CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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*Cahiers de topologie et géométrie différentielle catégoriques*, tome 12, nº 2 (1971), p. 197-212

<http://www.numdam.org/item?id=CTGDC_1971__12_2_197_0>

CAHIERS DE TOPOLOGIE
ET GEOMETRIE DIFFERENTIELLE

Vol. XII,2

# ON THE HOLONOMITY OF HIGHER ORDER CONNECTIONS

by Juraj VIRSIK

Higher order connections in Lie groupoids and their prolongations, notions first introduced by C. Ehresmann in [3], are studied in this note from their formal «holonomity» point of view. It is shown in particular, that the prolongation of an $r$-th order connection is semi-holonomic only if the connection is simple (i. e. is obtained by subsequent prolongations of a first order connection); it is holonomic if and only if this first order connection is curvature-free. A sequence of $r$ connections of order $r-1$ is attached to each $r$-th order connection. This sequence determines the $r$-th order connections uniquely up to an $r$-th order «covariant tensor on the base» with values in the isotropy Lie algebra bundle of the groupoid. The Appendix describes the intuitively obvious characterization of simple connections as those which give rise to «parallel transport».

The term manifold represents always a $C^{\infty}$-differentiable finite dimensional manifold and similar restrictions apply to related notions. The notions of non-holonomic, semi-holonomic and holonomic jets and the formalism of their calculus are those introduced by Ehresmann (cf. [2]), with the following notational conventions: If $f: M \to N$ is a local map, we write sometimes $j_x^r(u \to f(u))$ instead of $j_x^r f$, and $j_x^r[y] = j_x^r(u \to y)$ for a fixed $y \in N$. If $M = N$, $j_x^r = j_x^r(u \to u)$, $\hat{j}_x^r = j_x^r[x]$. A fibred manifold is given by a surjection of maximal rank $p_E: E \to B$ between two manifolds; it will be simply denoted by $E$. The fibred manifolds of non-holonomic, semi-holonomic, holonomic jets of local sections in the fibred manifold $E$ are denoted by $\tilde{D}^r E$ or $\bar{D}^r E$ or $D^r E$ respectively. A section $g$ in $\tilde{D}^r E$ is called an $s$-wave (flot) if $g(x) = j_x^s f$ for some section $f$ in $D^{r-s} E$; we write briefly $g = j^s f$. If $f: E \to F$ is a bundle morphism, then $\tilde{D}^r f: \tilde{D}^r E \to \tilde{D}^r F$ defined as $(\tilde{D}^r f)(X) = (j_\beta^r{}_X f)(X)$, will be also denoted by $j^r f$; we shall agree that $j^r f$ is to be regarded as either a section or a bundle morphism depending on whether $f$ was a section or a bundle morphism. Also $j^0$ is understood to be the identity functor. We shall identify $D^r(M\times N)$ with $J^r(M,N)$, thus regarding $M\times N$ always as a fibred manifold $M\times N\to M$ with the natural surjection. Given a fibred manifold $E$ and integers $r\geqslant q\geqslant 0$, we denote by $\rho^r_q:\tilde{D}^rE\to\tilde{D}^qE$ the natural (target) surjections with $\rho^r_r$ being the identity on $\tilde{D}^rE$. Together with $\rho^r_q$, we have also the surjections $(j^k\rho^{r-k}_{q-k}):\tilde{D}^rE\to\tilde{D}^qE$.

LEMMA 1. *The element $X\in\tilde{D}^rE$ is semi-holonomic if and only if*

$$(1)\qquad (j^k\rho^{r-k}_{q-k})(X)=\rho^r_q(X)$$

*for any integers $1\leqslant k\leqslant q<r$. Especially if $X\in\bar{D}^rE$, then*

$$(2)\qquad \rho^r_1(X)=(j^1\rho^{s-1}_0)\rho^r_s(X)\ \text{for each}\ s=1,2,\dots,r.$$

PROOF: $X\in\bar{D}^rE$ means that $X=j^1_x\xi$, where $\xi$ is a local section in $\bar{D}^{r-1}E$ and (cf. [2])

$$j^1_x(\rho^{r-1}_{r-2}\xi)=\xi(x),\ \text{i.e.}\ (j^1\rho^{r-1}_{r-2})(X)=\rho^r_{r-1}(X).$$

Thus the Lemma is evident for $r=2$, and we can proceed by induction:

(A) Let $X$ be semi-holonomic. Then (1) holds with $q=r-1$ and $k=1$. We shall first show that

$$(j^1\rho^{r-1}_{q-1})(X)=\rho^r_q(X)\ \text{for}\ q=1,\dots,r-1.$$

Since $\rho^r_{q+1}(X)$ is also semi-holonomic, we know that

$$\rho^r_q(X)=\rho^{q+1}_q\ \rho^r_{q+1}(X)=(j^1\rho^q_{q-1})\ \rho^r_{q+1}(X).$$

This is a recurrent formula for $\rho^r_q(X)$ and so having the desired formula for $q=r-1$ one derives it immediately for all $q=1,\dots,r-1$.

As to the case $k>1$, having $X=j^1_x\xi$, where $\xi$ is a semi-holonomic section, we apply the induction assumption to it. This gives for $u$ in a neighbourhood of $x$, and for integers $2\leqslant k\leqslant q<r$:

$$(3)\qquad (j^{k-1}\rho^{r-k}_{q-k})\ \xi(u)=\rho^{r-1}_{q-1}\ \xi(u),$$

and so taking the one-jets at $x$ of both the sides here, we establish the rest of (1) for $X\in\bar{D}^rE$.

