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HIGHER ORDER TORSIONS OF SPACES WITH CARTAN CONNECTION

by Ivan KOLAR

1. Introduction.

Let $E(B, F, G, P)$ be a fiber bundle associated with a principal fiber bundle $P(B, G)$ and let $\mathfrak{S}$ be a local cross section of E defined in a neighbourhood of $x \in B$. Let $\Phi = PP^{-1}$ be the groupoid associated with P, let C be a connection of the first order on Φ and let $C, C', \ldots, C^{(r)}, \ldots$ be the sequence of its prolongations according to Ehresmann, [4]. Then $C^{(r-1)}(x)$ is a semi-holonomic element of connection of order r on Φ at x and the prolongation of the partial composition law $(\theta, z) \to \theta . z$, $\theta \in \Phi$, $z \in E$, determines an element $\mathfrak{S}^{(r)}(x) = [C^{(r-1)}]^{-1}(x) . \mathfrak{S} \in \bar{J}^r_x(B, E_x)$. The mapping $\mathfrak{S}^{(r)}$ is a local cross section of

$$\bigcup_{x \in B} \bar{J}^r(B, E_x) = (B, \bar{J}^r(B, F), G, P)$$

and it will be called the r-th prolongation of $\mathfrak{S}$ with respect to C. We are interested in the following problem: under what conditions are the values of $\mathfrak{S}^{(r)}$ holonomic r-jets? For $r = 2$, this problem is solved by Theorem 1 of [5] and by Proposition 1 of this paper. The answer is that $\mathfrak{S}^{(2)}(x)$ is a holonomic 2-jet if and only if the torsion form at x vanishes. In the present paper, we treat the case of arbitrary r for a space with Cartan connection and we deduce the following result. Let G_x be the isotropy group of Φ over x and let $\mathfrak{g}_x$ be its Lie algebra, then the curvature form $\Omega(x)$ of C at x can be considered as an element of $\mathfrak{g}_x \otimes \wedge^2 T^*_x(B)$. Let $\mathfrak{S}$ be the fundamental section of our space with Cartan connection. We define the isotropy group H^r_x of order r of the homogeneous space E_x at $\mathfrak{S}(x)$ as the set of all $g \in G_x$ satisfying $j^r_{\mathfrak{S}(x)} g = j^r_{\mathfrak{S}(x)}$ $(= j^r_{\mathfrak{S}(x)} id_{E_x})$. Let $\mathfrak{h}^r_x$ be the Lie algebra of H^r_x, then we introduce the torsion form $\tau^r(x)$ of order r at x as the canonical projection of $\Omega(x)$ into $(\mathfrak{g}_x / \mathfrak{h}^r_x) \otimes \wedge^2 T^*_x(B)$; $\tau^0(x)$ coincides with the usual torsion form at x. Our main conclusion is that the values of $(r+2)$-nd prolongation of $\mathfrak{S}$ are holonomic $(r+2)$-jets if and only if τ^r vanishes identically. Since $dim\ H^r_x \neq 0$ implies

$$dim\ H^{r+1}_x < dim\ H^r_x,$$

there exists a smallest integer q satisfying $dim\ H^q = 0$, this number is called the *order of isotropy* of the homogeneous space F. Thus, if the values of the $(q+2)$-nd prolongation of the fundamental section of a space with Cartan connection are holonomic jets, then the curvature form of C vanishes, so that C is integrable and the values of any prolongation of $\mathfrak{S}$ are holonomic.- We remark finally that Proposition 2, which is our main tool in these investigations, has a general character and will be used later for the study of an analogous problem for submanifolds of a space with Cartan connection.

Unless otherwise stated, our considerations are in the category C^∞.

