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CHARACTERISTIC CLASSES AS NATURAL TRANSFORMATIONS AND TOPOLOGICAL INDEX OF CLASSICAL ELLIPTIC OPERATORS*)

by SHIH WEISHU

In this paper we point out that the formulas used in the computation of the topological index of the classical elliptic operators on Riemann or complex analytic manifolds are in fact valid for any vector bundle over a *CW*-complex. It turns out that their proofs become quite evident when a natural interpretation of the Todd class is given; in particular, we do not need the « weight ». Here is a sketch of what follows.

It is well known that a characteristic class may be defined as a natural transformation [4] from the functor K to H^* (i.e. Grothendieck group and ordinary cohomology). We propose to extend this study to the non-stable case. That is, we determine the semi-group of all natural transformations $\mathcal{H}om(\mathcal{E}, H)_{\mu}$, $\mu = (+, \times)$, *ring*, etc..., where $\mathcal{E}$ may be any one of the semi-groups of isomorphism classes of vector bundles, and H may be taken to be the groups K or KO as well as H^*. Here μ indicates the algebraic conditions which should be satisfied by the natural transformations, e.g. $\mu = (+, \times)$, $H = K$ means those which carry Whitney sum to tensor product (for example, the alternate exterior algebra $\Lambda' = \sum_i (-1)^i \Lambda^i$ of a vector bundle defines such a natural transformation). We prove that each of these different semi-groups is isomorphic to a suitable type of semi-groups of formal power series, and we determine some relationships between them, such as the canonical composition map

$$\mathcal{H}om(K, H^{2*})_{ring} \times \mathcal{H}om(\mathcal{E}, K)_{+,\times} \to \mathcal{H}om(\mathcal{E}, H^{2*})_{+,\times},$$

arising from natural constructions of vector bundles.

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*) Ce texte, multigraphié en 1963, a été exposé en 1963-64 aux Séminaires de Monsieur Ehresmann (Paris) et de Monsieur Palais (I.A.S. Princeton). Une partie des résultats a été reprise dans le livre de Hüsemoller «Topology of fiber bundles».

If we define the Euler class χ in general as the one which corresponds to the power series t (when it exists), we may consider the inverse of a natural transformation (when it exists) with respect to the power of χ. Then the Todd class is just the inverse of the alternate exterior algebra Λ' composed with the Chern character, similarly for the L-genus; and some propositions which indicate the relation between these classes (c.f. Chapter III) follow directly from this fact. At the same time, the stable elements $\mathcal{H}om(K, H)_\mu \subseteq \mathcal{H}om(\mathcal{E}, H)_\mu$ are also obtained as a consequence; it gives the classical results of Atiyah-Hirzebruch [4] and the Adams' operations [1].

Although the method is exactly the same for every $\mathcal{E}$ and H, we give the details only for the case of complex vector bundles and even dimensional oriented real vector bundles, because this is directly used in this paper. We mention, as a final remark, several other cases which are more or less related to that one, and leave the complete formulation for elsewhere. For reasons of convenience all the proofs are given in the last paragraph. They follow Borel's work [6] for H^*, Atiyah-Hirzebruch [3] for K, and Anderson [2] for KO.

Nothing is supposed to be known in this paper, hence it may be used as a survey of characteristic classes for non experts. The parts related to the topological index of classical operators are contained in §6 and §7, which are written in such a way as to be independent of the other paragraphs.

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1. Semi-groups.

We recall that a semi-group (abelian) is a set with an operation which satisfies all the axioms of an abelian group except the existence of inverse elements, i. e.: a commutative monoid. The following semi-groups are useful for our purpose.

(i) Let X be a *CW*-complex and denote by $\mathcal{E}_{\mathbf{C}}(X)$ the semi-group (resp. semi-ring) of isomorphism classes of complex vector bundles over X with respect to the Whitney sum (resp. and the tensor product). Similarly, we have the semi-group $\mathcal{E}^{+}_{\mathbf{R}}(X)$ of even dimensional oriented real vector bundles over X, the oriented real vector bundles, and real vector bundles $\mathcal{E}^{o}_{\mathbf{R}}(X)$, $\mathcal{E}_{\mathbf{R}}(X)$. The « dual bundle» defines an endomorphism of $\mathcal{E}(X)$, which is equal to the *complex conjugation* of a bundle in the complex case, i. e.

$$* = - : \mathcal{E}_{\mathbf{C}}(X) \to \mathcal{E}_{\mathbf{C}}(X),$$

where ξ^* (resp. $\overline{\xi}$) is the *dual* of ξ (resp. conjugate of ξ). Similarly, the complexification of a real vector bundle defines a homomorphism

$$\otimes \mathbf{C} : \mathcal{E}_{\mathbf{R}}(X) \to \mathcal{E}_{\mathbf{C}}(X)$$

and the following diagram is commutative

$$\begin{array}{ccc} \mathcal{E}_{\mathbf{R}}(X) & \xrightarrow{\otimes \mathbf{C}} & \mathcal{E}_{\mathbf{C}}(X) \\ {\scriptstyle 1 = *}\downarrow & & \downarrow{\scriptstyle *} \\ \mathcal{E}_{\mathbf{R}}(X) & \xrightarrow{\otimes \mathbf{C}} & \mathcal{E}_{\mathbf{C}}(X). \end{array}$$

(ii) Let Λ be a fixed integral domain and consider the ring $\Lambda[[t]]$ of formal power series with coefficients in Λ. Then the following subsets of $\Lambda[[t]]$ defined by

$$\Lambda^{+}[[t]] = \{ f \mid f(-t) = f(t) \},$$

$$\Lambda^{\pm}[[t]] = \{ f \mid f(-t) = f(t) \text{ or } f(-t) = -f(t) \},$$

$$\Lambda^{+}_{e}[[t]] = \{ f \mid f(t) = f(\tfrac{1}{1+t} - 1) \},$$

$$\Lambda^{\pm}_{e}[[t]] = \{ f \mid f(t) = f(\tfrac{1}{1+t} - 1) \text{ or } f(t) = -f(\tfrac{1}{1+t} - 1) \}$$

are subsemi-groups of the semi-group of $\Lambda[[t]]$ with respect to the multiplication of power series.

In the case $\Lambda = \mathbf{Z}$ = the ring of integers, we find that $\mathbf{Z}_e^{+}[[t]]$, $\mathbf{Z}_e^{\pm}[[t]]$ are just the semi-groups obtained as the inverse image of $\mathbf{Q}^{+}[[t]]$ (resp. $\mathbf{Q}^{\pm}[[t]]$), with $\mathbf{Q}$ the rationals, by the homomorphism of substitution of the power series « $e^t - 1$ » :

$$\gamma : \mathbf{Z}[[t]] \to \mathbf{Q}[[t]], \qquad \gamma(f) = f(e^t - 1).$$

Finally, we remark that, for each $f(t) \in \Lambda[[t]]$ the first non-vanishing coefficient $\alpha_k \in \Lambda$ of which is an invertible element of Λ, there exists a unique τf in $\Lambda[[t]]$ such that $f(t).\tau f(t) = t^k$; $k = \omega(f)$ is called the *order* of f, and τf its *inverse modulo the order.*

(iii) For each CW-complex X the even dimensional product

$$H^{2*}(X, \Lambda) = \prod_i H^{2i}(X, \Lambda)$$

of its cohomology is a semi-group with respect to the cup-product. The order of an element in $H^{2*}(X, \Lambda)$ is k, if its first non-vanishing component is in $H^{2k}(X, \Lambda)$. In the case $\Lambda = \mathbf{Z}$, the group of invertible elements of the semi-group $H^{2*}(X, \mathbf{Z})$ is just the $G^*(X, \mathbf{Z})$ used by Atiyah-Hirzebruch in [4].

It is clear that the set of homomorphisms from a semi-group A into a semi-group B is itself a semi-group : $Hom(A, B)$.

For each semi-group A there exists a unique (within isomorphism) abelian group $\overline{A}$ and a homomorphism ρ from A into $\overline{A}$ satisfying the usual universal property, i.e.

(1) the image $\rho(A)$ generates $\overline{A}$,

(2) for any abelian group B and homomorphism $\rho' : A \to B$, there exists a unique homomorphism $\overline{\rho} : \overline{A} \to B$ such that $\rho' = \overline{\rho} \circ \rho$, i.e. ρ defines an isomorphism of abelian groups

$$Hom(\overline{A}, B) \xrightarrow[\approx]{\rho} Hom(A, B).$$

In particular, the $K(X)$ of a finite CW-complex may be defined as $\overline{\mathcal{E}_{\mathbf{C}}(X)}$.

Every subset of a commutative ring with unity, which is closed under multiplication and contains the unity, is a semi-group; conversely any semi-group can be obtained in this way.

2. Characteristic classes as elements of *Hom* ($\mathfrak{S}$, *H*).

We recall that, if $\mathfrak{S}$ and H are two functors

$$\mathfrak{S},\ H : \mathfrak{F} \to \mathfrak{G}$$

from the category $\mathfrak{F}$ of finite CW-complexes into that of semi-groups $\mathfrak{G}$, then the set of natural transformations [8] from $\mathfrak{S}$ into H, which we shall denote by $\mathcal{H}om(\mathfrak{S}, H)$, is again a *semi-group*. If $f \in \mathcal{H}om(\mathfrak{S}, H)$, $X \in \mathfrak{F}$, we write f_X (or simply f if there is no confusion) for the homomorphism from $\mathfrak{S}(X)$ into $H(X)$ defined by f.

Now, if we associate to each finite CW-complex X the semi-group $\mathfrak{S}_{\mathbf{C}}(X)$, etc. (cf. § 1 for notation), we obtain contravariant functors $\mathfrak{S}_{\mathbf{C}}$, $\mathfrak{S}^{+}_{\mathbf{R}}$, K, H^{2*}, etc., from $\mathfrak{F}$ into $\mathfrak{G}$, and we use

$$\mathcal{H}om(\mathfrak{S}, H)_{\mu}, \text{ where } \mu = (+, \times), (+, +), \text{ etc.}$$

to indicate the semi-group of natural transformations; for example if $\mu = (+, \times)$ the first functor $\mathfrak{S}$ is the semi-group with respect to the Whitney sum and the second H with respect to the cup-product or tensor product.

DEFINITION 2.1. Any element of the semi-group $f \in \mathcal{H}om(\mathfrak{S}, H)_{\mu}$ will be called a *characteristic class* (or cohomology operation in the case $\mathfrak{S} = H$), and f is said to be *stable* if it is contained in the image of the canonical monomorphism

$$\mathcal{H}om(\tilde{K}, H)_{\mu} \to \mathcal{H}om(\mathfrak{S}_{\mathbf{C}}, H)_{\mu}$$

(similarly for KO, etc.). It is said to be *invertible* if f is an invertible element of the semi-group $\mathcal{H}om(\mathfrak{S}, H)_{\mu}$. In the case $\mathfrak{S} = \mathfrak{S}_{\mathbf{C}}$ and $H = H^{2*}$, we define the *order* of f, and denote it by $\omega(f)$, to be the minimum integer $k = \omega(f)$ for which there exists a complex line bundle ξ such that the $2k$-dimensional component of $f(\xi)$ in H^{2*} does not vanish. Similar definition for the case $\mathfrak{S}^{+}_{\mathbf{R}}$ H^{*}.

REMARK 2.1. The «order» is also defined for the other cases: $H = K$, KO, cf. § 4 for detail.

We recall some notations which shall be used later (cf. §1 (i)).

(i) The conjugation of a complex vector bundle defines a homomorphism of semi-groups

$$\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H)_{\mu} \to \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H)_{\mu},$$

which we shall denote by $f \to \overline{f}$ for any H and μ. And we have by definition

$$\overline{f}_X(\xi) = f_X(\overline{\xi}), \quad \xi \in \mathcal{E}_{\mathbf{C}}(X).$$

Similarly, in the case where $H = K$ and $\mathcal{E}$ is arbitrary, the conjugation endomorphism of K induces also a homomorphism

$$\mathcal{H}om(\mathcal{E}, K)_{\mu} \to \mathcal{H}om(\mathcal{E}, K)_{\mu}$$

for any μ. We shall call again this homomorphism the conjugation and denote it by the same notation as before. In fact, when $\mathcal{E} = \mathcal{E}_{\mathbf{C}}$ the two definitions of conjugation coincide, as we shall see later.