(B) Conversely, suppose $X=j^1_x\xi\in\tilde{D}^rE$ satisfies (1). Then especially $(j^1\rho^{r-1}_{r-2})(X)=\rho^r_{r-1}(X)$, and we only have to show that $\xi$ can be chosen as a semi-holonomic section. We have, equating the left hand sides of (1) corresponding to $k > 1$ and $k = 1$,

$$j^1_x(u \to (j^{k-1}\rho^{r-k}_{q-k})\,\xi(u)) = j^1_x(u \to \rho^{r-1}_{q-1}\xi(u)).$$

We deduce (3) from this by applying a result proved e.g. in [1] : If $a, b: N \to V$ is a transversal pair of maps (i.e. $da - db$ is surjective everywhere where it is defined), and $f: M \to N$ a local map such that $j^1_x(af) = j^1_x(bf)$ for some $x \in M$, then there is a local map $f': M \to N$ such that $j^1_x f = j^1_x f'$ and $af' = bf'$ in a neighbourhood of $x$. Now it is an exercise in the coordinate expressions of jets to see that $j^{k-1}\rho^{r-k}_{q-k},\ \rho^{r-1}_{q-1}: \tilde{D}^{r-1}E \to \tilde{D}^{q-1}E$ is a transversal pair, and so we can suppose that $\xi$ satisfies (3). But then by the induction assumption $\xi$ is a semi-holonomic section and so $X \in \bar{D}^r E$.

As to (2), it is a special case of (1) with $k = q = 1$ if $s = r$. If $s < r$, then (2) follows from $\rho^r_s(X) = (j^1\rho^{r-1}_{s-1})(X)$. This completes the proof.

REMARK. In general there are more than $q$ natural surjections $\tilde{D}^r E \to \tilde{D}^q E$, namely all those that can be constructed by various «compositions» of the surjections (1). There are however $\binom{r}{q}$ independent surjections $\tilde{D}^r E \to \tilde{D}^q E$ among them, constructed as follows:

Let $C \equiv \{ r \geq c_1 > \ldots > c_{r-q} \geq 1 \}$ be a decreasing sequence of integers. Note that the number of such sequences is exactly $\binom{r}{q}$, and that for each $c_t$ $(t = 1, \ldots, r-q)$, we have a surjection

$$\text{(1a)} \qquad j^{(r-t)-(c_t-1)}\rho^{c_t}_{c_t-1}: \tilde{D}^{r-t+1}E \to \tilde{D}^{r-t}E$$

of the form (1). We define the surjection $\rho^{r,\,C}_q: \tilde{D}^r E \to \tilde{D}^q E$ associated to the sequence $C$ as the composition

$$(j^{q-c_t+1}\rho^{c_{r-q}}_{c_{r-q}-1}) \ldots (j^{r-c_2-1}\rho^{c_2}_{c_2-1})(j^{r-c_1}\rho^{c_1}_{c_1-1})$$

of the maps (1a) with $t = 1, \ldots, r-q$. It is not difficult to see that $\rho^r_q$ corresponds to the sequence $\{c_t = r+1-t\}$ and the surjection (1) to the sequence $\{c_t = r+1-t-k\}$. Also the maps on the right hand side of (2) correspond to the sequence $\{1, 2, \ldots, r\}$ with $t = s$ omitted. Applying an argument analogous to that in the proof of Lemma 1 one could show that, if $X \in \bar{D}^r E$, then all $\rho^{r,\,C}_q(X) \in \bar{D}^q E$ are equal for each fixed $q < r$.

The following lemma is obvious.

LEMMA 2. *If $f, g$ are local sections in $E$ such that $j^1_x f = j^1_x g$, then also*

$$j^1_x(u \to j^r_u [f(u)]) = j^1_x(u \to j^r_u [g(u)]) .$$

The notion of a Lie groupoid $\Phi$ over $B$ is taken from [5] , and means the same as a locally trivial differentiable groupoid introduced by Ehresmann in [2] . Especially $a, b : \Phi \to B$ denote the right and left unit projections respectively, and $\sim : B \to \Phi$ (written also as $x \to \tilde{x}$) is the natural inclusion of the manifold of units into the groupoid. Further let $G = G(\Phi)$ be the isotropy group bundle, and $L = L(\Phi)$ the isotropy Lie algebra bundle attached to $\Phi$, i.e.

$$G_x = \{ \theta \in \Phi \mid a\theta = b\theta = x \} \text{ and } L_x = T_{\tilde{x}}(G_x)$$

as in [5] . We denote by $\tilde{D}^r_q G \subset \tilde{D}^r G$ the kernel of all the surjections $\rho^{r,C}_q$, i.e. $\tilde{D}^r_q G$ consists of those $X \in \tilde{D}^r G$ for which $\rho^{r,C}_q(X) = j^q_{\alpha X}(\sim)$ for all decreasing sequences $C = \{c_t\}$ as in the above remark. Analogously $\tilde{D}^r_q L \subset \tilde{D}^r L$ is the kernel of all the surjections $\rho^{r,C}_q$ regarded as vector bundle morphisms. Put

$$\bar{D}^r_q G = \tilde{D}^r_q G \cap \bar{D}^r G, \quad D^r_q G = \tilde{D}^r_q G \cap D^r G,$$

and

$$\bar{D}^r_q L = \tilde{D}^r_q L \cap \bar{D}^r L, \quad D^r_q L = \tilde{D}^r_q L \cap D^r L .$$

Note that $\bar{D}^r_q G$ and $\bar{D}^r_q L$ are actually the kernels of $\rho^r_q$ restricted to semi-holonomic jets.

