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COSTACKS - THE SIMPLICIAL PARALLEL OF SHEAF THEORY

by YUH-CHING CHEN

INTRODUCTION

Parallel to sheaf theory, costack theory is concerned with the study of the homology theory of simplicial sets with general coefficient systems. A coefficient system on a simplicial set K with values in an abelian category $\mathfrak{A}$ is a functor from K to $\mathfrak{A}$ (K is a category of simplexes); it is called a precostack; it is a simplicial parallel of the notion of a presheaf on a topological space X. A costack on K is a «normalized» precostack, it is realized as a simplicial set over K; it is a simplicial «espace étalé».

The theory developed here is functorial; it implies that «all» homology theories are derived functors. Although the treatment is completely independent of Topology, it is however almost completely parallel to the usual sheaf theory, and many of the same theorems will be found in it though the proofs are usually quite different.

In Chapter I we define the direct image functor $f_{\#}$ and the inverse image functor $f^{\#}$ of a simplicial map f and show how this pair of adjoint functors supply projectives and injectives to a category of precostacks. It turns out that, if the coefficient category $\mathfrak{A}$ has enough projectives, then there are enough projective precostacks on K. Thus the usual homological algebra applies to categories of precostacks.

Chapter II begins with the definition of the homology of K with a general system of coefficients; it is defined in the same way as singular homology is defined. The simplicial parallel of the Vietoris-Begle theorem follows immediately from the definition. The homology theory so defined is proved to be the left derived functor of the 0-th homology functor H_o (and so H_o is called a cosection functor). This enables us to give a con-

ceptual proof of a theorem of Moore and Puppe on the equivalence of homo-
logy and homotopy theories for simplicial abelian groups. It also fol-
lows that the chain homology functor, denoted H^C, is the left derived
functor of H_o^C. Thus any homology theory that is defined by chain homo-
logy of complexes of modules is a derived functor. In section 7, spectral
sequences of a simplicial map are constructed. A theorem which is the
simplicial parallel of the Leray spectral sequence of a map is proved.

Chapter III starts with functorial relations between precostacks and
costacks (see Theorem 8.2). It is proved that a projective costack is pro-
jective as a precostack and that an exact sequence of costacks is exact as
a sequence of precostacks. It also proves that the associated costack of
a projective precostack is projective. Thus, when homology is concerned,
we choose to regard costacks just as precostacks and make no attempt to
set up a separated homology theory for costacks. Finally, we define relative
costacks and precostacks for computing relative homology.

0 . Categories, functors, and modules.

Let $\mathcal{C}$ be a category and let A, B be objects of $\mathcal{C}$; the set of morphisms in $\mathcal{C}$ with source A and target B is denoted by $\mathcal{C}(A, B)$. If the class of objects of $\mathcal{C}$ is a set, then we say that $\mathcal{C}$ is a *small* category. The following symbols are fixed throughout this paper :

$\mathcal{A}$, an abelian category;

$\mathcal{A}b$, the category of abelian groups;

Λ, a commutative ring with unit;

$\mathcal{M}$, the category of unitary Λ - modules;

$\partial\mathcal{A}$, the category of chain complexes in $\mathcal{A}$ of the form

$$... \to X_n \to X_{n-1} \to ... \to X_o \to A \to 0, \text{ or } X_* \to A.$$

Let $\mathcal{C}$ be a small category and let $\mathcal{A}^{\mathcal{C}}$ be the category of functors (covariant) from $\mathcal{C}$ to $\mathcal{A}$; if morphisms are natural transformations of functors, then $\mathcal{A}^{\mathcal{C}}$ is an abelian category; exactness in $\mathcal{A}^{\mathcal{C}}$ is fiberwise, i.e. a sequence of functors

$$... \to F_{n+1} \to \dot{F}_n \to F_{n-1} \to ...$$

is exact if and only if, for each object x in $\mathcal{C}$, the sequence

$$... \to F_{n+1}(x) \to F_n(x) \to F_{n-1}(x) \to ...$$

is exact in $\mathcal{A}$. Products, coproducts (sums), and limits in $\mathcal{A}^{\mathcal{C}}$ are obtained from that of $\mathcal{A}$ by fiberwise constructions, e.g.

$$(\Pi F_i)(x) = \Pi F_i(x), \quad x \in \mathcal{C}.$$

By [6] the category $\mathcal{A}$ is said to be $AB(3)$ (resp. $AB^*(3)$) if it has arbitrary coproducts (resp. products). If the coproduct (resp. product) functor is exact on $\mathcal{A}$, then $\mathcal{A}$ is said to be $AB(4)$ (resp. $AB^*(4)$). An $AB(3)$ category $\mathcal{A}$ is said to be $AB(5)$ if the direct limit of any directed system of objects of $\mathcal{A}$ exists and if the direct limit functor, *lim*, is exact on $\mathcal{A}$. For example the category of modules $\mathcal{M}$ is $AB(3)$, $AB^*(3)$, $AB(4)$, $AB^*(4)$ and $AB(5)$. Since products, coproducts, and limits of

the functor category $\mathcal{A}^{\mathcal{C}}$ are fiberwise, we have

LEMMA 0.1. $\mathcal{A}^{\mathcal{C}}$ *inherits the following properties of* $\mathcal{A}$: *AB(3), AB*(3),*
AB(4), AB*(4), AB(5).

An object P is a *projective* (object) of $\mathcal{A}$ if the functor $\mathcal{A}(P,-)$:
$\mathcal{A} \rightarrow \mathcal{A}b$ is exact; P is *small* if, for any coproduct $\Sigma_i P$ of copies of P,

$$\mathcal{A}(\Sigma_i P, A) \approx \Sigma_i \mathcal{A}(P, A).$$

A *generator* of $\mathcal{A}$ is an object U of $\mathcal{A}$ such that $\mathcal{A}(U, A) \neq 0$ for any
$A \neq 0$ in $\mathcal{A}$. Injectives and cogenerators are defined dually. For example
Λ is a projective generator of $\mathfrak{M}$; Q/Z (rationals *mod 1*) is an injective
cogenerator of $\mathcal{A}b$. $\mathcal{A}$ *has enough projectives* (resp. *injectives*) if, for
each object A, there is a projective P (resp. injective E) and an exact
sequence $P \rightarrow A \rightarrow 0$ (resp. $0 \rightarrow A \rightarrow E$).

LEMMA 0.2 $[6, Theorem\ 1.10.1]$. *An AB(5) abelian category with a gene-*
rator has enough injectives.

For example $\mathfrak{M}$ has enough injectives. More precisely, for each
module A, there is an injective envelope E of A [8, p. 103]. It is well
known that $\mathfrak{M}$ has enough projectives : let P be the free module generated
by the *set A;* then one gets an exact sequence $P \rightarrow A \rightarrow 0$.