(ii) The *complexification* of a real vector bundle defines a homomorphism

$$\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H)_{\mu} \to \mathcal{H}om(\mathcal{E}_{\mathbf{R}}, H)_{\mu}$$

for any H and μ, which is denoted by $f \to f \otimes \mathbf{C}$, and by definition, for any real vector bundle η, we have $f \otimes \mathbf{C}(\eta) = f(\eta \otimes \mathbf{C})$. Similarly, the complexification induces a homomorphism from KO into K which gives

$$\mathcal{H}om(\mathcal{E}, KO)_{\mu} \xrightarrow{\otimes \mathbf{C}} \mathcal{H}om(\mathcal{E}, K)_{\mu}$$

for any $\mathcal{E}$ and μ. We call it again the complexification and use the same notation as before to denote it.

(iii) The decomplexification of a complex vector bundle and the canonical inclusion give homomorphisms

$$\mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, H)_{\mu} \to \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H)_{\mu}, \quad \mathcal{H}om(\mathcal{E}^{o}_{\mathbf{R}}, H)_{\mu} \to \mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, H)_{\mu}$$

for any H and μ.

(iv) The *composition* of natural transformations induces a canonical map

$$\mathcal{H}om(K, H)_{\mu} \times \mathcal{H}om(\mathcal{E}, K)_{\nu} \to \mathcal{H}om(\mathcal{E}, H)_{\delta}$$

which is denoted by $(g, f) \to g \circ f$; when μ, ν and δ are suitably chosen " $\circ$ " is bilinear.

The most trivial characteristic class which is not stable is the Euler class of a real oriented vector bundle, or the top Chern class of a complex vector bundle, and the following examples which are useful later.

EXAMPLE 2. 1. Consider the *alternate exterior algebra* (resp. *exterior algebra)*

$$\Lambda', \Lambda_X : \mathcal{E}_{\mathbf{C}}(X) \to K(X)$$

defined for each complex vector bundle ξ over X by the virtual bundle

$$\Lambda'_X(\xi) = \sum_i (-1)^i \Lambda^i \xi, \qquad \Lambda_X(\xi) = \sum_i \Lambda^i \xi,$$

when $\Lambda^i \xi$ is the i- th exterior power of ξ. This gives us two elements $\Lambda', \Lambda \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+, \times}$. It is easy to see that they are not stable. Similarly we have the analogous real cases.

EXAMPLE 2. 2. Let η be a real oriented plane bundle over X and consider the elements in $K(X)$ given by the virtual bundle

$$\tilde{\Lambda}^+_X(\eta) = \xi + \overline{\xi}, \qquad \tilde{\Lambda}^-_X(\eta) = \xi - \overline{\xi},$$

where ξ is the unique (within isomorphism) complex line bundle which is isomorphic to η as an oriented real bundle, and where $\overline{\xi}$ is its conjugate. We shall see later that this gives rise to unique maps

$$\tilde{\Lambda}^+_X : \mathcal{E}^o_{\mathbf{R}}(X) \to K(X), \qquad \tilde{\Lambda}^-_X : \mathcal{E}^+_{\mathbf{R}}(X) \to K(X)$$

transforming the Whitney sum into tensor product; hence we obtain

$$\tilde{\Lambda}^+, \tilde{\Lambda}^- \in \mathcal{H}om(\mathcal{E}^+_{\mathbf{R}}, K)_{+, \times}, \qquad \tilde{\Lambda}^+ \in \mathcal{H}om(\mathcal{E}^o_{\mathbf{R}}, K)_{+, \times}$$

which are non stable.

LEMMA 2.1. *Two characteristic classes are equal if and only if their restrictions on compact complex analytic manifolds are equal.*

LEMMA 2. 2. *Every additive class is stable, i. e. the following canonical inclusion is an isomorphism of groups*

$$\mathcal{H}om(K, H)_{+,+} \approx \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H)_{+,+}$$

(similarly for real cases), hence induces a bijective map on the subset of $\mathcal{H}om(K, H)_{+,+}$:

$$\mathcal{H}om(K, H)_{ring} \underset{\approx}{\to} \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H)_{ring}.$$

3. $\mathcal{H}om(\mathcal{E}, H^{2*})$-characteristic classes in H^{2*}.

Let $P_\infty(\mathbf{C})$ denote the infinite dimensional complex projective space (i.e. $K(\mathbf{Z}, 2)$) which may be considered as a classifying space for $U(1)$ and $SO(2)$. We recall that the cohomology ring of $P_\infty(\mathbf{C})$ is a polynomial ring with a distinguished generator α_o of dimension 2. Hence for each complex line bundle ξ (resp. oriented plane bundle η) over $X \in \mathcal{F}$, there exists a unique class, namely the *fundamental class* of ξ (resp. η), denoted by $\alpha_\xi \in H^2(X, \Lambda)$ (resp. $\alpha_\eta \in H^2(X, \Lambda)$), which is the image of α_o under a classifying map from X into $P_\infty(\mathbf{C})$ inducing ξ (resp. η), where Λ is any integral domain. Moreover, remark that, given a power series $g(t) \in \Lambda[[t]]$, then $g(\alpha_\xi) \in H^{2*}(X, \Lambda)$ (resp. $g(\alpha_n)$) is a well defined element. Then we have

THEOREM 3. 1. *There exists unique isomorphisms of semi-groups*

$$\varphi : \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times} \overset{\approx}{\to} \Lambda[[t]]_\times$$
$$\Psi : \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,+} \overset{\approx}{\to} \Lambda[[t]]_+$$

such that, for each complex line bundle ξ, *we have*

$$f(\xi) = \varphi(f)(\alpha_\xi), \quad f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times},$$
$$g(\xi) = \Psi(g)(\alpha_\xi), \quad g \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,+}.$$

If Λ is a field of characteristic zero, then there is a bijective map

$$\delta : \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{ring} \overset{\approx}{\to} \Lambda \cup \{e_o\}$$

(where the right side is the disjoint union of Λ with a point e_o) such that $\delta(0) = e_o$, where 0 is the unity of the semi-group $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,+}$; and for any $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{ring}$ with $f \neq 0$, we have $\delta(f).\alpha_\xi =$ the two dimensional component of $f(\xi)$. Moreover, the canonical inclusion $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{ring} \subseteq \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,+}$ is given by

$$\beta \to e^{\beta t}, \quad \beta \in \Lambda,\ e^{\beta t} \in \Lambda[[t]],$$

under the identification δ and Ψ.

REMARK. We write $\Lambda[[t]]_{\times}$ to indicate the multiplication of power series is used as the structure of semi-group, similarly for $\Lambda[[t]]_{+}$. By Lemma 2. 1 we may replace $\mathfrak{S}_{\mathbf{C}}$ by K in the case $(+, +)$ and ring. The notations here are the same as in § 1 (i) and (ii). It is clear that φ preserves the «order» which is now defined on both sides. We have the same remark for the following theorem.

THEOREM 3. 2. *If the coefficient ring Λ of H^{2*} contains $\frac{1}{2}$, then we have the unique isomorphisms of semi-groups*

$$\varphi : \mathcal{H}om(\mathfrak{S}^{+}_{\mathbf{R}}, H^{2*})_{+,\times} \to \Lambda^{\pm}[[t]]_{\times},$$
$$\Psi : \mathcal{H}om(\mathfrak{S}^{+}_{\mathbf{R}}, H^{2*})_{+,\times} \to \Lambda^{+}[[t]]_{+}$$

such that, for each oriented plane bundle η, we have

$$f(\eta) = \varphi(f)(\alpha_{\eta}), \quad f \in \mathcal{H}om(\mathfrak{S}^{+}_{\mathbf{R}}, H^{2*})_{+,\times},$$
$$g(\eta) = \Psi(g)(\alpha_{\eta}), \quad g \in \mathcal{H}om(\mathfrak{S}^{+}_{\mathbf{R}}, H^{2*})_{+,+}.$$

If the coefficient of H^{2*} is the rational numbers $\mathbf{Q}$ (or real numbers), then there is a unique bijective map

$$\delta : \mathcal{H}om(\mathfrak{S}^{+}_{\mathbf{R}}, H^{2*})_{ring} \overset{\approx}{\to} \mathbf{Q}^{+} \cup \{e_o\}$$

(where $\mathbf{Q}^{+}$ denotes the non negative rationals) such that the unity 0 of $\mathcal{H}om(\mathfrak{S}^{+}_{\mathbf{R}}, H^{2*})_{+,+}$ is mapped into e_o, and for any $f \in \mathcal{H}om(\mathfrak{S}^{+}_{\mathbf{R}}, H^{2*})_{ring}$, $f \neq 0$, we have $\delta(f).\alpha_{\eta}^{2}$ = the four-dimensional component of $f(\eta)$. Moreover, the canonical inclusion $\mathcal{H}om(\mathfrak{S}^{+}_{\mathbf{R}}, H^{2*})_{ring} \subseteq \mathcal{H}om(\mathfrak{S}^{+}_{\mathbf{R}}, H^{2*})_{+,+}$ is given by

$$\beta \to e^{\beta t} + e^{-\beta t}, \quad \beta \in \mathbf{Q}^{+}, \ e^{\beta t} + e^{-\beta t} \in \mathbf{Q}^{+}[[t]]$$

under the identification δ and Ψ.

CONVENTION 3. 1. In the case of $\mathcal{H}om(\mathfrak{S}_{\mathbf{C}}, H^{2*})_{ring} = \mathcal{H}om(K, H^{2*})_{ring}$, we shall denote by $ch : \mathbf{Q} \to \mathcal{H}om(K, H^{2*})_{ring}$ the restriction of the inverse of δ on $\mathbf{Q}$, and write $ch^{\beta} = ch(\beta)$, $\beta \in \mathbf{Q}$. This is reasonable because $ch^{1} = ch$ is just the classical Chern character, and ch^{o} may be identified with the augmentation in K-theory. We shall see later that ch^{-1}

and $ch^{1/2}$ will also be useful.

CONVENTION 3.2. We shall use the same operation for the semi-group $\mathcal{H}om(\mathcal{E}, H)_\mu$ as the one which is used in H, e.g. for $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$, we shall write multiplicatively its operation «$f.g$» for $f, g \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$. At the same time we shall use the same notation to denote the operations transported from that of power series by the bijective map φ (resp. Ψ); e.g. if $f, g \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$ and $\lambda \in \Lambda$, then $f \pm g$, λf, $f+\lambda$, λ are well defined elements of $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$. We have the substitution $f(\lambda g)$ and division f/g when this has meaning. We shall simply write f^λ for the substitution of λt into f, i. e. $f(\lambda t)$.

COROLLARY 3.1. *(Atiyah-Hirzebruch [4]). A characteristic class in* $\mathcal{H}om(\mathcal{E}, H^{2*})_{+,\times}$ *is stable if and only if its corresponding power series by* φ *is an inversible element hence of order zero. Also the identification map «h», given by Hirzebruch's multiplicative sequence of polynomials [15], commutes with* φ, *i. e. the following diagram is commutative*

$$\begin{array}{ccc} \Lambda_1[[t]] & \xleftrightarrow{\;h\;} & \mathcal{H}om(\tilde{K}, H^{2*})_{+,\times} \\ \downarrow & & \downarrow \\ \Lambda[[t]] & \xleftrightarrow{\;\varphi\;} & \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times} \end{array}$$

where $\Lambda_1[[t]]$ *denotes the set of power series with leading coefficient* $1 \in \Lambda$, *and where the two verticals are canonical inclusions.*

COROLLARY 3.2. *Under the identification by* φ *(resp.* Ψ*), the conjugation (and the dual* $*$, *cf. § 1 (i))*

$$\begin{array}{ccc} \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_\mu & \xrightarrow{\;«-» \text{ or } «*»\;} & \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_\mu \\ \downarrow & & \downarrow \\ \Lambda[[t]] & \longrightarrow & \Lambda[[t]] \end{array}$$

is given by $\bar{f}(t) = f(-t) = f^*(t)$, *where* $\mu = (+, \times)$ *or* $(+, +)$.

