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YUH-CHING CHEN

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# GERMS OF QUASI-CONTINUOUS FUNCTIONS

*by YUH-CHING CHEN*

## Introduction.

The notion of quasi-topological spaces was first introduced by Kowalsky [12] under the German name «Limesräume». Since then it has been applied to various branches of mathematics such as differential Geometry [1], [8], functional analysis [1], [2], [4], [6], [7], theory of differentiations [1], [3], [10], [14], [15], and algebraic topology [16]. It was Bastiani [1] who first applied this notion to differentiable manifolds and introduced the French term -quasi-topologie- which is not related to the quasi-topology defined by Spanier [18]. Since this work is inspired by some works of Ehresmann's school [8], [14], [15], [16], the term *quasi-topology* here is a translation of the French «quasi-topologie».

In this paper, we try to generalize the notions of germs of functions and sheaves in topological sense to that of $\pi$-germs of functions and $\pi$-sheaves in quasi-topological sense and to study the relations between these notions. We begin with a brief review of some basic definitions and properties on quasi-topologies and the introduction of the notion of germs and $\pi$-germs of functions using inductive limits. Then we generalize the notions of pre-sheaves and sheaves over a topological space to that of $\pi$-presheaves and $\pi$-sheaves over a quasi-topological space and show that every $\pi$-sheaf $E$ is reflected by the sheaf of germs of quasi-continuous local sections of $E$. In fact the category of $\pi$-sheaves over a quasi-topological space $(X, \pi)$ contains a reflective full subcategory isomorphic to the category of abelian sheaves over the underlying topological space $(X, T_\pi)$ of $(X, \pi)$. Finally, we show that the canonical injective structure of this reflective subcategory determines an effacement structure [13] on the category of $\pi$-sheaves. The homological Algebra of this effacement structure appears more complicated than the relative homological Algebra of Eilenberg-Moore [9] and Maranda [17]. We shall defer this pending further investigations.

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## 1. Germs and $\pi$-germs of functions.

$\pi$ will always stand for a quasi-topology on a set $X$. It is a function that associates to each $x \in X$ a family $\pi(x)$ of filters of subsets of $X$ satisfying the conditions:

(1) $F_1, F_2 \in \pi(x)$ implies $F_1 \cap F_2 \in \pi(x)$,

(2) $F_1 \in \pi(x)$ and $F_2 \supset F_1$ implies $F_2 \in \pi(x)$,

(3) the filter $x^{\epsilon}$ of all subsets of $X$ containing $x$ is in $\pi(x)$.

If $F \in \pi(x)$, we say that $F$ *converges to* $x$ in $\pi$. The pair $(X, \pi)$ is called a *quasi-topological space.*

Let $(X, \pi)$ and $(E, \tau)$ be quasi-topological spaces. A function $f:(X,\pi) \to (E,\tau)$, often written $f: X \to E$, is $(\pi,\tau)$*-continuous* (called *quasi-continuous* in [1], [15]) if, for every $x \in X$ and every $F \in \pi(x)$, the images of the sets in $F$ under $f$ generate a filter $f(F) \in \tau(f(x))$. $f$ is *quasi-open* if for every $x \in X$ and every $G \in \tau(f(x))$, there is $F \in \pi(x)$ with $f(F) \subset G$. If $A$ is a subset of $X$, the quasi-topology induced on $A$ by $\pi$ is denoted $\pi | A$ and we say that $(A, \pi | A)$ is a (quasi-topological) *subspace* of $(X, \pi)$.

A topology $T$ on $X$ is identified with the quasi-topology $\pi_T$ on $X$ in which the filter of neighborhoods of $x \in X$ is the smallest filter in $\pi_T(x)$. The category $\mathfrak{J}$ of topological spaces and continuous functions is identified with a full subcategory of the category $\mathfrak{Q}\mathfrak{J}$ of quasi-topological spaces and quasi-continuous functions (see e.g. [15]).

The *underlying topology* $T_\pi$ of $\pi$ is defined as follows: A set $U \subset X$ is open (in $T_\pi$) if and only if, for every $x \in U$, $F \in \pi(x)$ implies

$U \in F$. Thus every $(\pi, \tau)$-continuous function $f: X \to E$ is continuous in the underlying topologies $T_\pi$ and $T_\tau$. Note that $1: (X, \pi) \to (X, T_\pi)$ is $(\pi, T_\pi)$-continuous. In fact, the underlying topology functor $T: \mathfrak{Q}\mathfrak{T} \to \mathfrak{T}$ is left adjoint to the inclusion functor $\mathfrak{T} \subset \mathfrak{Q}\mathfrak{T}$, i.e., $\mathfrak{T}$ is identified with a coreflective subcategory of $\mathfrak{Q}\mathfrak{T}$ (in French term, $T$ is a projector functor). For further definitions and properties concerning quasi-topologies we refer the readers to [1], [16].