The notion of adjoint functors plays an important role in this paper;
here are a few preliminary remarks. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories, a functor
$F : \mathcal{C} \rightarrow \mathcal{C}'$ is a *left adjoint functor* of the functor $G : \mathcal{C}' \rightarrow \mathcal{C}$ if $\mathcal{C}'(Fx, x')$
is naturally equivalent to $\mathcal{C}(x, Gx')$, $x \in \mathcal{C}$, $x' \in \mathcal{C}'$. The left adjoint of
G is unique up to an equivalence of functors; we shall say that F is *the*
left adjoint of G and G is *the* right adjoint of F and write $F \dashv G : (\mathcal{C}, \mathcal{C}')$.
Left (resp. right) adjoint functors preserve epimorphisms, cokernels, and
right limits (resp. monomorphisms, kernels, and left limits).

LEMMA 0.3. *Let* $\mathcal{A}$ *and* $\mathcal{A}'$ *be abelian categories, and* $F \dashv G : (\mathcal{A}, \mathcal{A}')$. *If*
G *is exact (resp.* F *is exact) then* F *preserves projectives (resp.* G *pre-*
serves injectives).

PROOF. Let P be a projective in $\mathcal{A}$; then $\mathcal{A}(P,-)$ is exact. Since
$\mathcal{A}'(FP,-) \approx \mathcal{A}(P, G-)$ and G is exact, $\mathcal{A}'(FP,-)$ is exact and FP

is projective.

For $F \dashv G : (\mathcal{C}, \mathcal{C}')$, consider the natural equivalences

(0.1) $\mathcal{C}'(Fx, Fx) \approx \mathcal{C}(x, GFx)$ and $\mathcal{C}(Gx', Gx') \approx \mathcal{C}'(FGx', x')$,

the morphisms $x \to GFx$ and $FGx' \to x'$ corresponding to $1 : Fx$ and $1 : Gx'$ give natural transformations

(0.2) $\theta : 1 \to GF$ and $\rho : FG \to 1$.

Let $\mathcal{C}'$ be a *faithful* subcategory of $\mathcal{C}$, i.e. $\mathcal{C}'(x'_1, x'_2) = \mathcal{C}(x'_1, x'_2)$, and $J : \mathcal{C}' \to \mathcal{C}$ be the inclusion functor of $\mathcal{C}' \subset \mathcal{C}$. If J has a left (resp. right) adjoint functor R, then $\mathcal{C}'$ is *coreflective* (resp. *reflective)* in $\mathcal{C}$; R is the *coreflector* (resp. *reflector)* and $R(x)$ is the coreflection (resp. *reflection)* of x in $\mathcal{C}'$. Thus, for any x' in $\mathcal{C}'$ and morphism $f : x \to x'$ (resp. $f' : x' \to x$), there is a unique morphism $g : R(x) \to x'$ (resp. $g' : x' \to R(x)$) such that $g\theta_x = f$ (resp. $\rho_x g' = f'$). For example, the category of abelian sheaves on a space X is coreflective in the category of abelian presheaves on X; the associated sheaf functor is the coreflector. For other informations on coreflective and reflective subcategories, see Freyd [5]; notice that our coreflections are the reflections of [5].

CHAPTER I

PRECOSTACKS

1. Definition and examples of precostacks.

Let $K = \bigcup_{n \geq 0} K_n$ be a simplicial set (a semi-simplicial complex);
K_n is the set of n-simplexes of K, the face operators are $d^i : K_n \to K_{n-1}$,
$0 \leq i \leq n$, the degeneracy operators are $s^j : K_n \to K_{n+1}$, $0 \leq j \leq n$. K is
a small category : objects are simplexes; morphisms are determined by d^i
and s^j in the covariant manner, e.g. if the i-th face of $x_n \in K_n$ is $x_{n-1} \in$
K_{n-1}, then there is a morphism $d^i : x_n \to x_{n-1}$. A *precostack* over K, with
values in $\mathcal{C}$, is a functor (covariant) $A : K \to \mathcal{C}$. The category of precos-
tacks of modules over K is the functor category $\mathfrak{M}^K$. By lemma 0.2 we have

PROPOSITION 1.1. $\mathfrak{M}^K$ *is abelian,* $AB(3)$, $AB^*(3)$, $AB(4)$, $AB^*(4)$,
and $AB(5)$.

Let $\mathcal{C}_K$ be the category of simplicial sets over K : objects are
simplicial maps $L \to K$ with target K; morphisms are commutative dia-
grams

$$L \longrightarrow L'$$
$$\searrow \quad \swarrow$$
$$K$$

If $\mathfrak{M}_K$ denotes the subcategory of Λ-module objects of $\mathcal{C}_K$, then $\mathfrak{M}_K \approx \mathfrak{M}^K$,
see [2]. The «geometrical realization» of a precostack of modules over K
is a simplicial set over K with fibers over simplexes of Λ-modules; face
operators and degeneracy operators are Λ-homomorphisms on fibers.

EXAMPLE 1.1. Let $<0>$ be a simplicial point : a collapsed simplicial set
with one simplex in each dimension. Then a simplicial set is the geome-
trical realization of a precostack of sets over $<0>$. The category of sim-
plicial Λ-modules is identified with the category of precostacks $\mathfrak{M}<0>$.

EXAMPLE 1.2. Let X be a topological space and let $\Delta(X)$ be the total
singular complex of X (a simplicial set), then a local system of coeffi-
cients on X gives rise to a precostack over $\Delta(X)$.

EXAMPLE 1.3. Let $E \to X$ be a sheaf of modules over X; then $S(E) \to S(X)$

is (the geometrical realization of) a precostack of modules.

2. Direct and inverse images.

Let $f : K \to L$ be a simplicial map; f is a functor when K and L are regarded as categories of simplexes. Let B be a precostack of modules over L, i.e. $B \in \mathfrak{M}^L$; then the composite functor Bf is a precostack of modules over K. Bf is called the *inverse image* of B under f and is denoted by $f^{\#}B$. f induces a functor $f^{\#} : \mathfrak{M}^L \to \mathfrak{M}^K$ on precostacks. Also, f induces a functor $f_{\#} : \mathfrak{M}^K \to \mathfrak{M}^L$ in the following manner. For A in $\mathfrak{M}^K$, let $f_{\#}A$ in $\mathfrak{M}^L$ be the precostack over L that assigns to each $y \in L$ the module $\Sigma_{f(x) = y} A(x)$; the values of $f_{\#}A$ acting on morphisms of L are obtained in the natural way by the universal property of coproducts. For example for a morphism $d : y \to dy$ of L given by a face operator d, $(f_{\#}A)(d)$ is the unique homomorphism that renders the diagram

$$
\begin{array}{ccc}
A(x) & \xrightarrow{\ \ A(d)\ \ } & A(dx) \\
{\scriptstyle inj}\downarrow & & \downarrow{\scriptstyle inj} \\
\Sigma_{f(x) = y} A(x) & \xrightarrow{\ \ (f_{\#}A)(d)\ \ } & \Sigma_{f(x) = dy_1} A(x)
\end{array}
$$

commutative. $f_{\#}A$ is called the *direct image* of A under f.