This corollary leads to the following definition.

DEFINITION 3.1. Let $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, H^{2*})_{+,\times}$; then the conjugation $\bar{f}$ of f is defined to be the one which under the correspondence φ is $f(-t)$

COROLLARY 3.3. *If Λ contains $\frac{1}{2}$, then under the identification φ the complexification*

$$\begin{array}{ccc} \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times} & \xrightarrow{\otimes \mathbf{C}} & \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, H^{2*})_{+,\times} \\ \downarrow \varphi & & \downarrow \varphi \\ \Lambda[[t]] & \longrightarrow & \Lambda^{\pm}[[t]] \end{array}$$

is given by $f \otimes \mathbf{C}(t) = f(t).f(-t)$.

Hence, in particular, we have (cf. Corollary 3.2) $\bar{f} \otimes \mathbf{C} = f \otimes \mathbf{C}$. Similarly we have, under the identification

$$\Psi : \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, H^{2*})_{+,+} \approx \Lambda^{+}[[t]],$$

$f \otimes \mathbf{C}(t) = f(t) + f(-t)$; in particular, if $\Lambda = \mathbf{Q}$ = the rational field, then the complexification maps

$$\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{ring} \xrightarrow{\otimes \mathbf{C}} \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, H^{2*})_{ring}$$

surjectively and, in fact, under the identification δ, we have

$$(\beta) \otimes \mathbf{C} = \beta^{2}, \quad \beta \in \mathbf{Q};$$

hence ch^{β} and $ch^{-\beta}$ have the same image under $\otimes \mathbf{C}$. Moreover the decomplexification is just the canonical inclusion of $\Lambda^{+}[[t]]$ (resp. $\Lambda^{\pm}[[t]]$) into $\Lambda[[t]]$.

COROLLARY 3.4. *If $f_o \in \Lambda[[t]]$ has order $\omega(f_o) \geq 1$, then the ring homomorphism of $\Lambda[[t]]$ into itself, defined by the substitution of f_o into a power series, gives rise by φ (resp. Ψ) to an endomorphism of $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$ (resp. $(+,+)$). In particular, if $f_o(t) = \beta t$, where $\beta \in \Lambda$, we have, for each $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{\mu}$, $\mu = (+,\times)$ or $(+,+)$, and for any complex vector bundle ξ, the equality*

the $2k$-dimensional component of $f^{\beta}(\xi) = \beta^{k}$ times the $2k$-dimensional component of $f(\xi)$,

for all $k \geq 0$, where f^{β} is the image of f by the endomorphism of substitution βt (cf. Convention, and similar results for the real cases).

EXAMPLE 3.1. Consider the inclusion map $\Lambda \subseteq \Lambda[[t]]$ given by

$$\lambda \to \lambda + 0t + \ldots + 0t^n + \ldots, \quad \lambda \in \Lambda.$$

Then the corresponding characteristic class λ' defined by φ (resp. Ψ) is given by

$$\lambda'(\xi) = \lambda^n \in H^o(X, \Lambda), \quad \xi \in \mathcal{E}_{\mathbf{C}}(X), \quad n = \dim \xi,$$

$$\lambda'(\eta) = \lambda^n \in H^o(X, \Lambda), \quad \eta \in \mathcal{E}^+_{\mathbf{R}}(X), \ 2n = \dim \eta$$

(resp. n, λ), i.e. by the constant cohomology class represented by $\lambda^n \in \Lambda$.

The corollaries 3.2, 3.3, and 3.4 imply :

the $2k$-dimensional component of $f(\xi^*) = (-1)^k$ times the $2k$-dimensional component of $f(\xi)$,

for all $\xi \in \mathcal{E}_{\mathbf{C}}$.

EXAMPLE 3.2. The *Euler-class* χ with coefficient in Λ of an even dimensional oriented real vector bundle corresponds by φ to

$$t \in \Lambda^{\pm}[[t]] \approx \mathcal{H}om(\mathcal{E}^+_{\mathbf{R}}, H^{2*})_{+,\times}.$$

In particular, (cf. Definition 3.1) $\overline{\chi}(\eta) = (-1)^n \chi(\eta)$, $2n = \dim \eta$.

The total *Chern* class corresponds to

$$1 + t \in \Lambda[[t]] \approx \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times},$$

and the top Chern class corresponds to t (which may be called also the Euler class), hence its complexification (cf. Corollary 3.2) is the negative of the square of the real Euler class in $\mathcal{H}om(\mathcal{E}^+_{\mathbf{R}}, H^{2*})_{+,\times}$.

The Pontrjagin class corresponds to $1 + t^2$, the top Pontrjagin class corresponds to t^2, which is the square of the Euler class in the even dimensional case (this is true in general, but we state it only for the even dimensional case because of our limit on $\mathcal{E}^+_{\mathbf{R}}$). The equality $1 + t^2 = 1 - (it)^2$ in $\mathbf{C}[[t]]$, where $i^2 = -1$, and Corollaries 3.2, 3.5 give the relation between the Pontrjagin class of a real oriented bundle (cf. §8) and the Chern class of its complexification [16]. Similarly, $1 + t^2 = (1 + it)(1 + (-it))$ gives the relation between the Pontrjagin class of the decomplexification of a complex vector bundle and its Chern class.

In $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$ with rational coefficients, $\frac{t}{\sinh t}$ corresponds to the *A-genus* class, $\frac{t}{\tanh t}$ corresponds to the *L*-genus class, and $\frac{t}{1-e^{-t}}$ to the Todd class. The power series

$$\cos t \in \Lambda^{+}[[t]] \approx \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, H^{2*})_{+,+}$$

gives an f which satisfies $f(\eta \cdot \eta') = 2f(\eta) \cdot f(\eta')$. The derivative of a characteristic class is always defined, e.g. the derivative of the Pontrjagin class is twice the Euler class.

If $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{\times,\times}$ is such that $f(\xi + 1) = f(\xi) + 1$ for every complex line bundle ξ, where 1 is the trivial complex line bundle, then f is additive, i.e. $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{ring}$.

4. $\mathcal{H}om(\mathcal{E}, K)$ characteristic class in K.

We recall that if ξ is a complex line bundle over a finite *CW*-complex X and if $f(t) \in \mathbf{Z}[[t]]$ is a power series with integer coefficients, then $f(\xi - 1) \in K(X)$ is a *well defined element of* $K(X)$, obtained by substituting the virtual bundle $\xi - 1$ (where 1 is the trivial complex line bundle) into $f(t)$. This is possible, because the Chern character of $\xi - 1$ has evidently a vanishing zero-dimensional component, hence [3] it is nilpotent in $K(X)$. Similarly, if Λ is an integral domain and $f(t) \in \Lambda[[t]]$, then $f(\xi - 1) \in K(X) \otimes_{\mathbf{Z}} \Lambda$ is also well defined. Using the same notation as in § 1 (i) and (ii) we have :

THEOREM 4.1. *There exists unique isomorphisms of semi-groups*

$$\varphi : \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times} \xrightarrow{\approx} \Lambda[[t]]_{\times},$$
$$\Psi : \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times} \xrightarrow{\approx} \Lambda[[t]]_{+}$$

such that, for each complex line bundle ξ, *we have*

$$f(\xi) = \varphi(f)(\xi - 1), \quad f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times},$$
$$g(\xi) = \Psi(g)(\xi - 1), \quad g \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,+}.$$

In the case $\Lambda = \mathbf{Z}$, *there is a unique bijective map*

$$\delta : \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{ring} \xrightarrow{\approx} \mathbf{Z} \cup \{e_o\}$$

such that the unity 0 *of* $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,+}$ *is mapped by* δ *into* e_o, *and, for any* $f \neq 0$ *in* $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{ring}$, *we have* $f(\xi) = \xi^{\delta(f)}$ *(when* $\delta(f) < 0$,

$\xi^{\delta(f)} = \bar{\xi}^{|\delta(f)|}$). *Moreover, the canonical inclusion*

$$\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{ring} \subseteq \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,+}$$

is given by $n \to (1+t)^n$, $n \in \mathbf{Z}$, *under the identifications* δ *and* Ψ.

THEOREM 4.2. *There exists unique isomorphisms of semi-groups*

$$\varphi : \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times} \xrightarrow{\approx} \Lambda^{\pm}[[t]]_{\times},$$
$$\Psi : \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K \otimes_{\mathbf{Z}} \Lambda)_{+,+} \xrightarrow{\approx} \Lambda^{+}[[t]]_{+}$$

such that, for each oriented real plane bundle η, *we have*

$$f(\eta) = \varphi(f)(\xi - 1), \quad f \in \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times},$$
$$g(\eta) = \Psi(g)(\xi - 1), \quad g \in \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K \otimes_{\mathbf{Z}} \Lambda)_{+,+},$$

where ξ *is the unique (within isomorphism) complex line bundle isomorphic to* η *as real bundles. In the case where* $\Lambda = \mathbf{Z}$, *there is a unique bijective map*

$$\delta : \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K)_{ring} \xrightarrow{\approx} \mathbf{Z}^{+} \cup \{e_o\}$$

(where $\mathbf{Z}^{+}$ *the non-negative integers) such that the unity* 0 *of the semi-group* $\mathcal{H}om(\mathcal{E}_{\mathbf{R}}, K)_{+,+}$ *is mapped by* δ *into* e_o, *and, for each* $f \neq 0$ *in* $\mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K)_{ring}$, *we have* $f(\eta) = \xi^{\delta(f)} + \bar{\xi}^{\delta(f)}$. *Moreover the canonical inclusion* $\mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K)_{ring} \subseteq \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K)_{+,+}$ *is given by*

$$n \to (1+t)^n + (1+t)^{-n}, \quad n \in \mathbf{Z}^{+},$$

under the identifications δ *and* Ψ.

DEFINITION 4.1. The *order of a characteristic class* f in $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times}$ (resp. $\mathcal{E}_{\mathbf{R}}^{+}$) is defined to be the order of the power series corresponding to f by φ and will be denoted by $\omega(f)$, $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times}$ (resp. $\mathcal{E}_{\mathbf{R}}^{+}$). The *Euler-class* $\chi \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times}$ is defined to be the one which corresponds to «t» by φ.

From Lemma 2.2, it follows immediatly

COROLLARY 4.1 *(T. Dieck)* [10]. *The Adams' operations* [7] *are the only non-trivial cohomology operations of rings in K-theory.*

COROLLARY 4.2. *Under the identification* φ *(resp.* Ψ*), the conjugation* *and the dual (cf. § 1(i))*

$$\begin{array}{ccc} \mathcal{H}om(\mathcal{E}_{\mathbf{C}},\, K \otimes_{\mathbf{Z}} \Lambda)_{\mu} & \xrightarrow{\;\overline{} \,=\, *\;} & \mathcal{H}om(\mathcal{E}_{\mathbf{C}},\, K \otimes_{\mathbf{Z}} \Lambda)_{\mu} \\ \downarrow & & \downarrow \\ \Lambda[[t]] & \longrightarrow & \Lambda[[t]] \end{array}$$

are given by $\overline{f}(t) = f(\frac{1}{1+t} - 1) = f^*(t)$, *where* $\mu = (+, \times), (+, +)$.

COROLLARY 4.3. *Under the identification* φ, *the conjugation induced by that of* K

$$\begin{array}{ccc} \mathcal{H}om(\mathcal{E}^+_{\mathbf{R}},\, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times} & \longrightarrow & \mathcal{H}om(\mathcal{E}^+_{\mathbf{R}},\, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times} \\ \downarrow & & \downarrow \\ \Lambda^{\pm}_{e}[[t]] & \longrightarrow & \Lambda^{\pm}_{e}[[t]] \end{array}$$

is given by $\overline{f}(t) = f(\frac{1}{1+t} - 1)$.

