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DIAGRAM LEMMAS IN SEMI-EXACT *JTK*-CATEGORIES

by J.V. MICHALOWICZ

1. Introduction

The concept of a *JTK*-category was introduced in [5] to provide a procedure for dealing in categories with the non-categorical concepts of subobject, quotient object, one-to-one mapping, onto mapping, embedding and quotient map.

DEFINITION. A category $\mathcal{Q}$ is called a *JTK-category* if there is a class T of morphisms in $\mathcal{Q}$ such that T and the classes J, K, L, N of morphisms defined by

$$(\mathrm{T1}) \quad \begin{cases} J = \{ m \mid m = gt \text{ implies } g \in M \text{ and } t \in S \} \\ K = \{ e \mid e = th \text{ implies } h \in E \text{ and } t \in S \} \end{cases}$$

$$(\mathrm{T2}) \quad \begin{cases} L = \{ m \mid m = gt \text{ implies } g \in M \} \\ N = \{ e \mid e = th \text{ implies } h \in E \} \end{cases}$$

satisfy the following conditions

(T3) $\quad T = \{ b \mid b = t_1 f t_2 \text{ implies } f \in B \}.$

(T4) $\quad JJ \subset J;\ KK \subset K;\ TT \subset T.$

(T5) $\quad J \cap T = T \cap K = J \cap K = S.$

(T6) $\quad TJ \subset L;\ KT \subset N.$

(T7) Every morphism f in $\mathcal{Q}$ has a representation $f = jtk$ which is unique up to isomorphism.

We use M, E, B, S, I, R, C for the classes of monomorphisms, epimorphisms, bimorphisms, isomorphisms, identity morphisms, retractions, coretractions, respectively, in any category and we use lower case letters as indicators; e.g., $m \in M$, $t \in T$, $t_1 \in T$, etc. In the *JTK*-category, J is designed to be an abstraction of the class of embeddings in $\mathcal{Q}$, K of the quotient maps, T of the one-to-one onto mappings, L of the one-to-one mappings and N of the onto mappings. In [4, 5] it is shown that these classes of morphisms do indeed retain the desired properties of their progenitors, examples of JTK-categories are given, and the basic theory of the JTK-category is developed.

A J-normal and K-conormal JTK-category (i.e., every morphism in J is a kernel and every morphism in K is a cokernel) with kernels and cokernels is called a semi-exact JTK-category. A few basic properties of the semi-exact JTK-category are obtained in [4, 5] and it is shown that every exact category is a balanced semi-exact JTK-category in which the JTK-categorical classes of morphisms are just the corresponding categorical classes. However, there are semi-exact JTK-categories which are not exact categories; an example is given in [4, 5].

We begin this paper by giving several examples of semi-exact JTK-categories which are not exact categories. Some of these are important categories and thus a deeper investigation of the semi-exact JTK-category is justified. In this paper we generalize various diagram lemmas and isomorphism theorems from the exact category to the semi-exact JTK-category, with special attention being paid to conditions under which our results can be strengthened. It will be clear from these results that the essential difference between the exact category and the semi-exact JTK-category lies in the fact that each morphism in the exact category has a two-part decomposition $f = me$ whereas in the semi-exact JTK-category the decomposition is in three-parts as $f = jtk$.

2. Examples

Our first example is the category $\mathcal{C}_1$ of abelian topological groups and continuous homomorphisms, where by a topological group we mean a set with a group structure and a topology compatible with the group structure. Any single-point group is a zero object for this category and kernels and cokernels are constructed in the obvious way. It follows that $M = \overline{M}$ (the class of one-to-one morphisms), $E = \overline{E}$ (the class of onto morphisms), $B = \overline{B}$ (the class of one-to-one onto morphisms) and $S \subsetneq B$. This category becomes a JTK-category with $T = B$, $L = M$, $N = E$, J the class of embeddings and K the class of quotient maps; each morphism $f: G \to G'$ has the decomposition $G \to G/Ker\, f \to Im\, f \to G'$. It is easily seen that $\mathcal{C}_1$ is in fact J-normal and K-conormal and thus a semi-exact JTK-category, but it is not an exact category since it is not balanced.

We note further that $\mathcal{C}_1$ is additive and has products (Cartesian product) and thus $\mathcal{C}_1$ is a semi-exact additive JTK-category with products which is not an abelian category. (The full subcategory of $\mathcal{C}_1$ the objects of which are the compact (Hausdorff) abelian topological groups is an abelian category). For the most part, the construction of categorical concepts such as intersections, pullbacks, equalizers, cointersections, etc., in $\mathcal{C}_1$ is achieved by combining the corresponding constructions in the categories of topological spaces and abelian groups.

Similar examples are the category of real linear topological spaces and, more generally, the category of topological (left) A-modules over a topological ring A with identity.

Another example is the full subcategory $\mathcal{C}_2$ of the category of pointed topological spaces which has as objects those with one or two points. It is easily checked that $M = \overline{M}$, $E = \overline{E}$, $B = \overline{B}$, and $S \subsetneq B$ and that $\mathcal{C}_2$ is a semi-exact JTK-category, with $T = B$, $L = M$, $N = E$, J the class of embeddings and K the class of quotient maps, which is not an exact category.

3. Diagram lemmas.

Various results will now be presented, including versions of the nine lemma, five lemma, and Noether isomorphism theorems for the semi-exact JTK-category, which reduce to the familiar theorems in the special case of an exact category. The proofs involve similar technics as used for the exact category (although the line of reasoning differs in some cases) and hence will usually be omitted. Associated with each of these diagram lemmas is the important problem of keeping the assumptions as weak as possible and further, of obtaining conditions under which the conclusions can be strengthened. We begin with

PROPOSITION 1. (Nine Lemma) *For a commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A' & & A & & A'' & & \\
 & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & B' & \xrightarrow{h_1} & B & \xrightarrow{h_2} & B'' & \longrightarrow & 0 \\
 & & {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g''} & & \\
0 & \longrightarrow & C' & \xrightarrow{h_3} & C & \xrightarrow{h_4} & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in a semi-exact JTK-category $\mathfrak{A}$ *where all the rows and columns are semi-exact and* $f, f', f'', h_1 \in J$ *and* $g, h_4 \in K$, *there are unique morphisms* $f_1 : A' \to A$ *and* $f_2 : A \to A''$ *which keep the diagram commutative. Moreover,* $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ *is semi-exact and* $f_1 \in J$.