2. Second prolongation of a cross section.

Let V, W be two manifolds and let X be a semi-holonomic 2-jet of V into W, $\alpha X = v$, $\beta X = w$. In [5], we have introduced the difference tensor $\Delta(X)$ of X, $\Delta(X) \in T_w(W) \otimes \wedge^2 T^*_v(V)$, by means of its expression in local coordinates. Since this tensor plays an important role in our considerations, we present also the following invariant definition of this concept. First of all, we remark that every r-jet can be identified with a homomorphism, see e.g. [1]. Now, consider the canonical projection of 2-jets onto 1-jets as well as the injection of holonomic 2-jets into semi-holonomic 2-jets. If we add the corresponding kernels and factor-spaces, we get the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & J^1_{(v,w)}(V,W) & \leftarrow & J^2_{(v,w)}(V,W) & \leftarrow & T_w(W)\otimes S^2 T^*_v(V) & \leftarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow{\scriptstyle 2} & & \\
0 & \leftarrow & J^1_{(v,w)}(V,W) & \leftarrow & \bar{J}^2_{(v,w)}(V,W) & \leftarrow & T_w(W)\otimes(\otimes T^*_v(V)) & \leftarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow & & \\
 & & 0 & \leftarrow & \bar{J}^2_{(v,w)} / J^2_{(v,w)} & \overset{\psi}{\leftarrow} & T_w(W)\otimes \wedge^2 T^*_v(V) & \leftarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The last row shows that ψ is an isomorphism and $\psi^{-1}\phi=\Delta$ is the mapping which assigns to every $X \in \bar{J}^2_{(v,w)}(V,W)$ its *difference tensor* $\Delta(X) \in T_w(W) \otimes \wedge^2 T_v^*(V)$. It is clear that $\Delta(X)=0$ *if and only if* X *is holonomic.*

The difference tensor of a product of two semi-holonomic 2-jets is determined by

LEMMA 1. *Let* U, V, W *be manifolds. Let* $X \in \bar{J}^2(V,W)$, $Y \in \bar{J}^2(U,V)$, $\alpha Y=u$, $\beta Y=\alpha X=v$, $\beta X=w$ *and let* X_1 *or* Y_1 *denote the underlying 1-jet of* X *or* Y *respectively, then*

$$\Delta(XY)=X_1\Delta(Y)+\Delta(X)Y_1 \in T_w(W)\otimes\wedge^2 T_u^*(U),$$

where X_1 *or* Y_1 *is considered as a homomorphism* $X_1: T_v(V)\to T_w(W)$ *or* $Y_1: T_u(U)\to T_v(V)$.

REMARK 1. In particular, if f is a mapping of V into W, then $\Delta(fY)=f_*\Delta(Y)$, where f_* means the differential of f.

PROOF. For the sake of simplicity, we shall express all considered objects by means of local coordinates. Let h_1 or h_2 or h_3 be a holonomic 2-frame on U or V or W at u or v or w respectively, then

$$h_2^{-1}Yh_1=(y^i_r, y^i_{rs}),\quad h_3^{-1}Xh_2=(x^a_i, x^a_{ij})$$

and

$$h_3^{-1}XYh_1=(x^a_i y^i_r,\quad x^a_{ij}y^i_r y^j_s+x^a_i y^i_{rs}),$$

$a=1,\dots,\dim W$, $i,j=1,\dots,\dim V$, $r,s=1,\dots,\dim U$, cf. [3]. According to [5], Proposition 10, the corresponding coordinates of $\Delta(X)$ or $\Delta(Y)$ are $x^a_{[ij]}$ or $y^i_{[rs]}$, where the square brackets denote antisymmetrization. Now, we find directly that the coordinates of $\Delta(XY)$ are

$$x^a_{[ij]}y^i_r y^j_s+x^a_i y^i_{[rs]},$$

QED.

LEMMA 2. *Let* $Z \in \bar{J}^2(U, V\times W)$, $\alpha Z=u$, $\beta Z=(v,w)$ *and let* $Z=(X,Y)$, $X\in\bar{J}^2(U,V)$, $Y\in\bar{J}^2(U,W)$, *then* $\Delta(Z)=i_{1*}\Delta(X)+i_{2*}\Delta(Y)$, *where* i_1 *is the injection of* V *as* $V\times\{w\}$ *into* $V\times W$ *and* i_2 *is the injection of* W *as* $\{v\}\times W$ *into* $V\times W$.