For simplicial maps $f : K \to L$ and $g : L \to M$ we have $(gf)_{\#} = g_{\#}f_{\#}$ and $(gf)^{\#} = f^{\#}g^{\#}$.

LEMMA 2.1. $f_{\#}$ *and* $f^{\#}$ *are exact functors, i.e. they take exact sequences to exact sequences.*

This follows from the definition of $f_{\#}$ and $f^{\#}$ and the fact that categories of precostacks of modules are $AB(4)$.

LEMMA 2.2. $f_{\#}$ *is the left adjoint of* $f^{\#}$, *i.e. there is a natural isomorphism*

(2.1) $$\mathfrak{M}^K(A, f^{\#}B) \to \mathfrak{M}^L(f_{\#}A, B)$$

of abelian groups.

PROOF. An element of $\mathfrak{M}^K(A, f^{\#}B)$ is a natural transformation

$$\varphi = \{ \varphi_x : x \in K \} : A \to f^{\#}B = Bf$$

of functors. For each such φ there corresponds a natural transformation $\Psi = \{\Psi_y : y \in L\} : f_{\#}A \to B$ defined by the universal mapping diagram of $\Sigma_{f(x) = y} A(x)$:

(2.2)

$$A(x) \xrightarrow{\ i_x\ } \Sigma_{f(x) = y} A(x) = (f_{\#}A)(y)$$

with maps φ_x, Ψ_y, Ψ_y down to

$$(f^{\#}B)(x) = B(y),$$

$\varphi_x = \Psi_y i_x$. This correspondence gives rise to an isomorphism.

PROPOSITION 2.3. $f_{\#}$ *preserves projectives*, $f^{\#}$ *preserves injectives.*

This follows from Lemmas 0.3, 2.1, and 2.2.

3. Projectives and injectives of $\mathfrak{M}^K$.

Let $<n>$ be the simplicial set of the standard n-simplex and let δ be its only non degenerate n-simplex; $<0>$ is a simplicial point as indicated before; the simplicial standard 1-simplex $<1>$ plays in simplicial homotopy theory a similar role as the unit interval I plays in the homotopy theory of topological spaces. Given a simplicial set $K = \bigcup_{n \geq 0} K_n$, for each $x \in K_n$ the correspondence $\delta \to x$ determines uniquely a simplicial map $x^{\delta} : <n> \to K$ [8, p. 237]. This shows that, if Λ^n is the constant precostack over $<n>$ with value Λ, then for any A in $\mathfrak{M}^{<n>}$

(3.1) $\mathfrak{M}^{<n>}(\Lambda^n, A) \approx \mathfrak{M}(\Lambda, A(\delta)) \approx A(\delta)$.

THEOREM 3.1. $\mathfrak{M}^K$ *has enough projectives and injectives.*

PROOF. Let $U = \Sigma_{x \in K}(x_{\#}^{\delta}\Lambda^n)$, $n = dim\ x$, then U is a projective generator of $\mathfrak{M}^K$. As exactness in $\mathfrak{M}^K$ is fiberwise, it follows from (3.1) that $\mathfrak{M}^{<n>}(\Lambda^n, -)$ is exact and so Λ^n is projective in $\mathfrak{M}^{<n>}$; $x_{\#}^{\delta}\Lambda^n$ is projective in $\mathfrak{M}^K$ since $x_{\#}^{\delta}$ preserves projectives. Now U as a coproduct of projectives is itself projective. U is a generator since, for $A \neq 0$ in $\mathfrak{M}^K$,

(3.2) $\mathfrak{M}^K(U, A) = \mathfrak{M}^K(\Sigma x_{\#}^{\delta}\Lambda^n, A) \approx \Pi \mathfrak{M}^K(x_{\#}^{\delta}\Lambda^n, A)$

$$\approx \Pi \mathfrak{M}^{<n>}(\Lambda^n, x^{\delta\#}A) \approx \Pi(x^{\delta\#}A)(\delta)$$

$$\approx \Pi A(x^{\delta}(\delta)) = \Pi_{x \in K} A(x) \neq 0.$$

$\mathfrak{M}^K$ has enough projectives since each A in $\mathfrak{M}^K$ is a quotient of $\Sigma_I U$,

the index set is the *set* $\mathfrak{M}^K(U, A) \approx \prod_{x \in K} A(x)$. Finally, by Lemmas 0.1 and 0.2, $\mathfrak{M}^K$ has enough injectives since it is $AB(5)$ and has a generator U.

   This theorem tells us that there is a homology theory on $\mathfrak{M}^K$ defined by derived functors and computed by resolutions. In particular, when $K = \, = <0>$ is a simplicial point, $0 : <n> \to <0>$ is the only simplicial map. $0_\# : \mathfrak{M}^{<n>} \to \mathfrak{M}^{<0>}$ maps Λ^n onto the n-dimensional *model* simplicial module also denoted by Λ^n (we adopt the term «model» from [8, p. 237]). Thus in $\mathfrak{M}^{<0>}$ the coproduct of model simplicial modules of all dimensions, $U = \Sigma_{n \geq 0} \Lambda^n$, is a projective generator. We shall show that the classical homology theory of simplicial modules defined by chain modules is a derived functor.

CHAPTER II

HOMOLOGY

4. Homology of precostacks.

For each A in $\mathfrak{M}^K$, let ∂A be the chain module of A: the module
of n-chains is $\Sigma_x A(x)$, where x ranges over all n-simplexes of K, the
boundary operator ∂_n is the alternating sum of the face operators d_n^i. The
homology of K with *coefficients* in A is by definition the chain homology
of ∂A and is denoted by $H(K;A)$, or sometimes by HA. Since the cor-
respondence $A \to \partial A$ defines an (additive) exact functor, $H = \{H_n : n \geq 0\}$
is a homological functor (a connected sequence of functors in the sense
of $[1, p. 43]$).

Let $f : K \to L$ be a simplicial map, it is easily seen that the chain
modules

(4.1) $$\partial A = \partial(f_\# A), \quad A \text{ in } \mathfrak{M}^K.$$

We have the dual version of the Vietoris-Begle Theorem :

LEMME 4.1. $H(K;A) \approx H(L;f_\# A)$.