Now $f_2 \in N$ but in general $f_2 \notin K$. In fact, $f_2 \in K$ for all h_2 in K if and only if $KJ \subset JK$. The sufficiency of this condition will be clear to the reader who works through the proof of Proposition 1. For the necessity, consider $kj \in KJ$, and construct a diagram to use in Proposition 1 by letting $h_2 = k$, $f = j$, $g =$ *cokernel of* j, $h_1 =$ *kernel of* k, $h_4 =$ *cokernel of* $g h_1$ and $h_3 =$ *kernel of* h_4. Then g'' in K and g' in N exist automatically and we let f'' and f' be the kernels of g'' and g', respectively. Thus $kj = f'' f_2$ is in JK.

The condition $KJ \subset JK$ holds in the finite semi-exact JTK-category of [4,5], in $\mathcal{C}_2$, and, of course, in any exact category; it does not hold in $\mathcal{C}_1$. For example, let G be $\mathbf{R}^2$, H the subgroup given by a straight line through the origin with slope α where α is irrational, and N the sub-group consisting of all points with integral coordinates. Then

$$h : H \to G \to G/N$$

is in KJ, where $H \to G$ is the inclusion map and $G \to G/N$ the natural surjection, but h is not open onto its image and thus is not in JK, for a morphism $f = jtk$ in $\mathcal{C}_1$ has $t \in S$ iff f is open onto its image, which follows from the fact that each natural surjection in $\mathcal{C}_1$ is open.

It can be shown that $f_2 \in K$ for all h_2 in Proposition 1 iff $\mathfrak{A}$ is an exact category. Another version of the Nine Lemma is

PROPOSITION 2. *Given a commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{f_1} & A & \xrightarrow{f_2} & A'' & \longrightarrow & 0 \\
 & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow & & \\
0 & \longrightarrow & B' & \xrightarrow{h_1} & B & \xrightarrow{h_2} & B'' & \longrightarrow & 0 \\
 & & {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g''} & & \\
0 & \longrightarrow & C' & \xrightarrow{h_3} & C & \xrightarrow{h_4} & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in a semi-exact JTK-category where the columns and middle row are semi-exact and $f, f', f'', h_1 \in J$ and $g, g', g'', h_2 \in K$, then the top row is semi-exact iff the bottom row is semi-exact.

Consider a pullback diagram

$$\begin{array}{ccc}
P & \xrightarrow{g_2} & A_2 \\
{\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle f_2} \\
A_1 & \xrightarrow[f_1]{} & A
\end{array}$$

Now $f_2 \in M$ implies $g_1 \in M$ in any category and $f_2 \in J$ implies $g_1 \in J$ in any JTK-category. Moreover, in a semi-exact JTK-category $\mathfrak{A}$, $g_1 \in M$ implies $f_2 \in M$, which follows from the fact that, if u is a kernel of g_1 then $g_2 u$ is a kernel of f_2. On the other hand, $g_1 \in J$ implies $f_2 \in J$ for each pullback diagram in $\mathfrak{A}$ iff $\mathfrak{A}$ is exact.

PROPOSITION 3. (First Noether Isomorphism Theorem) *For $j: B \to A_2$ and $j': A_2 \to A_1$ (i.e., $B \subset A_2 \subset A_1$) in a semi-exact JTK-category, there is a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_2 & \xrightarrow{j'} & A_1 & \longrightarrow & A_1/A_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A_2/B & \longrightarrow & A_1/B & \xrightarrow[k]{} & A_1/A_2 & \longrightarrow & 0
\end{array}$$

with semi-exact rows.

PROOF. Apply Proposition 1* to the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & = & B & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'j} & & \downarrow & & \\
0 & \longrightarrow & A_2 & \xrightarrow{j'} & A_1 & \longrightarrow & A_1/A_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
 & & A_2/B & & A_1/B & & A_1/A_2 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that in the resulting semi-exact sequence

$$0 \longrightarrow A_2/B \xrightarrow{g_1} A_1/B \xrightarrow{g_2} A_1/A_2 \longrightarrow 0$$

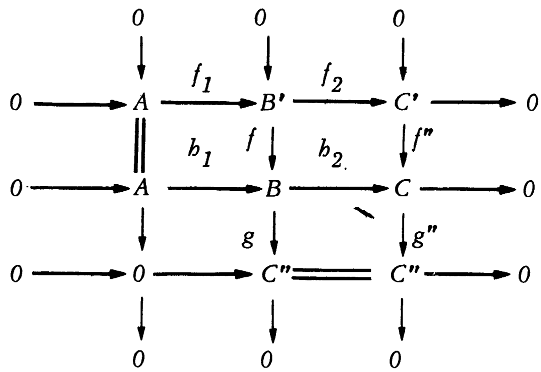
we have $g_2 \in K$ so we can write $(A_1/B)/(A_2/B) = A_1/A_2$.

Now $g_1 \in L$ but it does not appear necessary that $g_1 \in J$. In fact, $g_1 \in J$ for all j, j' iff $\{kj \mid (\textit{kernel of } k) \leqslant j\} \subset JK$. For if this condition holds, then in the above proof we have $k'j' = g_1 k''$ where k' is the cokernel of $j'j$, k'' the cokernel of j, and (kernel of k')$=j'j \leqslant j'$ so that $g_1 k'' \in JK$ and $g_1 \in J$. The converse follows by constructing, for any morphism $k'j'$ with (kernel of k')$=j'j$, the diagram in the above proof for j, j'. This condition is satisfied in all the examples of semi-exact JTK-categories which have been presented. I do not have an example of a semi-exact JTK-category in which it does not hold, although it is not satisfied in the JTK-category of topological spaces with distinguished points, which is J-normal with kernels and cokernels. It might be noted that the intersection of the class $\{kj \mid (\textit{kernel of } k) \leqslant j\}$ with T (resp. L, N) is S (resp. J, K) in any semi-exact JTK-category.

Two more corollaries of Proposition 1 in the semi-exact JTK-category $\mathfrak{A}$ are as follows. Any pullback diagram

$$\begin{array}{ccc}
B' & \xrightarrow{f_2} & C' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\
B & \xrightarrow{h_2} & C
\end{array}$$

with $h_2 \in K$ and $f'' \in J$ can be extended to a commutative diagram

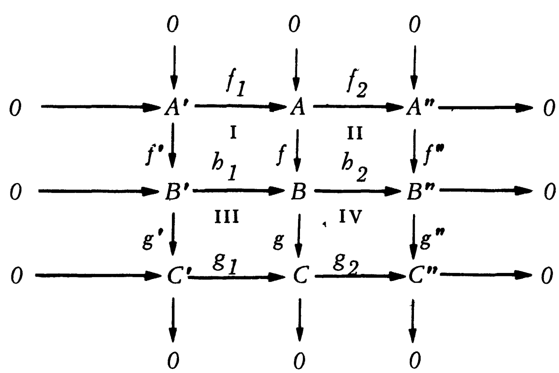


with semi-exact rows and columns where $g, g'' \in K$ and $f_1, h_1 \in J$. In fact, if we change $h_2 \in K$ to $h_2 \in N$, then the same conclusion is valid if the assertion $f_2 \in N$ is omitted and $g \in K$ is weakened to $g \in N$. As a consequence of this, for any $f: A \to B$ and $j: B' \to B$, there is a semi-exact sequence

$$0 \longrightarrow f^{-1}(B') \xrightarrow{g} A \xrightarrow{h} J\text{-}Im(f)/(J\text{-}Im(f) \cap B') \longrightarrow 0$$

where $g \in J$. Moreover, if $f \in JK$, then $h \in K$ and there is a morphism $n: f^{-1}(B') \to J\text{-}Im(f) \cap B'$ in N; if also $\{kj \mid (\text{kernel of } k) \leqslant j\} \subset JK$ in $\mathfrak{A}$, then this n is in K. Finally, if $f \in K$ then $f^J(f^{-1}(B')) = B'$.