*Convention.* Let $A$ be a subset of $X$. A $(\pi, \tau)$-continuous function $f: A \to E$ is the restriction to $A$ of a $(\pi \mid U, \tau)$-continuous function from an open neighborhood $U$ of $A$ to $E$. Thus if $\alpha$ is the directed set (by inclusion) of open neighborhoods of $A$ and if $\mathfrak{Q}\mathfrak{T}(U, E)$ denotes the set of $(\pi, \tau)$-continuous functions from $U \in \alpha$ to $E$, then the *restriction map*

$$(1.1) \qquad r: \lim_{\to \alpha} \mathfrak{Q}\mathfrak{T}(U, E) \to \mathfrak{Q}\mathfrak{T}(A, E)$$

is a surjection, where $\{\mathfrak{Q}\mathfrak{T}(U, E) \mid U \in \alpha\}$ forms a direct system of sets of $(\pi, \tau)$-continuous functions with genuine restriction maps.

Let $Q(x) = \bigcap \{F_i \mid F_i \in \pi(x)\}$ be the filter that is the intersecsection of all filters $F_i$ in $\pi(x)$. Then each $A_x \in Q(x)$ is the union of a family of subsets of $X$ one from each filter $F_i \in \pi(x)$. A set $A_x \in Q(x)$ is called a *$\pi$-neighborhood* of $x$ (thus every neighborhood is a $\pi$-neighborhood). Notice that: (1) each $A_x \in Q(x)$ contains $x$, but $Q(x)$ may not converge to $x$ in $\pi$, and (2) $A_x$ may not be a $\pi$-neighborhood of another point $y \in A_x$.

We proceed now to define germs and $\pi$-germs of functions using inductive limits. Let $O(x)$ be the set of open neighborhoods of $x$ (open in the underlying topology $T_\pi$ of $\pi$). Then $O(x) \subset Q(x)$. Order both sets $O(x)$ and $Q(x)$ by inclusion. Then $O(x)$ is a directed subset of $Q(x)$. The inductive limit

$$\lim_{\to} \mathfrak{Q}\mathfrak{T}(A, E), \ A \in Q(x) \ \text{(resp. } \lim_{\to} \mathfrak{Q}\mathfrak{T}(U, E), \ U \in O(x)\text{)},$$

is the set of *$\pi$-germs* (resp. *germs*) *of $(\pi, \tau)$-continuous functions at* $x$. Each $f: A \to E$ in $\mathfrak{Q}\mathfrak{T}(A, E)$ (resp. $f: U \to E$ in $\mathfrak{Q}\mathfrak{T}(U, E)$) determines a $\pi$-germ (resp. germ) $f_x$ of a $(\pi, \tau)$-continuous function at $x$. Often, we simply call $f_x$ the *limit of* $f$ *at* $x$. Since $O(x) \subset Q(x)$ as directed sets, there is a map

$$(1.2) \quad \gamma_x : \varinjlim \mathfrak{Q}\mathfrak{T}(U, E) \to \varinjlim \mathfrak{Q}\mathfrak{T}(A, E)$$

that associates to each germ $f_x$ at $x$ a $\pi$-germ $f'_x = \gamma_x(f_x)$ at $x$. It follows from (1.1) that $\gamma_x$ is surjective. If $O(x)$ is cofinal in $Q(x)$, then the notions of germs and $\pi$-germs coincide. In particular, this is the case when $\pi$ is topological.

## 2. $\pi$-sheaves.

A map $p: E \to X$ of quasi-topological spaces $(E, \tau)$ and $(X, \pi)$ defines a *$\pi$-sheaf* $E$ if the following conditions are satisfied (see [5]):

(S1) For every point $f_x \in E$ with $p(f_x) = x$, there exists a subset $U_f \subset E$ containing $f_x$ such that the map $p \mid U_f$ is a $(\tau, \pi)$-homeomorphism of $(U_f, \tau \mid U_f)$ onto an open neighborhood $U_x$ of $x$;

(S2) $\tau$ is the final quasi-topology determined by all $\tau \mid U_f$ via inclusion maps;

(S3) For every $x \in X$, the *stalk* $E_x = p^{-1}(x)$ is an abelian group and the group operations are quasi-continuous in $\tau$.

In particular, if all $U_f$ in (S1) can be choosen open in the underlying topology $T_\tau$ of $\tau$, then we say that $p$ *spreads* $E$ over $X$ and $E$ is a *$\pi$-spreading space.* It is easy to see that:

PROPOSITION 2.1. *If* $E$ *is a $\pi$-sheaf, then* $p$ *is* $(\tau, \pi)$*-continuous and quasi-open* (cf. proposition 1.2.16 of [15]).

COROLLARY 2.2. *If* $(E, \tau)$ *is a $\pi$-spreading space, then* $(E, T_\tau)$ *is an abelian sheaf over the topological space* $(X, T_\pi)$. (We assume that the readers are familiar with the general theory of abelian sheaves).

Let $p: (E, \tau) \to (X, \pi)$ be a $\pi$-sheaf. A *section of* $E$ *over a subset* $A$ *of* $X$ is a function $s: A \to E$ which is the restriction to $A$ of a section $s'$ of $E$ over an open neigboorhood $U$ of $A$ (i.e. $s': U \to E$ is a $(\pi \mid U, \tau)$-continuous function such that $ps'$ is the identity of $U$); in particular, $s$ is $(\pi, \tau)$-continuous on $A$. The set $\Gamma(A, E)$ of sections of $E$

over $A$ is an abelian group ( the addition is pointwise). It follows that the restriction map

$$(2.1) \qquad r : \lim_{\to \alpha} \Gamma(U, E) \to \Gamma(A, E)$$

is an epimorphism for every $\pi$-neighborhood $A$ of $x \in X$. This will be referred to as property (PS) in the definition of $\pi$-presheaves in the next section.