There is a natural diffeomorphism of fibred manifolds

$$\tilde{D}^r_0 G \to \tilde{D}^r_0 L : X \to (j^r exp^{-1})\, X, \tag{4}$$

where $exp : L \to G$ is the exponential map on fibres, $exp^{-1}$ being defined in a neighbourhood of $\sim(B) \subset \Phi$ . Note that (4) preserves holonomity and semi-holonomity, and also takes each $\tilde{D}^r_q G$ onto $\tilde{D}^r_q L$, $q = 0, 1, \ldots, r-1$. Especially (4) maps $\bar{D}^r_{r-1} G$ onto a vector bundle canonically isomorphic with $L \otimes (\otimes^r T(B)^*)$. We shall identify these two bundles and write simply

$$\bar{D}^r_{r-1} G = L \otimes (\otimes^r T(B)^*) \text{ as well as } D^r_{r-1} G = L \otimes (S^r T(B)^*).$$

In this sense one can regard elements of $\bar{D}^r_{r-1} G$ as covariant tensors on $B$

with values in $L$.

Let us recall here the notion of higher order connections in $\Phi$ as introduced in [3].

DEFINITION 1. A *non-holonomic* or *semi-holonomic* or *holonomic infinitesimal connection* (to be abbreviated as IC) *of order* $r \geqslant 1$ *in* $\Phi$ is a $C^\infty$-map $\Gamma: B \to \tilde{J}^r(B,\Phi)$ or $\bar{J}^r(B,\Phi)$ or $J^r(B,\Phi)$ satisfying

$$\rho_0^r \Gamma = \sim, \quad (j^r a)\Gamma(x) = \tilde{j}_x^r, \quad (j^r b)\Gamma(x) = j_x^r \text{ for all } x \in B.$$

It is well known that for $r = 1$ this corresponds to the standard notion of a connection in any of the principal bundles determined by $\Phi$, as defined e.g. in [4]. It is also evident that, if $\Gamma$ is an $r$-th order IC, then $\rho_q^r \Gamma$ $(q = 1, \dots, r-1)$ is a $q$-th order IC.

Let now $\zeta = \zeta^{(1)}: x \to j_x^1 \zeta_x$ be a first order IC in $\Phi$ and define for each integer $r \geqslant 1$ the map

$$\zeta^{(r)}: B \to \tilde{J}^r(B,\Phi), \quad x \to j_x^1(u \to j_u^{r-1}[\zeta_x(u)]).$$

It follows from Lemma 2 that $\zeta^{(r)}$ is well defined. Denoting by a dot the composition in $\Phi$ as well as its prolongation to each $\tilde{J}^r(B,\Phi)$ (cf. [2]; see also [6] for more explicite rules), we easily derive that for any $r$-th order IC $\Gamma$ and any first order IC $\zeta$ the map

$$\Gamma * \zeta : B \to \tilde{J}^{r+1}(B,\Phi), \quad x \to j_x^1 \Gamma . \zeta^{(r+1)}(x)$$

is a well defined IC of order $r+1$ in $\Phi$. Note that $\Gamma' = \Gamma * \rho_1^r \Gamma$ is called in [3] the prolongation of $\Gamma$.

Given $r$ first order connections $\zeta_1, \zeta_2, \dots, \zeta_r$ in $\Phi$, we can define recurrently their composition $\zeta_1 * \dots * \zeta_r$, which is an $r$-th order IC in $\Phi$, defined explicitly as

$$x \to j_x^{r-1}\zeta_1^{(1)} . j_x^{r-2}\zeta_2^{(2)} \dots j_x^1 \zeta_{r-1}^{(r-1)} . \zeta_r^{(r)}(x). \tag{5}$$

Conversely, if $\Gamma$ is an $r$-th order IC in $\Phi$ admitting a decomposition in the form (5), then all the $\zeta_s$ $(s = 1, \dots, r)$ are first order connections uniquely determined by $\Gamma$. This is established by the

LEMMA 3. *If* $\Gamma$ *is a* $r$*-th order* IC *in* $\Phi$, *then each of the maps*

(6) $$B \to J^1(B,\Phi): x \to (j^1 \rho_0^{s-1})\, \rho_s^r \Gamma(x)$$

*$s = 1, \dots, r$, is a first order connection in $\Phi$. If $\Gamma$ is given as in (5), then the maps (6) coincide with the generating connections $\zeta_s$ ($s = 1, \dots, r$).*

PROOF: We have

$$\begin{aligned}(j^1 a)(j^1 \rho_0^{s-1})\, \rho_s^r \Gamma(x) &= (j^1 \rho_0^{s-1})(j^s a)\, \rho_s^r \Gamma(x)\\ &= (j^1 \rho_0^{s-1})\, \rho_s^r (j^r a) \Gamma(x)\\ &= (j^1 \rho_0^{s-1})\, \rho_s^r j_x^r [x]\\ &= j_x^1 [x],\end{aligned}$$

and analogously

$$(j^1 b)(j^1 \rho_0^{s-1})\, \rho_s^r \Gamma(x) = j_x^1.$$

So evidently the maps (6) represent connections.

