take $f = s$).

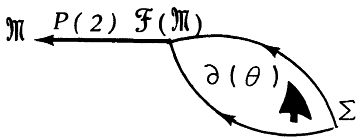
b) Let G be a category in V. It remains to show that $\overline{\partial}(G)(2)$ is equivalent to the functor $\overline{\partial}(G)(-)(2): \Sigma \to V$ assigning $\overline{\partial}(G)(x)(2)$ to $x \in \Sigma$. Indeed, if F is an object of $\mathcal{F}(\mathfrak{M})$ and C the category $\eta(F)$ determined by F, there exists an equivalence

$$T(F) \text{ from } (\overline{P}(2)\partial)(F) = \partial(F)(2) \text{ to } P(2)\partial(F)$$

such that $\eta(T(F))$ is the canonical isomorphism from $\boxminus C$ to $\square\square C$. This defines an equivalence T from the functor $\overline{P}(2)\partial$ to the functor

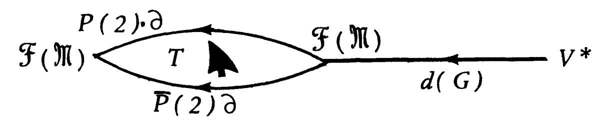
$$P(2)\cdot\partial : \mathcal{F}(\mathfrak{M}) \to \mathcal{F}(\mathfrak{M}) \text{ assigning } P(2)\partial(\theta) \text{ to } \theta .$$

$$\mathcal{F}(\mathfrak{M}) \xleftarrow{\overline{P}(2)} \mathcal{F}(\mathcal{F}(\mathfrak{M})) \xleftarrow{\partial} \mathcal{F}(\mathfrak{M})$$



We have the equivalence

$$Td(G) : (\overline{P}(2)\partial)d(G) \to (P(2)\cdot\partial)d(G),$$



from A to A'. Since

$$A(f) = \overline{P}(2)\partial d(G)(f) = \partial(V(G-, f))(2) =$$
$$= V(-, f)\overline{\partial}(G)(2) = d(\overline{\partial}(G)(2))(f),$$

for any f in V, it follows $A = d(\overline{\partial}(G)(2))$. On the other hand,

$$A'(f)(x) = P(2)\partial(d(G)(f))(x) = P(2)\partial(V(G-, f))(x) =$$
$$= \partial(V(G-, f))(x)(2) = V(-, f)\overline{\partial}(G)(x)(2),$$

for any $x \in \Sigma$; so

$$A'(f) = V(-, f)\overline{\partial}(G)(-)(2) = d(\overline{\partial}(G(-)(2))(f)$$

for each f in V; this implies $A' = d(\overline{\partial}(G)(-)(2))$. Hence, $Td(G)$ belongs to $\mathfrak{L}$ and $d^{-1}(Td(G)): \overline{\partial}(G)(2) \to \overline{\partial}(G)(-)(2)$ is an equivalence.∇

DEFINITION. With the notations of Proposition 28, we call $\overline{\partial}(G)$ *the double category in* V *of quartets of* G, while $\overline{\partial}(G)(2)$ (resp. $\overline{\partial}(G)(-)(2)$) is called *the lateral* (resp. the *longitudinal*) category of quartets of G, and denoted by $\boxminus G$ (resp. by $\boxbox G$).

The preceding proof shows that the categories determined by

$$V(-,s)\boxminus G \quad\text{and}\quad V(-,s)\boxbox G$$

are isomorphic to the lateral and to the longitudinal categories of quartets of the category determined by $V(G-,s)$, for any object s of V. Moreover,

- $\boxminus G$ and $\boxbox G$ are isomorphic,
- $\boxminus G(1)$ and $\boxbox G(1)$ are isomorphic to $G(2)$,
- $\boxminus G(2)$ and $\boxbox G(2)$ are pullbacks of $(G(\kappa), G(\kappa))$ in V.

