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SOME REMARKS ON SHEAF COHOMOLOGY

by YUH-CHING CHEN

Introduction. Perhaps the most important theorem that makes sheaf theory an essential tool in the study of algebraic geometry and several complex variables is the well-known comparison theorem of Leray that says:
If $H^q(U_\sigma; \mathfrak{F}) = 0$ for $q \geq 1$ and all $\sigma \in N(\mathfrak{U})$, then $H^*(\mathfrak{X}; \mathfrak{F}) \approx H^*(N(\mathfrak{U}); \mathfrak{F})$.
The crucial point is that the theorem enables one to compute sheaf cohomology (by Čech complexes) in some given situations. In this note we shall study a general simplicial cohomology with a system of coefficients, then apply it to obtain some simplicial interpretation of sheaf cohomology.

Section 1 contains some technical terminologies and results which enable us to argue the main results in simple terms. Main theorems are in sections 2 and 3.

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1. Stacks and costacks. Let $K = \bigcup_{q \geq 0} K_q$ be a simplicial set with q-simplexes $\sigma \in K_q$, face operators $d^i : K_q \to K_{q-1}$, degenaracy operators $s^j : K_q \to K_{q+1}$. In this paper, a simplicial set K is often considered as a category with objects simplexes σ, τ, ..., and morphisms $d^i : \sigma \to d^i\sigma$, $s^j : \tau \to s^j\tau$ and their compositions. A (cohomological) system of coefficients on K with values in a category $\mathfrak{A}$ is then a contravariant functor $A : K \to \mathfrak{A}$. We shall call such a contravariant functor A a *prestack* over the simplicial set K. For example, if $\mathfrak{F}$ is an abelian sheaf over a space $\mathfrak{X}$ and if $\mathfrak{U}$ is an open cover or a locally finite closed cover of $\mathfrak{X}$, then $\mathfrak{F}$ gives rise to the prestack of abelian groups $S\mathfrak{F}$ over the nerve $K = N(\mathfrak{U})$ of $\mathfrak{U}$ defined by $(S\mathfrak{F})(\sigma) = \Gamma(U_\sigma, \mathfrak{F})$, where U_σ is the support of the simplex σ. Note that here $K = N(\mathfrak{U})$ is regarded as a category of simplexes (non-degenerate ones and degenerate ones) and that $(S\mathfrak{F})(s^j\sigma) = (S\mathfrak{F})(\sigma)$ for every degeneracy operator s^j. The system of coefficients $\mathcal{H}^q(\mathfrak{F})$ of Godement

[2, p. 209] is another example of a prestack over $N(\mathfrak{U})$. If a prestack A has the property that $A(\sigma)\approx A(s^j\sigma)$ for every s^j, then A is called a *stack*. Therefore $S\mathfrak{F}$ and $\mathcal{H}^q(\mathfrak{F})$ are indeed stacks.

A (covariant) functor $A: K\to\mathfrak{A}$ is called a *precostack* over K with values in $\mathfrak{A}$. Let A be a precostack of abelian groups. Then the graph of A, the set $\bigcup_{\sigma\in K} A(\sigma)$, is a simplicial set and there is a simplicial projection $\pi: \bigcup A(\sigma)\to K$ such that $\pi^{-1}(\sigma)=A(\sigma)$. A precostack is often identified with its graph. For example, the singular complex of the abelian sheaf $\mathfrak{F}$ is a precostack, or rather the graph of a precostack of groups, over the singular complex of the base space $\mathfrak{X}$. A precostack can also be viewed as a (homological) system of coefficients.

Let $\mathfrak{A}b$ be the category of abelian groups and let $\mathfrak{A}b_K$ be the category of abelian prestacks over K (the category of group-valued contravariant functors on K, or the category of systems of coefficient groups over K). Then $\mathfrak{A}b_K$ is an abelian category in which sums and products of exact sequences are exact. The category $\mathfrak{A}b^K$ of abelian precostacks is also an abelian category with exact sums and products. It is proved in [1] that $\mathfrak{A}b^K$ has enough projectives and injectives. We shall prove that $\mathfrak{A}b_K$ has enough injectives.