Let now $\Gamma$ be as in (5). We first have to make sure that the map $(j^1 \rho_0^{s-1})\, \rho_s^r$ commutes with the prolonged multiplications in (5). According to the definition of the latter, it is sufficient to show that - denoting by $\psi$ the composition rule in $\Phi$ - for any jet $X \in \tilde{J}^r(B, \Phi \times \Phi)$ we have

$$(j^1 \rho_0^{s-1})\, \rho_s^r (j_x^r \psi)(X) = j_x^1 \psi (j^1 \rho_0^{s-1})\, \rho_s^r (X),$$

provided the left hand side is defined. But this relation can be easily established by induction on $r$. Thus we conclude that the expression in (6) is a «prolonged multiple» of expressions of the form

$$(j^1 \rho_0^{s-1})\, \rho_s^r j_x^{r-q} \zeta_q^{(q)} \quad (q = 1, \dots, r).$$

If $q > s$, this expression equals

$$(j^1 \rho_0^{s-1})\, \rho_s^q \zeta_q^{(q)}(x) = (j^1 \rho_0^{s-1}) j_x^s [\zeta_{q,x}(x)] = j_x^1 [\tilde{\tilde{x}}].$$

If $q < s$, the expression is equal to

$$(j^1 \rho_0^{s-1}) j_x^{s-q} \zeta_q^{(q)} = (j^1 \rho_0^{s-1}) j_x^1 (u \to j_u^{s-q-1} \zeta_q^{(q)}) = j_x^1 (\sim).$$

Finally for $q = s$ we obtain

$$(j^1 \rho_0^{s-1})\, \rho_s^r j_x^{r-s} \zeta_s^{(s)} = (j^1 \rho_0^{s-1})\, \zeta_s^{(s)}(x) = j_x^1 (u \to \zeta_{s,x}(u)) = \zeta_s(x).$$

From there we easily conclude that

$$(j^1 \rho_0^{s-1})\, \rho_s^r (\zeta_1 * \dots * \zeta_r) = \zeta_s,$$

and this completes the proof.

We shall denote the connections in (6) by $\zeta_s(\Gamma)$. If $\Gamma$ is decomposable (i.e. of the form $\zeta_1 * \dots * \zeta_r$), then the connections $\zeta_s(\Gamma) = \zeta_s$ $(s = 1, \dots, r)$ are called its *generating connections.*

We derive immediately from Lemma 1 and the semi-holonomity of simple connections (cf. [3]) the

THEOREM 1. *If $\Gamma$ is a semi-holonomic infinitesimal connection in $\Phi$, then all the $\zeta_s(\Gamma)$ are equal. A decomposable infinitesimal connection is semi-holonomic if and only if it is simple, i.e. all its generating connections are equal.*

More generally, extending only formally the argument in the proof of Lemma 1, one can see that if $\Gamma$ is an IC of order $r$, then all the $\rho_q^{r,C}\Gamma$ are IC of order $q$. Especially $\rho_q^{r,k}\Gamma = (j^k \rho_{q-k}^{r-k})\Gamma$, $k = 1, \dots, q$, are infinitesimal connections in $\Phi$, and evidently

THEOREM 2. *An infinitesimal connection $\Gamma$ in $\Phi$ is semi-holonomic if and only if $\rho_q^{r,k}\Gamma = \rho_q^r\Gamma$ for all $1 \leqslant k \leqslant q < r$,*

A formally straightforward but rather awkward manipulation with jets leads to the following lemma, the proof of which will be omitted.

LEMMA 4. *If $\Gamma$ is a section in $\tilde{J}^r(B, \Phi)$ and if $\zeta_s(\Gamma)$ is defined by (6), then*

$$(j^1 \rho_0^{s-1})\, \rho_s^q (j^k \rho_{q-k}^{r-k})\Gamma = \begin{cases} \zeta_s(\Gamma), & \text{if } 1 \leqslant s \leqslant q-k \\ \zeta_{r-q+s}(\Gamma), & \text{if } q-k < s \leqslant q \end{cases}$$

*for any integers $1 \leqslant k \leqslant q < r$.*

If $\zeta$ is a section in $J^1(B, \Phi)$ and if $\zeta^{(t)}$ is defined as above, then

$$(j^k \rho_{q-k}^{r-k})\, j_x^{r-t}\, \zeta^{(t)} = \begin{cases} j^{r-t}\, \zeta^{(q-r+t)}, & \text{if } r-k < t \leqslant r \\ j^k(u \to j_u^{r-k}[\tilde{u}]), & \text{if } q-k < t \leqslant r-k \\ j^{q-t}\, \zeta^{(t)}, & \text{if } 1 \leqslant t \leqslant q-k \end{cases}$$

for any integers $1 \leqslant k \leqslant q < r$.

As a corollary of this lemma we have

THEOREM 3. *If $\Gamma$ is an $r$-th order infinitesimal connection in $\Phi$, then*

$$\zeta_s(\rho_q^{r,k}\Gamma) = \begin{cases} \zeta_s(\Gamma) & \text{for } 1 \leqslant s \leqslant q-k \\ \zeta_{r-q+s}(\Gamma) & \text{for } q-k < s \leqslant q \end{cases}$$

*If* $\Gamma$ *is a decomposable connection* $\zeta_1 * \dots * \zeta_r$, *then the connections* $\rho_q^{r,k}\Gamma$ *are given by*

$$\rho_q^{r,k}(\zeta_1 * \dots * \zeta_r) = \zeta_1 * \zeta_2 * \dots * \zeta_{q-k} * \zeta_{r-k+1} * \zeta_{r-k+2} * \dots * \zeta_r.$$

*Especially all the* $\rho_q^{r,k}\Gamma$ *are also decomposable.*

REMARK. One could derive again in the general case that $\zeta_s(\rho_q^{r,C}\Gamma) = \zeta_{d_s}(\Gamma)$, where $(d_1, \dots, d_q)$ is the increasing sequence obtained by deleting the elements $c_t$, $t = 1, \dots, r-q$, from $(1, \dots, r)$. Also the connection $\rho_q^{r,C}(\zeta_1 * \dots * \zeta_r)$ is equal to the expression obtained by deleting from $\zeta_1 * \dots * \zeta_r$ the members $\zeta_{c_t}$, $t = 1, \dots, r-q$.