We shall now prove the following main theorem on the homology of
precostacks.

THEOREM 4.2. $H = H(K;-)$ is «the» (unique up to natural equivalence of
functors) left derived functor of $H_0(K;-)$, i.e. $H(K;A)$ or HA is the
chain homology of the chain module obtained by applying the functor $H_0(K;-)$
to a projective resolution of A in $\mathfrak{M}^K$.

PROOF. Since $H_0(K;-)$ is right exact, by the isomorphism criterion of
Cartan and Eilenberg $[1, p. 87]$ it suffices to show that $H_q(K;P) = 0$
for $q > 0$ and P projective. Since a projective P is a summand of a co-
product ΣU of copies of U and since $H(K;-)$ preserves coproducts, it
suffices to show that $H_q(K;U) = 0$ for $q > 0$. Now

(4.2) $$\partial U = \partial(\Sigma_{x \in K} x_\#^\delta \Lambda^n) = \Sigma_{x \in K} \partial(x_\#^\delta \Lambda^n) = \Sigma_{n \geq 0} \partial \Lambda^n$$

by Lemma 4.1; $H_q(K;U)$ is isomorphic to the coproduct of the q-th homo-

logy group of $\partial\Lambda^n$, $n = 0, 1, 2, \ldots$. Since the $\partial\Lambda^n$ are acyclic [8, p. 238], $H_q(K; U) = 0$ for $q > 0$ and the theorem is proved.

$H_o(K; A)$ is the *cosection* of A over K (dual to the notion of *section* in sheaf theory). Since H_o is right exact and preserves coproducts, it is a «tensor product» functor; its left derived functor H is therefore a «torsion product» functor. Sometimes we write $H_n(K; A) = Tor_n^K(A)$ for $K \neq <0>$, and $H_n(<0>; A) = Tor_n A$ (c.f. [10]).

Applying this theorem to the examples in Section 1, we see that the classical homology theory of simplicial modules, the singular theory, and the homology with local coefficients are all derived functors; they can be computed by resolutions.

Finally let $f : K \to L$ be a simplicial map and let B be a precostack in $\mathfrak{M}^L$; f induces a homomorphism

(4.3) $f_* : H(K; f^{\#}B) \to H(L; B)$

in the following way. Let P_* be a projective resolution of $f^{\#}B$; then $f_{\#}P_*$ is a projective resolution of $f_{\#}f^{\#}B$, since $f_{\#}$ is exact and preserves projectives. By «the comparaison theorem» the map $\rho : f_{\#}f^{\#}B \to B$ of (0.2) lifts uniquely (up to homotopy) to a chain map from $f_{\#}P_*$ to a projective resolution of B. This chain map induces the homomorphism f_* of (4.3). f_* can also be obtained from a naturally defined chain map from the chain module $\partial(f^{\#}B)$ to the chain module ∂B; we leave the details to the reader.

5. A remark on simplicial homotopy.

A simplicial set $K = \cup_{n \geq 0} K_n$ is a Kan complex if, for each set of n $(n-1)$-simplexes

$$\sigma^o, \sigma^1, \ldots, \sigma^{k-1}, \sigma^{k+1}, \ldots, \sigma^n, \quad 0 \leq k \leq n,$$

with

$$d^{j-1}\sigma^i = d^i\sigma^j \text{ for } 1 \leq j, \ i \neq k \neq j,$$

there is a n-simplex σ with $d^i\sigma = \sigma^i$ for $i \neq k$. A Kan complex with a fixed 0-simplex e as its «base point» is called a pointed Kan complex. For example, a simplicial group is a pointed Kan complex with base point the

identity element of the 0-dimension group $[9]$. Since we are interested primarily in simplicial abelian groups, we shall restrict the general homotopy theory on pointed Kan complexes to the category of simplicial abelian groups $\mathcal{A}b^{<0>}$.

Let $\pi = \{\pi_q ; q \geq 0\}$ be a homotopy theory on $\mathcal{A}b^{<0>}$ (π is characterized by axioms analogue to that of the homology theory of Eilenberg and Steenrod, see $[7, \text{Chap. II}]$); then each π_q is an additive functor from $\mathcal{A}b^{<0>}$ to $\mathcal{A}b$. The q-th homotopy group $\pi_q(K, e)$ is defined, according to J.C. Moore, as the quotient group

$$\bigcap_{i \leq q} Kernel\ d_q^i / d_{q+1}^{q+1} (\bigcap_{j \leq q+1} Kernel\ d_{q+1}^j),$$

see $[7, \text{Chap. IV}]$. We shall give a conceptual proof of the following theorem of Moore and Puppe.

PROPOSITION 5.1. *The homotopy groups and homology groups of a simplicial abelian group are naturally isomorphic.*

PROOF. Let A be a simplicial abelian group. Since $H(<0>; A)$ is a left derived functor of $H_o(<0>, A)$ and since $H_o(<0>; A)$ is equivalent to $\pi_o(A)$, we shall show that π is a left derived functor of π_o on $\mathcal{A}b^{<0>}$. Since every short exact sequence of simplicial abelian groups is a fiber sequence of Kan complexes, there is associated to it a homotopy sequence of homotopy groups connected by connecting homomorphisms. The correspondence is functorial and so π is a homological functor. To complete the proof we shall show that $\pi_q(P, e) = 0$ for $q > 0$ and P projective in $\mathcal{A}b^{<0>}$. As in the proof of Theorem 4.1, $\pi_q(P, e)$ is a summand of $\pi_q(\Sigma U, e) = \pi_q(\Sigma\Sigma Z^n, e)$. But $\pi_q(\Sigma\Sigma Z^n, e) \approx \Sigma\Sigma \pi_q(Z^n, e)$ (this follows from the definition of π_q) and Z^n is «contractible» for each $n \geq 0$, $\pi_q(\Sigma U, e) = 0$ for $q > 0$, so is $\pi_q(P, e)$.

The original proof of the theorem is a consequence of the equivalence between simplicial modules and chain modules; we shall establish the equivalence (without proof) and some of its consequences for use in later sections.

For A in $\mathcal{M}^{<0>}$, let CA be the normalized chain module of A;

CA is the quotient of ∂A by the chain submodule generated by the degenerate simplexes of A [8, p. 236]; then the correspondence $A \to CA$ defines a functor $C : \mathfrak{M}^{<0>} \to \partial\mathfrak{M}$, where $\partial\mathfrak{M}$ is the category of left (positive) chain modules.

LEMMA 5.2. *The functor C is a homotopy preserving equivalence of categories (C carries simplicial homotopy to chain homotopy).*

For details see, e.g., [3].