Let X be a simplicial set and let $\varphi: X\to K$ be a simplicial map. Then φ induces two functors $\varphi^{\#}: \mathfrak{A}b_K\to\mathfrak{A}b_X$ and $\varphi_{\#}: \mathfrak{A}b_X\to\mathfrak{A}b_K$ defined as

$$\varphi^{\#}B=B\varphi \quad\text{and}\quad (\varphi_{\#}A)(\sigma)=\Pi_x A(x),\quad x\in\varphi^{-1}(\sigma).$$

Both functors $\varphi^{\#}$ and $\varphi_{\#}$ are exact and $\varphi^{\#}$ is (left) adjoint to $\varphi_{\#}$. Therefore, $\varphi_{\#}$ preserves injectives (cf. [1]). If $X=\Delta^n$ is the standard simplicial n-simplex, then since the constant stack $Q^{(n)}$ over Δ^n with value the group of rationals mod 1 is injective, $\varphi_{\#}Q^{(n)}$ is injective in $\mathfrak{A}b_K$. Let $\varphi_\sigma: \Delta^n\to K$ be the simplicial map that sends the only non-degenerate n-simplex δ^n of Δ^n onto $\sigma\in K_n$ and let $Q=\Pi_{\sigma\in K}(\varphi_\sigma)_{\#}Q^{(n)}$, $n=\dim\sigma$. Then Q is an injective generator of $\mathfrak{A}b_K$ and so $\mathfrak{A}b_K$ has enough injectives.

2. Representation of cohomology by generalized Eilenberg-MacLane complexes. Let K be a fixed simplicial set. For each abelian prestack $A\in\mathfrak{A}b_K$, let

C^*A be the *cochain complex* of A with $C^qA = \Pi_\sigma A(\sigma)$, $\sigma \in K_q$ and with coboundary maps alternating sums of the homomorphisms $A(d^i)$. Then C^* is an exact functor from $\mathfrak{A}b_K$ to the category of cochain complexes of abelian groups. The homology groups of C^*A, denoted by $H^*(K;A)$ or $H^*(A)$, are cohomology groups of K with coefficients in A (*a system of coefficients*). Let $\Gamma_K = Hom(Z,-)$ (where Z is the constant stack of integers over K) be the *section functor* on $\mathfrak{A}b_K$ and let $R^n\Gamma_K$ be the n-th derived functor of Γ_K. Then it is not hard to show that

THEOREM 2.1. *$H^*(K;-) \approx R^*\Gamma_K(-) \approx Ext^*_K(Z,-)$, where $Ext^n_K(Z,A)$ is the group of equivalence classes of n-fold extensions of A by Z in $\mathfrak{A}b_K$.*

Let $\varphi: X \to K$ be a simplicial map. It is easily seen that $H^*(X;-) \approx H^*(K;\varphi_\#(-))$. φ induces a homomorphism

$$\varphi^*: H^*(K;A) \to H^*(X;\varphi^\# A),\quad A \in \mathfrak{A}b_K,$$

defined by $\varphi^*([c]) = [cf]$, where the cocycle $c \in \Pi_\sigma A(\sigma)$, $\sigma \in K_n$, is regarded as a function on K_n. Note that φ^* can also be obtained from the morphism $\rho_A: A \to \varphi_\#\varphi^\# A$ of the adjoint transformation $\rho: 1 \to \varphi_\#\varphi^\#$.

Let C_K be the category of simplicial sets over K, objects X_φ are simplicial maps $\varphi: X \to K$; morphisms $f: X_\varphi \to Y_\psi$ are simplicial maps $f: X \to Y$ such that $\varphi = \psi f$. For a given *stack* A over K, the cohomology groups of X_φ with coefficients in A are defined as

$$H^*(X_\varphi;A) = H^*(X;\varphi^\# A).$$

This defines a cohomology functor $H^*(-;A)$ on C_K. We shall show that this cohomology on C_K is representable by the *generalized Eilenberg-MacLane complexes* $K(A,n)_\theta \in C_K$ of the system of coefficients A.