THEOREM 4. *If* $\Gamma$ *is an r-th order infinitesimal connection in* $\Phi$ *and* $\zeta$ *a first order connection in* $\Phi$, *then*

$$\zeta_s(\Gamma * \zeta) = \zeta_s(\Gamma) \text{ for } s = 1, \dots, r, \text{ and } \zeta_{r+1}(\Gamma * \zeta) = \zeta.$$

The proof follows directly from the definition of $\Gamma * \zeta$. This together with Theorem 1 leads to

THEOREM 5. *If* $\Gamma$ *is an r-th order infinitesimal connection, and* $\zeta$ *a first order connection in* $\Phi$ *such that* $\Gamma * \zeta$ *is semi-holonomic, then necessarily* $\zeta = \zeta_1(\Gamma)$, *i.e.* $\Gamma * \zeta = \Gamma'$.

THEOREM 6. *If* $\Gamma$ *is an r-th order infinitesimal connection such that its prolongation* $\Gamma'$ *is semi-holonomic, then* $\Gamma$ *is necessarily simple, i.e.* $\Gamma = \zeta * \dots * \zeta$.

PROOF: According to Theorem 2, the semi-holonomity conditions for $\Gamma * \zeta$ imply

$$\begin{aligned} \rho_{q+1}^r \Gamma = \rho_{q+1}^{r+1}(\Gamma * \zeta) &= (j^1 \rho_q^r)(\Gamma * \zeta) \\ &= (j^1 \rho_q^r)\, j_x^1(u \to \Gamma(u) . j_u^r [\zeta_x(u)]) \\ &= j_x^1(u \to \rho_q^r \Gamma . j_u^q [\zeta_x(u)]) = \rho_q^r \Gamma * \zeta \end{aligned}$$

for all $q = 1, \dots, r-1$. But this means that

$$\Gamma = \rho^r_r \Gamma = \zeta * \dots * \zeta .$$

THEOREM 7. *A simple infinitesimal connection $\zeta * \dots * \zeta$ in $\Phi$ is holonomic if and only if the generating first order connection $\zeta$ is curvature-free.*

To prove this we first state an almost obvious

LEMMA 5. *Let $\Phi$ and $\bar{\Phi}$ be two Lie groupoids over $B$, and let $F: \Phi \to \bar{\Phi}$ be a $C^\infty$-functor which is the identity on $B$. For $\Gamma: B \to \tilde{J}^r(B, \Phi)$, denote $F\Gamma: B \to \tilde{J}^r(B, \bar{\Phi})$ the map $x \to (j^r F)\Gamma(x)$. If now $\Gamma$ is an* IC *in $\Phi$, then $F\Gamma$ is an* IC *in $\bar{\Phi}$. If $\Gamma$ is semi-holonomic or holonomic, then so is $F\Gamma$. If $\tilde{\Gamma} = \Gamma * \zeta$ then $F\tilde{\Gamma} = F\Gamma * F\zeta$. Especially $F$ takes decomposable* IC-*s into decomposable ones and simple* IC-*s into simple ones.*

PROOF OF THEOREM 7. If $\Gamma$ is holonomic, then so is $\rho^r_2 \Gamma = \zeta * \zeta$ and the curvature of $\zeta$ vanishes according to the result in [3].

Conversely, let $\zeta$ be curvature-free. If $U \subset B$ is open, denote by $\Phi|_U \subset \Phi$ the corresponding full subgroupoid and by $\zeta|_U$ the restriction of $\zeta$ to $U$. Evidently $\zeta|_U$ is a connection in $\Phi|_U$, and if $\zeta|_U * \dots * \zeta|_U$ is holonomic for each element $U$ of an open cover of $B$, then so is $\Gamma = \zeta * \dots * \zeta$. But a well known result about curvature-free connections, (see e.g. [4]) states that $B$ can be covered by open substes $U$, each $U$ admitting an invertible $C^\infty$-functor from $\Phi|_U$ onto the trivial groupoid $U \times G_z \times U$ ($z \in B$ being fixed) which takes $\zeta|_U$ into the trivial connection $x \to j^1_x(u \to (x, \tilde{z}, u))$. Now it is very easy to see that all the prolongations of this trivial connection are holonomic. Hence, according to Lemma 4, so are those of $\zeta|_U$ and this completes the proof.

Let now $\Gamma$ and $\bar{\Gamma}$ be two $r$-th order infinitesimal connections in $\Phi$. Then $\bar{\Gamma}.\Gamma^{-1}: x \to \bar{\Gamma}(x).\Gamma^{-1}(x)$ (or $\Gamma.\bar{\Gamma}^{-1}$) is a section in $\tilde{D}^r G$. We shall say that $\Gamma$ and $\bar{\Gamma}$ are *equivalent in the q-th order* ($1 \leqslant q < r$) if $\rho^{r,C}_q \Gamma = \rho^{r,C}_q \bar{\Gamma}$ for all decreasing sequences $C \equiv \{r > c_1 > \dots > c_{r-q} \geqslant 1\}$. Especially they are equivalent in the first order if

$$\zeta_s(\Gamma) = \zeta_s(\bar{\Gamma}), \quad s = 1, \dots, q;$$

they are equivalent in the $(r-1)$-st order if

$$(j^k \rho^{r-k}_{r-k-1})\Gamma = (j^k \rho^{r-k}_{r-k-1})\bar{\Gamma}, \quad k=0,1,\dots,r-1.$$