Let H^C denote the chain homology functor; then by the normalization theorem of Eilenberg -Mac Lane we have $H^C(\partial A) \approx H^C(CA)$. Thus Lemma 5.2 shows that $\mathfrak{M}^{<0>}$ is identified with $\partial\mathfrak{M}$ when homology is concerned. As a direct consequence of Theorem 3.1 and Lemma 5.2 we have :

LEMMA 5.3. $\partial\mathfrak{M}$ *has enough projectives and injectives; the chain homology functor H^C is the left derived functor of H^C_o .*

This is true for all $\partial\mathfrak{A}$, when $\mathfrak{A}$ has enough projectives and injectives. In such a case, one constructs projectives (resp. injectives) as a coproduct (resp. product) from some «basic» projectives (resp. injectives).

6. Hyperhomology and derived spectral functors.

Let $\partial\mathfrak{M}^K$ be the category of complexes of precostacks of modules over K of the form

$$\ldots \to X_n \overset{\partial}{\to} X_{n-1} \to \ldots \to X_o \overset{\varepsilon}{\to} A \to 0, \text{ or } X_* \to A;$$

then $\partial\mathfrak{M}^K$ has enough projectives and injectives. X_* is projective in $\partial\mathfrak{M}^K$ if and only if it is a coproduct of the «basic» projectives of the form

$$\ldots \to 0 \to P \overset{1}{\to} P \to 0 \to \ldots ,$$

with a projective precostack P in positions n and $n-1$ (see, e.g., [4]). It is easy to see that a «basic» projective is projective and that X_* is projective if and only if

(1) each X_n is projective in $\mathfrak{M}^K$, and

(2) X_* is *split exact*, i.e. $H^C_q(X_*) = 0$ for $q \neq 0$ and ∂X_n is a summand of X_{n-1} (recall that H^C denotes the chain homology functor). Thus for any additive functor $F : \mathfrak{M}^K \to \mathfrak{A}$ we have $H^C_q(FX_*) \approx FH^C_q(X_*) = 0$

for $q \neq 0$, if X_* is projective (here FX_* is the complex obtained by applying F to X_* termwise).

REMARKS. a) For A_* in $\partial \mathfrak{M}^L$, let X_{**} be a left complex over A_* ; then there results a bicomplex X of the form

$X_{p,q} \in \mathfrak{M}^K$, and homology precostacks $H_p^I(X_{*,q})$ and $H_q^{II}(X_{p,*})$. The total homology precostack of X is denoted by $H^{tot}(X)$. X_{**} is called a *strong projective resolution* of A_* if it is a «projective resolution» in the sense of Cartan and Eilenberg, i.e. if, for each $p \geq 0$, $H_p^I(X)$ is a projective resolution of $H_p^C(A_*)$ in $\mathfrak{M}^L$, and if $B_p^I(X)$, the p -boundaries of the first boundary operation, is a projective resolution of $B_p(A_*)$, see [1, p. 363].

b) Let $F : \mathfrak{M}^K \to \mathfrak{A}$ be an additive functor, let

$$(6.1) \qquad E^I(FX) = H_p^I H_q^{II}(FX), \quad E^{II}(FX) = H_p^{II} H_q^I(FX)$$

be the spectral homology sequences of FX . If X_{**} is a strong projective resolution of A_* , then (6.1) becomes

$$(6.2) \quad E^I(FA_*) = H_p^C((\mathfrak{L}_q F)A_*), \quad E^{II}(FA_*) = (\mathfrak{L}_p F)(H_q^C A_*),$$

where $\mathfrak{L}_n F$ is the n -th left derived functor of F . Both spectral sequences of (6.2) converge to the hyperhomology $(HF)(A_*) = H^{tot}(FX)$; HF is the *hyperhomology functor* of F .

We shall show that the hyperhomology functor $HF : \partial \mathfrak{M}^K \to \mathfrak{A}$ is

a derived functor.

LEMMA 6.1. *Let* P_{**} *be a projective resolution of* A_* *and let* X_{**} *be a strong projective resolution, then*

$$(6.3) \qquad H^{tot}(FP) \approx H^{tot}(FX) = (HF)(A_*).$$

Indeed, the identity chain map of A_* lifts uniquely (up to homotopy) to a map $P \to X$ of bicomplexes. This map induces a morphism

$$E^I(FP) \to E^I(FX)$$

of spectral sequences, which is an isomorphism since the two spectral sequences are isomorphic to $H_p^C((\mathcal{L}_q F)A_*)$.

THEOREM 6.2. *The hyperhomology functor HF is the left derived functor of the composite functor* $FH_o^C : \partial \mathfrak{M}^K \to \mathfrak{A}$, *i.e.* $HF \approx \mathcal{L}(FH_o^C)$.

PROOF. Let P_{**} be a projective resolution of A_*; the first spectral sequence $E^I(FP)$ converges to $H^{tot}(FP) \approx (HF)(A_*)$ by (6.3). Compute the second spectral sequence

$$E^{II}(FP) = H_p^{II} H_q^I(FP);$$

since each column of P is a projective in $\partial \mathfrak{M}^K$, it is split exact and so $H_q^I(FP) \approx F(H_q^I P) = 0$ for $q \neq 0$. Thus the second spectral sequence collapses to

$$H_p^{II}(FH_o^C P) = \mathcal{L}_p(FH_o^C)(A_*),$$

and yields

$$\mathcal{L}(FH_o^C)(A_*) \approx H^{tot}(FP) = (HF)(A_*).$$

This completes the proof.

Thus for obtaining the spectral sequences (6.2) one may use either a projective resolution of A_* or a strong projective resolution of A_*; in either case the hyperhomology is $\mathcal{L}(FH_o^C)(A_*)$.

The argument in this section remains valid when $\mathfrak{M}^K$ is replaced by an abelian category with enough projectives. For example, let A_* be a complex over A in $\mathfrak{M}^K$; then ∂A_* (∂ applies to A_* termwise) is a complex over ∂A in $\partial \mathfrak{M}$ and yields a bicomplex of Λ-modules. (6.2) becomes

(6.4) $\ E^I(F(\partial A)) = H_p^C((\mathfrak{L}_q F)(\partial A)), \quad E^{II}(F(\partial A)) = (\mathfrak{L}_p F)(H_q A);$

both converge to $\mathfrak{L}(FH_o)A$. If A_* is a projective resolution of A, then ∂A_* is a projective resolution of ∂A in $\partial \mathfrak{M}$. In this case the second spectral sequence collapses and yields

$$H_p^C(H_q^C F(\partial A_*)) = \mathfrak{L}(FH_o)A.$$

Moreover, if F is the identity functor of $\mathfrak{M}$, then both spectral sequences collapse and yield $H(K;A) = H(A) \approx (\mathfrak{L} H_o)(A)$ of Theorem 4.2.