COROLLARY 4.4. *Under the identification* φ, *the complexification*

$$\begin{array}{ccc} \mathcal{H}om(\mathcal{E}^+_{\mathbf{C}},\, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times} & \xrightarrow{\;\otimes\, \mathbf{C}\;} & \mathcal{H}om(\mathcal{E}^+_{\mathbf{R}},\, K \otimes \Lambda)_{+,\times} \\ \downarrow & & \downarrow \\ \Lambda[[t]] & \longrightarrow & \Lambda^{\pm}_{e}[[t]] \end{array}$$

is given by

$$f \otimes \mathbf{C}(t) = f(t) \,.\, f(\frac{1}{1+t} - 1).$$

Similarly, under the identification

$$\Psi : \mathcal{H}om(\mathcal{E}^+_{\mathbf{R}},\, K \otimes_{\mathbf{Z}} \Lambda)_{+,+} \approx \Lambda^{+}_{e}[[t]],$$

the complexification is given by

$$f \otimes \mathbf{C}(t) = f(t) + f(\frac{1}{1+t} - 1).$$

In particular, under the identification δ, *the complexification*

$$\mathcal{H}om(\mathcal{E}_{\mathbf{C}},\, K)_{ring} \to \mathcal{H}om(\mathcal{E}^+_{\mathbf{R}},\, K)_{ring}$$

is given by $n \to |n|$, $n \in \mathbf{Z}$, *hence it is surjective.*

COROLLARY 4.5. *Under the identification* φ: $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times} \xrightarrow{\approx} \mathbf{Z}[[t]]$, *the alternate exterior algebra (cf. Example 2.1)*

$$\Lambda' = \sum_i (-1)^i \Lambda^i = \sum_i \Lambda^{2i} - \sum_i \Lambda^{2i+1}$$

and the exterior algebra $\Lambda = \sum_i \Lambda^i$ *correspond respectively to* $-t$ *and* $2+t$.

COROLLARY 4.6. *Under the identification* φ :

$$\mathcal{H}om(\mathcal{E}^+_{\mathbf{R}}, K)_{+,\times} \xrightarrow{\approx} \mathbf{Z}^{\pm}_e[[t]],$$

the power series $(1+t) - (1+t)^{-1}$, $(1+t) + (1+t)^{-1}$ *correspond respectively to the characteristic classes* $\tilde{\Lambda}^-$ *and* $\tilde{\Lambda}^+$ *introduced in Example 2.2. The image of* $\tilde{\Lambda}^-$ *on an even dimensional oriented real vector bundle is exactly the one obtained by the construction given in Chapter III, §6, by using Riemann metric.*

REMARK 4.1. It follows immediately from Corollaries 4.4 and 4.5 that the complexification of the alternate exterior algebra $\Lambda' \otimes \mathbf{C}$ and that of the exterior algebra $\Lambda \otimes \mathbf{C}$ correspond, under φ: $\mathcal{H}om(\mathcal{E}^+_{\mathbf{R}}, K)_{+,\times} \approx \mathbf{Z}^{\pm}_e[[t]]$, to the power series

$$\varphi(\Lambda' \otimes \mathbf{C}) = 2 - ((1+t) + (1+t)^{-1}),$$

$$\varphi(\Lambda \otimes \mathbf{C}) = 2 + (1+t) + (1+t)^{-1};$$

hence we have

$$\Lambda' \otimes \mathbf{C} = 2 - \tilde{\Lambda}^+ \quad \text{and} \quad \Lambda \otimes \mathbf{C} = 2 + \tilde{\Lambda}^+$$

respectively (cf. Convention 3.2). Moreover, the above correspondence between $\mathcal{H}om(\mathcal{E}, K)_{\mu}$ and the power series $\mathbf{Z}[[t]]$ shows that *there are other methods,* besides the exterior powers, *for obtaining a natural transformation from bundles to virtual bundles.* For example, in the case $\Lambda = \mathbf{Q}$ = the rational numbers, the power series $log(1+t)$ corresponds to an element of $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \mathbf{Q})_{+,\times}$ which will be useful later.

EXAMPLE 4.1. The characteristic class defined by «associate to each complex vector bundle its virtual class in K» corresponds, under φ:

$\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,+} \approx \mathbf{Z}[[t]]$, to $1+t$. And the Corollary 4.2 implies that the power series $(1+t)^{-1} = \frac{1}{1+t}$ corresponds to the characteristic class which associates to each complex vector bundle the virtual class of its dual (or conjugate) bundle.

5. Relation between $\mathcal{H}om(\mathcal{E}, K)$ and $\mathcal{H}om(\mathcal{E}, H^{2*})$.

We shall interpret the canonical map, defined by the composition of natural transformations as indicated in §2 (iv), after the identifications of the last two paragraphs. Here we use the same notations as before.

THEOREM 5.1. *Let the coefficients in H^{2*} be the rational numbers. Then under the identifications by φ and δ, the canonical compositions*

$$\mathcal{H}om(K, H^{2*})_{ring} \times \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times} \xrightarrow{\circ} \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times},$$

$$\mathcal{H}om(K, H^{2*})_{ring} \times \mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, K)_{+,\times} \xrightarrow{\circ} \mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, H^{2*})_{+,\times}$$

are given by $\beta \circ f(t) = f(e^{\beta t} - 1)$, the substitution of the power series $e^{\beta t} - 1$ into f, where $\beta \in \mathbf{Q}$, $f \in \mathbf{Z}[[t]]$ or $\mathbf{Z}^{\pm}_{e}[[t]]$. Moreover, it is additive with respect to f when β is kept fixed. Similarly, we may replace K by $K \otimes_{\mathbf{Z}} \mathbf{Q}$.

COROLLARY 5.1. *The canonical composition and the complexification commute with each other, i.e. the following diagram is commutative*

$$\begin{array}{ccc}
\mathcal{H}om(K, H^{2*})_{ring} \times \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times} & \xrightarrow{\circ} & \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times} \\
\downarrow {\scriptstyle 1 \times (\otimes \mathbf{C})} & & \downarrow {\scriptstyle \otimes \mathbf{C}} \\
\mathcal{H}om(K, H^{2*})_{ring} \times \mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, K)_{+,\times} & \xrightarrow{\circ} & \mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, H^{2*})_{+,\times}
\end{array}$$

Similarly we may replace K by $K \otimes_{\mathbf{Z}} \mathbf{Q}$.

COROLLARY 5.2. *The canonical composition and the conjugation commute with each other, i. e. the following diagram is commutative*

$$\begin{array}{ccc}
\mathcal{H}om(K, H^{2*})_{ring} \times \mathcal{H}om(\mathfrak{S}_{\mathbf{R}}^{+}, K)_{+,\times} & \xrightarrow{\circ} & \mathcal{H}om(\mathfrak{S}_{\mathbf{R}}^{+}, H^{2*})_{+,\times} \\
\Big\downarrow {\scriptstyle -\times 1 \text{ or } 1\times -} & & \Big\downarrow {\scriptstyle -} \\
\mathcal{H}om(K, H^{2*})_{ring} \times \mathcal{H}om(\mathfrak{S}_{\mathbf{R}}^{+}, K)_{+,\times} & \xrightarrow{\circ} & \mathcal{H}om(\mathfrak{S}_{\mathbf{R}}^{+}, H^{2*})_{+,\times}
\end{array}$$

(cf. Definition 3. 1 and Corollary 4. 3). In particular, if $f \in \mathcal{H}om(\mathfrak{S}_{\mathbf{R}}, K)_{+,\times}$ *is self-conjugate, then, for any* $\beta \in \mathbf{Q} \approx \mathcal{H}om(K, H^{2*})_{ring}$, *we have* $ch^{\beta} \circ f = ch^{-\beta} \circ f$ *(cf. Convention 3. 1 for notation).*

COROLLARY 5. 3. *For any* $f \in \mathcal{H}om(\mathfrak{S}_{\mathbf{C}}, K)_{+,\times}$, $ch^{\beta} \in \mathcal{H}om(K, H^{2*})_{ring}$, *we have* $ch^{\beta} \circ f = (ch \circ f)^{\beta}$ *(here we use the convention 3. 2 for the substitution of* βt, *cf. Corollary 3. 4).*

EXAMPLE 5.1. Consider the power series $log(1+t) \in \mathbf{Q}[[t]]$ and the characteristic class corresponding by φ:

$$f \in \mathcal{H}om(\mathfrak{S}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \mathbf{Q})_{+,\times}, \qquad \varphi(f) = log(1+t);$$

then it follows from Theorem 5.1 that $ch \circ f =$ top Chern class. Similarly, the composition of the Chern character $ch = ch^{1}$ with the class corresponding to the power series $1 + log(1+t)$ is the total Chern class.

REMARK 5. 1. The other composition maps can be obtained also; for example, the composition (cf. Lemma 2. 2 and Corollary 3. 1)

$$\mathcal{H}om(K, H^{2*})_{+,\times} \times \mathcal{H}om(\mathfrak{S}_{\mathbf{C}}, K)_{+,+} \longrightarrow \mathcal{H}om(\mathfrak{S}_{\mathbf{C}}, H^{2*})_{+,\times},$$

under the identifications φ and $\Psi : \Lambda_1[[t]] \times \mathbf{Z}[[t]] \to \Lambda[[t]]$, is given by the infinite product

$$g \circ f(t) = \prod_{m \overset{n}{\leq} n}^{\infty} g(mt)^{(-1)^m C_m^n \alpha_n},$$

where $g \in \Lambda_1[[t]]$, $f \in \mathbf{Z}[[t]]$,

$$f(t) = \alpha_o + \alpha_1 t + \ldots + \alpha_n t^n + \ldots,$$

and the C_m^n are the binomial coefficients. Upon restricting this map to $\mathcal{H}om(K, K)_{ring} \subseteq \mathcal{H}om(\mathfrak{S}_{\mathbf{C}}, K)_{+,+}$, we obtain the *characteristic class of* *a virtual bundle under the Adams' operation.* Similarly, the composition

$$\mathcal{H}om(K, H^{2*})_{+,+} \times \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,+} \xrightarrow{\circ} \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,+},$$

$$\mathcal{H}om(K, K)_{ring} \times \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times} \xrightarrow{\circ} \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times},$$

etc., are also determined under the identifications by φ, Ψ, where the first gives the relation between *Chern character and Adams' operations.* However, these are not directly related to our purpose, hence we leave the detail for elsewhere.

6. L-genus and Todd class.

We shall give another interpretation of the Todd class (resp. A-, L-genus) which is more closely related to the Λ', $\tilde{\Lambda}^{+}$, and $\tilde{\tilde{\Lambda}}^{-}$ elements of $\mathcal{H}om(\mathcal{E}, K)_{+,\times}$ obtained from the exterior power of a vector bundle (cf. Corollaries 4.5 and 4.6). Consider first the sub-semi-group of $\mathcal{H}om(\mathcal{E}, H)_{+,\times}$ defined in the following way: denoting by k a non-negative integer and by ξ (resp. η) any complex line bundle (resp. any oriented real plane bundle),

$$\mathcal{H}om^{\#}(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times} = \{ f \mid \exists\, \bar{f}, k; \mathfrak{m}(f.\bar{f})(\xi) = (\xi - 1)^k \},$$

$$\mathcal{H}om^{\#}(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times} = \{ f \mid \exists\, \bar{f}, k; \mathfrak{m}(f.\bar{f})(\xi) = (\alpha_{\xi})^k \},$$

$$\mathcal{H}om^{\#}(\mathcal{E}_{\mathbf{R}}^{+}, H^{2*})_{+,\times} = \{ f \mid \exists\, \bar{f}, k; \mathfrak{m}(f.\bar{f})(\eta) = (\alpha_{\eta})^k \},$$

where α_{ξ} (resp. α_{η}) the fundamental class of ξ (resp. η), and where we use the multiplication for the semi-group operation in $\mathcal{H}om$ (cf. Convention 3.2).