LEMMA 3. *Let* $\bar{J}^2 E$ *be the second semi-holonomic prolongation of a fibered manifold* (E, p, B). *If* $X \in \bar{J}^2 E$, $\alpha X = x$, $\beta X = z$, *then* $\Delta(X) \in T_z(E_x) \otimes \wedge^2 T_x^*(B)$.

LEMMA 4. *Let* (E_1, p_1, B) *and* (E_2, p_2, B) *be two fibered manifolds over the same base and let* (E_3, p_3, B) *be their fiber product.* **Let** $X_i \in \bar{J}^2 E_i$, $i = 1, 2, 3$, $X_3 = (X_1, X_2)$, $\alpha X_3 = x$, $\beta X_3 = (z_1, z_2)$, *then* $\Delta(X_3) = i_{1*}\Delta(X_1) + i_{2*}\Delta(X_2)$, *where* i_1 *is the injection of* E_{1x} *as* $E_{1x} \times \{z_2\}$ *into* E_3 *and* i_2 *is the injection of* E_{2x} *as* $\{z_1\} \times E_{2x}$ *into* E_3.

PROOFS of all the three lemmas are obvious.

Further, let Φ be a Lie groupoid of operators on a fibered manifold (E, p, B), see [8]. Let X be a non-holonomic element of connection of order r on Φ at $x \in B$, let V be a manifold and Z an element of $\tilde{J}^r(V, E)$ such that $p(\beta Z) = x$; then the prolongation of the partial composition law $(\theta, z) \to \theta . z$, $\theta \in \Phi$, $z \in E$, determines an element $X^{-1}(Z) = (X^{-1} p Z). Z \in \tilde{J}^r(V, E_x)$, see [4]. We shall say that $X^{-1}(Z)$ is the *development of* Z *into* E_x *by means of* X, cf. [5] (from another point of view, $X^{-1}(Z)$ may be called «the absolute differential of Z with respect to X», [4]). In particular, if $\mathfrak{S}$ is a local cross section of (E, p, B), then we write only $X^{-1}(\mathfrak{S})$ instead of $X^{-1}(j_x^r \mathfrak{S})$ and this element will be said the *development of* $\mathfrak{S}$ *into* E_x *by means of* X. Moreover, if C is a connection of the first order on Φ and if $C, C', \dots, C^{(r)}, \dots$ is the sequence of its prolongations, then $[C^{(r)}]^{-1}(x)(\mathfrak{S})$ is also called the *$(r+1)$-st development of* $\mathfrak{S}$ *into* E_x *by means of* C and $\mathfrak{S}^{r+1} : x \to [C^{(r)}]^{-1}(x)(\mathfrak{S})$ is a local cross section of $\bigcup_{x \in B} \tilde{J}_x^{r+1}(B, E_x)$, which will be called the *$(r+1)$-th prolongation of* $\mathfrak{S}$ *with respect to* C.

Let $P(B, G)$ be a principal fiber bundle, let $\Phi = PP^{-1}$ be the groupoid associated with P and let G_x be the isotropy group of Φ over $x \in B$. Let C be a connection of the first order on Φ and let Γ be the representant of the connection C on P, [5]. The curvature form $(\Omega)_u$ of Γ at $u \in P_x$ is an element of $\mathfrak{g} \otimes \wedge^2 T_x^*(B)$, $\mathfrak{g}$ being the Lie algebra of G. The frame u can be considered as a mapping $\tilde{u} : G \to G_x$ given by

$$g \to (ug)\, u^{-1}.$$

Let $\tilde{u}_*$ be the differential of $\tilde{u}$ at the unit of G, then $\Omega(x) = \tilde{u}_*(\Omega)_u \in \mathfrak{g}_x \otimes \wedge^2 T^*_x(B)$ does not depend on the choice of $u \in P_x$. The form $\Omega(x)$ will be called the *curvature form of* C *at* $x \in B$.