7. Spectral sequences of a simplicial map.

Let A_* be a positive complex of precostacks of modules over K, $A_* \in \partial \mathfrak{M}^K$, and let $F = H_o : \mathfrak{M}^K \to \mathfrak{M}$ be the cosection functor over K. Then, since $H \approx \mathfrak{L} H_o$, the spectral sequences of the previous section become

(7.1) $\quad E^I(H_o A_*) = H_p^C(H_q A_*), \quad E^{II}(H_o A_*) = H_p(K; H_q^C A_*),$

where $H_q A_*$ is obtained by applying $H_q = \mathfrak{L}_q H_o$ to A_* termwise. There are two interesting cases corresponding to the degeneracy of one or the other of the spectral sequences.

CASE 1. Assume that $H_p^C(H_q A_*) = 0$ for $q > 0$, e.g. when each term of A_* is an acyclic precostack. Then the first spectral sequence collapses and yields

$$E_{p,o}^I(H_o A_*) = H_p^C(H_o A_*).$$

There is a spectral sequence (the second spectral sequence).

$$E_{p,q}^2 = H_p(K; H_q^C A_*) \Longrightarrow H_{p+q}^C(H_o A_*).$$

CASE 2. Assume that $H_q^C(A_*) = 0$ for $q > 0$, e.g. when A_* is a resolution of A. Then the second spectral sequence collapses and yields

$$E_{p,o}^{II}(H_o A_*) = H_p(K;A), \quad A = H_o^C A_*.$$

There is a spectral sequence (the first spectral sequence)

$$E_{p,q}^2 = H_p^C(H_q A_*) \Longrightarrow H_{p+q}(K;A).$$

As an application of case 1 we shall construct a simplicial dual of the Leray spectral sequence of a map.

Let $f : K \to L$ be a simplicial map. For each A in $\mathfrak{M}^K$, let $f_o A$ be an object in $\mathfrak{M}^L$ defined as follows: For $y \in L$, let $A_y(x)$ be the constant precostack over K with value the module $\Sigma_{f(x) = y} A(x)$. Let

$$(f_o A)(y) = H_o(K; A_y(x))$$

be the 0-th (classical) homology module of K with coefficients in $\Sigma_{f(x) = y} A(x)$. Then, since f, H_o, A are functors, $f_o A$ is made a functor from L to $\mathfrak{M}$ in the natural way; $u : y \to uy$ in L determines a homomorphism

$$u : \Sigma_{f(x) = y} A(x) \to \Sigma_{f(x) = uy} A(x)$$

which extends to a morphism $u : A_y(x) \to A_{uy}(x)$, which induces the morphism $u : (f_o A)(y) \to (f_o A)(uy)$. It is easily seen that the correspondence $A \to f_o A$ defines a functor $f_o : \mathfrak{M}^K \to \mathfrak{M}^L$, called the *direct image functor* (with respect to the cosection functor H_o). Since H_o is right exact, we have :

PROPOSITION 7.1. f_o *is right exact.*

We shall show that $f_o A$ is acyclic whenever A is projective.

PROPOSITION 7.2. *If A is projective in* $\mathfrak{M}^K$, *then* $H_q(L; f_o A) = 0$ *for* $q > 0$.

PROOF. As usual let $\partial(f_o A)$ be the chain module of $f_o A$,

$$\ldots \to \partial_{n+1}(f_o A) \to \partial_n(f_o A) \to \partial_{n-1}(f_o A) \to \ldots ;$$

then $\partial_n(f_o A) = \Sigma_y H_o(K; A_y(x))$, where y ranges over all n-simplexes of L. Since H_o preserves coproducts, a simple computation shows that

$$\partial_n(f_o A) \approx H_o(K; \partial_n A) = H_o(\partial_n A / K),$$

where $\partial_n A / K$ is the constant precostack over K with value the n-chain module $\partial_n A$ of ∂A. Thus ∂A is a chain of coefficients that gives the chain of constant precostacks

$$(7.2) \qquad \ldots \to \partial_{n+1} A / K \to \partial_n A / K \to \partial_{n-1} A / K \to \ldots$$

from which $\partial(f_o A)$ is obtained by applying the cosection functor H_o termwise.

If A is projective, so is ∂A. Thus each $\partial_n A$ is a projective module and ∂A is split exact, see the first paragraph of section 6. This

gives the split exactness of (7.2), therefore $\partial(f_o A)$ is exact. This proves the proposition.

For A in $\mathfrak{M}^K$, the cosection of $f_o A$ is computed in the following proposition.

PROPOSITION 7.3. $H_o(f_o A) \approx H_o(K; H_o A)$, the (classical) 0-th homology module of K with coefficients in the module $H_o A$.

PROOF. Since H_o is right exact and preserves coproducts, we have the commutative diagram with exact rows and columns

$$
\begin{array}{ccccccc}
\Sigma_1 \partial_1 A & \longrightarrow & \Sigma_1 \partial_o A & \longrightarrow & \Sigma_1 H_o A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Sigma_o \partial_1 A & \longrightarrow & \Sigma_o \partial_o A & \longrightarrow & \Sigma_o H_o A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_o(\partial_1 A/K) & \longrightarrow & H_o(\partial_o A/K) & \longrightarrow & H_o(K; H_o A) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

where $\Sigma_i \partial_j A$ is the coproduct indexed by K_i of copies of $\partial_j A$, $i, j = 0, 1$. The last row of the diagram shows that

$$(7.3) \qquad\qquad H_o(K; H_o A) \approx H_o(f_o A).$$

The following theorem is a simplicial dual of the Leray spectral sequence of a map.

THEOREM 7.4. Let $f: K \to L$ be a simplicial map, let A be in $\mathfrak{M}^K$, and let P_* be a projective resolution of A. If K is connected (any two 0-simplexes can be joined by a chain of 1-simplexes), then there is a spectral sequence

$$E^2_{p,q} = H_p(L; H^C_q(f_o P_*)) \Longrightarrow H_{p+q}(K; A),$$

where $f_o P_*$ is obtained by applying f_o to P_* termwise.

PROOF. Since each term of P_* is projective, each term of $f_o P_*$ is acyclic by proposition 7.2. Case 1 applies and yields a spectral sequence

$$(7.4) \qquad E^2_{p,q} = H_p(L; H^C_q(f_o P_*)) \Longrightarrow H^C_{p+q}(H_o(f_o P_*)).$$

Since K is connected, (7.3) yields $H_o(f_o P_*) \approx H_o P_*$. The right hand side of (7.4) becomes $H_{p+q}(A) = H_{p+q}(K;A)$. This proves the theorem.