LEMMA 6.1. *For each $f \in \mathcal{H}om^{\#}$, there is a unique $\bar{f}$ in $\mathcal{H}om^{\#}$ which gives the minimum value of k and, therefore, we obtain a homomorphism of semi-groups*

$$\tau : \mathcal{H}om^{\#}(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times} \to \mathcal{H}om^{\#}(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times},$$

$$\tau : \mathcal{H}om^{\#}(\mathcal{E}, H^{2*})_{+,\times} \to \mathcal{H}om^{\#}(\mathcal{E}, H^{2*})_{+,\times}$$

(where $\mathcal{E} = \mathcal{E}_{\mathbf{C}}, \mathcal{E}_{\mathbf{R}}^{+}$). Moreover, the image « τf » is invertible, hence $\tau^2 f . \tau f = 1$, where $\tau^2 f = \tau(\tau f)$ and where 1 is the unity of the semi-group $\mathcal{H}om^{\#}$.

REMARK 6.1. The minimum k is just the order $\omega(f)$ of f (cf. Definition 4.1). $\mathcal{H}om^{\#}$ contains all the invertible elements of $\mathcal{H}om$ and, if Λ is a field, then $\mathcal{H}om^{\#}$ and $\mathcal{H}om$ coincide. We define the τ for $\mathcal{H}om(\mathcal{E}_{\mathbf{R}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times}$ in §8 .

It follows from the definition of τ that we have (cf. Definition 4.1):

COROLLARY 6.1. *For any* $f \in \mathcal{H}om^{\#}(\mathcal{E}_{\mathbf{C}}, K)_{+,\times}$ *and any complex vector bundle* ξ, *we have* $(f.\tau f)(\xi) = \chi(\xi)^{\omega(f)}$, *where* $\chi \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times}$ *is the Euler class,* $\omega(f)$ *the order of* f. *Similarly, if* $f \in \mathcal{H}om^{\#}(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$, *then we have* $(f.\tau f)(\xi) = (\text{top Chern class of } \xi)^{\omega(f)}$. *Finally we have* $\chi^{\omega(f)}.\tau^2 f = f$ *for any* $f \in \mathcal{H}om^{\#}(\mathcal{E}_{\mathbf{C}}, H)_{+,\times}$, *where* χ *is the Euler class if* $H = K \otimes_{\mathbf{Z}} \Lambda$, *and the top Chern class if* $H = H^{2*}$.

COROLLARY 6.2. *The complexification and* τ *are anti-commutative, i.e. the following diagram is commutative :*

$$\begin{array}{ccc}
\mathcal{H}om^{\#}(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times} & \xrightarrow{\otimes \mathbf{C}} & \mathcal{H}om^{\#}(\mathcal{E}^{+}_{\mathbf{R}}, H^{2*})_{+,\times} \\
\Big\downarrow \tau & & \Big\downarrow (-1)^{\frac{[\omega(f)]}{2}}.\tau \\
\mathcal{H}om^{\#}(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times} & \xrightarrow{\otimes \mathbf{C}} & \mathcal{H}om^{\#}(\mathcal{E}^{+}_{\mathbf{R}}, H^{2*})_{+,\times} .
\end{array}$$

REMARK 6.2. Here we use Convention 3.2 for $(-1)^k g$, $g \in \mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, H^{2*})_{+,\times}$, i.e. for every oriented real plane bundle η we have $((-1)^k g)(\eta) = (-1)^k g(\eta)$ in H^{2*}.

Let us recall that «o» denotes the canonical composition

$$\circ : \mathcal{H}om(K, H^{2*})_{ring} \times \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times} \to \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$$

and that, when the field of rational numbers $\mathbf{Q}$ is taken for coefficients in H^{2*},

$$ch^{\beta} \in \mathcal{H}om(K, H^{2*})_{ring}, \quad \beta \in \mathbf{Q},$$

denotes an arbitrary element (cf. Convention 3.1). Then we propose the following

DEFINITION 6.1. For each pair of elements $ch^{\beta} \in \mathcal{H}om(K, H^{2*})_{ring}$,

$f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times}$, the *$\tau$-class of (ch^{β}, f)* is defined to be the class $\tau(ch^{\beta} \circ f)$, i.e. the image of (ch^{β}, f) under the composition

$$\mathcal{H}om(K, H^{2*})_{ring} \times \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times} \xrightarrow{\circ} \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times} \xrightarrow{\tau} \\ \longrightarrow \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$$

(similar definition for $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \mathbf{Q})_{+,\times}$).

And this definition is justified by the

THEOREM 6.1. *The following equalities hold in* $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$

$$(1) \quad \tau(ch^{-1} \circ \Lambda') = \mathfrak{J}, \qquad (2) \quad \tau(ch \circ \Lambda') = -\overline{\mathfrak{J}},$$

where $\mathfrak{J}$ *is the Todd class,* Λ' *the alternate exterior algebra. In* $\mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, H^{2*})_{+,\times}$

$$(3) \quad 2\tau(ch \circ \tilde{\Lambda}^{-}) = A\text{-genus}, \qquad (4) \quad \tau(ch \circ \tilde{\Lambda}^{-} . \tau \tilde{\Lambda}^{+}) = L\text{-genus},$$

where $\tilde{\Lambda}^{-}, \tilde{\Lambda}^{+}$ *is defined in Example 2.2.*

REMARK 6.3. We may replace ch by ch^{β} in Theorem 6.1, e.g.

$$\tau(ch^{\beta} \circ \tilde{\Lambda}^{-} . \tau \tilde{\Lambda}^{+}) = \frac{1}{\beta} L^{\beta}, \quad \beta \neq 0, \ \beta \in \mathbf{Q},$$

but the present statement is convenient for later use.

Now the following corollary follows immediately from the definition of τ or Corollary 6.1.

COROLLARY 6.3. *In* $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$, $(ch^{-1} \circ \Lambda') . \mathfrak{J}$ = *top Chern class; similar equality for A-, L-genus.*

The following lemma gives the relation between «$\circ$» and «τ».

LEMMA 6.2. *If the rationals are taken for coefficients in* H^{2*}, *then the following diagram is commutative*

$$\begin{array}{ccc}
\mathcal{H}om(K, H^{2*})_{ring} \times \mathcal{H}om^{\#}(\mathcal{E}_{\mathbf{C}}, K)_{+,\times} & \xrightarrow{\quad\circ\quad} & \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times} \\
\Big\downarrow {\scriptstyle 1 \times (\tau \times \omega) \circ \Delta} & & \Big\downarrow {\scriptstyle \tau} \\
\mathcal{H}om(K, H^{2*})_{ring} \times \mathcal{H}om^{\#}(\mathcal{E}_{\mathbf{C}}, K)_{+,\times} \times \mathbf{Z}^{+} & \xrightarrow{\quad\circ'\quad} & \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}
\end{array}$$

where Δ *is the diagonal of* $\mathcal{H}om^{\#}$, *where* ω, *the «order», maps* $\mathcal{H}om^{\#}$ *into* *the non-negative integers* $\mathbf{Z}^+$ *and where* $\circ'$ *is defined by*

$$\circ'(ch^{\beta}, f, n) = \tau(ch^{\beta}\circ\chi)^n.(ch^{\beta}\circ f),$$

χ *being the Euler class. In particular, if* f *is inversible, then we have* $\tau(ch^{\beta}\circ f) = ch^{\beta}\circ\tau f$.

This implies, in particular, the (cf. Remark 6.3)

COROLLARY 6.4. *In* $\mathcal{H}om(\mathcal{E}^+_{\mathbf{R}}, H^{2*})_{+,\times}$ *we have*

$$\begin{aligned} 2L^{\beta/2} &= \tau(ch^{\beta/2}\circ\tilde{\Lambda}^-.\tau\tilde{\Lambda}^+) \\ &= \tau^2(ch^{\beta}\circ\tilde{\Lambda}^-).\tau(ch^{\beta}\circ(\tilde{\Lambda}^+-2)) \\ &= -\tau^2(ch^{\beta}\circ\tilde{\Lambda}^-).\tau(ch^{-\beta}\circ\Lambda'\otimes\mathbf{C}). \end{aligned}$$

7. Application to topological index of classical elliptic operators.

Given a $\mathcal{E}^+_{\mathbf{R}}$-pair (resp. $\mathcal{E}_{\mathbf{C}}$-pair) (X,η), where η is a $2n$-dimensional oriented real vector bundle (resp. complex vector bundle) over a finite CW-complex X, we shall denote by $t(\eta)$ and

$$\Phi = \Phi_{\eta} : H^*(t(\eta)) \to H^*(X)$$

the Thom space and the isomorphism of η (resp. decomplexification of η), and by $p : K(t(\eta)) \to K(X)$ the homomorphism induced by the zero section of η. We shall define a subset $U(t(\eta)) \subseteq K(t(\eta))$ of $K(t(\eta))$, namely the *Universal elements of the pair* (X,η), as follows : Consider those representations $\rho : G \to SO(2n)$ (resp. $U(n)$), where G is a compact connected Lie group such that :

(1) the image $\rho(G)$ has its rank maximum,

(2) ρ induces η, i.e. in $\mathcal{E}^+_{\mathbf{R}}(X)$, $\eta = h^*_{\rho}(\rho^*\eta_o)$ for some map h_{ρ} from X into the classifying space B_G, where η_o is the universal $2n$-oriented (resp. n-complex) bundle over $B_{SO(2n)}$. Then we take the union

$$U(t(\eta)) = \cup\, h^*_{\rho} K(t(\rho^*\eta_o))$$

over all ρ and h_{ρ} of the image of h^*_{ρ}. Remark that $U(t(\eta))$ is non-empty, because one can take $G = SO(2n)$ or a maximal torus of $SO(2n)$. Now let $\theta \in \mathcal{H}om(\mathcal{E}^+_{\mathbf{R}}, K)_{\mu}$ (resp. $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{\mu}$), then the subset of $U(t(\eta))$

$$U_\theta(t(\eta)) = \cup\, h^*_\rho K_\theta(t(\rho^*\eta_o)),$$
$$\text{where } K_\theta = p^{-1}(\theta(\rho^*\eta_o)) \subseteq K(t(\rho^*\eta_o)),$$

is called *θ-universal elements.*

REMARK 7.1. We may replace K by $K \otimes_{\mathbf{Z}} \Lambda$ or KO, etc..

We can define an *elliptic operator to be θ-universal* if its symbol is contained in $U_\theta(t(\eta))$. Then it is easy to see that, for an even dimensional oriented Riemann manifold, the operator $d+\delta$ is $\Lambda' \otimes \mathbf{C} = 2-\tilde{\Lambda}^+$ universal (cf. Corollary 4.5 and Chapter III). Similarly, for a complex analytic manifold, the operator $\bar{\partial} + \mathcal{V}$ (cf. Chapter IV) is Λ'-universal, where Λ' is the alternate exterior algebra $\Lambda' \in \mathcal{H}om(\mathfrak{S}_{\mathbf{C}}, K)_{+,\times}$.

Now given a characteristic class f in $\mathcal{H}om(K, H^{2*})$ and an $\mathfrak{S}$-pair (X,η), we want to compute the composite

$$\Phi . f : K(t(\eta)) \xrightarrow{f_{t(\eta)}} H^{2*}(t(\eta)) \xrightarrow{\Phi_\eta} H^{2*}(X)$$

of f with Thom's isomorphism. For θ-universal elements, this can be done by finding another $f^{(\theta)} \in \mathcal{H}om(\mathfrak{S}, H^{2*})$ which gives the required value on $\mathfrak{S}(X)$ without passing to the Thom space; more precisely, we need :

(3) For each $\mathfrak{S}$-pair (X,η) and each $\delta \in U_\theta(t(\eta))$, $\Phi . f(\delta) = f^{(\theta)}(\eta)$, i.e. on θ-universal elements $\Phi . f$ is constant, as one might have expected.