PROPOSITION 4. *Suppose in a semi-exact JTK-category we are given the diagram*



with semi-exact middle row and middle column, and $f, h_1 \in J$ *and* $g, h_2 \in K$. *This diagram is commutative with semi-exact rows and columns and with* $f' \in J$, $g'' \in K$, $f'' \in J$ *(or* $f_2 \in K$*) and* $g_1 \in J$ *(or* $g' \in K$*) iff* I *is a pullback,* IV *is a pushout, and* II *and* III *are factorizations of* $h_2 f$ *and* $g h_1$, *respectively, through their* J*-images (or* K*-coimages). In this case it follows that* $f_1 \in J$ *and* $g_2 \in K$.

As a consequence we have

PROPOSITION 5. *For* $j_1 : A_1 \to A$ *and* $j_2 : A_2 \to A$ *in a semi-exact* JTK*-category, there is a commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 \cap A_2 & \longrightarrow & A_2 & \longrightarrow & A_2 / A_1 \cap A_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle j_2} & & \downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\ j_1\ } & A & \longrightarrow & A/A_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1/A_1 \cap A_2 & \longrightarrow & A/A_2 & \longrightarrow & A/A_1 \overset{J}{\cup} A_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with semi-exact rows and columns.

Next we consider the version of the Second Noether Isomorphism Theorem which is valid in the semi-exact JTK-category.

PROPOSITION 6. *If* $j_1 : A_1 \to A$ *and* $j_2 : A_2 \to A$ *are morphisms in* J *in a semi-exact* JTK*-category, then there is a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 \cap A_2 & \longrightarrow & A_2 & \longrightarrow & A_2 / A_1 \cap A_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle t} & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_1 \overset{J}{\cup} A_2 & \longrightarrow & A_1 \overset{J}{\cup} A_2 / A_1 & \longrightarrow & 0
\end{array}$$

with semi-exact rows where the left-hand square is a pullback and $t \in T$.

The morphism t in this diagram is in general not an isomorphism. In fact, we can show that this $t \in S$ for all j_1, j_2 in a semi-exact JTK-category $\mathfrak{A}$ iff $\{kj \mid j : D \to B,\ k : B \to C$ and i_B is the J-union of j and the kernel of $k\} \subset JK$ in $\mathfrak{A}$. This condition holds in the finite semi-exact JTK-category of [4,5], in $\mathcal{C}_2$, and certainly in any exact category; on the other hand, it fails in $\mathcal{C}_1$. For instance, if we look back at the example following Proposition 1, we have $H \overset{J}{\cup} N = H + N$ which is dense in G and $H \cap N$ is the origin since α is irrational. Now $N/H \cap N$ is isomorphic to N and so is discrete but $H \overset{J}{\cup} N/H$, although in one-to-one correspondence with N, is not discrete since the point H in $H \overset{J}{\cup} N/H$ is not open. Hence $N/H \cap N$ and $H \overset{J}{\cup} N/H$ are not isomorphic so that the morphism t of Proposition 6 cannot be an isomorphism.

For $j_1 : A_1 \to A$, $j_2 : A_2 \to A$ in a semi-exact JTK-category $\mathfrak{A}$ we will call the J-union $j : A_1 \overset{J}{\cup} A_2 \to A$ a direct J-union if $A_1 \cap A_2 = 0$. Clearly this definition is motivated by the notion of direct sum for abelian topological groups. It now follows from Proposition 6 that if $j : A_1 \overset{J}{\cup} A_2 \to A$ is a direct J-union of j_1 and j_2, then there is a morphism

$$t : A_2 \longrightarrow A_1 \overset{J}{\cup} A_2 / A_1$$

in T. Again t need not be an isomorphism, by the above example.

It is not surprising that the morphisms in JK should play such an important role in the discussion of the preceding results, for in the category $\mathcal{C}_1$ these are precisely the strict morphisms, as in Bourbaki [1,p. 236]. Hence in any semi-exact JTK-category $\mathfrak{A}$ we will call the morphisms in JK strict morphisms. For example, any zero morphism is strict. Also if $f : A \to B$ and $g : B \to C$ are strict morphisms in $\mathfrak{A}$, then gf is strict if $f \in N$ or $g \in L$.

PROPOSITION 7. *For $f : A \to C$ and $j : B \to A$ in a semi-exact JTK-category there is induced a morphism $n : A/B \to f^J(A)/f^J(B)$ in N. Moreover, if f is strict, then so is n; that is, $n \in K$.*

PROOF. Let $u : f^J(A) \to C$ be the J-image of f and $v : f^J(B) \to C$ the J-image of fj. Then $f = uh$ and $fj = vh'$ where $h, h' \in N$. Now $fj = uhj$ implies $v = uj'$ so that $uhj = uj'h'$. Thus we have the commutative diagram with semi-exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{\;j\;} & A & \xrightarrow{\;k\;} & A/B & \longrightarrow & 0 \\
 & & \downarrow h' & & \downarrow h & & & & \\
0 & \longrightarrow & f^J(B) & \xrightarrow{\;j'\;} & f^J(A) & \xrightarrow{\;k'\;} & f^J(A)/f^J(B) & \longrightarrow & 0
\end{array}$$

and so there is a morphism $g: A/B \to f^J(A)/f^J(B)$ with $gk = k'h$. Now $g \in N$ since $h \in N$; and, if f is strict, then $h \in K$ which implies that $g \in K$. This result, as well as the next one; are generalizations of familiar results in $\mathcal{C}_1$.