A map $\phi : E \to F$ of $\pi$-sheaves $(E, \tau)$ and $(F, \sigma)$ is called a *$\pi$-homomorphism* if: (1) $\phi$ is $(\tau, \sigma)$-continuous, and (2) for every $x \in X$ the map $\phi_x = \phi \mid E_x$ is a group homomorphism of $E_x$ into $F_x$. We write $\phi = \{ \phi_x \mid x \in X \}$. The classes of $\pi$-sheaves and $\pi$-homomorphisms form a category $\mathfrak{L}_\pi$ called the *category of $\pi$-sheaves.*

PROPOSITION 2.3. *The class of $\pi$-spreading spaces form a full subcategory $\mathfrak{L}_X$ of $\mathfrak{L}_\pi$. If $\mathfrak{L}_T$ denotes the category of abelian sheaves over the topological space $(X, T_\pi)$, then the underlying topology functor $T : \mathfrak{Q}\mathfrak{T} \to \mathfrak{T}$ induces an isomorphism $T_X : \mathfrak{L}_X \to \mathfrak{L}_T$ of categories.*

Indeed, $T_X$ sends a $\pi$-spreading space $p : (E, \tau) \to (X, \pi)$ to an abelian sheaf $p : (E, T_\tau) \to (X, T_\pi)$. The inverse of $T_X$ is defined as follows. Let $\pi : (E, \mathfrak{U}) \to (X, T_\pi)$ be an abelian sheaf, where $\mathfrak{U}$ is a topology on $E$. Then, by definition, for every point $f_x \in E$ with $p(f_x) = x$, there is a $U_f \in \mathfrak{U}$ such that $p \mid U_f$ is a homeomorphism of $U_f$ onto an open neighborhood $U_x$ of $x$. Endow each $U_f$ with a quasi-topology $\tau_f$ that makes $p \mid U_f$ a $(\tau, \pi)$-homeomorphism and let $\tau$ be the final quasi-topology on $E$ determined by all $\tau_f$ via inclusion maps. Then $p : (E, \tau) \to (X, \pi)$ is a $\pi$-spreading space. $T_X^{-1}$ carries $p : (E, \mathfrak{U}) \to (X, T_\pi)$ to $p : (E, \tau) \to (X, \pi)$.

## 3. Construction of $\pi$-sheaves.

Let $Q_\pi$ be the category whose class of objects is the set

$$\{\emptyset\} \cup \bigcup_{x \in X} Q(x) = \{ A_x \in Q(X) \mid x \in X \} \cup \{\emptyset\}$$

of $\pi$-neighborhoods of points of $X$ and whose morphisms are inclusion maps, and let $Ab$ be the category of abelian groups and homomorphisms. A $\pi$-presheave is a contravariant functor $P : Q_\pi \to Ab$ satisfying the condition:

(PS) For every $A \in Q(x)$, the restriction map $r : \lim_{\to \alpha} P(U) \to P(A)$ is an epimorphism, where $\alpha$ is the set of all open neighborhoods $U$ of $A$ directed by inclusion. A *homomorphism* of $\pi$-presheaves is a natural transformation of functors. $\pi$-presheaves and their homomorphisms form a category $\mathcal{P}_\pi$ of functors.

A typical example of a $\pi$-presheaf is the $\pi$-presheaf $\Gamma E$ of local sections of a $\pi$-sheaf $E$ defined as follows. For every inclusion map $i : B \to A$ in $Q_\pi$, the map $(\Gamma E)(i) : \Gamma(A, E) \to \Gamma(B, E)$ is the restriction map of sections of $E$ over $A$ to that of $E$ over $B$. The property (PS) is verified by (2.1). In fact, there is a functor $\Gamma : \mathcal{L}_\pi \to \mathcal{P}_\pi$ called a *local section functor.*

Let $P$ be a $\pi$-presheaf. We shall construct the *associated $\pi$-sheaf* $SP$ of $P$ as follows. For every $x \in X$ let

$$(3.1) \qquad (SP)_x = \lim_{\to} P(A_x), \quad A_x \in Q(x)$$

be the set of limits $f_x$ of $f \in P(A_x)$ at $x$, and let

$$(3.2) \qquad SP = \bigcup_{x \in X} (SP)_x .$$

We shall endow $SP$ with a quasi-topology $\tau$ so that the projection $p : SP \to X$ defined by $p(f_x) = x$ is a $\pi$-sheaf: For each open set $U$ of $X$ and for each $f \in P(U)$ let

$$(3.3) \qquad U_f = \{ f_x \in (SP)_x \mid x \in U \text{ and } f_x = \textit{limit of } f \textit{ at } x \}$$

be the set of points of $SP$ which are the limits of $f$ at points of $U$. Endow each $U_f$ with a quasi-topology that makes $p \mid U_f : U_f \to U$ a $(\tau_f, \pi)$-homeomorphism. Then we have

$$SP = \bigcup \{ U_f \mid U \text{ open in } X,\ f \in P(U) \},$$

and $\tau_f$ and $\tau_g$ agree on $U_f \cap V_g$ for any two sets $U_f$ and $V_g$ defined by (3.3). $\tau$ is the final quasi-topology on $SP$ determined by all $\tau_f$ via inclusion maps. Then

PROPOSITION 3.1. *$p : (SP, \tau) \to (X, \pi)$ is a $\pi$-sheaf called the associated $\pi$-sheaf of the $\pi$-presheaf $P$.*

In practice, most of $\pi$-sheaves are constructed in this way from $\pi$-

presheaves of $\pi$-germs of functions satisfying some prescribed properties such as quasi-continuous, quasi-holomorphic [15], etc... In fact the introduction of the notion of $\pi$-sheaves is motivated by this sort of examples. We should point out that in the construction above the sets $U_f$ are not open in $T_\tau$ in general. Since the limit $f_x$ of $f \in P(U)$ is taken on the directed set $Q(x)$, not on $O(x)$, there may exist $f$, $g \in P(U)$ with $U_f \cap U_g$ not open in $U_f$ or $U_g$.