Suppose $\Lambda = \mathbf{Q}$ for the rest of this paragraph; then we have :

PROPOSITION 7.1. *Let $\theta \in \mathcal{H}om(\mathfrak{S}, K)_\mu$ and $f \in \mathcal{H}om(K, H^{2*})_\nu$ be such that their composite $f \circ \theta$ is in $\mathcal{H}om(\mathfrak{S}, H^{2*})_{+,\times}$ and of order greater than one : $\omega(f\circ\theta) \geq 1$. Then there exists a unique $f^{(\theta)} \in \mathcal{H}om(\mathfrak{S}, H^{2*})_{+,\times}$ satisfying the condition (3). In fact, it is given by*

$$f^{(\theta)} = \chi^{\omega-1} . \tau^2(f\circ\theta), \qquad \omega = \omega(f\circ\theta),$$

where χ is the Euler class in $\mathcal{H}om(\mathfrak{S}, H^{2*})_{+,\times}$, $\tau^2 = \tau\circ\tau$ the homomorphism in Lemma 6.1 and $\mathfrak{S} = \mathfrak{S}_{\mathbf{C}}, \mathfrak{S}^+_{\mathbf{R}}$. In particular, if the order $\omega(f\circ\theta)$ is equal to one, we have $f^{(\theta)} = \tau^2(f\circ\theta)$.

On the other hand, we have

LEMMA 7.1. *Let f be any element in $\mathcal{H}om(K, H^{2*})_{ring}$ and (X,η) a*

$\mathcal{E}$-pair ($\mathcal{E} = \mathcal{E}_{\mathbf{C}}, \mathcal{E}_{\mathbf{R}}^{+}$). Then for any $\xi \in K(\eta)$, $\delta \in K(t(\eta))$, we have

$$\Phi \circ f(\xi . \delta) = f(\pi^{-1}\xi) \cup \Phi . f(\delta)$$

in $H^{2*}(X)$, where $\pi : K(X) \underset{\approx}{\to} K(\eta)$ is the isomorphism induced by the the projection of the bundle η and where we use the «$\cup$» for the H^{2*} product.

Now this gives another approach to the relation between the topological index i_t of classical elliptic operators and the characteristic classes of a manifold X. Remark, first, that in the definition of i_t we can replace the factor of the Todd class of the tangent bundle by the *cotangent bundle* $T^*(K)$ because they are isomorphic; hence if D is an elliptic operator from a complex vector bundle E_1 to another E_2, the topological index may be defined as

$$i_t(D) = \{ch(D) . \mathcal{J}(T^*(X) \otimes \mathbf{C})\}[X].$$

Moreover, the above lemma shows that *it is sufficient to study the operators without coefficient bundle* (cf. Chapter III, § 5.2). In fact, if F is another complex vector bundle and D is of order 1, then from the relation of symbols (where we use the same notation in Chapter IV, §8)

$$\sigma_1(D \hat{\otimes} 1_F) = \sigma_1(D) \hat{\otimes} \sigma_o(1_F),$$

we deduce (cf. Chapter II, § 3.2) that, in $K(t(T^*(X)))$,

$$[\sigma_1(D \hat{\otimes} 1_F)] = \pi^* F . [\sigma_1(D)],$$

where we use the same notation in Chapter I-4. Taking $\eta = T^*(X)$, $f = ch$, $\xi = \pi^* F$, $\delta = [\sigma_1(D)]$ in Lemma 7.1, we obtain

$$ch(D \hat{\otimes} 1_F) = ch(F) \cup ch(D);$$

and our assertion follows.

EXAMPLE 7.1. Consider the following special case of Theorem 7.1 :

$$\mathcal{E} = \mathcal{E}_{\mathbf{R}}^{+}, \ f = ch \ \text{and} \ \theta = \Lambda' \otimes \mathbf{C} \in \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K)_{+,\times}$$

the complexification of the alternate exterior algebra. Then it is easy to see, from Corollaries 4.3, 4.4, and Theorem 5.1, that the order of $f \circ \theta$ is equal to 2. Take any $\mathcal{E}_{\mathbf{R}}^{+}$-pair (X, η) and any elements $\delta \in U_{\theta}(t(\eta))$, then it follows from Proposition 7.1, Corollaries 3.3, 5.1, 6.2, Theorem 6.1, and Lemma 6.1 that the following equality is valid in $H^{2*}(X, \mathbf{Q})$:

$$\begin{aligned}\Phi \circ ch(\delta) \cup \mathfrak{I}(\eta \otimes \mathbf{C}) &= \chi(\eta) \cup \tau^2 (ch \circ \Lambda' \otimes \mathbf{C})(\eta) \cup \overline{\mathfrak{I}}(\eta \otimes \mathbf{C}) \\ &= \chi(\eta) \cup (-\tau^2 (ch \circ \Lambda')(\eta \otimes \mathbf{C})) \cup \overline{\mathfrak{I}}(\eta \otimes \mathbf{C}) \\ &= \chi(\eta) \cup (\tau^2 (ch \circ \Lambda').(-\overline{\mathfrak{I}}))(\eta \otimes \mathbf{C}) \\ &= \chi(\eta) \cup 1(\eta \otimes \mathbf{C}) = \chi(\eta),\end{aligned}$$

where we use $\cup$ for the cup-product in $H^{2*}(X)$ to distinguish the multiplication of the semi-group $\mathcal{H}om(\mathcal{E}^+_{\mathbf{R}}, H^{2*})_{+,\times}$, and 1 for its unity which, applied to any bundle, is equal to the $1 \in H^0(X)$. In particular we have :

THEOREM 7.1. *If X is an even dimensional oriented Riemann manifold and D a $\Lambda' \otimes \mathbf{C}$- universal elliptic operator, then the topological index of D is given by*

$$i_t(D) = \chi(T(X))[X].$$

PROOF. Take $\eta = T^*(X)$ in the above example.

EXAMPLE 7.2. Consider the special case of Proposition 7.1 :

$$\mathcal{E} = \mathcal{E}_{\mathbf{C}}, \quad f = ch \quad \text{and} \quad \theta = \Lambda' \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times}$$

the alternate exterior algebra. Then it is easy to see, from Corollary 4.4, Theorem 5.1, that the order of $f \circ \theta = ch \circ \Lambda'$ is equal to 1. Hence for any $\mathcal{E}_{\mathbf{C}}$-pair (X, η), and any elements $\delta \in U_\theta(t(\eta))$, we have, from Propositions 7.1 and 6.1:

$$\begin{aligned}\Phi \circ ch(\delta) \cup \mathfrak{I}(\eta \otimes \mathbf{C}) &= \Phi \circ ch(\delta) \cup \overline{\mathfrak{I}}(\eta) \cup \mathfrak{I}(\eta) \\ &= (\tau^2 (ch \circ \Lambda') . \overline{\mathfrak{I}} . \mathfrak{I})(\eta) = (-\mathfrak{I})(\eta)\end{aligned}$$

in $H^{2*}(X)$. In particular, we have :

THEOREM 7.2. *If X is a complex analytic manifold and D a Λ'-universal elliptic operator, then the topological index of D is given by $i_t(D) =$ $= \mathfrak{I}(T(X))[X]$.*

PROOF. Take $\eta = T^*(X) =$ the complex dual of the complex tangent bundle of X in the above example. Then it follows from Example 3.1 that

$$(-\mathcal{J})(T^*(X)) = (-1)^n \mathcal{J}(T^*(X)), \quad n = \dim_{\mathbf{C}} X$$

and (or Corollaries 3.2, 3.4) the $2n$-dimensional component of $\mathcal{J}(T^*(X))$ equals $(-1)^n$ times the $2n$-dimensional component of $\mathcal{J}(T(X))$; hence

$$(-\mathcal{J})(T^*(X))[X] = \mathcal{J}(T(X))[X].$$

EXAMPLE 7.3. Consider the special case of Proposition 7.1 :

$$\mathcal{E} = \mathcal{E}^+_{\mathbf{R}}, \quad f = ch \quad \text{and} \quad \theta = \tilde{\Lambda}^- \in \mathcal{H}om(\mathcal{E}^+_{\mathbf{R}}, K \otimes_{\mathbf{Z}} \mathbf{Q})_{+,\times}$$

(cf. Corollary 4.5). Then it follows easily from Theorem 5.1 and Corollary 4.5 that the order of $f \circ \theta = ch \circ \tilde{\Lambda}^-$ is equal to 1. Hence, for any $\mathcal{E}^+_{\mathbf{R}}$-pair (X, η) and every element $\delta \in U_\theta(t(\eta))$, we have, according to Proposition 7.1 and Corollaries 6.2, 5.2,

$$\Phi \circ ch(\delta) \cup \mathcal{J}(\eta \otimes \mathbf{C}) = \tau^2 (ch \circ \tilde{\Lambda}^-)(\eta) \cup \tau(ch^{-1} \circ \Lambda' \otimes \mathbf{C})(\eta)$$
$$= (\tau^2 (ch \circ \tilde{\Lambda}^-) . \tau(ch^{-1} \circ \Lambda' \otimes \mathbf{C}))(\eta) = (-2 L^{\frac{1}{2}})(\eta)$$

in $H^{2*}(X)$. In particular, we have :

THEOREM 7.3. *If X is an even dimensional oriented Riemann manifold and D a $\tilde{\Lambda}^-$-universal elliptic operator on X, then the topological index of D is given by $i_t(D) = L(T(X))[X]$.*

PROOF. Take $\eta = T^*(X)$ the cotangent bundle of X in the above example. Then from Example 3.1 and Corollary 3.4, we have, in $H^{2n}(X)$, $2n = \dim X$:

$2n$-dimensional component of $(-2L^{\frac{1}{2}})(T^*(X))$

$= (-2)^n$ times the $2n$-dimensional component of $L^{\frac{1}{2}}(T^*(X))$

$= (-2)^n . (-\frac{1}{2})^n$ times the $2n$-dimensional component of $L(T(X))$

$=$ $2n$-dimensional component of $L(T(X))$.

8. Final remarks.

We like to point out that the method introduced before may be equally useful to study the characteristic classes of other kinds of vector bundles and the relationship between them. However we shall restrict ourselves only to some cases which are more or less related to the forthcoming paragraphs. The complete details and the proof of what follows will appear elsewhere in the future.

The $\mathcal{H}om(\mathcal{E}, H)_{\mu}$ can be determined in general : For $\mathcal{E}^{o}_{\mathbf{R}}$, the functor of real oriented bundles, H^{2*}, and Λ the coefficient ring which is an integral domain and contains $\frac{1}{2}$, we have the analogue of theorem 3.2 :

Let $(\Lambda[[t]] \times \Lambda)^{o}_{\times}$ be the sub-semi-group of the product

$$\Lambda[[t]]_{\times} \times \Lambda_{\times}$$

of semi-groups, defined by the union

$$(\Lambda[[t]] \times \Lambda)^{o}_{\times} = (\Lambda^{\pm}[[t]] \times \{0\}) \cup (\Lambda^{+}[[t]] \times \Lambda^{*}),$$

where $\{0\}$ is the zero of Λ and $\Lambda^{*} = \Lambda - \{0\}$ (c.f. §1 (ii)). Then there exists a unique isomorphism of semi-groups

$$\varphi = \varphi_1 \times \varphi_2 : \mathcal{H}om(\mathcal{E}^{o}_{\mathbf{R}}, H^{2*})_{+,\times} \overset{\approx}{\to} (\Lambda[[t]] \times \Lambda)^{o}_{\times}$$

such that, for each $f \in \mathcal{H}om(\mathcal{E}^{o}_{\mathbf{R}}, H^{2*})_{+,\times}$, we have

$$f(\eta) = \varphi_1(f)(\alpha_{\eta}), \qquad f(1) = \varphi_2(f),$$

where η is an arbitrary oriented plane bundle, α_{η} its fundamental class, and 1 the trivial real line bundle.

The relation between $\mathcal{E}^{+}_{\mathbf{R}}$ and $\mathcal{E}^{o}_{\mathbf{R}}$ follows from the fact that the canonical homomorphism

$$\mathcal{H}om(\mathcal{E}^{o}_{\mathbf{R}}, H^{2*})_{+,\times} \to \mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, H^{2*})_{+,\times},$$

induced by the inclusion $\mathcal{E}^{+}_{\mathbf{R}} \subseteq \mathcal{E}^{o}_{\mathbf{R}}$, is given under the identification φ (c.f. Theorem 3.2) by

$$p_1 : (\Lambda[[t]] \times \Lambda)^{o}_{\times} \to \Lambda^{\pm}[[t]]_{\times},$$

the canonical projection of the first factor.