It is not difficult to see that $\Gamma$ and $\bar{\Gamma}$ are equivalent in the $q$-th order iff $\bar{\Gamma}.\Gamma^{-1}$ is a section in $\tilde{D}^r_q G \subset \tilde{D}^r G$. (One has only to verify that $\rho^{r,C}_q$ commutes with the prolonged composition in $\Phi$. This can be done similarly as for the special case in the proof of Lemma 3). Especially they are equivalent in the $(r-1)$-st order if $\bar{\Gamma}.\Gamma^{-1}$ is a section in

$$\tilde{D}^r_{r-1} G = \bar{D}^r_{r-1} G = L \otimes (\otimes^r T(B)^*).$$

THEOREM 8. *Let $\Phi$ be a Lie groupoid over $B$ satisfying the second axiom of countability. Then there is a natural correspondence between $(r-1)$-st order equivalence classes of $r$-th order infinitesimal connections in $\Phi$, and $r$-tuples of $(r-1)$-st order infinitesimal connections in $\Phi$. If $\Gamma$ and $\bar{\Gamma}$ belong to the same equivalence class, then they differ by a covariant tensor on $B$ with values in the Lie algebra bundle $L$ of $\Phi$, i.e. $\bar{\Gamma}.\Gamma^{-1}$ is a section in $L \otimes (\otimes^r T(B)^*)$.*

PROOF. If $\Gamma$ is an $r$-th order IC, then $(j^k \rho^{r-k}_{r-k-1})\Gamma$, $k=0,1,\dots,r-1$, is an $r$-tuple of IC of order $r-1$. We only have to show the converse, i.e. that given $(r-1)$-st order connections $\gamma_1, \dots, \gamma_r$ there is an $r$-th order IC $\Gamma$ such that $(j^k \rho^{r-k}_{r-k-1})\Gamma = \gamma_{k+1}$, $k=0,1,\dots,r-1$. Let $\tilde{Q}^s \subset \tilde{J}^s(B,\Phi)$ be the submanifold of those $X$ which satisfy

$$(j^s a)X = \hat{j}^s_x, \quad (j^s b)X = j^s_x, \quad \rho^s_0 X = \tilde{x}$$

with $x = \alpha X$; $\tilde{Q}^s$ is a fibre bundle (cf. [3]), and an IC of order $s$ is a section in $\tilde{Q}^s$. Denote $\Pi : \tilde{Q}^r \to \tilde{Q}^{r-1} \times \dots \times \tilde{Q}^{r-1}$ ($r$ times) the map

$$X \to (\rho^r_{r-1} X, (j^1 \rho^{r-1}_{r-2})X, \dots, (j^{r-1} \rho^1_0)X).$$

It is now sufficient to show that $\Pi$ admits a section, i.e. a right inverse in the category of differentiable manifolds. But the second countability of $\Phi$ implies that of $\tilde{Q}^{r-1} \times \dots \times \tilde{Q}^{r-1}$ (especially its paracompactness), and the fibre of $\Pi$ is diffeomorphic to $\mathbf{R}^{m^r+n}$, where $m = dim\ B$ and $n$ is the dimension of the isotropy group of $\Phi$. Thus by a well known result (cf. [4]) $\Pi$ admits a section. The rest of the Theorem follows from the previous results.

The following is obvious.

THEOREM 9. *An r-th order infinitesimal connection which is equivalent in the (r−1)-st order to a semi-holonomic connection is itself semi-holonomic. Such an equivalence class consists of semi-holonomic infinitesimal connections if and only if all the corresponding (r−1)-st order connections* $\gamma_1, \dots, \gamma_r$ *are semi-holonomic and equal.*

One could also show that, if $\Gamma$ and $\overline{\Gamma}$ are equivalent in the $q$-th order, then so are $\Gamma * \zeta$ and $\overline{\Gamma} * \zeta$ for any first order IC $\zeta$.

Each IC $\Gamma$ is equivalent in the first order to $\zeta_1(\Gamma) * \dots * \zeta_r(\Gamma)$; especially for $r=2$, these two connections differ by a quadratic tensor on $B$ with values in $L$. Hence we also have

THEOREM 10. *Every second order infinitesimal connection* $\Gamma$ *in* $\Phi$ *is uniquely determined by two first order connections* $\zeta_1(\Gamma)$, $\zeta_2(\Gamma)$, *and a linear map* $A(\Gamma): T(B) \otimes T(B) \to L(\Phi)$. $\Gamma$ *is semi-holonomic iff*

$$\zeta_1(\Gamma) = \zeta_2(\Gamma).$$

*Any two of the following conditions imply the third:*

(a) $\Gamma$ *is holonomic,*

(b) $A(\Gamma)$ *is symmetric,*

(c) $\rho_1^2 \Gamma$ *is curvature-free.*

In general we put

$$A(\Gamma) = \Gamma . [\zeta_1(\Gamma) * \dots * \zeta_r(\Gamma)]^{-1} \tag{7}$$

for any $r$-th order IC $\Gamma$. This is a section in $D_1^r G$ and can be regarded as the obstacle to the decomposability of $\Gamma$. The map $A$ commutes with all our «natural surjections» (especially preserves semi-holonomity), takes prolongations of connections into waves of sections, and carries the «irreducible» part of the non-holonomity of a higher order connection. All this is more precisely stated in the

THEOREM 11. *Let* $\Gamma$ *be an infinitesimal connection of order* $r$ *in* $\Phi$. *Then* $A(\Gamma)$ *defined by (7) is a section in* $\tilde{D}_1^r G$ *satisfying the conditions:*

(I) $\Gamma$ *is uniquely determined by* $\zeta_1(\Gamma), \dots, \zeta_r(\Gamma)$, *and* $A(\Gamma)$.