PROPOSITION 1. *Let* $\mathfrak{S}$ *be a local cross section of an associated fiber bundle* $E(B, F, G, P)$ *defined in a neighbourhood of* $x \in B$. *Let* $H_x \subset G_x$ *be the stability group of* $\mathfrak{S}(x) \in E_x$ *and let* $\mathfrak{h}_x$ *be its Lie algebra, then the second development* $\mathfrak{S}^{(2)}(x) = (C')^{-1}(x)(\mathfrak{S})$ *of* $\mathfrak{S}$ *into* E_x *by means of* C *is a holonomic 2-jet if and only if the canonical projection of* $\Omega(x)$ *into* $(\mathfrak{g}_x / \mathfrak{h}_x) \otimes \wedge^2 T^*_x(B)$ *vanishes.*

REMARK 2. If G acts on F transitively, then Proposition 1 coincides with Theorem 1 of [5].

PROOF. Consider Φ as a fibered manifold with projection a (as usual, $a(\theta)$ means the source of $\theta \in \Phi$). Let K be the fiber product of (Φ, a, B) and (E, p, B); then the action of Φ on E is a mapping $\mathcal{H} : K \to E$ such that $\mathcal{H}(\theta, z) = \theta . z$. We have $(C')^{-1}(x)(\mathfrak{S}) = \mathcal{H}((C')^{-1}(x), j^2_x \mathfrak{S})$ and Lemmas 1 and 4 give

$$\Delta((C')^{-1}(x)(\mathfrak{S})) = \mathcal{H}_* [i_{1*} \Delta(C'^{-1}(x)) + i_{2*} \Delta(j^2_x \mathfrak{S})],$$

where i_1 or i_2 is the injection of $\Phi_x = a^{-1}(x)$ as $\Phi_x \times \{ \mathfrak{S}(x) \}$ or of E_x as $\{ e_x \} \times E_x$ into K respectively and e_x denotes the unit of G_x. But we have deduced in [5], Proposition 12, that

$$\Delta(C'(x)) = \Omega(x) \in \mathfrak{g}_x \otimes \wedge^2 T^*_x(B)$$

and one sees easily that $\Delta((C')^{-1}(x)) = -\Delta(C'(x))$. Further, consider the injection i_3 of G_x into Φ_x; then

$$i_{1*} \Delta((C')^{-1}(x)) = i_{1*} i_{3*} \Delta((C')^{-1}(x)) = -(i_1 i_3)_* \Omega(x).$$

On the other hand, since $i_{2*} \Delta(j^2_x \mathfrak{S}) = 0$, we can replace it by

$$0 = i_{4*} \Delta(j^2_{\mathfrak{S}(x)}),$$

where $j^2_{\mathfrak{S}(x)}$ means the 2-jet of the identity mapping of the one-element manifold $\{ \mathfrak{S}(x) \}$ and i_4 is the injection of $\{ \mathfrak{S}(x) \}$ as $\{(e_x, \mathfrak{S}(x))\}$ into K. But the composition of $\mathcal{H}$ and $(i_1 i_3, i_4)$ is the restriction of $\mathcal{H}$ to $G_x \times \{ \mathfrak{S}(x) \}$ and the kernel of its differential at $(e_x, \mathfrak{S}(x))$ can be identified with the Lie algebra of the stability group H_x of $\mathfrak{S}(x)$. It follows that $\Delta(C'^{-1}(x)(\mathfrak{S}))$ vanishes if and only if the canonical projection of $\Omega(x)$ into $(\mathfrak{g}_x / \mathfrak{h}_x) \otimes \wedge^2 T^*_x(B)$ vanishes; QED.