Similarly, the case $\mathcal{E}^{+}_{\mathbf{R}}$ and KO is given by the following analogue of Theorem 4.2 :

There is a unique isomorphism of semi-groups

$$\varphi : \mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, KO)_{+,\times} \overset{\approx}{\to} \mathbf{Z}[[t]]_{\times}$$

satisfying the following condition : For each oriented real plane bundle η and any $f \in \mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, KO)_{+,\times}$, we have $f(\eta) = \varphi(f)(\eta - 2)$, when 2 is the trivial real plane bundle. Moreover the complexification $KO \to K$ induces a homomorphism

$$\begin{array}{ccc} \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, KO)_{+,\times} & \xrightarrow{\otimes \mathbf{C}} & \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K)_{+,\times} \\ \downarrow & & \downarrow \\ \mathbf{Z}[[t]] & \longrightarrow & \mathbf{Z}_e^{\pm}[[t]] \end{array}$$

which is given by

$$f \otimes \mathbf{C}(t) = f(1 + t + \frac{1}{1+t} - 2)$$

under the identification φ (c.f. Theorem 4.2).

The alternate exterior algebra of a real bundle η,

$$\Lambda'_{\mathbf{R}}(\eta) = \sum_i (-1)^i \Lambda^i \eta,$$

(resp. the exterior algebra $\Lambda_{\mathbf{R}}$), corresponds, under φ, to the power series

$$\varphi(\Lambda'_{\mathbf{R}}) = -t, \qquad \varphi(\Lambda_{\mathbf{R}}) = 4 + t,$$

Hence $\Lambda'_{\mathbf{R}} \otimes \mathbf{C}$ corresponds to $2 - (1 + t + \frac{1}{1+t})$, which is equal to $\Lambda' \otimes \mathbf{C}$, the complexification of the (complex) alternate exterior algebra Λ' (c.f. Corollaries 4.3 and 4.4)(Similarly for $\Lambda_{\mathbf{R}}$).

Several relationships between the semi-groups of natural transformations can be obtained just as Theorem 5.1; for example, consider the case $\mathcal{E} = \mathcal{E}_{\mathbf{C}}$ and $H = K$; then under the identification φ and δ (c.f. Theorem 4.1) the canonical composition

$$\begin{array}{ccc} \mathcal{H}om(K, K)_{ring} \times \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times} & \xrightarrow{\circ} & \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times} \\ \downarrow & & \downarrow \\ (\mathbf{Z} \cup \{e_o\}) \times \mathbf{Z}[[t]] & \xrightarrow{\circ} & \mathbf{Z}[[t]] \end{array}$$

is given by

$$\begin{cases} (k \circ f)(t) = f((1+t)^k - 1), & k \in \mathbf{Z}, \\ (e_o \circ f)(t) = 0. \end{cases}$$

And one may replace $\mathcal{E}_{\mathbf{C}}$ by $\mathcal{E}_{\mathbf{R}}^{+}$ and $\mathbf{Z}[[t]]$ by $\mathbf{Z}_e^{\pm}[[t]]$ without modification on the resulting equation.

Now for each $\mathcal{E}_{\mathbf{C}}$-pair (X, ξ), let us denote by u_ξ the analogue Thom class in K-theory and the Thom homomorphism $i_! : K(X) \to K(t(\xi))$ defined by Bott [7]. If f is a stable characteristic class $f \in \mathcal{H}om(K, H)$, then the value on u_ξ of the composite of f with the corresponding Thom homomorphism for H on u_ξ defines an element of $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H)$. In particular take $H = K$; we obtain a map u :

$$\begin{array}{ccc} \mathcal{H}om(K, K)_{ring} & \xrightarrow{\ u\ } & \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times} \\ \Big\downarrow \delta & & \Big\downarrow \varphi \\ \mathbf{Z} \cup \{e_o\} & \longrightarrow & \mathbf{Z}[[t]] \end{array}$$

defined by $u(f)(\xi) = i_!^{-1} f(u_\xi)$, where ξ is any complex vector bundle and $f \in \mathcal{H}om(K, K)_{ring}$. Then, under the identifications δ and φ of Theorem 4.1, u is given by :

$$u(k) = 1 + (1+t) + \ldots + (1+t)^{k-1}, \quad k > 0,$$

$$u(k) = 1 + (1+t)^{-1} + \ldots + (1+t)^{k+1}, \quad k < 0,$$

$$u(0) = 1, \qquad u(e_o) = 0 .$$

And the following «cocycle condition» [7] is immediate from the composition «$\circ$» :

$$u(k.h) = (k \circ u(h)).u(k),$$

where $k, h, k.h \in \mathbf{Z}$, and $u(k.h), k \circ u(h), u(k) \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times}$ (c.f. Remark 3.2).

Similarly, the analogue of *Wu*'s construction given in [12] can be written down easily from the composition «$\circ$», i.e. for each $f \in \mathcal{H}om(K,K)_{ring}$ invertible, we define

$$Wu(f, \xi) = f^{-1} \circ i_!^{-1} \circ f(u_\xi),$$

for each complex vector bundle ξ. This gives a map

$$\begin{array}{ccc} (\mathcal{H}om(K, K)_{ring} - \{0\}) & \xrightarrow{\ Wu\ } & \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times} \\ \delta \Big\downarrow & & \Big\downarrow \varphi \\ \mathbf{Z} & \longrightarrow & \mathbf{Z}[[t]] \end{array}$$

where «0» is the unity of the semi-group $\mathcal{H}om(K, K)_{+,+}$. Then, under the identifications δ and φ, Wu is given by

$$Wu(k) = 1 + \frac{1}{(1+t)^{|k|}} + \ldots + \frac{1}{(1+t)^{|k|(|k|-1)}},$$

where $|k|$ denotes the absolute value of the integer $k \neq 0$; and some other parts of Bott, Atiyah-Hirzebruch works can also be interpreted in this way.

The homomorphism τ of §6 can be introduced in other cases. We shall consider here the $\mathcal{E}^+_{\mathbf{R}}$. For simplicity, let us consider only those $f \in \mathcal{H}om(\mathcal{E}^+_{\mathbf{R}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times}$ whose order is even, $\omega(f) = 2q$, and the $2q$-th coefficient in $\varphi(f)$ is invertible in Λ. All such f form evidently a sub-semi-group called $\mathcal{H}om^{\#}(\mathcal{E}^+_{\mathbf{R}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times}$. Then a homomorphism τ, analogue to that of Theorem 6.1,

$$\tau : \mathcal{H}om^{\#}(\mathcal{E}^+_{\mathbf{R}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times} \to \mathcal{H}om^{\#}(\mathcal{E}^+_{\mathbf{R}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times}$$

can be defined directly by the isomorphism φ of Theorem 4.2 as

$$\varphi(\tau(f)) = \frac{t^{2q}}{\varphi(f)} \cdot (-1 + t - t^2 + t^3 - \ldots)^q .$$

And the following relation is satisfied for all $f \in \mathcal{H}om^{\#}(\mathcal{E}^+_{\mathbf{R}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times}$:

$$\tau f . f = (\Lambda' \otimes \mathbf{C})^q ,$$

where Λ' is the complex alternate exterior algebra and where product is used for the operation of the semi-group $\mathcal{H}om^{\#}$. Similarly the other properties in §6 can also be obtained; for example the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{H}om^{\#}(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times} & \xrightarrow{\otimes \mathbf{C}} & \mathcal{H}om^{\#}(\mathcal{E}^+_{\mathbf{R}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times} \\
\downarrow \tau & & \downarrow \tau \\
\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times} & \xrightarrow{\otimes \mathbf{C}} & \mathcal{H}om^{\#}(\mathcal{E}^+_{\mathbf{R}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times}
\end{array}$$

where the first τ is that defined in §6.

It can be seen that, between $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$ (resp. $+,+$) and $\Lambda[[t]]$ (c.f. Theorems 3.1 and 4.1 and their proofs) the isomorphism corresponds to the elementary symmetric functions

$$\sigma_n = x_1 \dots x_n \quad (\text{resp. } \sigma_1 = x_1 + x_2 + \dots + x_n).$$

In fact, to each elementary symmetric function σ_k, there corresponds a map φ'_{σ_k} from $\Lambda[[t]]$ into $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})$ (resp. K), hence there is defined characteristic classes $\varphi'_{\sigma_k}(f)$ which are useful to study the relations between classical characteristic classes.

Characteristic classes of two or several variables of vector bundles (e.g. tensor product of two bundles) can also be studied in this manner, as well as other kinds of canonical algebraic constructions of associated bundles can be written down by the identifications φ or Ψ (e.g. the k-th symmetric product of a vector bundle,etc.), and several analogues of proposition 7.1 may be established (e.g. given two characteristic classes, find the corresponding lifting class into the Thom space). Hence it seems not a bad idea to formulate the characteristic classes under this form, in detail.

9. The proofs.

Before the proofs, the following elementary remarks will be useful.

Let $\mathcal{C}$ be the category of CW-complexes and $j : \mathcal{F} \to \mathcal{C}$ the inclusion functor of finite CW-complexes $\mathcal{F}$ into $\mathcal{C}$. The functors $\mathcal{E}_{\mathbf{C}}$, $\mathcal{E}^+_{\mathbf{R}}$, H^{2*} given in § 1 are again well defined on $\mathcal{C}$. As in [3] we denote by $\mathcal{K}$ the functor in $\mathcal{C}$ defined by

$$\mathcal{K}(X) = \varprojlim_n K(X^n), \quad X \in \mathcal{C},$$

where X^n is the n-skeleton of X. Denote for a moment by $\mathcal{H}om_{\mathcal{C}}$ (resp. $\mathcal{H}om_{\mathcal{F}}$) the natural transformations to distinguish in which category the functors $\mathcal{E}$ and H are considered. Denote by

$$\gamma : \mathcal{H}om_{\mathcal{C}}(\mathcal{E}, H)_\mu \to \mathcal{H}om_{\mathcal{F}}(\mathcal{E}, H)_\mu$$

the restriction defined by the composition $f \circ j$ for $f \in \mathcal{H}om(\mathcal{E}, H)_\mu$.

LEMMA 9.1. *The restriction γ is an isomorphism of semi-groups, where $\mathcal{E} = \mathcal{E}_{\mathbf{C}}$, $\mathcal{E}^+_{\mathbf{R}}$, etc. and $H = \mathcal{K}$, H^{2*}, etc...*

PROOF. As the reasoning is exactly the same for every case, it is sufficient to treat the case $H = \mathcal{K}$, i.e. to prove the isomorphism

$$\gamma : \mathcal{H}om_{\mathcal{C}}(\mathcal{E}, \mathcal{K})_{\mu} \approx \mathcal{H}om_{\mathcal{F}}(\mathcal{E}, K)_{\mu},$$

because the restriction of $\mathcal{K}$ on $\mathcal{F}$ is just K. Let p denote the canonical homomorphism $p_X : \mathcal{E}(X) \to \varprojlim_n \mathcal{E}(X^n)$ and define

$$l : \mathcal{H}om_{\mathcal{F}}(\mathcal{E}, K)_{\mu} \to \mathcal{H}om_{\mathcal{C}}(\mathcal{E}, \mathcal{K})_{\mu}$$

by

$$l(g)_X = (\varprojlim_n g_{X^n}) \circ p_X, \quad X \in \mathcal{C},$$

where

$$\varprojlim_n g_{X^n} : \varprojlim_n \mathcal{E}(X^n) \to \varprojlim_n K(X^n) = \mathcal{K}(X), \quad g \in \mathcal{H}om_{\mathcal{F}}.$$

Then it is clear that $\gamma \circ l = 1$. Moreover γ is injective, because the naturality implies that every $f \in \mathcal{H}om_{\mathcal{C}}(\mathcal{E}, \mathcal{K})_{\mu}$ satisfies $f_X = (\varprojlim_n f_{X^n}) \circ p_X$. Now $\gamma \circ l = 1$ implies $\gamma \circ l \circ \gamma = \gamma \circ 1$, and the injectivity of γ gives $l \circ \gamma = 1$.