Let A be a *stack* of groups over K. $K(A,n)_\theta$, or $\theta: K(A,n) \to K$, is defined as follows. For $\tau = d^i\sigma$ the i-th face of $\sigma \in K_q$, let f be the the morphism in C_K defined by $f(\delta^{q-1}) = d^i\delta^q$, see the diagram

$$\begin{array}{ccc} \Delta^{q-1} & \xrightarrow{\ f\ } & \Delta^q \\ & {\scriptstyle\varphi_\tau}\searrow \quad \swarrow{\scriptstyle\varphi_\sigma} & \\ & K & \end{array}$$

Then f induces a homomorphism of groups of normalized n-cocycles

$$Z^n(f): Z^n(\Delta^q; \varphi_\sigma^\# A) \to Z^n(\Delta^{q-1}; f^\# \varphi_\sigma^\# A) = Z^n(\Delta^{q-1}; \varphi_\tau^\# A).$$

Define the simplicial set $K(A,n) = \bigcup K_q(A,n)$ by letting $K_q(A,n) = \bigcup Z^n(\Delta^q; \varphi_\sigma^\# A)$, $\sigma \in K_q$, with face operators defined by $Z^n(f)$ and degeneracy operators defined in a similar way. Let $\theta: K(A,n) \to K$ be the obvious simplicial projection with $\theta^{-1}(\sigma) = Z^n(\Delta^q; \varphi_\sigma^\# A)$. Then $K(A,n)_\theta$ is a well-defined object in C_K. Note that by the remark on precostack in section 1, $K(A,n)$ is an abelian precostack over K. If $K = \Delta^0$ is a simplicial point, then the *stack* A is (isomorphic to) a constant stack with value group π. In this case $K(A,n)$ is the classical Eilenberg-MacLane complex $K(\pi, n)$.

In stating the cohomology representation theorem, we need the concept of homotopy in C_K. In the diagram

$$\begin{array}{ccccc} X & \xleftarrow{\;p\;} & X \times I & \xrightarrow{\;F\;} & Y \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \varphi p} & \swarrow_{\psi} & \\ & & K & & \end{array}$$

$I = \Delta^1$ is the standard simplicial 1-simplex, p is the projection $p(x,d) = x$, $F: (X \times I)_{\varphi p} \to Y_\psi$ is called a *K-homotopy*. Two maps $f, g = X_\varphi \to Y_\psi$ are *K-homotopic* if they are connected by a K-homotopy F. For each $\sigma \in K$ let Δ^σ be the simplicial subset of K generated by σ. The K-homotopy is a system of simplicial homotopies

$$F = \{ F_\sigma : \varphi^{-1}(\Delta^\sigma) \times I \to \psi^{-1}(\Delta^\sigma) \mid \sigma \in K \}$$

related by the simplicial operators d^i, s^j of K (a stack of simplicial homotopies). Let $[X_\varphi, Y_\psi]$ denote the set of equivalence classes of K-homotopic maps from X_φ to Y_ψ. If Y is the graph of a precostack, then $[X_\varphi, Y_\psi]$ is an abelian group.

THEOREM 2.2. *For any stack A («normalized prestack») there is a natural isomorphism*

$$\varphi^n : [X_\varphi, K(A,n)_\theta] \to H^n(X_\varphi; A) \quad \textit{for} \quad X_\varphi \in C_K.$$