PROPOSITION 8. *Let $\mathfrak{A}$ be a semi-exact JTK-category. If $\mathfrak{A}$ has products, then for a family $\{j_\lambda : A_\lambda \to B_\lambda \mid \lambda \in \Lambda\}$ of morphisms in J, there is induced a morphism $l: \chi B_\lambda / \chi A_\lambda \to \chi (B_\lambda / A_\lambda)$ in L. If $\mathfrak{A}$ is semi-additive with finite products, this morphism l is in T for each finite index set Λ. On the other hand, if $\mathfrak{A}$ has products and if the product of epimorphisms is an epimorphism in $\mathfrak{A}$, then $l \in T$ in all cases. Finally, if $\mathfrak{A}$ has products and if the product of morphisms in K is again in K, then this l is always an isomorphism.*

PROOF. Let $A = \chi A_\lambda$ with projections $\{p_\lambda : A \to A_\lambda \mid \lambda \in \Lambda\}$ and $B = \chi B_\lambda$ with projections $\{q_\lambda : B \to B_\lambda \mid \lambda \in \Lambda\}$. Then we have the commutative diagram with semi-exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{u} & B & \xrightarrow{k} & B/A & \longrightarrow & 0 \\
 & & \downarrow p_\lambda & & \downarrow q_\lambda & & & & \\
0 & \longrightarrow & A_\lambda & \xrightarrow[j_\lambda]{} & B_\lambda & \xrightarrow[k_\lambda]{} & B_\lambda/A_\lambda & \longrightarrow & 0
\end{array}$$

for each λ where $u = \chi j_\lambda \in J$. Let $g = \chi k_\lambda$. Then u is the kernel of g since the product preserves kernels. Therefore, we can represent g as $j't'k$, since k is the cokernel of u and thus the K-coimage of g and $l = j't' : B/A \to \chi (B_\lambda / A_\lambda)$ is in L. Note that for each λ, $p'_\lambda j' t'$ is the unique morphism which completes the above diagram, where the p'_λ's are the projections for $\chi (B_\lambda / A_\lambda)$.

If $\mathfrak{A}$ is semiadditive with finite products and Λ is finite, let

$$\{u_\lambda : B_\lambda \to B \mid \lambda \in \Lambda\} \text{ and } \{u'_\lambda : B_\lambda/A_\lambda \to \chi (B_\lambda/A_\lambda) \mid \lambda \in \Lambda\}$$

be the appropriate injections. Then for each $\mu \in \Lambda$,

$$g u_\mu = \sum_\lambda \mu'_\lambda p'_\lambda g u_\mu = \sum_\lambda u'_\lambda k_\lambda q_\lambda u_\mu = u'_\mu k_\mu .$$

It now follows that $Coker(g) = 0$ and so $g \in N$ and $l \in T$. For if $vg = 0$, then $0 = v g u_\mu = v u'_\mu k_\mu$ so $v u'_\mu = 0$ for all $\mu \in \Lambda$ which implies $v = 0$. The rest of the statement is obvious.

For example, in $\mathcal{C}_1$ the product of morphisms in K is always in K

so we get $l \in S$ in this proposition.

Now consider $j_1 : A_1 \to A$, $j_2 : A_2 \to A$ in a semi-exact JTK-category $\mathfrak{A}$ where we assume that $A_1 \oplus A_2$ exists with $u_1 : A_1 \to A_1 \oplus A_2$ and $u_2 : A_2 \to A_1 \oplus A_2$ as the injections. Then there exists a unique morphism $f : A_1 \oplus A_2 \to A$ with $fu_1 = j_1$ and $fu_2 = j_2$. This morphism f may give us quite a bit of information about $A_1 \overset{J}{\cup} A_2$ and $A_1 \cap A_2$ as we will now discuss. We can show that the J-union of j_1 and j_2 is the J-image of f. Hence $f \in N$ iff i_A is the J-union of j_1 and j_2; i.e., $A_1 \overset{J}{\cup} A_2 = A$. Also $f \in J$ iff f is the J-union of j_1 and j_2; i.e., $A_1 \overset{J}{\cup} A_2 = A_1 \oplus A_2$. Furthermore, $f \in L$ implies $A_1 \cap A_2 = 0$ for in this case

$$\begin{array}{ccc} 0 & \longrightarrow & A_1 \\ \big\downarrow & & \big\downarrow j_1 \\ A_2 & \underset{j_2}{\longrightarrow} & A \end{array}$$

is a pullback diagram since if $j_1 h_1 = j_2 h_2$, then $fu_1 h_1 = fu_2 h_2$ which implies $u_1 h_1 = u_2 h_2$, so that $h_1 = p_1 u_1 h_1 = p_1 u_2 h_2 = 0$ and likewise $h_2 = 0$, where p_1, p_2 are the projections. If $\mathfrak{A}$ is additive, the converse of this statement is also true. For if the above diagram is a pullback and $ff_1 = ff_2$, then

$$f = f(u_1 p_1 + u_2 p_2) = j_1 p_1 + j_2 p_2$$

so that

$$0 = f(f_1 - f_2) = j_1 p_1 (f_1 - f_2) + j_2 p_2 (f_1 - f_2).$$

Hence $j_1 p_1 (f_1 - f_2) = j_2 p_2 (f_2 - f_1)$ so $p_1 (f_1 - f_2) = 0$ and $p_2 (f_2 - f_1) = 0$ from the pullback. Then $p_1 (f_1 - f_2) = 0$ and $p_2 (f_1 - f_2) = 0$ implies

$$f_1 - f_2 = 0 \text{ or } f_1 = f_2 ,$$

so that $f \in M = L$. To summarize, we have for any semi-exact additive JTK-category, like $\mathcal{C}_1$, that $f \in N$ iff $A_1 \overset{J}{\cup} A_2 = A$, $f \in L$ iff $A_1 \cap A_2 = 0$, and $f \in T$ iff $A_1 \overset{J}{\cup} A_2 = A$ and $A_1 \cap A_2 = 0$; that is, i_A is a direct J-union of j_1 and j_2.

Another result we'd like to include is that if $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ and $0 \to A' \xrightarrow{f'} B \xrightarrow{g'} C' \to 0$ are semi-exact sequences in a semi-exact JTK-category with f, f' in J and g, g' in K, then $A' \xrightarrow{f'} B \xrightarrow{g} C$ is in $N(L, T)$ if and only if $A \xrightarrow{f} B \xrightarrow{g'} C'$ is in $N(L, T)$.

We will now consider a few results concerning split semi-exact sequences in a semi-exact JTK-category $\mathcal{A}$; for the most part these are simple generalizations of the corresponding statements in the exact category. We say that a short semi-exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$ splits if $g \in R$.

PROPOSITION 9. *If $\mathcal{A}$ is additive and the semi-exact sequence*

$$0 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

splits with $f \in J$ (say $gf' = i_C$), then $B = A \oplus C$ with f and f' as the injections; further, $g' : B \to A$ can be chosen so that g' and g are the projections for this coproduct.