(II) *$A(\Gamma)$ is the trivial section $j^r(\sim)$ iff $\Gamma$ is decomposable.*

(III) $A((j^k \rho^{r-k}_{q-k})\Gamma) = (j^k \rho^{r-k}_{q-k}) A(\Gamma)$ *for all* $0 \leqslant k \leqslant q < r$.

(IV) $A(\Gamma * \zeta) = j^1 A(\Gamma)$ *for any first order connection* $\zeta$. *Conversely, if* $A(\Gamma)$ *is a one-wave, then* $\Gamma = \rho^r_{r-1}\Gamma * \zeta_r(\Gamma)$.

(V) $\Gamma$ *is semi-holonomic iff all the* $\zeta_s(\Gamma)$ *are equal, and* $A(\Gamma)$ *is semi-holonomic.*

(VI) *Any two of the following properties imply the third:*

(a) $\Gamma$ *is holonomic,*

(b) $A(\Gamma)$ *is holonomic,*

(c) $\rho^r_1 \Gamma$ *is curvature-free.*

PROOF: (I) and (II) are evident. As for (III), knowing that $\rho^{r,k}_q$ commutes with the prolonged multiplication in $\Phi$, we have to prove

$$\rho^{r,k}_q \Gamma . [\zeta_1(\rho^{r,k}_q \Gamma) * \dots * \zeta_r(\rho^{r,k}_q \Gamma)]^{-1} = \rho^{r,k}_q \Gamma . [\rho^{r,k}_q (\zeta_1(\Gamma) * \dots * \zeta_r(\Gamma))]^{-1},$$

but this is an immediate consequence of Theorem 3. Now according to Theorem 4 we have

$$A(\Gamma * \zeta) = (\Gamma * \zeta) . (\zeta_1 * \dots * \zeta_r * \zeta)^{-1} = j^1(\Gamma . (\zeta_1 * \dots * \zeta_r))$$

and this proves the first part of (IV). Suppose now that $A(\Gamma) = j^1 \Omega$, i.e.

$$\Gamma = j^1 \Omega . (\zeta_1 * \dots * \zeta_r) = j^1(\Omega . j^{r-2} \zeta^{(1)}_1 \dots \zeta^{(r-1)}_{r-1}) . \zeta^{(r)}_r$$
$$= [\Omega . j^{r-2} \zeta^{(1)}_1 \dots \zeta^{(r-1)}_{r-1}] * \zeta_r$$

(cf. (5)). Applying now $\rho^r_{r-1}$ to both sides of this relation we find

$$\rho^r_{r-1} \Gamma = \Omega . j^{r-2} \zeta^{(1)}_1 \dots \zeta^{(r-1)}_{r-1}.$$

As to (V), if $\Gamma$ is semi-holonmic, then all the $\zeta_s(\Gamma)$ are equal by Theorem 1, hence $\zeta_1(\Gamma) * \dots * \zeta_r(\Gamma)$ is semi-holonomic and so is also $A(\Gamma)$. The converse is trivial, and so is (VI). This completes the proof.

Note here again that (III) could have been replaced by the more general law

$$A(\rho^{r,C}_q \Gamma) = \rho^{r,C}_q A(\Gamma).$$

## APPENDIX

For an arbitrary manifold $M$ denote by $\Lambda(M)$ the set of all smooth paths in $M$, i.e. all $C^{\infty}$-maps from $[0,1]$ into $M$. Denote also by $\bar{\Lambda}(M)$ the set of all piecewise smooth and continuous paths.

DEFINITION 2. Let $\Phi$ be a Lie groupoid over $B$. A *path connection* $\sigma$ in $\Phi$ is a map

$$\sigma : \Lambda(B) \to \Lambda(\Phi),\ \lambda \to \sigma^{\lambda}$$

satisfying the relations

$$a\sigma^{\lambda}(t) = \lambda(0),\quad b\sigma^{\lambda}(t) = \lambda(t),\quad t \in [0,1], \tag{8}$$

and the following «transport» condition:

If $\psi : [0,1] \to [t_0, t_1] \subset [0,1]$ is a diffeomorphism, then

$$\sigma^{\lambda} \circ \psi = \sigma^{\lambda\psi} \cdot \sigma^{\lambda}(\psi(0)). \tag{9}$$

It follows immediately from (9) that $\sigma^{\lambda}(0) = \sim(\lambda(0))$. Moreover, if $\lambda, \bar{\lambda} \in \Lambda(B)$ are such that $\bar{\lambda}(0) = \lambda(1)$, i.e. the path $\lambda_*\bar{\lambda}$ is defined, then we have

$$\lambda = (\lambda_*\bar{\lambda}) \circ \psi_0 \text{ and } \bar{\lambda} = (\lambda_*\bar{\lambda}) \circ \psi_1,$$

where $\psi_0(t) = \frac{1}{2}t$, $\psi_1(t) = \frac{1}{2}t + \frac{1}{2}$. If now $\lambda_*\bar{\lambda} \in \Lambda(B)$, we can apply (9) to the path $\lambda_*\bar{\lambda}$ and $\psi_0$ or $\psi_1$. This gives

$$\sigma^{\lambda*\bar{\lambda}}(t) = \begin{cases} \sigma^{\lambda}(2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ \sigma^{\bar{\lambda}}(2t-1) \cdot \sigma^{\lambda}(1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \end{cases} \tag{10}$$

and especially

$$\sigma^{\lambda*\bar{\lambda}}(1) = \sigma^{\bar{\lambda}}(1) \cdot \sigma^{\lambda}(1). \tag{11}$$

Analogously (9) yields $\sigma^{(\lambda^{-1})}(1) = [\sigma^{\lambda}(1)]^{-1}$, and shows that $\sigma^{\lambda}$ is the constant path if $\lambda$ is constant.