Finally, we shall see that there is a functor $S: \mathcal{P}_\pi \to \mathcal{L}_\pi$ that sends a homomorphism $\rho: P \to P'$ of $\pi$-presheaves to a $\pi$-homomorphism $\phi: SP \to SP'$ defined as follows. Since $\rho$ is a natural transformation of functors, it consists of a family of group homomorphisms $\rho_A: P(A) \to P'(A)$ indexed by the objects $A$ of $Q_\pi$. For a point $f_x \in SP$, that is the limit of $f \in P(A)$ at $x$, let $\phi(f_x)$ be the limit of $\rho_A(f) \in P'(A)$ at $x$, i.e.

$$(3.4) \qquad \phi(f_x) = g_x, \text{ where } g = \rho_A(f) \in P'(A).$$

It is obvious that $\phi$ is a $\pi$-homomorphism and that $S$ is a functor.

## 4. The functors $S$ and $\Gamma$.

THEOREM 4.1. *The functor $S: \mathcal{P}_\pi \to \mathcal{L}_\pi$ is left adjoint to the functor $\Gamma: \mathcal{L}_\pi \to \mathcal{P}_\pi$.*

The proof will follow two lemmas.

LEMMA 1. *There is a natural transformation from the composite functor $S\Gamma$ of $S$ and $\Gamma$ to the identity functor of $\mathcal{L}_\pi$.*

PROOF. Let $E$ be a $\pi$-sheaf. Then for every $x \in X$,

$$(4.1) \qquad (S\Gamma E)_x = \varinjlim (\Gamma E)(A) = \varinjlim \Gamma(A, E), \; A \in Q(x).$$

That is, $(S\Gamma E)_x$ is the group of $\pi$-germs at $x$ of local sections of $E$; a point in $(S\Gamma E)_x$ is the $\pi$-germ $s_x$ at $x$ represented by a section $s \in \Gamma(A, E)$. Let $\theta(s_x) = s(x)$. Then $\theta: S\Gamma E \to E$ so defined is a $\pi$-homomorphism. The class of $\theta$ (indexed by the objects $E$ of $\mathcal{L}_\pi$) form a natural transformation from $S\Gamma$ to the identity functor of $\mathcal{L}_\pi$. Moreover, it is easily shown that:

COROLLARY. *The quasi-topology of $E$ is the final quasi-topology deter-*

*mined by that of* $S\Gamma E$ *via* $\theta$, *i.e.*, $\theta$ *is a* $\pi$*-epimorphism.*

LEMMA 2. *There is a natural transformation from the identity functor of* $\mathcal{P}_\pi$ *to the composite functor* $\Gamma S$ *of* $\Gamma$ *and* $S$.

PROOF. Let $P$ be a $\pi$-presheaf. Then $(\Gamma SP)(A)=\Gamma(A,SP)$ for every $A\in|Q_\pi|$. Define a map $h_A:P(A)\to\Gamma(A,SP)$ by $h_A(f)=s$ with $s(x)=f_x$ for every $x\in A$. This is well defined since, for every $f\in P(A)$, the family $\{f_x\,|\,x\in A\}$ do define a section $s$ of $SP$ over $A$. It is easily checked that $h_A$ is a homomorphism and that

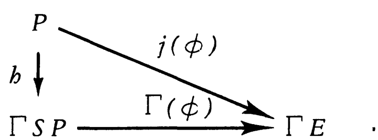
$$h=\{h_A:P(A)\to(\Gamma SP)(A)\,|\,A\in|Q_\pi|\}\qquad(4.2)$$

is a $\pi$-homomorphism from $P$ to $\Gamma SP$. The family of such $h$ (indexed by the objects $P$ of $\mathcal{P}_\pi$) form a natural transformation from the identity functor of $\mathcal{P}_\pi$ to $\Gamma S$.