REMARK. $\mathfrak{F}(V)$ is the category of 1-morphisms of the 2-category $\mathfrak{N}(V)$ of *natural transformations in* V: If $\theta: G \to G'$ and $\theta': G \to G'$ are functors in V, a natural transformation in V from θ to θ' is a functor Θ in V, from G to $\boxminus G'$, such that $\overline{\partial}(G')(\alpha)\boxbox\Theta = \theta$ and $\overline{\partial}(G')(\beta)\boxbox\Theta = \theta'$ (by construction, we may clearly identify $\overline{\partial}(G')(1)$ with G'). When V admits pullbacks, it is known [G1] that $\mathfrak{N}(V)$ is a representable 2-category, a representation of the category G in V being precisely the lateral category $\boxminus G$ in V of quartets of G.

PROPOSITION 29. *Let* $\mathcal{V} = (V, \tau, i, a, b, c, m, D)$ *be a symmetric monoidal closed category, where* V *admits pullbacks and kernels of pairs.*

1° There exists a functor $E: \mathfrak{F}(V)\times\mathfrak{F}(V)^* \to \mathfrak{F}(V)$ *such that, for a pair* (G', G) *of categories in* V*, we have:*

$$E(G', G) = \int_{x'x} D(-, G(x))\,\overline{\partial}(G')(x').$$

2° If the conditions of Proposition 26 are satisfied, E *is equivalent to the closure functor* $\hat{D}'$ *of* $\mathfrak{F}(\mathcal{V})$ *and* $\overline{\partial}(G)$ *is equivalent to the category* $\hat{D}'(G, qY-)$ *in* $\mathfrak{F}(V)$*, for any category* G *in* V.

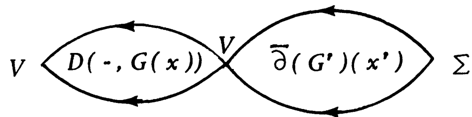
Δ. 1° The Lemma of Proposition 26 shows that the existence of pullbacks and kernels in V implies there exist Σ-ends in V. It follows that there exist also Σ-ends in $\mathfrak{F}(V)$, which are computed evaluationwise. We choose a Σ-end-functor $\int: \mathfrak{F}(V)^{\Sigma\times\Sigma^*} \to \mathfrak{F}(V)$.

a) Let G and G' be categories in V. There exists a functor A from $\Sigma\times\Sigma\times\Sigma^*$ to V which assigns

$$D(\overline{\partial}(G')(x')(y), G(x)) \quad\text{to}\quad (y, x', x).$$

The corresponding functor $A' : \Sigma \times \Sigma^* \to V^\Sigma$, which assigns

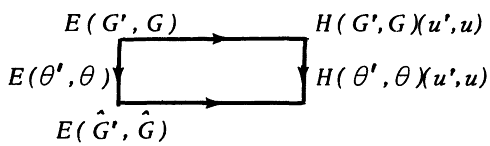
$$D(-, G(x))\, \overline{\partial}(G')(x') \text{ to } (x', x),$$

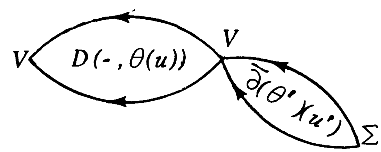


takes its values in $\mathfrak{F}(V)$, since $A'(u', u)$ is, for a pair (u', u) of objects of Σ, the composite of the category $\overline{\partial}(G')(u')$ in V with the functor $D(-, G(u))$ which commutes with pullbacks. So, there exists a functor $H(G', G) : \Sigma \times \Sigma^* \to \mathfrak{F}(V)$, restriction of A'. We denote by $E(G', G)$ the canonical end $\int H(G', G)$ in $\mathfrak{F}(V)$.

b) Let $\theta : \hat{G} \to G$ and $\theta' : G' \to \hat{G}'$ be functors in V. If u and u' are objects of Σ, we have the natural transformation

$$H(\theta', \theta)(u', u) = D(-, \theta(u))\, \overline{\partial}(\theta')(u'),$$





from $H(G', G)(u', u)$ to $H(\hat{G}', \hat{G})(u', u)$. Assigning this natural transformation to (u', u), we get a natural transformation

$$H(\theta', \theta) : H(G', G) \to H(\hat{G}', \hat{G}) : \Sigma \times \Sigma^* \rightrightarrows \mathfrak{F}(V).$$

We write $E(\theta', \theta) = \int H(\theta', \theta)$.