3. Recurrence formula.

We shall show how to extend our previous result to higher orders. Since Φ is a groupoid of operators on $(B, \bar{J}^r(B, F), G, P)$, we can consider the second development $(C')^{-1}(x)(\mathfrak{S}^{(r)})$ of $\mathfrak{S}^{(r)}$ by means of C.

PROPOSITION 2. *Let the values of $\mathfrak{S}^{(r+1)}$ be holonomic, then $\mathfrak{S}^{(r+2)}(x)$ is a holonomic $(r+2)$-jet if and only if $\Delta((C')^{-1}(x)(\mathfrak{S}^{(r)}))$ vanishes.*

PROOF. In [5], Proposition 1, we have deduced that

$$\mathfrak{S}^{(r+2)}(x) = C^{-1}(x)(\mathfrak{S}^{(r+1)}) = (C')^{-1}(x)(\mathfrak{S}^{(r)}).$$

But $\mathfrak{S}^{(r+2)}(x) \in J^{r+2}(B, E_x)$ implies $C'^{-1}(x)(\mathfrak{S}^{(r)}) \in J^2(J^r(B, E_x))$, i.e. $\Delta((C')^{-1}(x)(\mathfrak{S}^{(r)})) = 0$. Conversely, we have

$$\mathfrak{S}^{(r+2)}(x) \in \bar{J}^{(r+2)}(B, E_x), \quad \mathfrak{S}^{(r+2)}(x) \in J^1(J^{r+1}(B, E_x)),$$

$$\mathfrak{S}^{(r+2)}(x) \in J^2(J^r(B, E_x)),$$

but this implies that $\mathfrak{S}^{(r+2)}(x) \in J^{(r+2)}(B, E_x)$. Indeed, take a local coordinate system and let $a_i, \dots, a_{i_1 \dots i_{r+1}}, a_{i_1 \dots i_{r+2}}$ be the corresponding coordinates of $\mathfrak{S}^{(r+2)}(x)$, [3]. From $\mathfrak{S}^{(r+2)}(x) \in J^1(J^{(r+1)}(B, E_x))$ we deduce that $a_i, \dots, a_{i_1 \dots i_{r+1}}$ are symmetric in all subscripts and $a_{i_1 \dots i_{r+1} i_{r+2}}$ are symmetric in the first $r+1$ subscripts. In addition, $\mathfrak{S}^{(r+2)}(x) \in J^2(J^r(B, E_x))$ implies $a_{i_1 \dots i_{r+1} i_{r+2}} = a_{i_1 \dots i_{r+2} i_{r+1}}$, so that $a_{i_1 \dots i_{r+2}}$ are symmetric in all subscripts, QED.

From Propositions 1 and 2 we obtain immediately

COROLLARY 1. *Suppose that the values of $\mathfrak{S}^{(r+1)}$ are holonomic. Let K_x be the stability group of $\mathfrak{S}^{(r)}(x)$ and let $\mathfrak{k}_x$ be its Lie algebra, then $\mathfrak{S}^{(r+2)}(x)$ is a holonomic $(r+2)$-jet if and only if the canonical projection of $\Omega(x)$ into $(\mathfrak{g}_x / \mathfrak{k}_x) \otimes \wedge^2 T^*_x(B)$ vanishes.*

4. Isotropy groups of higher orders.

Let F be a homogeneous space with fundamental group G (which implies that G acts effectively on F). Fix a point $c \in F$, then the *isotropy group* H^r *of order* r *at* c is the set of all $g \in G$ satisfying $j^r_c g = j^r_c$ $(= j^r_c id_F)$; $H^0 = H$ is the stability group of c. The Lie algebra $\mathfrak{h}^r$ of H^r is determined by

PROPOSITION 3. *For* $r \geqslant 1$, $\mathfrak{h}^r \subset \mathfrak{h}$ *is characterized by* $[\mathfrak{h}^r, \mathfrak{g}] \subset \mathfrak{h}^{r-1}$, *i.e. an element* $X \in \mathfrak{h}$ *belongs to* $\mathfrak{h}^r$ *if and only if* $[X, \mathfrak{g}] \subset \mathfrak{h}^{r-1}$.