PROOF OF THEOREM 4.1. We want to show that there is a natural equivalence

$$j:\mathcal{L}_\pi(SP,E)\to\mathcal{P}_\pi(P,\Gamma E)\qquad(4.2)$$

of the set of $\pi$-homomorphisms from $SP$ to $E$ to the set of homomorphisms from $P$ to $\Gamma E$. For any $\phi\in\mathcal{L}_\pi(SP,E)$, define $j(\phi)=\Gamma(\phi)h$ as in the diagram




Then $j(\phi)$ consists of a set of homomorphisms $j(\phi)_A:P(A)\to(\Gamma E)(A)$ defined by

$$j(\phi)_A(f)=(\Gamma(\phi)_A h_A)(f)=\Gamma(\phi)_A(s)=\phi s,\qquad(4.4)$$

where $s\in\Gamma(A,E)$ is the section $s(x)=f_x$. We claim that $j$ is a bijection with inverse $k$ defined by $k(\rho)=\theta S(\rho)$ for every $\rho\in\mathcal{P}_\pi(P,\Gamma E)$. Indeed,

$$(jk(\rho))_A(f)(x)=\theta S(\rho)(s(x))=\theta S(\rho)(f_x)=\theta(\rho_A(f))_x=\rho_A(f)(x)\qquad(4.5)$$

for every $f \in P(A)$ shows that $jk(\rho)=\rho$. On the other hand,

$$(4.6) \qquad kj(\phi)(f_x)=\theta S(j(\phi))(f_x)=\theta(j(\phi)_A(f))_x = (j(\phi)_A(f))(x)=\phi(f_x)$$

shows that $kj(\phi)=\phi$. Since $j$ and $k$ are defined by functors and natural transformations, the bijection $j$ is natural.

REMARK. (1) $\mathcal{P}_\pi$ and $\mathcal{L}_\pi$ are additive categories and $j$ is indeed a natural isomorphism of groups.

(2) $\mathcal{P}_\pi$ and $\mathcal{L}_\pi$ are not abelian categories. For example, $\mathcal{P}_\pi$ is not closed under the formation of kernels since the property (PS) which is defined by colimits is not preserved by kernels.

## 5. The subcategory $\mathcal{L}_X$ of $\mathcal{L}_\pi$.

Recall that $T_\pi$ is the underlying topology of $\pi$. Regard $T_\pi$ as a category with morphisms inclusion maps; then it is a full subcategory of $Q_\pi$. Let $\mathcal{P}_X$ be the category of presheaves over $(X, T_\pi)$, i.e., the category of contravariant functors from $T_\pi$ to $Ab$. Then there is a functor $R': \mathcal{P}_\pi \to \mathcal{P}_X$ defined by $R'(P)=P\,|\,T_\pi$. On the other hand, we define a functor $J': \mathcal{P}_X \to \mathcal{P}_\pi$ as follows. For every presheaf $G: T_\pi \to Ab$, let $J'G$ be a mapping on $Q_\pi$ to $Ab$ with

$$(5.1) \qquad (J'G)(A)=\lim_{\to \alpha} G(U), \quad \forall A \in |Q_\pi|,$$

where $\alpha$ is the set of open neighborhoods of $A$ directed by inclusion. Then $J'G$ verifies the property (PS) and thus defines a $\pi$-presheaf. By a routine limit argument in category theory, one shows that the correspondence $G \to J'G$ defines a functor $J'$ from $\mathcal{P}_X$ to $\mathcal{P}_\pi$ and that

PROPOSITION 5.1. *$J'$ is left adjoint to $R'$. Moreover, the composite functor $R'J'$ of $R'$ and $J'$ is naturally equivalent to the identity functor of $\mathcal{P}_X$ (and therefore $J'$ is a full embedding).*

Recall that $\mathcal{L}_X$ is a full subcategory of $\mathcal{L}_\pi$ (see proposition 2.3). $\Gamma\,|\,\mathcal{L}_X$ defines a functor $\Gamma'$ from $\mathcal{L}_X$ to $\mathcal{P}_X$ that can be identified with the composite functor $R'\Gamma J$, where $J$ is the inclusion functor of $\mathcal{L}_X$ in $\mathcal{L}_\pi$. On the other hand, a functor $S': \mathcal{P}_X \to \mathcal{L}_X$ with

$$(5.2) \qquad S'G = \bigcup_{x \in X} (S'G)_x, \text{ where } (S'G)_x = \varinjlim G(U),\ U \in O(x),$$

can be defined by replacing $Q_\pi$ by $T_\pi$ in the construction of $S$ in section 3. Notice that here the limit $f_x$ of $f$ at $x$ is taken on $O(x)$ instead of $Q(x)$; contrary to the remark of section 3, $U_f \cap V_g$ is always open in $\tau_f$ and $\tau_g$. Therefore, the quasi-topology $\tau$ on $S'G$ is the only quasi-topology that renders each $U_f$ an open subset of $S'G$ (cf. [16], p.28). In fact, the family of all subsets $U_f$ form a basis for the underlying topology $T_\tau$ on $S'G$. Similar to theorem 4.1 we have

PROPOSITION 5.2. *$S'$ is left adjoint to $\Gamma'$. Moreover, the composite functor $S'\Gamma'$ is naturally equivalent to the identity functor of $\mathcal{L}_X$.*

If $\mathcal{L}_X$ is identified with the category $\mathcal{L}_T$ of abelian sheaves over $(X, T_\pi)$ by the functor $T_X$ of proposition 2.3, then the functors $S'$ and $\Gamma'$ are identified with the associated sheaf functor and the local section functor, respectively, of the theory of sheaves.

Like $\mathcal{L}_T$, $\mathcal{L}_X$ is an abelian category with enough injectives; it is AB5 (see [11]). The injective structure on $\mathcal{L}_X$ is called *the canonical injective structure on* $\mathcal{L}_X$.