c) It is easily verified that we have so defined a functor

$$H : \mathfrak{F}(V) \times \mathfrak{F}(V)^* \to \mathfrak{F}(V)^{\Sigma \times \Sigma^*},$$

and a fortiori a functor

$$E = \int H : \mathfrak{F}(V) \times \mathfrak{F}(V)^* \to \mathfrak{F}(V).$$

2° We suppose moreover that the conditions of Proposition 26 are satisfied, i.e. V admits sums of pairs and also either τ commutes with pullbacks or the insertion functor from $\mathfrak{F}(V)$ to V^Σ admits a left adjoint. Then there exists a symmetric monoidal closed category $\mathfrak{F}(V)$ whose closure functor $\hat{D}'$ is defined by $\hat{D}'(G', G) = \int H'(G', G)$, (Proposition 23), the functor $H'(G', G) : \Sigma \times \Sigma^* \to \mathfrak{F}(V)$ assigning

$$D(-, G(x))\hat{D}'(G', qY-)(x') \text{ to } (x', x),$$

where q is a «partial adjoint» of $V(-, i)$.

For any category G in V, we denote the category $\hat{D}'(G, qY-)$ in $\mathfrak{F}(V)$ by $\delta(G)$.

a) Let G be a category in V. Then $\delta(G)$ is equivalent to $\overline{\partial}(G)$. Indeed, according to the proof of Proposition 23 (Part 1), for each object u of Σ, the category $\delta(G)$ is such that $V(-, s)\delta(G)(u)$ is canonically equivalent to $\partial(V(G-, s))(u)$, for any object s of V.

$$\partial(V(G-, s))(u) \approx V(-, s)\overline{\partial}(G)(u),$$

by Proposition 28. Hence, denoting yet by $d: \mathfrak{F}(V) \to \mathfrak{L}$ the isomorphism defined in Part 1, Proposition 28, we deduce that

$$d(\delta(G)(u)) \text{ and } d(\overline{\partial}(G)(u))$$

are equivalent; a fortiori there exists an equivalence $\xi(G)(u): \delta(G)(u) \to \overline{\partial}(G)(u)$. More precisely, we get an equivalence $\xi(G): \delta(G) \to \overline{\partial}(G)$.

b) Let G and G' be categories in V. We define an equivalence

$$X(G', G): H'(G', G) \to H(G', G): \Sigma \times \Sigma^* \rightrightarrows \mathfrak{F}(V)$$

assigning the equivalence

$$D(-, G(u'))\xi(G)(u) \text{ to } (u', u) \in \Sigma_0 \times \Sigma_0 .$$

Moreover, there exists

- a functor $H': \mathfrak{F}(V) \times \mathfrak{F}(V)^* \to \mathfrak{F}(V)^{\Sigma \times \Sigma^*}$, defined as in Part 1, such that $\hat{D}'(\theta', \theta)$ is an end of $H'(\theta', \theta)$, for each pair (θ', θ) of functors in V;

- an equivalence $X: H' \to H$ assigning $X(G', G)$ to (G', G).

Hence $\int X: \int H' \to \int H$ is an equivalence, and $\hat{D}'$ is equivalent to E. ∇

The construction of E does not depend upon the existence of sums in V. This suggests that E could always be a closure functor on $\mathfrak{F}(V)$. In fact, we have:

PROPOSITION 30. *Let $\mathcal{V}$ be a symmetric monoidal closed category*

$$(V, \tau, i, a, b, c, m, D),$$

If V admits pullbacks and kernels of pairs and if τ commutes with pullbacks, then there exists a symmetric monoidal closed category

$$(\mathfrak{F}(V),\ \hat{\tau}',\ i\hat{\ },\ \hat{a}',\ \hat{b}',\ \hat{c}',\ \hat{m}',\ E),$$

where E is the functor defined in Proposition 29 and where $\hat{\tau}$ assigns the category $\tau[G', G]$ to the pair (G', G) of categories in V.