PROOF. To define φ^n, let $c \in C^n(\theta^{\#}A)$ be the n-cochain on $K(A,n)$ defined by $c(\gamma)=\gamma(\delta^n)$ for every $\gamma \in K_n(A,n)$. Then c is a cocycle called the *fundamental cocycle* on $K(A,n)$. The cohomology class

$$[c] \in H^n(K(A,n)_\theta; A)=H^n(K(A,n); \theta^{\#}A)=H^n(C^*(\theta^{\#}A))$$

is said to be *characteristic* for $K(A,n)_\theta$. For each homotopy class $[f]$ $\in [X_\varphi, K(A,n)_\theta]$, let $\varphi^n([f])=f^*[c]=[cf] \in H^n(X_\varphi; A)$. Then φ^n is a homomorphism independent of the representative f. φ^n has an inverse that sends each cohomology class $[b] \in H^n(X_p; A)$ onto the homotopy class of the K-map $f: X \to K(A,n)$ defined by $(f(x))(\delta^n)=b(x)$. Thus φ^n is an isomorphism.

If K is a simplicial point, then A is isomorphic to a constant stack with value group π and the theorem becomes the classical representation theorem of simplicial cohomology by $K(\pi, n)$.

3. Applications to sheaf cohomology. Let C be the category of abelian sheaves over a topological space $\mathfrak{X}$, let $\mathfrak{U}=\{U_\alpha\}$ be an open cover of X, and let $K=N(\mathfrak{U})$ be the nerve of $\mathfrak{U}$. For each sheaf $\mathfrak{F}$ in C, let $S\mathfrak{F}$ be the stack over K defined by $(S\mathfrak{F})(\sigma)=\Gamma(U_\sigma, \mathfrak{F})$, the local sections of $\mathfrak{F}$ over the support U_σ of σ. Then $S: C \to \mathfrak{A}b_K$ is a left exact functor. Note that $C^*(S\mathfrak{F})$ is the usual Čech complex of $\mathfrak{U}$ with coefficients in $\mathfrak{F}$. Consider left exact functors

$$C \xrightarrow{\ S\ } \mathfrak{A}b_K \xrightarrow{\ \Gamma_K\ } \mathfrak{A}b, \quad \Gamma_K = Hom(Z, -),$$

where $\Gamma_K S = \Gamma$ is the section functor of sheaves; we claim that

THEOREM 3.1. *There is a spectral sequence*

$$E_2^{p,q} = H^p(K; R^q S\mathfrak{F}) \Rightarrow H^n(\mathfrak{X}; \mathfrak{F}),$$

where R^qS is the right q-th derived functor of S.

Since C, $\mathfrak{A}b_K$ and $\mathfrak{A}b$ are abelian categories with enough injectives, the theorem follows from the

LEMMA. *S takes injective sheaves into Γ_K-acyclic stacks, i. e. $H^q(K; S\mathfrak{F})$ $=0$ for $p>0$ and $\mathfrak{F}$ an injective sheaf (cf. Theorem 2.1.).*

PROOF. Let $\mathfrak{G}^*$ be an injective resolution of $\mathfrak{F}$. Then the double complex

$C^*(S\mathfrak{S}^*)=\Sigma C^p(S\mathfrak{S}^q)$ gives rise to two spectral sequences of which the second one degenerates and the first one yields an isomorphism $H^p(\Gamma\mathfrak{S}^*)$ $\approx H^p(C^*(S\mathfrak{S}^*))$. If $\mathfrak{F}$ is injective, then $H^p(\Gamma\mathfrak{S}^*)=H^p(\mathfrak{X};\mathfrak{F})=0$ for $p>0$ and $H^p(C^*(S\mathfrak{S}^*))=H^p(K;S\mathfrak{F})$. Thus $H^p(K;S\mathfrak{F})\approx H^p(\mathfrak{X};\mathfrak{F})=0$ for $p>0$ and $\mathfrak{F}$ injective.