## 6. Germs of local sections of a $\pi$-sheaf.

In the diagram

$$\begin{array}{ccc}
\mathcal{L}_X & \underset{R}{\overset{J}{\rightleftarrows}} & \mathcal{L}_\pi \\
\Gamma'\downarrow\ \uparrow S' & & \Gamma\downarrow\ \uparrow S \\
\mathcal{P}_X & \underset{J'}{\overset{R'}{\leftrightarrows}} & \mathcal{P}_\pi
\end{array}$$

of categories and functors, let $R = S'R'\Gamma$. Then for any $\pi$-sheaf $E$,

$$(6.1) \qquad (RE)_x = (S'R'\Gamma E)_x = \varinjlim \Gamma(U, E),\ U \in O(x),$$

since $(R'\Gamma E)(U) = \Gamma(U, E)$ for every $U$ in $T_\pi$. Thus $RE = \bigcup_{x \in X} (RE)_x$ is the $\pi$-sheaf (indeed a sheaf) of germs of local sections of $E$. Since

(6.2) $$S\Gamma E = \bigcup_{x \in X} \varinjlim \Gamma(A, E),\ A \in Q(x)$$

is the $\pi$-sheaf of $\pi$-germs of local sections of $E$, formula (1.2) and property (PS) show that there is a surjective $\pi$-homomorphism $\zeta : RE \to S\Gamma E$ defined by the set of group epimorphisms:

(6.3) $$\zeta_x : \varinjlim \Gamma(U, E) \to \varinjlim \Gamma(A, E),\ x \in X.$$

We claim that the quasi-topology $\sigma'$ on $S\Gamma E$ is the final quasi-topology determined by the quasi-topology $\sigma$ on $RE$ via $\zeta$ and therefore $\zeta$ is a $\pi$-epimorphism. Indeed, since $\sigma$ (resp. $\sigma'$) is the final quasi-topology determined by the family $\sigma_f = \sigma \mid U_f$ (resp. $\sigma'_f = \sigma' \mid U_f$) via inclusion maps $U_f \subset RE$ (resp. $U_f \subset S\Gamma E$), every $p \mid U_f$ is a $(\sigma, \pi)$-homeomorphism (resp. $(\sigma', \pi)$-homeomorphism) of $U_f$ onto $U$. Now, let $(Y, \sigma'')$ be a quasi-topological space and let $\phi : S\Gamma E \to Y$ be a map such that $\phi\zeta$ is $(\sigma, \sigma'')$-continuous. Then, since each $\phi \mid U_f = (\phi\zeta \mid U_f)(\zeta \mid U_f)^{-1}$ is $(\sigma', \sigma'')$-continuous, so is $\phi$ (cf. [16]). This shows that $\sigma'$ is the final quasi-topology determined by $\sigma$ via $\zeta$.

Recall (corollary of lemma 1 of section 4) that $\theta : S\Gamma E \to E$ is a $\pi$-epimorphism; so is

(6.4) $$\psi : RE \to E,\quad \psi = \theta\zeta.$$

We identify $\mathfrak{L}_X$ with $\mathfrak{L}_T$ and see that every $\pi$-sheaf $E$ is a quotient of the sheaf of germs of local sections of $E$. More generally, we shall prove that $\mathfrak{L}_X$ is a reflective subcategory of $\mathfrak{L}_\pi$ and that $\psi$ of (6.4) is a reflection. Thus every $\pi$-sheaf is *reflected* by the sheaf of germs of its local sections.

THEOREM 6.1. *$R : \mathfrak{L}_\pi \to \mathfrak{L}_X$ is right adjoint to the inclusion functor $J : \mathfrak{L}_X \to \mathfrak{L}_\pi$, i.e., $R$ is a reflector.*

PROOF. We want to show that for $E'$ in $\mathfrak{L}_X$ and $E$ in $\mathfrak{L}_\pi$, there is a natural bijection

(6.5) $$\mathfrak{L}_\pi(JE', E) \to \mathfrak{L}_X(E', RE).$$

We observe that $J = SJ'\Gamma'$. Indeed, since

$$(J'\Gamma' E')(A) = \varinjlim_{\alpha} \Gamma(U, E'), \qquad \forall A \in |Q_\pi|,$$

we have

$$(SJ'\Gamma'E')_x = \varinjlim (J'\Gamma'E')(A) = \varinjlim \Gamma(U, E') = E'_x .$$

Therefore $SJ'\Gamma'E' = E'$. Now, by theorem 4.1 and proposition 5.1,

$\mathfrak{L}_\pi(JE', E) = \mathfrak{L}_\pi(SJ'\Gamma'E', E) \approx \mathfrak{P}_\pi(J'\Gamma'E', \Gamma E) \approx \mathfrak{P}_X(\Gamma'E', R'\Gamma E)$.

Since $S' | \Gamma'(\mathfrak{L}_X)$ is a full embedding, proposition 5.2 shows that

$$\mathfrak{P}_X(\Gamma'E', R'\Gamma E) \approx \mathfrak{L}_X(S'\Gamma'E', S'R'\Gamma E) \approx \mathfrak{L}_X(E', RE).$$

Thus, $\mathfrak{L}_\pi(JE', E)$ is naturally isomorphic to $\mathfrak{L}_X(E', RE)$.