PROPOSITION 10. *If $\mathcal{A}$ is additive, $A \xrightarrow{f} B \xrightarrow{g} C$ semi-exact, $A \xleftarrow{u} B \xleftarrow{v} C$ of order two and $uf = i_A$ and $gv = i_C$, then $B = A \oplus C$ with f and v as injections and u and g as projections.*

We also point out that, if $\mathcal{A}$ is additive, any sequence $A \xrightarrow{f_1} B \xrightarrow{f_2} C$ of order two for which there are morphisms $g_1 : B \to A$ and $g_2 : C \to B$ with

$$f_1 g_1 + g_2 f_2 = i_B$$

is semi-exact.

Again consider $j_1 : A_1 \to A$ and $j_2 : A_2 \to A$ in a semi-exact JTK-category $\mathcal{A}$. It can now be shown that if $A = A_1 \oplus A_2$ and j_1 and j_2 are the injections, then i_A is the direct J-union of j_1 and j_2. If $\mathcal{A}$ is also additive, we have a partial converse in that if i_A is the direct J-union of j_1 and j_2, then $A = A_1 \oplus A_2$ with j_1 and j_2 as injections iff $k_2 j_1$ is strict, where k_2 is the cokernel of j_2. For, using the diagram in Proposition 5 with $A_1 \cap A_2 = 0$ and $A/A_1 \overset{J}{\cup} A_2 = 0$, we see that $k_2 j_1 \in T$. Hence $k_2 j_1$ strict is equivalent to $k_2 j_1 \in S$. So if $k_2 j_1$ is strict, then $k_2 j_1 = s \in S$ and $0 \longrightarrow A_2 \xrightarrow{j_2} A \xrightarrow{s^{-1} k_2} A_1 \longrightarrow 0$ splits, so $A = A_1 \oplus A_2$ with injections j_1 and j_2, by Proposition 9. On the other hand, if $A = A_1 \oplus A_2$ with injections j_1 and j_2, then we can take k_2 to be the projection $p_1 : A \to A_1$ so $k_2 j_1 = i_{A_1} \in S$. Note that this does not generally hold in $\mathcal{C}_1$, for example.

In a semi-exact additive JTK-category with finite products, it can be shown that in a pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{g_2} & A_2 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow[f_1]{} & A \end{array}$$

$f_1 \in K$ implies $g_2 \in N$. I have not been able to strengthen this general result, even though in $\mathcal{C}_1$ the stronger statements that $f_1 \in N$ implies $g_2 \in N$ and $f_1 \in K$ implies $g_2 \in K$ are valid.

We now give the «5 lemma».

PROPOSITION 11. *Suppose*

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ {\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle h_3}\downarrow & & {\scriptstyle h_4}\downarrow & & {\scriptstyle h_5}\downarrow \\ B_1 & \xrightarrow[g_1]{} & B_2 & \xrightarrow[g_2]{} & B_3 & \xrightarrow[g_3]{} & B_4 & \xrightarrow[g_4]{} & B_5 \end{array}$$

is a commutaive diagram with semi-exact rows in a semi-exact JTK-category. Then we have the following implications:

(i) If $h_1 \in N$, $h_2 \in J$, $h_4 \in L$ *and* f_2 *is strict, then* $h_3 \in L$.

(ii) If $h_5 \in L$, $h_2 \in N$, $h_4 \in K$ *and* g_3 *is strict, then* $h_3 \in N$.

(iii) If $h_1 \in N$, $h_5 \in L$, h_2 *and* $h_4 \in S$, f_2 *and* g_3 *are strict, then* $h_3 \in T$.

Pullback and pushout diagrams are related to semi-exact sequences in a semi-exact additive JTK-category $\mathfrak{A}$ with finite products much like they are to exact sequences in the special case of an abelian category (see Freyd [2, p. 52]). That is, suppose we have a square

$$\begin{array}{ccc} P & \xrightarrow{g_2} & A_2 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow{f_1} & A \end{array}$$

in $\mathfrak{A}$. Consider the composition $P \xrightarrow{\binom{g_1}{g_2}} A_1 \oplus A_2 \xrightarrow{(f_1, -f_2)} A$. Then

$P \to A_1 \oplus A_2 \to A$ is 0 iff the square commutes; $0 \to P \to A_1 \oplus A_2 \to A$ is semi-exact with $P \to A_1 \oplus A_2$ in J iff the square is a pullback; the sequence $P \to A_1 \oplus A_2 \to A \to 0$ is semi-exact with $A_1 \oplus A_2 \to A$ in K iff the square is a pushout; and $0 \to P \to A_1 \oplus A_2 \to A \to 0$ is semi-exact with $P \to A_1 \oplus A_2$ in J and $A_1 \oplus A_2 \to A$ in K iff the square is both a pullback and a pushout. As a corollary, we note that if at least one of $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ is in K, then the corresponding pullback diagram is also a pushout.

Of course many other diagram lemmas can be formulated for semi-exact JTK-categories as the need arises; however, the results given in this section should give the reader a good idea of the type of answers to expect.

4. The connecting morphism

In this section we will construct (under certain conditions) the connecting morphism in the semi-exact JTK-category. Our approach will follow the same general lines as the corresponding construction in the abelian category as presented, for example, by U. Oberst at the NSF Advanced Seminar in Category Theory at Bowdoin College in the summer of 1969. We will need some preliminary lemmas, where $\mathfrak{A}$ denotes a semi-exact JTK-category.

LEMMA 12. *In $\mathfrak{A}$ we have:*

(i) For $f : A \to B$ in K and $j : B' \to B$, $f^J(f^{-1}(B')) = B'$.

(ii) For $f : A \to B$ in J and $j : A' \to A$, $f^{-1}(f^J(A')) = A'$.

PROOF. (i) was given previously; (ii) is valid in any JTK-category.

LEMMA 13. *For any JTK-category which has inverse images for morphisms in J and which satisfies the assumption:*

(S1) For $f : A \to B$ in L, $j_1 : A_1 \to A$, $j_2 : A_2 \to A$, if

$$J\text{-}Im(fj_1) \subset J\text{-}Im(fj_2),$$

then

$$J\text{-}Im(j_1) \subset J\text{-}Im(j_2),$$

$f^{-1}(f^J(A')) = A'$ for $j : A' \to A$ whenever $f : A \to B$ is in L.