Again $f: K \to L$ is a simplicial map. This time we consider the inverse image functor $f^{\#}: \mathfrak{M}^L \to \mathfrak{M}^K$. Let B be in $\mathfrak{M}^L$ and let X_* be a projective resolution of B. Since $f^{\#}$ is exact, $f^{\#}X_*$ is a resolution of $f^{\#}B$. Case 2 applies.

THEOREM 7.5. *Let* $f: K \to L$ *be a simplicial map, let* B *be in* $\mathfrak{M}^L$, *and let* X_* *be a projective resolution of* B. *There is a spectral sequence*

(7.5) $\qquad E^2_{p,q} = H^C_p(H_q(f^{\#}X_*)) \implies H_{p+q}(K; f^{\#}B).$

In particular when $B = G$ is a constant precostack over L with value the module G (a local system of coefficient modules), then $f^{\#}G$ is constant (a local system) over K and $H(K; f^{\#}G)$ is the classical homology of K with (local) coefficients in G, i.e. $H(K; f^{\#}G) = H(K; G)$. In this case (7.5) becomes

(7.6) $\qquad E^2_{p,q} = H^C_p(H_q(f^{\#}X_*)) \implies H_{p+q}(K; G).$

<div align="center">

CHAPTER III

COSTACKS

</div>

8. Subcategory of costacks:

Recall that a precostack of modules over K, $A \in \mathfrak{M}^K$, is characterized by :

(1) For each $x \in K$, $A(x)$ is a module, and

(2) For each simplicial operator u (u can be decomposed in a unique way as a composite of face operators d^i and degeneracy operators s^j), $A(u_x) : A(x) \to A(ux)$ is a homomorphism of modules.

It is realized as a simplicial set over K. If for each degeneracy s of K the homomorphism $A(s_x) : A(x) \to A(sx)$ is an isomorphism, then A is a *costack* (of modules over K). By the definition of K,

$$d^i s^j = 1 \text{ for } j = i, \, i+1,$$

we have for $A(s_x) : A(x) \to A(sx)$ and $A(d_{sx}) : A(sx) \to A(x)$,

(8.1) $\qquad A(d_{sx})A(s_x) = A(d_{sx}s_x) = 1 : A(x).$

Thus we have :

PROPOSITION 8.1. *A precostack A is a costack if and only if $A(d_{sx})$ is an isomorphism for all $x \in K$.*

Examples 1.2 and 1.3 of Section 1 are two examples of costacks.

Costacks over K form a subcategory $\mathfrak{A}_\Lambda^K$ of $\mathfrak{M}^K$. The following properties of $\mathfrak{A}_\Lambda^K$ are easy to check :

(C_1) $\mathfrak{A}_\Lambda^K$ is a faithful, exact subcategory of $\mathfrak{M}^K$;

(C_2) it is closed under the formation of subobjects, quotient objects, and extensions, i.e. it is a Serre subcategory of $\mathfrak{M}^K$;

(C_3) it is closed under the formation of products and coproducts. Therefore, by a theorem of Freyd [5],

(C_4) $\mathfrak{A}_\Lambda^K$ is reflective and coreflective in $\mathfrak{M}^K$ (c.f. section 0).

Let $R : \mathfrak{M}^K \to \mathfrak{A}_\Lambda^K$ be the coreflector; R is the left adjoint functor of the inclusion functor $J : \mathfrak{A}_\Lambda^K \to \mathfrak{M}^K$; the coreflection RA of $A \in \mathfrak{M}^K$ is called the *associated costack* of A. (Strictly speaking, a coreflection is

a pair (RA, θ_A), where $\theta_A : A \to RA$ is the canonical map displayed in (0.2)). For example, let A be a simplicial module, $A \in \mathfrak{M}^{<0>}$; the associated costack of A is the constant simplicial module with $H_o(A)$ in each dimension. Indeed,

$$\theta_A = (..., \beta, \alpha, \varepsilon) : A \to RA$$

is the simplicial homomorphism

with $\varepsilon = $ cokernel of $(d^o - d^1)$,

$$\alpha = \varepsilon d^o = \varepsilon d^1, \text{ and } \beta = \alpha d^o = \alpha d^1 = \alpha d^2 = \varepsilon d^o d^o, \text{ etc...}$$

(that ε, α, β,... are compatible with degeneracy operators is obvious). By property C_1, the inclusion functor J is exact, therefore RA is projective in $\mathfrak{A}_\Lambda^K$ if A is projective in $\mathfrak{M}^K$ (see lemma 0.3).

CONSTRUCTION OF REFLECTIONS. For A in $\mathfrak{M}^K$, let $R'A$ be the costack defined as follows : $(R'A)(x) = A(x)$ for a nondegenerate simplex $x \in K$ and $(R'A)(sx) = A(x)$ for any degeneracy operator s and any $x \in K$. Then $R'A$ is the reflection of A in $\mathfrak{A}_\Lambda^K$. Indeed, the canonical map

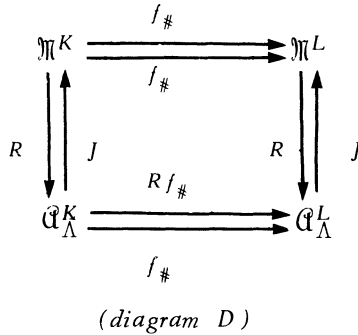$$\rho_A = \{ \rho_x : x \in K \} : R'A \to A$$

is the natural transformation such that $\rho_x = 1 : A(x)$ if x is nondegenerate, and $\rho_{sx} = A(s_x) : A(x) \to A(sx)$; ρ_{dx} is determined by the compatibility of a simplicial map with s and d. That ρ_A is well-defined follows from an easy but rather tedious computation which we shall not present here. Also, we omit the tedious proof that the $R'A$ so defined is actually the reflection of A. By recognizing all these, it is fairly easy to see that ρ_A is a monomorphism and that the reflector R' is an exact functor.

Since R' is the right adjoint functor of J and since both R' and

J are exact, R' preserves injectives and J preserves projectives by Lemma 0.3.

We conclude this section with a theorem that summarizes some results scattered over the previous sections.

THEOREM 8.2. *Let* $f : K \to L$ *be a simplicial map. There is a diagram* D

$$
\begin{array}{ccc}
\mathfrak{M}^K & \xrightarrow[\;\;f_{\#}\;\;]{f_{\#}} & \mathfrak{M}^L \\[4pt]
R \downarrow \uparrow J & \quad Rf_{\#} \quad & R \downarrow \uparrow J \\[4pt]
\mathfrak{A}_{\Lambda}^K & \xrightarrow[\;\;f_{\#}\;\;]{} & \mathfrak{A}_{\Lambda}^L
\end{array}
$$

(diagram D)

in which

(1) *All functors but* R *and* $Rf_{\#}$ *are exact,* R *and* $Rf_{\#}$ *are right exact;*

(2) $f_{\#}$, R *and* $Rf_{\#}$ *are the left adjoint functors of* $f^{\#}$, J, *and* $f^{\#}/\mathfrak{A}_{\Lambda}^L$ *respectively.*

(3) $f_{\#}$, R, $Rf_{\#}$, *and* J *preserve projectives.*

NOTE. $f^{\#}$ takes costacks to costacks but $f_{\#}$ usually not.