COROLLARY 6.2. *$R | \mathfrak{L}_X$ is the identity functor of $\mathfrak{L}_X$.*

## 7. An effacement structure on $\mathfrak{L}_\pi$.

Let the canonical injective structure of $\mathfrak{L}_X$ be denoted $(\mathfrak{M}, \varepsilon)$ and identify $\mathfrak{L}_X$ with $\mathfrak{L}_T$. Then $\mathfrak{M}$ is the class of sheaf monomorphisms and $\varepsilon$ is the class of injective sheaves [11]. $(\mathfrak{M}, \varepsilon)$ induces an effacement structure $(\mathfrak{F}', \mathfrak{H}')$ on $\mathfrak{L}_X$, where $\mathfrak{F}' = \mathfrak{M}$ and $\mathfrak{H}'$ is the class of $\pi$-homomorphisms that factor through an injective object of $\mathfrak{L}_X$ (see proposition 2.13 of [13]).

The notion of an *effacement structure* on a category $C$ was first defined by Zimmermann [19] under the German name «Erweiterungspaare». It consists of a pair $(\mathfrak{F}, \mathfrak{H})$ of two classes of morphisms of $C$ satisfying the following three conditions:

(1) $\mathfrak{H}$ is the class of all morphisms $h: A \to B$ such that, for every $f: X \to Y$ in $\mathfrak{F}$ and for every given $u: X \to A$ in $C$, there is a morphism $v: Y \to B$ such that $vf = hu$;

(2) $\mathfrak{F}$ is the class of all morphisms $f: X \to Y$ such that, for every $h: A \to B$ in $\mathfrak{H}$ and for every given $u: X \to A$ in $C$, there is a morphism $v: Y \to B$ such that $vf = hu$;

(3) for every object $A$ in $C$, there is a morphism $f \in \mathfrak{F} \cap \mathfrak{H}$ with domain $A$.

We need also the following definition from [13]. $\mathfrak{F}' \subset \mathfrak{F}$ is a *subbasis* for $\mathfrak{F}$ if $\mathfrak{F}$ is the class of morphisms $f: X \to Y$ such that there exists a push-out

$$\begin{array}{ccc} X' & \xrightarrow{\ f'\ } & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ \phi\ } & Z \end{array}$$

with $f' \in \mathfrak{F}'$ and a morphism $k : Y \to Z$ with $kf = \phi$.

Let $(\mathfrak{F}', \mathfrak{H}')$ be the effacement structure on $\mathfrak{L}_X$ induced by $(\mathfrak{M}, \varepsilon)$ as mentioned in the first paragraph. We have

THEOREM 7.1. *There is an effacement structure* $(\mathfrak{F}, \mathfrak{H})$ *on* $\mathfrak{L}_\pi$ *in which* $\mathfrak{H} = R^{-1}(\mathfrak{H}')$ *and* $\mathfrak{F}$ *has* $\mathfrak{F}'$ *as a subbasis.*

In view of theorem 4.6 of [13], $(\mathfrak{F}, \mathfrak{H})$ is the inverse transfer of $(\mathfrak{F}', \mathfrak{H}')$ by the pair of adjoint functors $J$ and $R$ provided that push-outs exist in $\mathfrak{L}_\pi$.

Given $\pi$-homomorphisms $\phi : E \to F$ and $\psi : E \to G$, we shall construct the push-out $K$ of $\phi$ by $\psi$. For every $x \in X$, let $K_x$ be the push-out of $\phi_x : E_x \to F_x$ by $\psi_x : E_x \to G_x$ (since push-outs exist in $Ab$) and let $K = {}_{\prime}\bigcup_{x \in X} K_x$. Then we have a diagram

$$\begin{array}{ccc} E & \xrightarrow{\ \phi\ } & F \\ \psi \downarrow & & \downarrow i \\ G & \xrightarrow{\ j\ } & K \end{array}$$

Endow $K$ with the final quasi-topology $\tau$ determined by the quasi-topologies of $F$ and $G$ via $i$ and $j$. Then $(K, \tau)$ is a $\pi$-sheaf which is the push-out of $\phi$ by $\psi$.

Finally, we shall generalize the notion of injective resolutions in sheaf theory to that of $\pi$-injective resolutions. Recall that for every sheaf $E'$ (e.g. $E' = RE$ of a $\pi$-sheaf $E$) there is an injective resolution

$$0 \longrightarrow E' \longrightarrow Q'_*$$

defined as follows: Choose a map $i' : E' \to Q'_0$ in $\mathfrak{F}' \cap \mathfrak{H}'$ (i.e. $i'$ is a monomorphism with $Q'_0$ injective) and let $G_0$ be the cokernel of $i'$. Then we obtain an exact sequence

$$0 \longrightarrow E' \xrightarrow{i'} Q'_0 \xrightarrow{j'} G_0 \longrightarrow 0$$

Choose a map $k' : G_0 \to Q'_1$ in $\mathfrak{F}' \cap \mathfrak{H}'$ and let $G_1$ be the cokernel of $k'$. Then an exact sequence

$$0 \longrightarrow E' \xrightarrow{i'} Q'_0 \xrightarrow{d'_0} Q'_1 \longrightarrow G_1 \longrightarrow 0, \quad d'_0 = k'j',$$

is obtained. Continuing in this way, we get an injective resolution

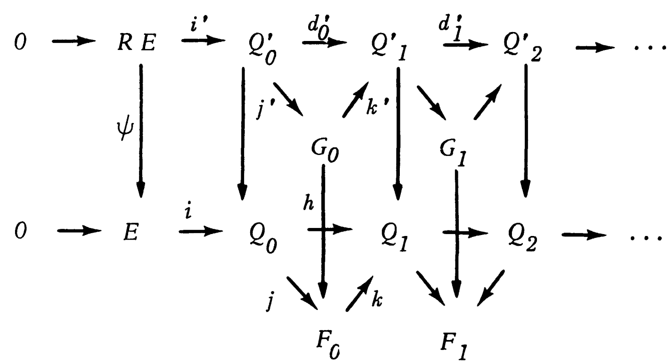
$$0 \longrightarrow E' \xrightarrow{i'} Q'_0 \xrightarrow{d'_0} Q'_1 \xrightarrow{d'_1} Q'_2 \longrightarrow \cdots$$