PROOF. For $f : A \to B$ in L and $j : A' \to A$ we have $fj = j't' \in L$. Then

$$\begin{array}{ccc} A' & \xrightarrow{t'} & J\text{-}Im(fj) \\ j \downarrow & & \downarrow j' \\ A & \xrightarrow[f]{} & B \end{array}$$

is a pullback, for if $fh_1 = j'h_2$ then, if j'' is the J-image of h_1, we have $J\text{-}image\,(fj'') = J\text{-}image\,(fh_1) \leqslant j' = J\text{-}image\,(fj)$ which implies $j'' \leqslant j$ by (S1). So $h_1 = jg$ for some g. Also $j't'g = fh_1 = j'h_2$ implies $h_2 = t'g$, and this g is unique. Hence $A' = f^{-1}(f^J(A'))$.

We note that a semi-exact JTK-category has inverse images for morphisms in J. Furthermore, any concrete JTK-category with J the class of embeddings, $L = \bar{\bar{M}}$ and J-image the intuitive notion of Image satisfies (S1); likewise, any concrete JTK-category with K the class of quotient maps, $N = \bar{\bar{E}}$ and K-Coimage the intuitive notion of Coimage satisfies the dual assumption

(S1) For* $f: A \to B$ *in* N, $k_1 : B \to B_1$, $k_2 : B \to B_2$, *if*

$$K\text{-}Coim\,(k_1 f) \subset K\text{-}Coim\,(k_2 f),$$

then

$$K\text{-}Coim\,(k_1) \subset K\text{-}Coim\,(k_2).$$

In particular, all of the examples given previously satisfy (S1) and (S1)*. Examples of JTK-categories not satisfying (S1), say, can be constructed by omitting the morphism which produces the desired inclusion.

LEMMA 14. *If* $\mathfrak{A}$ *satisfies (S1), then for* $f: A \to B$ *and* $j: A' \to A$,

$$f^{-1}(f^J(A')) = A' \overset{J}{\cup} Ker\, f.$$

PROOF. First consider the case where $f \in K$ and $Ker\, f \subset A'$ (so we have $A' \overset{J}{\cup} Ker\, f = A'$). We obtain the commutative diagram with columns and top two rows semi-exact

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Ker\, f & \xrightarrow{j_2} & A' & \xrightarrow{g \in N} & f^J(A') & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} & & \\
0 & \longrightarrow & Ker\, f & \xrightarrow{j_1} & A & \xrightarrow{f} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle k'} & & \\
0 & \longrightarrow & 0 & \longrightarrow & A/A' & \xrightarrow[g' \in K]{} & Cok(j') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

so, by the 9 lemma, the bottom row is semi-exact or $g' \in K \cap T = S$. Now consider the commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
f^{-1}(f^J(A')) & \xrightarrow{h} & f^J(A') \\
{\scriptstyle j_0}\downarrow & f & \downarrow{\scriptstyle j'} \\
A & \longrightarrow & B \\
{\scriptstyle k'f}\downarrow & & \downarrow{\scriptstyle k'} \\
Cok(j') & = & Cok(j') \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where the top square is a pullback and the right-hand column is semi-exact. But then the left-hand column is also semi-exact so that $j_0 = kernel\,(k'f) = kernel\,(g'k) = kernel\,(k) = j$ and $A' = f^{-1}(f^J(A'))$ as desired.

Next assume only that $f \in K$. Then

$$f^J(A' \overset{J}{\cup} Ker\, f) = f^J(A') \overset{J}{\cup} f^J(Ker\, f) = f^J(A') \overset{J}{\cup} 0 = f^J(A').$$

Also $Ker\, f \subset A' \overset{J}{\cup} Ker\, f$ so $f^{-1}(f^J(A' \overset{J}{\cup} Ker\, f)) = A' \overset{J}{\cup} Ker\, f$ by the first part of the proof. Consequently,

$$f^{-1}(f^J(A')) = f^{-1}(f^J(A' \overset{J}{\cup} Ker\, f)) = A' \overset{J}{\cup} Ker\, f.$$

Finally, for arbitrary $f: A \to B$, write $f = l_0 k_0$; then by lemma 13,

$$f^{-1}(f^J(A')) = k_0^{-1}(l_0^{-1}(l_0^J(k_0^J(A')))) = k_0^{-1}(k_0^J(A'))$$
$$= A' \overset{J}{\cup} Ker\, k_0 = A' \overset{J}{\cup} Ker\, f.$$

Note that if the (S1) assumption is omitted, then, by Lemma 12, the conclusion of Lemma 14 is still valid for f strict.

To dualize these results we need some additional notation. For $f: A \to B$ and $e: B \to B'$ we define $(f^J)^*(B')$ to be $K\text{-}Coim\,(ef)$. Also for $f: A \to B$ and $e: A \to A'$ we use the notation

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle e}\downarrow & & \downarrow \\
A' & \longrightarrow & (f^{-1})^*(A')
\end{array}$$

for the pushout. In the same vein we will use the symbol $\overset{J}{\cup}{}^*$ for the K-

counion.

LEMMA 15. *The following hold in* $\mathfrak{A}$:

(i) For $f: A \to B$ *and* $j: B' \to B$, *in the pullback diagram*

$$\begin{array}{ccc} f^{-1}(B') & \longrightarrow & B' \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow{\ f\ } & B \end{array}$$

$j' = kernel\,(v f)$ *where* v *is the cokernel of* j.

(ii) For $f: A \to B$ *and* $k: A \to A'$, *in the pushout diagram*

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle k'} \\ A' & \longrightarrow & (f^{-1})^*(A') \end{array}$$

$k' = cokernel\,(f u)$ *where* u *is the kernel of* k.

PROOF. (i) is proved in [4, p.72]; (ii) is the dual result.

LEMMA 16. *For* $k_1: B \to C_1$ *and* $k_2: B \to C_2$ *in* $\mathfrak{A}$,

$$Ker\,(C_1 \uplus^* C_2) = Ker\, k_1 \cap Ker\, k_2.$$

PROOF. This is left to the reader.

LEMMA 17. *If* $\mathfrak{A}$ *satisfies (S1)*, then for* $f: A \to B$ *and* $j: B' \to B$,

$$f^J(f^{-1}(B')) = B' \cap J\text{-}Im\,(f).$$

PROOF. By Lemma 14* we have

$$(f^{-1})^*((f^J)^*(Cok\,(j))) = Cok\,(j) \uplus^* Cok\,(f).$$

Thus by Lemmas 15 and 16 and the pullback

$$\begin{array}{ccc} f^{-1}(B') & \longrightarrow & B' \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow{\ f\ } & B \end{array}$$

$$\begin{aligned} f^J(f^{-1}(B')) &= J\text{-}Im\,(fj') = Ker\,(Cok\,(fj')) = Ker\,((f^{-1})^*(Cok\,(j'))) \\ &= Ker\,((f^{-1})^*((f^J)^*(Cok\,(j)))) \\ &= Ker\,(Cok\,(j) \uplus^* Cok\,(f)) = B' \cap J\text{-}Im\,(f). \end{aligned}$$

Again, if the (S1)* assumption is omitted, Lemma 17 is still valid if f is strict.