Δ. Let $\hat{\tau}$ be the tensor-product functor on V^{Σ} such that

$$(\theta' \hat{\tau} \theta)(x) = \theta'(x) \tau \theta(x), \quad \text{for any } x \in \Sigma,$$

if θ and θ' are natural transformations. As τ commutes with pullbacks, $G' \hat{\tau} G$ is a category in V when such are G and G'. So, there exists a functor $\hat{\tau}': \mathfrak{F}(V) \times \mathfrak{F}(V) \to \mathfrak{F}(V)$, restriction of $\hat{\tau}$ and, $\mathfrak{F}(V)$ being a full subcategory of V^{Σ}, the canonical symmetric monoidal category on V^{Σ}, whose tensor-product is $\hat{\tau}$, admits a symmetric monoidal subcategory

$$(\mathfrak{F}(V),\ \hat{\tau}',\ i\hat{\ },\ \hat{a}',\ \hat{b}',\ \hat{c}',\ \hat{m}')$$

since $i\hat{\ }$ is a category in V, the category I indexing pullbacks being connected. Hence Proposition 30 will result from Theorem II-5-8 [EK] if we know that $E(G', G)$ is a cofree structure generated by G' relative to the functor $- \hat{\tau}' G: \mathfrak{F}(V) \to \mathfrak{F}(V)$, for each pair (G', G) of categories in V. We will only sketch the proof of this assertion, omitting the purely technical computations.

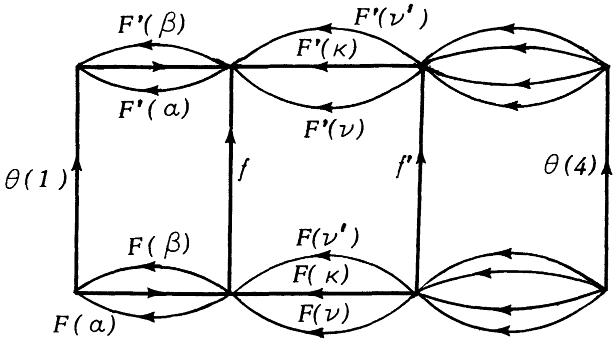
1° The following remarks will be useful:

a) Let F and F' be categories in V and $f: F(2) \to F'(2)$ a morphism of V. There exists a functor $\theta: F \to F'$ in V such that $\theta(2) = f$ iff f satisfies the equalities:

- $f . F(\iota . \alpha) = F'(\iota . \alpha) . f, \quad f . F(\iota . \beta) = F'(\iota . \beta) . f,$
- $f . F(\kappa) = F'(\kappa) . f'$, where f' is the «pullback» morphism such that $F'(\nu) . f' = f . F(\nu)$ and $F'(\nu') . f' = f . F(\nu')$ (it exists, the two first equalities implying $F'(\alpha) . f . F(\nu) = F'(\beta) . f . F(\nu')$, since $F'(\iota)$ is a monomorphism and $\alpha . \nu = \beta . \nu'$).

In this case, we have

$$\theta(3) = f', \quad \theta(1) = F'(\alpha) . f . F(\iota),$$



and $\theta(4)$ is defined by pullbacks. (The existence of θ means that σ admits an «idea» [E3], which is $(\kappa, \iota . \alpha, \iota . \beta)$.)

We will say that $\theta: F \to F'$ is the functor in V defined by f.

b) Let $B: \Sigma \times \Sigma^* \to V$ be a functor such that $B(-, u)$ is a category in V for each object u of Σ, and S an end of B, with canonical projections $p(u): S \to B(u,u)$. If $g: s \to B(2,2)$ is a morphism in V, there exists a morphism $\hat{g}: s \to S$ such that $p(2). \hat{g} = g$ iff:

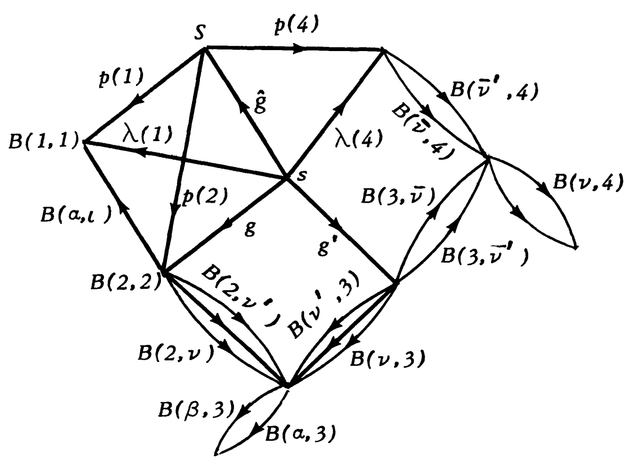
- $B(2, \iota . \alpha). g = B(\iota . \alpha, 2). g$, $B(2, \iota . \beta). g = B(\iota . \beta, 2). g$,
- $B(2, \kappa). g = B(\kappa, 3). g'$, where g' is the unique morphism such that

$$B(2, \nu). g = B(\nu, 3). g' \text{ and } B(2, \nu'). g = B(\nu', 3). g'$$

(its existence follows from the fact that $B(-, 3)$ is a category in V).