This proposition is direct consequence of the following lemmas.

Consider a manifold V; at a point $x \in V$ denote by $\mathfrak{L}^r_x$ the space of all germs X of vector fields on V at x satisfying $j^r_x X = 0$, cf. [6], § 3. If u^i, $i, j, k = 1, \dots, dim\, V$, are local coordinates on V in a neighbourhood of x, then $X = \xi^i(u)\frac{\partial}{\partial u^i}$ belongs to $\mathfrak{L}^r_x$ if and only if

$$j^r_x \xi^i(u) = 0.$$

LEMMA 5. *Let* X_i *be germs of vector fields on* V *at* x *such that their values at* x *form a basis of* $T_x(V)$. *Let* $X \in \mathfrak{L}^0_x$, *then* $X \in \mathfrak{L}^r_x$ *if and only if* $[X, X_i] \in \mathfrak{L}^{r-1}_x$, $r \geqslant 1$.

PROOF. Let $X = \xi^i(u)\frac{\partial}{\partial u^i}$, $\xi^i(x) = 0$, $X_i = \eta^j_i(u)\frac{\partial}{\partial u^j}$, $det\, \eta^j_i(x) \neq 0$ and let $[X, X_i] \in \mathfrak{L}^{r-1}_x$, so that

$$(*) \qquad \xi^k(u)\frac{\partial \eta^j_i(u)}{\partial u^k} - \eta^k_i(u)\frac{\partial \xi^j(u)}{\partial u^k} = f^j_i(u),$$

where $j^{r-1}_x f^j_i(u) = 0$. For $u = x$, $(*)$ gives $\frac{\partial \xi^j(x)}{\partial u^k} = 0$ and by successive differentiation of $(*)$ we get analogously $j^r_x \xi^i(u) = 0$, QED.

LEMMA 6. *Let* X *be an analytic vector field defined in a neighbourhood of a point* x *of an analytic manifold* V. *Then the germ of* X *at* x *belongs to* $\mathfrak{L}^r_x$ *if and only if every local transformation* ϕ_t *of the local one-parameter group determined by* X *satisfies* $j^r_x \phi_t = j^r_x$ $(= j^r_x id_V)$.

PROOF is based directly on the Taylor formula for ϕ_t:

$$\phi^i_t = (1 + tX + \dots + \frac{t^n}{n!} X^n + \dots) u^i.$$

PROPOSITION 4. *If $dim\, H^r \neq 0$, then $dim\, H^{r+1} < dim\, H^r$.*

PROOF. Suppose $dim\, H^{r+1} = dim\, H^r$, then $\mathfrak{h}^{r+1} = \mathfrak{h}^r$ and Proposition 3 would give $[\mathfrak{h}^r, \mathfrak{g}] \subset \mathfrak{h}^r$, which would imply that $\mathfrak{h}^r$ would be a non-trivial ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$, but this is a contradiction with the fact that G acts effectively on F, QED.

By Proposition 4, there exists a smallest q satisfying $dim\, H^q = 0$; this number is called the *order of isotropy of* F.

REMARK 3. An example by Lumiste, [7], p.445, shows that for every integer p there exists a homogeneous space the order of isotropy of which is p. (Although Lumiste has introduced the higher order isotropy groups in a different way, Proposition 3 shows that both definitions are equivalent.)

PROPOSITION 5. *Let ω^α, $\alpha = 1, \ldots, dim\, G$, be independent Maurer-Cartan forms of G such that $\omega^i = 0$, $i = 1, \ldots, dim\, F$, are differential equations of H. Let*

$$d\omega^i = \frac{1}{2} c^i_{jk}\, \omega^j \wedge \omega^k + c^i_{j\lambda}\, \omega^j \wedge \omega^\lambda, \quad \lambda = dim\, F + 1, \ldots, dim\, G,$$

then the differential equations of H^1 are $\omega^i = 0$, $c^i_{j\lambda}\, \omega^\lambda = 0$.