We shall generalize this construction to one for a $\pi$-sheaf $E$. Let $Q_0$ be the push-out of $i' : RE \to Q'_0$ by the reflection $\psi : RE \to E$

$$\begin{array}{ccc} RE & \xrightarrow{\quad i' \quad} & Q'_0 \\ \psi \big\downarrow & & \big\downarrow \psi_0 \\ E & \xrightarrow{\quad i \quad} & Q_0 \end{array} .$$

Then it is easy to show that, since $i' \in \mathfrak{F}' \cap \mathfrak{H}'$, $i \in \mathfrak{F} \cap \mathfrak{H}$. Moreover, $i$ is a $\pi$-monomorphism. In the diagram

(7.1)



let $F_0$ be the cokernel of $i$, $Q_1$ be the push-out of $k'$ by $h$, $F_1$ be the cokernel of $d_0 = kj$ and $Q_2$ be a push-out again. Then by repeating this construction we obtain an exact sequence

$$(7.2) \qquad 0 \xrightarrow{i} E \xrightarrow{d_0} Q_0 \xrightarrow{d_1} Q_1 \longrightarrow Q_2 \longrightarrow \cdots$$

called a *$\pi$-injective resolution* of $E$ in $\mathfrak{L}_\pi$. We remark that, in general, neither $Q_i$ is injective in $\mathfrak{L}_\pi$, nor is $RQ_i$ in $\mathfrak{L}_X$. We call (7.2) a «$\pi$-injective» resolution just because it is constructed out of an *injective* resolution of $RE$. Nevertheless, when $Q'{*}$ is replaced by another injective resolution of $RE$, the corresponding $\pi$-injective resolution of $E$ is chain homotopic to (7.2). Therefore, there is a cohomology theory on $\mathfrak{L}_\pi$ defined by resolutions that generalizes the sheaf cohomology. We shall study this further under a more general topic.

Department of Mathematics
Wesleyan University, Middletown, Conn.
Fordham University, Bronx, N.Y., U.S.A.

## REFERENCES


[1] BASTIANI A., Applications différentiables et variétés différentiables de dimension infinie, *Jour. Analyse Math. Jérusalem,* 13 (1964) 1-114.

[2] BINZ E., Bemerkungen zu limitierten Funktionenalgebren, *Math. Ann.* 175 (1968), 169-184.

[3] BINZ E., Ein Differenzierbarkeitsbegriff in limitierten Vektorräumen, *Comm. Math. Helv.* 41 (1966-67).

[4] BINZ E. and KELLER H., Funktionenräume in des Kategorie der Limesräume, *Ann. Acad. Sci. Fenn.* (Finlande) A1, 383 (1966).

[5] CHEN Y.C., Quasi-faisceaux sur les espaces quasi-topologiques, *C.R.A.S.* Paris 273 (1971).

[6] COOK C. and FISCHER H., Uniform convergent structures, *Math. Ann.* 173 (1967), 290-306.

[7] COOK C. and FISCHER M., On equicontinuity and continuous convergence, *Math. Ann.* 159 (1965), 94-104.

[8] EHRESMANN C., Catégories topologiques I, II, III, *Koninkl. Nederl. Akad. van Wetenschappen,* Proceeding series A 69, 1 (1966).

[9] EILENBERG-MOORE, Foundations of relative homological algebra, *Memoirs Amer. Math. Sci.* 55 (1965).

[10] FRÖLICHER-BUCHER, Calculus in vector spaces without norm, *Lecture Notes in Math.* 30, Springer (1966).

[11] GROTHENDIECK A., Sur quelques points d'algèbre homologique, *Tohoku Math. Jour.* 9 (1957), 119-221.

[12] KOWALSKY H., Limesräume und Kompletierung, *Math. Nach.* 12 (1954), 301-340.

[13] LEROUX P., Sur les structures d'effacement, (to appear).

[14] MACHADO A., Caractérisation à l'aide de quasi-topologies de la structure de variété analytique complexe, *C.R.A.S.* Paris 268 (1969), 1402-1405.

[15] MACHADO A., Quasi-variétés complexes, *Cahiers de Top. et Géom. Diff.* XI,3 (1969), 229-279.

[16] MACHADO A., Quasi-topologie algébrique, *Esquisses Math.* 10 (1970).

[17] MARANDA J., Injective structures, *Trans. Amer. Math. Sci.* 110 (1964), 98-135.

[18] SPANIER E., Quasi-topologies, *Duke Math. Jour.* 30 (1963).

[19] ZIMMERMANN W., *Injektive Strukturen und M-injektive Objekte,* Thesis, Ludwig-Maximilians-Universität, München (1969).