It is easily proved that there exists a cone $\lambda: s^{\wedge} \to \therefore B$, where

$$\lambda(2) = g, \quad \lambda(3) = g', \quad \lambda(1) = B(\alpha, \iota). g,$$



and $\lambda(4)$ is defined by pullback as being the morphism such that

$$B(3, \overline{\nu}).\, g' = B(\overline{\nu}, 4).\, \lambda(4),$$
$$B(3, \overline{\nu}').\, g' = B(\overline{\nu}', 4).\, \lambda(4).$$

Then $\hat{g}$ is the factor of λ through the cone $p: S^{\wedge} \to \therefore B$ defining the end.

2° Let G and G' be categories in V.

a) We consider:

- the longitudinal category $\square G' = \overline{\partial}(G')(\cdot)(2)$ of quartets of G', denoted by $\hat{G}'$ (definition page 97);

- the canonical projection $p(2)$ from the end

$$E(G', G)(2) = \int_{x'x} D(\hat{G}'(x'), G(x)) \text{ to } D(\hat{G}'(2), G(2));$$

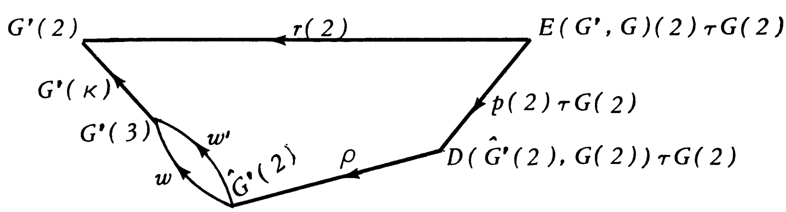
- the morphism

$$\rho: D(\hat{G}'(2), G(2)) \tau G(2) \to \hat{G}'(2)$$

defining $D(\hat{G}'(2), G(2))$ as a cofree structure generated by $\hat{G}'(2)$ relative to the functor $-\tau G(2): V \to V$;

- the canonical projections $w: \hat{G}'(2) \to G'(3)$ and w' defining $\hat{G}'(2)$ as a pullback of $(G'(\kappa), G'(\kappa))$.

It may be shown that the composite morphism $r(2)$:



satisfies the hypothesis of Part 1-a, so it defines a functor:

$$r: E(G', G) \hat{\tau}' G \to G'.$$

b) r defines $E(G', G)$ as a cofree structure generated by G' relative to $-\hat{\tau}' G: \mathfrak{F}(V) \to \mathfrak{F}(V)$. Indeed, let $\theta: G'' \hat{\tau}' G \to G'$ be a functor in V. To define the unique functor in V:

$$\theta': G'' \to E(G', G) \text{ such that } r \square (\theta' \hat{\tau}' G) = \theta,$$

we are going to construct a morphism $g: G''(2) \to D(\hat{G}'(2), G(2))$ satisfying the hypothesis of Part 1-b, applied to the functor B assigning

$$D(\hat{G}'(x'), G(x)) \text{ to } (x', x).$$

Then there exists a morphism

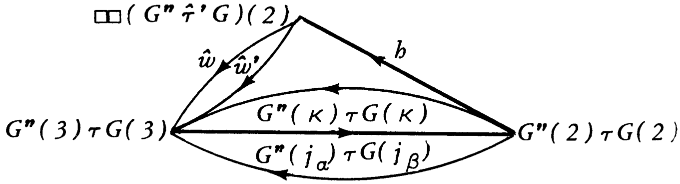
$$\hat{g}: G''(2) \to E(G', G)(2) \quad \text{such that} \quad p(2).\hat{g} = g,$$

and a technical argument proves that $\hat{g}$ defines a functor θ' in V, from G'' to $E(G', G)$, satisfying the wanted property.