PROOF. Let X_α be the basis of $\mathfrak{g}$ dual to ω^α. Since the tangent space of F at c can be identified with $\mathfrak{g}/\mathfrak{h}$ and $X_i + \mathfrak{h}$ is a basis of $\mathfrak{g}/\mathfrak{h}$, our assertion follows from Proposition 3 and from the formulae

$$[X_\lambda, X_i + \mathfrak{h}] = -c^j_{\lambda i} X_j + \mathfrak{h}.$$

REMARK 4. By Proposition 5, the order of isotropy of an affine space is *1*. One sees easily that the order of isotropy of a projective space is *2*.

5. Higher order torsions and their vanishing.

Let F be a homogeneous space with fundamental group G, let $P(B, G)$ be a principal fibre bundle, let $\Phi = PP^{-1}$ be the groupoid associated with P, let C be a connection of the first order on Φ and let $\mathfrak{S}$ be a global cross section of $E = E(B, F, G, P)$. A *space with Cartan connection of type* F can be defined as the quintuple $\mathcal{S} = \mathcal{S}(B, \Phi, E, \mathfrak{S}, C)$ satisfying the following conditions: a) $dim\, B = dim\, F$, b) $C^{-1}(x)(\mathfrak{S})$ is regular for every $x \in B$. (Indeed, b) is equivalent to condition 4) of § 5, [2]). The section $\mathfrak{S}^{(1)} = C^{-1}(\mathfrak{S}) : B \to (B, J^1(B, F), G, P)$ is a soldering of E to B, cf. [9] , p.4). Let $\Omega(x) \in \mathfrak{g}_x \otimes \wedge^2 T^*_x(B)$ be the curvature form of C at x and let H^r_x be the isotropy group of order r of E_x at $\mathfrak{S}(x)$, then the canonical projection $\tau^r(x)$ of $\Omega(x)$ into $(\mathfrak{g}_x / \mathfrak{h}^r_x) \otimes \wedge^2 T^*_x(B)$ will be called the *torsion form of order* r *of* $\mathcal{S}$ *at* x; $\tau^0(x) = \tau(x)$ is the usual torsion form of $\mathcal{S}$ at x. We put $\tau^{-1}(x) = 0$.

REMARK 5. Since the order of isotropy of an affine space is 1, the higher order torsions are trivial in the affine case: τ^0 is the usual torsion form and τ^1 coincides with the curvature form.

THEOREM 1. *Suppose that the torsion form* τ^{r-1} *of order* $r-1$ *of* $\mathcal{S}$ *vanishes identically. Then the* $(r+2)$*-nd development* $\mathfrak{S}^{(r+2)}(x)$ *of the fundamental section* $\mathfrak{S}$ *of* $\mathcal{S}$ *at* x *is a holonomic* $(r+2)$*-jet if and only if* $\tau^r(x)$ *vanishes.*

PROOF. We have only to show that the stability group of $\mathfrak{S}^{(r)}(x)$ coincides with H^r_x. Put $\mathfrak{S}(x) = z$, $\mathfrak{S}^{(r)}(x) = S$. From $j^r_z g = j^r_z$ we get

$$g S = j^r_z g S = j^r_z S = S;$$

conversely, let $gS = S$, then, by regularity of S, there exists S^{-1} such that $SS^{-1} = j^r_z$ and we have $j^r_z = g S S^{-1} = j^r_z g$, QED.

From Proposition 4 and Theorem 1, we deduce

COROLLARY 2. *Let* q *be the order of isotropy of the homogeneous space* F. *If the values of the* $(q+2)$*-nd prolongation* $\mathfrak{S}^{(q+2)}$ *of* $\mathfrak{S}$ *are holonomic, then* C *is integrable, so that the values of* $\mathfrak{S}^{(r)}$ *are holonomic for every* r.

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