LEMMA 18. *Suppose* $\mathfrak{A}$ *satisfies (S1). Then for* $j_1 : A_1 \to A$, $j_2 : A_2 \to A$, $j_3 : A_3 \to A$ *with* $j_1 \leq j_2$ *(i.e.,* $A_1 \subset A_2$*),*

$$A_1 \overset{J}{\cup} (A_2 \cap A_3) = A_2 \cap (A_1 \overset{J}{\cup} A_3).$$

PROOF. We have $j_1 = j_2 j$. Let k be the cokernel of j_3. Then

$$\begin{aligned} A_2 \cap (A_1 \overset{J}{\cup} A_3) &= J\text{-}Im(j_2) \cap (A_1 \overset{J}{\cup} A_3) = j_2^J(j_2^{-1}(A_1 \overset{J}{\cup} A_3)) \\ &= j_2^J(j_2^{-1}(A_1 \overset{J}{\cup} Ker(k))) = j_2^J(j_2^{-1}(k^{-1}(k^J(A_1)))) \\ &= j_2^J((kj_2)^{-1}(kj_2)^J(J\text{-}Im(j))) = j_2^J(J\text{-}Im(j) \overset{J}{\cup} Ker(kj_2)) \\ &= j_2^J(J\text{-}Im(j)) \overset{J}{\cup} j_2^J(Ker(kj_2)) = A_1 \overset{J}{\cup} j_2^J((kj_2)^{-1}(0)) \\ &= A_1 \overset{J}{\cup} j_2^J(j_2^{-1}(k^{-1}(0))) = A_1 \overset{J}{\cup} (k^{-1}(0) \cap J\text{-}Im(j_2)) \\ &= A_1 \overset{J}{\cup} (A_2 \cap A_3). \end{aligned}$$

It is clear from the above proof that the conclusion of Lemma 18 is still true, when the (S1) assumption is omitted, as long as kj_2 is strict. Hence the (S1) assumption could be replaced by the requirement that $KJ \subset JK$ in $\mathfrak{A}$. Consequently, in a semi-exact JTK-category satisfying (S1) or the condition $KJ \subset JK$, the equivalence classes of the JTK-categorical subobjects of any object form a modular lattice under $\overset{J}{\cup}$ and $\cap$; dually, in a semi-exact JTK-category satisfying (S1)* or the condition $KJ \subset JK$, the equivalence classes of the JTK-categorical quotient objects of any object form a modular lattice under $\overset{J}{\cup}{}^*$ (K-counion) and $\cap^*$ (cointersection).

Now suppose we are given a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{h_1} & A & \xrightarrow{h_2} & A'' & \longrightarrow & 0 \\ & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & B' & \xrightarrow{h_3} & B & \xrightarrow{h_4} & B'' & \longrightarrow & 0 \end{array}$$

with semi-exact rows in a semi-exact JTK-category $\mathfrak{A}$, where h_1 and h_3 are in J and h_2 and h_4 in K. This can be enlarged to the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Ker\ f' & \xrightarrow{g_1} & Ker\ f & \xrightarrow{g_2} & Ker\ f'' & & \\
 & & {\scriptstyle j_1}\downarrow & & {\scriptstyle j_2}\downarrow & & {\scriptstyle j_3}\downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{h_1} & A & \xrightarrow{h_2} & A'' & \longrightarrow & 0 \\
 & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow & & \\
0 & \longrightarrow & B' & \xrightarrow{h_3} & B & \xrightarrow{h_4} & B'' & \longrightarrow & 0 \\
 & & {\scriptstyle k_1}\downarrow & & {\scriptstyle k_2}\downarrow & & {\scriptstyle k_3}\downarrow & & \\
 & & Cok\ f' & \xrightarrow{g_3} & Cok\ f & \xrightarrow{g_4} & Cok\ f'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

with semi-exact rows and columns and with $g_1 \in J$ and $g_4 \in K$. We wish to construct a connecting morphism $\delta : Ker\ f'' \to Cok\ f'$.

Some conditions must be placed on $\mathfrak{A}$. We assume that $\mathfrak{A}$ has the property that in each inverse image

$$
\begin{array}{ccc}
f_1^{-1}(B') & \xrightarrow{g} & B' \\
\downarrow & \cdot & \downarrow{\scriptstyle j} \\
A & \xrightarrow{f_1} & B
\end{array}
$$

$f_1 \in K$ implies $g \in K$, and, dually, that in each

$$
\begin{array}{ccc}
A & \xrightarrow{f_1} & B \\
{\scriptstyle k}\downarrow & & \downarrow \\
A' & \xrightarrow{h} & (f_1^{-1})^*(A')
\end{array}
$$

$f_1 \in J$ implies $h \in J$. We further assume that $\mathfrak{A}$ satisfies (S1) and (S1)* (or that f in the above diagram is strict). All these conditions are satisfied in $\mathcal{C}_1$ (or, more generally, in the category of topological A-modules over a topological ring A with identity), in $\mathcal{C}_2$, and in any exact category.

Using a proposition of Mitchell [6, p.15], we obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \xrightarrow{\beta} & h_2^{-1}(Ker\, f'') & \xrightarrow{\rho} & Ker\, f'' & \longrightarrow & 0 \\
 & & \| & & \downarrow \alpha & & \downarrow j_3 & & \\
0 & \longrightarrow & A' & \xrightarrow{h_1} & A & \xrightarrow{h_2} & A'' & \longrightarrow & 0 \\
 & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\
0 & \longrightarrow & B' & \xrightarrow{h_3} & B & \xrightarrow{h_4} & B'' & \longrightarrow & 0 \\
 & & \downarrow k_1 & & & & & & \\
 & & Cok\, f' & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

with semi-exact rows where the upper right-hand square is a pullback and $\rho \in K$ by our assumption. Note that α and β are in J. Since $h_4 f\alpha = f'' j_3 \rho = 0$, there exists $\gamma : h_2^{-1}(Ker\, f'') \to B'$ with $f\alpha = h_3\gamma$. Then $f' = \gamma\beta$ so $k_1\gamma\beta = 0$ and there exists $\delta : Ker\, f'' \to Cok\, f'$ with $k_1\gamma = \delta\rho$, since ρ is the cokernel of β. This is the desired connecting morphism as we shall see.