REMARKS. (1) $S\mathfrak{S}^*$ is a complex of stacks over K from which $R^qS\mathfrak{F}=H^q(S\mathfrak{S}^*)$ is computed. It can be shown by a routine computation that $R^qS\mathfrak{F}$ is isomorphic to $\mathcal{H}^q(\mathfrak{F})$ defined by $\mathcal{H}^q(\mathfrak{F})(\sigma)=H^q(\mathfrak{U}_\sigma,\mathfrak{F})$ in [2]. Thus the spectral sequence in the theorem is isomorphic to the spectral sequence $E_2^{p,q}=H^p(K;\mathcal{H}^q(\mathfrak{F}))$ of Leray. Consequently, one has the well-known

COROLLARY. *$H^n(\mathfrak{X};\mathfrak{F})\approx H^n(K;S\mathfrak{F})$ if $H^q(U_\sigma;\mathfrak{F})=0$, for $q\geq 1$ and every $\sigma\in K$.* (This and Leray theorem are proved in [2] using the Cech resolution $C^*(\mathfrak{U};\mathfrak{F})$ called the canonical resolution of $\mathfrak{F}$.)

(2) Let O be a sheaf of commutative rings with identities and let $O(\mathfrak{X})$ be the ring of (global) sections of O. Then for each O-module $\mathfrak{F}$, $S\mathfrak{F}$ is a stack of $O(\mathfrak{X})$-modules over K. The theory on $\mathfrak{Ab}_K$ carries over to a theory on $\mathfrak{M}_K$, the category of prestacks of $O(\mathfrak{X})$-modules over K. In particular, we have $H^*(K;-)\approx Ext_K^*(R,-)$ on $\mathfrak{M}_K$, where R is the constant stack with value $O(\mathfrak{X})$. This and the corollary above show that, for the O-module $\mathfrak{F}$,

THEOREM 3.2. *If $H^q(U_\sigma;\mathfrak{F})=0$ for $q\geq 1$ and every $\sigma\in K$, then*

$$H^*(\mathfrak{X};\mathfrak{F})\approx Ext_K^*(R,S\mathfrak{F}).$$

For example, let $(\mathfrak{X},O)$ be a scheme (resp. a complex analytic space) and let $\mathfrak{U}$ be an open cover of $\mathfrak{X}$ by affine varieties (resp. by Stein spaces). Then for a quasi-coherent (resp. coherent) O-module $\mathfrak{F}$, $H^n(\mathfrak{X};\mathfrak{F})\approx Ext_K^n(R,S\mathfrak{F})$ is, by abuse of language, the module of «K-coherent n-fold extensions» of the system of modules $\{\mathfrak{F}(U_\sigma)\mid\sigma\in K\}$ by the module $O(\mathfrak{X})$.

Finally we shall prove a representation theorem for sheaf cohomology. In the representation Theorem 2.2, if X_φ is the identity map $1:K\to K$, simply denote this by K, we have $[K,K(A,n)_\Theta]\approx H^n(K;A)$, i.e. for a stack A, *$H^n(K,A)$ is isomorphic to the group of homotopy classes of sections of $K(A,n)$.*

If precostacks are identified with their graphs, then $\mathfrak{A}b^K$ can be identified with a subcategory of C_K. Two precostack homomorphisms are *homotopic* if they are K-homotopic as morphisms in C_K. The group of homotopy classes of precostack homomorphisms from A to B is denoted by $Hom_K[A, B]$. If Z^K denotes the constant costack of integers, then $[K, E_\Theta] \approx Hom_K[Z^K, E]$ for $E_\Theta \in C_K$ in which E is a precostack. In particular, $[K, K(A, n)_\Theta] \approx Hom_K[Z^K, K(A, n)]$. We have the

LEMMA. *$H^n(K; A)$ is naturally isomorphic to $Hom_K[Z^K, K(A, n)]$ for stacks A over K.*

This and Theorem 3.1 show that

THEOREM 3.3. *There is a spectral sequence*

$$E_2^{p,q} = Hom_K[Z^K, K(R^q S\mathfrak{F}, p)] \Longrightarrow H^n(\mathfrak{X}; \mathfrak{F}).$$

COROLLARY *(representation of sheaf cohomology). If $H^q(U_\sigma; \mathfrak{F}) = 0$ for $q \geq 1$ and every $\sigma \in K$, then*

$$H^n(\mathfrak{X}; \mathfrak{F}) \approx Hom_K[Z^K, K(S\mathfrak{F}, n)].$$

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