9. Costack homology.

Let A be a costack of Λ-modules over K, $A \in \mathfrak{A}_{\Lambda}^K$. The *homology* of K with *coefficients* in A is defined to be

$$H(K ; JA) = H(JA) = H^C(\partial JA)$$

as in the beginning of the section 4 and is denoted simply by $H(K ; A)$ or $H(A)$. By regarding A as a precostack, we proved in theorem 4.2 that $H(K ; A)$ is isomorphic to $(\mathfrak{L} H_o)(A)$ on $\mathfrak{M}^K$. We shall show that this can also be done in $\mathfrak{A}_{\Lambda}^K$ and that in computing $H(K ; A)$ by resolutions there is no difference in doing it in $\mathfrak{M}^K$ or in $\mathfrak{A}_{\Lambda}^K$.

THEOREM 9.1. *Let* NK *be the set of nondegenerate simplexes of* K, *let*

(9.1) $\qquad U' = \Sigma_{x \in NK} \, x_{\#}^{\delta} \Lambda^n \text{ and } \overline{U} = RU'.$

Then $\overline{U}$ is a projective generator of $\mathcal{C}_{\Lambda}^{K}$, and therefore $\mathcal{C}_{\Lambda}^{K}$ has enough projectives and injectives (by Lemma 0.2).

PROOF. U' being a coproduct of projectives is itself a projective, therefore, by part (3) of Theorem 8.2, $\overline{U} = RU'$ is projective. Compute $\mathcal{C}_{\Lambda}^{K}(\overline{U}, A)$, $A \in \mathcal{C}_{\Lambda}^{K}$; since R is the left adjoint of J, we have

(9.2) $\qquad \mathcal{C}_{\Lambda}^{K}(\overline{U}, A) \approx \mathfrak{M}^{K}(\Sigma \, x_{\#}^{\delta} \Lambda^n, JA) \approx \Pi \mathfrak{M}^{K}(x_{\#}^{\delta} \Lambda^n, A).$

By (3.2) the last term of (9.2) is isomorphic to $\Pi_{x \in NK} A(x)$. Thus

(9.3) $\qquad \mathcal{C}_{\Lambda}^{K}(\overline{U}, A) \approx \Pi_{x \in NK} A(x),$

and is nontrivial if $A \neq 0$. This shows that $\overline{U}$ is a generator and the theorem is proved.

Thus, let P_* be a projective resolution of A in $\mathcal{C}_{\Lambda}^{K}$, then since J is exact and preserves projectives, P_* is a projective resolution of A in $\mathfrak{M}^{K}$ and $H(K;A)$ is naturally isomorphic to $(\mathfrak{L}H_o)(A)$ where H_o is either the cosection functor on $\mathcal{C}_{\Lambda}^{K}$ or the cosection functor on $\mathfrak{M}^{K}$.

We shall now consider the relative homology of a pair of simplicial sets with coefficients in a costack. Let (K, K') be a simplicial pair; K' is a simplicial subset of K. Then the inclusion map $i : K' \to K$ induces the functors $i_{\#}$ and $i^{\#}$ on precostacks. For A' in $\mathfrak{M}^{K'}$, $i_{\#}A'$ is a functor (a precostack) on K to $\mathfrak{M}$ with « supports » in K', i.e.

$$(i_{\#}A')(x) = A'(x) \text{ for } x \in K',$$

$$(i_{\#}A')(x) = 0 \qquad \text{ for } x \in K - K'.$$

This shows that if A' is a costack, so is $i_{\#}A'$. Thus we have a pair of adjoint functors $i_{\#} \dashv i^{\#} : (\mathcal{C}_{\Lambda}^{K'}, \mathcal{C}_{\Lambda}^{K})$. It is clear that $i_{\#}$ is an exact full embedding of categories; we shall identify A' with $i_{\#}A'$ and regard $\mathcal{C}_{\Lambda}^{K'}$ as a subcategory of $\mathcal{C}_{\Lambda}^{K}$.

Let (K, K') be a simplicial pair; for A in $\mathcal{C}_{\Lambda}^{K}$, let qA be the quotient

$$qA = A/(i_{\#}i^{\#}A) = A/(i^{\#}A);$$

then qA has supports in $K - K'$. Any costack over K having supports in

$K - K'$ is of the form qA for some $A \in \mathcal{A}^K_\Lambda$ and is called a *relative co-stack* over K (relative to K'). All qA over K form a subcategory $q\mathcal{A}^K_\Lambda$ of $\mathcal{A}^K_\Lambda$. *Relative precostacks* and $q\mathcal{M}^K$ are defined in the same way.

For a simplicial pair (K, K') and a costack $A \in \mathcal{A}^K_\Lambda$, the *relative homology* of (K, K') with *coefficients* in A is defined to be the homology of qA and is written $H(K, K'; A) = H(qA)$. Thus $H(K, K'; A)$ can be computed by resolutions either in $\mathcal{M}^K$ or in $\mathcal{A}^K_\Lambda$ as $(\mathcal{L}H_o)(qA)$. In particular, if K' is empty, one identifies (K, K') with K and obtains the (absolute) homology $H(K; A)$.

Let $f : (K, K') \to (L, L')$ be a simplicial map of simplicial pairs; let B be a costack over L. Since $f^\#$ maps costacks onto costacks, the homomorphism f_* of (4.3) induces a homomorphism

$$f_* : H(K, K'; f^\# B) \to H(L, L'; B).$$

If $g = (M, M') \to (K, K')$ is another simplicial map, one has

$$(fg)_* = f_* g_* : H(M, M'; g^\# f^\# B) \to H(L, L'; B).$$

It is proved in [2] that this H is a unique homology theory on the category of simplicial pairs in the sense of Eilenberg and Steenrod (actually the uniqueness theorem in [2] is more general than this).

For computing $H(qA)$, we prefer to do it in $\mathcal{M}^K$. Define $H(K, K'; A)$ for $A \in \mathcal{M}^K$ and extend the homology theory of previous chapters to the relative case. The details will be left to the reader.

In particular, if K is finite (K has a finite number of non degenerate simplexes) and $\Lambda = Z$ is the ring of integers, the homology of a costack can be computed by torsion groups of modules (see [2]).

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