We must verify that the sequence

$$0 \longrightarrow Ker\, f' \xrightarrow{g_1} Ker\, f \xrightarrow{g_2} Ker\, f'' \xrightarrow{\delta} Cok\, f' \xrightarrow{g_3} Cok\, f \xrightarrow{g_4} Cok\, f'' \longrightarrow 0$$

is semi-exact where we have $g_1 \in J$ and $g_4 \in K$. We have

$$\begin{aligned}
Ker\, \delta &= \rho^J(\rho^{-1}(\delta^{-1}(0))) \\
&= \rho^J((\delta\rho)^{-1}(0)) \\
&= \rho^J((k_1\gamma)^{-1}(0)) \\
&= \rho^J(\gamma^{-1}(Ker\, k_1)) \\
&= \rho^J(\gamma^{-1}(J\text{-}Im\, f')) \\
&= \rho^J((h_3\gamma)^{-1}(f^J(A')))
\end{aligned}$$

from the pullback

$$\begin{array}{ccccc}
\gamma^{-1}(J\text{-}Im\, f') & \longrightarrow & J\text{-}Im\, f' & = & f^J(A') \\
\downarrow & & \downarrow v & & \downarrow h_3 v \\
h_2^{-1}(Ker\, f'') & \xrightarrow[\gamma]{} & B' & \xrightarrow[h_3]{} & B
\end{array}$$

$$
\begin{aligned}
Ker\ \delta &= \rho^J((f\alpha)^{-1}(f^J(A')))\\
&= \rho^J(\alpha^{-1}(f^{-1}(f^J(A'))))\\
&= \rho^J(\alpha^{-1}(A' \overset{J}{\cup} Ker\ f))\\
&= h_2^J(h_2^{-1}(Ker\ f'') \cap (Ker\ f \overset{J}{\cup} A'))
\end{aligned}
$$

from the pullback (intersection)

$$
\begin{array}{ccc}
\alpha^{-1}(A' \overset{J}{\cup} Ker\ f) & \longrightarrow & A' \overset{J}{\cup} Ker\ f\\
\downarrow & & \downarrow\\
h_2^{-1}(Ker\ f'') & \xrightarrow{\ \alpha\ } & A
\end{array}
$$

$$Ker\ \delta = h_2^J(Ker\,f \overset{J}{\cup}(h_2^{-1}(Ker\,f'') \cap A'))$$

by Lemma 18 since $Ker\ f \subset h_2^{-1}(Ker\,f'')$

$$
\begin{aligned}
&= h_2^J(Ker\,f) \overset{J}{\cup} h_2^J(h_2^{-1}(Ker\,f'') \cap A')\\
&= h_2^J(Ker\,f) \overset{J}{\cup} 0
\end{aligned}
$$

since $h_2^{-1}(Ker\,f'') \cap A' \subset A'$

$$
\begin{aligned}
&= h_2^J(Ker\,f)\\
&= J\text{-}Im(h_2\, j_2)\\
&= J\text{-}Im(j_3\, g_2)\\
&= J\text{-}Im(g_2).
\end{aligned}
$$

That $Cok\ \delta = K\text{-}Coim(g_3)$ can be verified by a dual argument, so our construction is finished.

As a consequence, we get the following result. Suppose $\mathfrak{A}$ is a semi-exact JTK-category satisfying the previous assumptions and there is given a commutative diagram

$$
\begin{array}{ccccccc}
 & & A' & \xrightarrow{h_1} & A & \xrightarrow{h_2} & A'' & \longrightarrow & 0\\
 & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow & & \\
0 & \longrightarrow & B' & \xrightarrow{h_3} & B & \xrightarrow{h_4} & B'' & &
\end{array}
$$

with semi-exact rows, where $h_2 \in K$, $h_3 \in J$, and h_1 and h_4 are strict. This can be enlarged to the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & Ker\, f' & \xrightarrow{g_1} & Ker\, f & \xrightarrow{g_2} & Ker\, f'' & & \\
 & & j_1 \downarrow & & j_2 \downarrow & & \downarrow j_3 & & \\
 & & A' & \xrightarrow{h_1} & A & \xrightarrow{h_2} & A'' & \longrightarrow & 0 \\
 & & f' \downarrow & & f \downarrow & & \downarrow f'' & & \\
0 & \longrightarrow & B' & \xrightarrow{h_3} & B & \xrightarrow{h_4} & B'' & & \\
 & & k_1 \downarrow & & k_2 \downarrow & & \downarrow k_3 & & \\
 & & Cok\, f' & \xrightarrow{g_3} & Cok\, f & \xrightarrow{g_4} & Cok\, f'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with semi-exact columns and middle two rows. We now apply the preceding construction to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J\text{-}Im\, h_1 & \xrightarrow{h_1'} & A & \xrightarrow{h_2} & A'' & \longrightarrow & 0 \\
 & & g' \downarrow & & f \downarrow & & \downarrow g'' & & \\
0 & \longrightarrow & B' & \xrightarrow{h_3} & B & \xrightarrow{h_4'} & K\text{-}Coim\, h_4 & \longrightarrow & 0
\end{array}$$

with semi-exact rows where $h_1', h_3 \in J$ and $h_2, h_4' \in K$ to get a connecting morphism $\delta : Ker\, g'' \to Cok\, g'$ such that the sequence

$$0 \to Ker\, g' \xrightarrow{g_1'} Ker\, f \xrightarrow{g_2'} Ker\, g'' \xrightarrow{\delta} Cok\, g' \xrightarrow{g_3'} Cok\, f \xrightarrow{g_4'} Cok\, g'' \to 0$$

is semi-exact with $g_1' \in J$ and $g_4' \in K$. But $Ker\, f'' = Ker\, g''$ and $Cok\, f' = Cok\, g'$, so we have $g_2' = g_2$ and $g_3' = g_3$. Furthermore, since $h_1 = h_1' k_1'$,

$$\begin{aligned} Ker\, g' &= k_1'^{\,J}(k_1'^{\,-1}(Ker\, g')) = k_1'^{\,J}((g' k_1')^{-1}(0)) \\ &= k_1'^{\,J}(Ker\, f') = J\text{-}Im(k_1' j_1) \end{aligned}$$

and so

$$j_2(J\text{-}im(g_1)) = J\text{-}im(h_1 j_1) = h_1'(J\text{-}im(k_1' j_1)) = h_1'(ker(g')) = j_2 g_1',$$

which implies that g_1' is the J-image of g_1. Dually, g_4' is the K-coimage of g_4 so that the sequence

$$Ker\, f' \xrightarrow{g_1} Ker\, f \xrightarrow{g_2} Ker\, f'' \xrightarrow{\delta} Cok\, f' \xrightarrow{g_3} Cok\, f \xrightarrow{g_4} Cok\, f''$$

is semi-exact.

The connecting morphism is frequently useful in diagram chasing situations.

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