The expression (10) can be used to define $\sigma$ even when the composition $\lambda_*\lambda$ is not a smooth path. Thus we can uniquely extend each path connection to a map $\sigma : \bar{\Lambda}(B) \to \bar{\Lambda}(\Phi)$, with the preservation of all its listed properties.

The relations (11) and the following show that

$$H(\sigma) = \bigcup_{\lambda \in \bar{\Lambda}(B)} \sigma^{\lambda}(1) \subset \Phi$$

defines a subgroupoid of $\Phi$ over $B$ called the *holonomy groupoid* of $\sigma$. The path connection $\sigma$ is called *flat* if $\sigma$ maps closed paths onto closed paths, or equivalently, if $H(\sigma)$ is isomorphic to the trivial groupoid $B \times B$.

The path connection thus defined is essentially a structure of only «topological character». One can namely modify very easily Definition 2 to replace the Lie groupoid by a topological groupoid. To make use of the differentiable structure on $\Phi$, we shall suppose that the $r$-jet at $0$ of $\sigma^{\lambda}$ is uniquely determined by that of $\lambda$, and that this correspondence is itself infinitesimal. More exactly, we shall say that a *path connection* $\sigma$ in $\Phi$ *is infinitesimal of order* $r$ if there is a map $\Gamma : B \to \tilde{J}^r(B, \Phi)$ such that

$$\lambda \in \Lambda(B),\ \lambda(0) = x,\ \text{imply}\ j_0^r \sigma^{\lambda} = \Gamma(x) j_0^r \lambda. \tag{12}$$

It is not difficult to see that $\Gamma$ is then necessarily an IC in $\Phi$, and that it is uniquely determined by the path connection $\sigma$ and the integer $r$. We shall be concerned with the converse: given an $r$-th order IC $\Gamma$, does there exists a path connection $\sigma$ such that (12) be satisfied? It follows from the standard theory of (first order) connections that for $r = 1$ the answer is always affirmative, i.e. each first order IC admits a unique path connection, and this path connection is flat iff the first order IC is curvature-free. As for the general case it is immediately clear that $\Gamma$ can admit at most one path connection, because (12) implies

$$j_0^1 \sigma^{\lambda} = (\rho_1^r \Gamma)(x) j_0^1 \lambda .$$

An easy application of Lemma 1 shows that, if $\Gamma$ admits a path connection, then it is necessarily semi-holonomic.

THEOREM 12. *Let $\Gamma$ be an $r$-th order infinitesimal connection in $\Phi$ admitting the path connection $\sigma$. Then the connections $\rho_s^r \Gamma$, $(s = 1, \ldots, r-1)$, and $\Gamma' = \Gamma * \rho_1^r \Gamma$ admit the same path connection $\sigma$.*

PROOF: The first part is trivial. As for $\Gamma'$ let us substitute $\lambda \circ \psi_u$ (for each fixed $u \in [0, 1]$) instead of $\lambda$ into (12) and apply (9) with

$$\psi(t) = \psi_u(t) = (1-u)t + u.$$

We get

(13) $\lambda \in \Lambda(M)$, $u \in I$ imply $j^r_u \sigma^\lambda \cdot j^r_u [\sigma^\lambda(u)]^{-1} = \Gamma(\lambda(u)) j^r_u \lambda$.

Now to show that $\Gamma'$ admits $\sigma$ we have to show explicitly that $\lambda \in \Lambda(M)$, $\lambda(0) = x$ and (12) imply

$j_0^{r+1} \sigma^\lambda = \{ j^1_x \Gamma \cdot j^1_x (u \to j^r_u [\zeta_x(u)]) \} j_0^{r+1} \lambda$, where $(\rho^r_1 \Gamma)(x) = j^1_x \zeta_x$.

Using (13) we have for a fixed $\lambda \in \Lambda(M)$,

$$j_0^{r+1} \sigma^\lambda = j^1_0 (u \to \Gamma(\lambda(u)) j^r_u \lambda \cdot j^r_u [\sigma^\lambda(u)])$$

$$= j^1_0 (u \to \Gamma(\lambda(u)) j^r_u \lambda) \cdot j^1_0 (u \to j^r_u [\zeta_x(\lambda(u))]),$$

because of Lemma 2 and the fact that (12) implies $j^1_0 \sigma^\lambda = j^1_0(\zeta_x \lambda)$. Now the first member in the last expression is clearly $j^1_x \Gamma j_0^{r+1} \lambda$ and the second is $j^1_x (u \to j^r_u [\zeta_x(u)]) j_0^{r+1} \lambda$, q.e.d.

COROLLARY. *The infinitesimal connection $\Gamma$ admits a path connection if and only if $\Gamma$ is simple. The path connection is flat if and only if $\Gamma$ is holonomic, i.e. generated by a curvature-free first order connection.*

PROOF: The first part follows from Theorem 6 and the fact that $\Gamma'$ admitting a path connection must be semi-holonomic. The second part is a consequence of Theorem 7.

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