To construct g, we consider:

- the morphism $\overline{\partial}(\theta)(2)(2): \square\square(G'' \hat{\tau}' G)(2) \to \hat{G}'(2)$,
- the morphisms $G''(j_\alpha)\tau G(j_\beta)$ and $G''(j_\beta)\tau G(j_\alpha)$ from $G''(2)\tau G(2)$ to $G''(3)\tau G(3)$,
- the projections $\hat{w}$ and $\hat{w}'$ of the pullback of

$$(G''(\kappa)\tau G(\kappa), G''(\kappa)\tau G(\kappa)) \text{ defining } \square\square(G'' \hat{\tau}' G)(2).$$



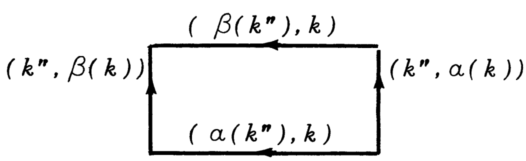
- the unique morphism $h: G''(2)\tau G(2) \to \square\square(G'' \hat{\tau}' G)(2)$ such that

$$\hat{w}.h = G''(j_\beta)\tau G(j_\alpha), \quad \hat{w}'.h = G''(j_\alpha)\tau G(j_\beta);$$

it exists, since

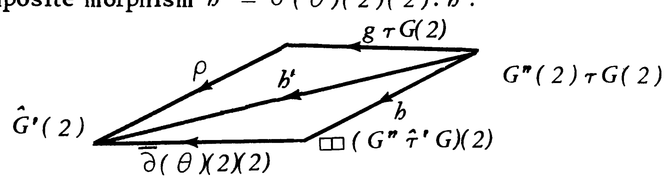
$$(G''(\kappa)\,\tau\, G(\kappa)).(G''(j_\beta)\tau G(j_\alpha)) = G''(\kappa.j_\beta)\tau G(\kappa.j_\alpha) =$$
$$G''(2)\,\tau\, G(2) = (G''(\kappa)\tau G(\kappa)).(G''(j_\alpha)\tau G(j_\beta)).$$

(For usual categories, this morphism corresponds to the map from the product category $C'' \times C$ to $\underline{\square}(C'' \times C)$ assigning to (k'', k) the quartet



.)

- the composite morphism $h' = \overline{\partial}(\theta)(2)(2).h$.



Then g is the unique morphism $g: G''(2) \to D(\hat{G}'(2), G(2))$ such that

$$\rho.(g \tau G(2)) = h'. \quad \nabla$$

COROLLARY. *If V admits pullbacks, kernels of pairs and a cartesian closed structure, then $\mathcal{F}(V)$ admits a cartesian closed structure.* ∇

REMARKS. 1° Proposition 30 (announced in [BE]) has been indicated by the first of the authors in 1971, in a lecture at the Séminaire Ehresmann (Paris). The proof given then was along the same arguments as above except that ends were not explicitely used (the authors did not know them) but constructed from kernels and pullbacks.

2° It may be asked whether Proposition 30 extends to more general cone-bearing categories. This does not seem true. Indeed, we denote now by σ any projective cone-bearing category (Σ, Γ). As in Part 1 Proposition 28, we prove that there exists an equivalence d from V^{σ}, for any category V, to the full subcategory $\mathfrak{L}'$ of $(\mathfrak{M}^{\sigma})^{V^*}$ defined as follows:

the objects of $\mathfrak{L}'$ are those functors H such that the functor $H(-)(u)$ is representable, for any object u of Σ

(in the case where σ is the prototype of categories, $\mathfrak{L}'$ is identical with $\mathfrak{L}$, since 1, 3 and 4 are constructed successively as projective limits).

But, even if σ is cartesian, there is no way to prove that, G being a σ-structure in V, the functor from Σ to $(\mathfrak{M}^{\sigma})^{V^*}$ associated to

$$\overline{M}(d(G)-, Y-): V^* \times \Sigma \to \mathfrak{M}^{\sigma}$$

takes its values in $\mathfrak{L}'$. However, if such is the case, we may extend the construction of $\overline{\partial}(G)$, and then the construction of E.

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