Q.e.d.

This lemma permits us to use freely $\mathcal{C}$ or $\mathcal{F}$ in the proofs; in fact we shall do this even without mentioning it each time.

LEMMA 9.2. *If $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H)$ denotes the set of all natural transformations, then we have a canonical identification*

$$j : \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H) \approx \prod_{n \geq 0} H(B_{U(n)}),$$

where $B_{U(n)}$ is the classifying space of the unitary group of dimension n (similarly for $\mathcal{E}^+_{\mathbf{R}}$, etc.. and H arbitrary K, H^{2*}, etc., i.e. contravariant functors), defined for each $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H)$ by

$$j(f) = \prod_{n \geq 0} f_{B_{U(n)}}(\xi^o_n),$$

where ξ^o_n is the n-dimensional universal complex vector bundle over $B_{U(n)}$. (Similarly η^o_{2n} the 2n-dimensional oriented real universal vector bundle over $B_{SO(2n)}$).

Although this lemma is a trivial consequence of the definition of classifying space, it tells us that what we have to do is to find the corresponding subset of $\mathcal{H}om(\mathcal{E}, H)$ which satisfies the additional property $\mu = (+,+),(+,\times)$ etc., by working out $B_{U(n)}$. In particular to construct a «f», it is sufficient to give its value on ξ^o_n (resp. η^o_{2n}) and check the property μ. Using the same notation as in lemma 9.2, we have :

COROLLARY 9.1. *Let f and g be two elements of $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H)_\mu$, where $\mu = (+,\times),(+,+)$ (resp. $\mathcal{E}^+_{\mathbf{R}}$); and suppose that the functor H satisfies the condition that the map induced by the inclusion of a maximal torus T_n, into $U(n)$*

$$(1) \qquad j : H(B_{U(n)}) \to H(B_{T_n})$$

(resp. $B_{SO(2n)}$) is a monomorphism. Then $f = g$ if and only if

$$f_{B_{U(1)}}(\xi^o_1) = g_{B_{U(1)}}(\xi^o_1) \quad (resp.\ \eta^o_2).$$

This is an immediate consequence of the (notation as in lemma 9.2)

LEMMA 9.3. *In $\mathcal{E}_{\mathbf{C}}(B_{T_n})$ (resp. $\mathcal{E}^+_{\mathbf{R}}(B_{T_n})$), we have*

$$j^*(\xi^o_n) = \xi^1_1 + \ldots + \xi^n_1 \quad (resp.\ j^*(\eta^o_{2n}) = \eta^1_2 + \ldots + \eta^n_2),$$

where j^* is induced by the inclusion of the maximal torus T_n into $U(n)$ (resp. $SO(2n)$) on the $\mathcal{E}_{\mathbf{C}}$ (resp. $\mathcal{E}^+_{\mathbf{R}}$) of the classifying spaces, and

$$\xi^i_1 = p^*_i(\xi^o_1) \quad (resp.\ \eta^i_2 = p^*_i(\eta^o_2)),$$

where p_i is the i-th projection (c.f. (A) below) :

$$B_{T_n} = \underbrace{B_{T_1} \times \ldots \times B_{T_1}}_{n} \to B_{T_1} = B_T$$

PROOF. This is a consequence of the definition of an associated bundle and of the fact : The pull back by j of the principal universal bundle of the group $U(n)$ (resp. $SO(2n)$) is isomorphic to the principal bundle obtained by «extension of the structure group by j to $U(n)$ from the principal universal bundle of the group T_n».

Q.e.d.

COROLLARY 9.2. *Let $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H)_{+,+}$ (resp. $\mathcal{E}^+_{\mathbf{R}}$) and let H be a functor of semi-rings satisfying the condition (1) of Corollary 9.1. Then f*

is a natural transformation of semi-rings if and only if

$$f_{B_{T_2}}(\xi_1^1 \otimes \xi_1^2) = f_{B_{T_2}}(\xi_1^1) \cdot f_{B_{T_2}}(\xi_1^2) \quad (resp.\ \eta_2^i),$$

where we use the same notation as in Lemma 9.3.

PROOF. From the bilinear property of tensor product and the additivity of f, i.e. $\mu = (+, +)$, it follows from the hypothesis that

$$f((\xi_1^1 + \dots + \xi_1^n) \otimes (\zeta_1^1 + \dots + \zeta_1^m)) = f(\xi_1^1 + \dots + \xi_1^n) \cdot f(\zeta_1^1 + \dots + \zeta_1^m)$$

for any complex line bundles ξ_1^i, ζ_1^j. Then the lemma 3, the condition 1 in Corollary 9.1 and the commutative diagram

$$\begin{array}{ccccc} B_{T_n} & \xrightarrow{\Delta} & B_{T_n} \times B_{T_n} & \xrightarrow{\otimes} & B_{T_{n^2}} \\ \big\downarrow j & & \big\downarrow j & & \big\downarrow \\ B_{U(n)} & \xrightarrow{\Delta} & B_{U(n)} \times B_{U(n)} & \xrightarrow{\otimes} & B_{U(n^2)} \end{array}$$

(where Δ denotes the diagonal) implies

$$f_{B_{U(n)}}(\xi_n^o \otimes \xi_n^o) = f_{B_{U(n)}}(\xi_n^o) \cdot f_{B_{U(n)}}(\xi_n^o),$$

because $\xi_n^o \otimes \xi_n^o$ is the pull back of the universal complex vector bundle over $B_{U(n^2)}$ by the composite map $\otimes \circ \Delta$. And the result follows from Lemma 2 and the commutativity of the diagram

$$\begin{array}{ccc} B_{U(m)} \times B_{U(n)} & \longrightarrow & B_{U(mn)} \\ \big\downarrow & & \big\downarrow \\ B_{U(n)} \times B_{U(n)} & \longrightarrow & B_{U(n^2)} \end{array}$$

for $m \leq n$.

Q.e.d.

LEMMA 9.4. *The formal power series* $f(t) \in \mathbf{Q}[[t]]$ *which satisfy the condition*

$$f(x) \cdot f(y) = f(x+y) + f(x-y)$$

are 0 *and* $f(t) = e^{\alpha t} + e^{-\alpha t}$, *for* α *non negative* : $\alpha \in \mathbf{Q}^+$. *The formal* *power series* $f(t) \in \mathbf{Z}[[t]]$ *which satisfy the condition*

$$(ii) \qquad f(x).f(y) = f((x+1)(y+1)-1)$$

are 0 *and* $(1+t)^k$, *where* k *is any integer* : $k \in \mathbf{Z}$. *Finally the formal power series* $f(t) \in \mathbf{Z}[[t]]$ *which satisfy the condition*

$$(iii) \qquad f(x).f(y) = f(\frac{x+1}{y+1} - 1) + f((x+1)(y+1)-1)$$

are 0 *and* $(1+t)^n + (1+t)^{-n}$, *where* n *is a non-negative integer* : $n \in \mathbf{Z}^+$.

PROOF. Take the second derivative with respect to y of the relation *(i)* ; we have

$$f(x)f''(y) = f''(x+y) + f''(x-y)$$

Put $y = 0$, and integrate the resulting differential equation; we obtain *(i)*. Now replacing the variable x (resp. y) by $tx-1$ (resp. $tx-1$), *(ii)* gives the relation

$$f(t^2 xy-1) = f(tx-1).f(ty-1).$$

Take the derivative with respect to t of this relation and substitute $t = \frac{1}{x}$ into the result obtained. And then using again the change of variable $z=\frac{y}{x}-1$, we get the differential equation

$$f(z) = k(z+1).f'(z), \qquad k \text{ constant.}$$

Integrate this equation, check the initial condition and we obtain *(ii)*. Finally, take the second derivative with respect to y of the relation *(iii)*, and then put $y = 0$; we obtain the differential equation

$$(x+1)^2 f''(x) + (x+1) f'(x) + kf(x) = 0,$$

where k is a constant. Integrate this equation and then take only the analytic solutions with integers coefficients, and we obtain *(iii)*.

LEMMA 9.5. *Let* $f : T_n \to T_m$ *be a continuous epimorphism of the n-dimensional torus* T_n *to the m-dimensional one. Then the induced map on the cohomology of the classifying space,*

$$f^* : H^*(B_{T_m}, \Lambda) \to H^*(B_{T_n}, \Lambda),$$

is a monomorphism, where Λ *is an integral domain.*

PROOF. As the cohomology of B_{T_m} is a polynomial ring generated by the ring generated by elements of order 2, hence it is sufficient to show the injectivity of

$$f^2 : H^2(B_{T_m}, \mathbf{Z}) \to H^2(B_{T_n}, \mathbf{Z}),$$

from which the injectivity of f^* follows by a purely algebraic reasoning. But this is clear, because of the identification (c.f. Chapter III)

$$H^2(B_{T_n}, \mathbf{Z}) = Hom(T_n, T_1),$$

where «*Hom*» means continuous homomorphisms.

Q.e.d.

LEMMA 9.6. *Let f be a continuous homomorphism of a compact connected Lie group into a compact connected Lie group G', such that the image $f(G)$ of f has the same maximal rank as G'. Then we can choose maximal tori $T \subseteq G$ and $T' \subseteq G'$ in such a way that $f(T) = T'$.*

PROOF. We may suppose f is an epimorphism without loss of generality. Take any maximal torus T' of G', and let $G_o = f^{-1}(T')$ be the inverse image of T'. Then the differential of f maps the Lie algebra of the compact group G_o onto the abelian algebra of T'. Since the Lie algebra of G_o is the direct sum of a semi-simple algebra and an abelian algebra, and since the semi-simple part must go to zero, there is a maximal torus T_o of G_o which is mapped onto T' by f. Any maximal torus of G which contains T_o satisfies the required condition.

Q.e.d.

Now we shall recall some well-known classical results which are needed for our purpose.

(A) The infinite dimensional complex projective space of $K(\mathbf{Z}, 2)$ may be taken as the classifying space for $U(1)$ and $SO(2)$. We shall denote it by B_T, where T is the one dimensional torus (in fact : we have $T = U(1) = SO(2)$ as topological group). Denote by

$$B_{T_n} = \underbrace{B_T \times \ldots \times B_T}_{n}$$

the classifying space of the n-dimensional torus $T_n = \underbrace{T \times \ldots \times T}_{n}$, and recall that the cohomology of $B_T = B_{T_1}$ is a polynomial ring with one generator α_o in $H^2(B_T)$. Under the correspondence $\alpha_o \to t$, we shall identify

$$H^{**}(B_T, \Lambda) = H^{2*}(B_T, \Lambda) = \Lambda[[t]];$$

similarly, under the correspondence $\alpha_1 \to x$, $\alpha_2 \to y$,

$$H^{2*}(B_{T_2}, \Lambda) \approx \Lambda[[x, y]], \qquad H^{2*}(B_{T_n}, \Lambda) = \Lambda[[\alpha_1, \alpha_2, \ldots, \alpha_n]],$$

where $\alpha_i = p_i^*(\alpha_o)$, with p_i the i-th canonical projection of B_{T_n} into B_T.

(B) As an Eilenberg-MacLane [13] space, B_T has an operation, say

$$\rho_+ : B_{T_2} = B_T \times B_T \to B_T,$$

which has a (homotopy) inverse $\rho_o : B_T \to B_T$, and we shall denote by $\rho_- : B_{T_2} \to B_T$ the difference, i.e. $\rho_-(a, b) = \rho_+(a, \rho_o b)$. The effect of the ρ's on cohomology is given by :

$$\rho_\pm(t) = x \pm y, \qquad\qquad \rho_o(t) = -t,$$

under the identification in *(A)*.

(C) The map induced by the inclusion of the maximal torus T_n into $U(n)$,

$$j^* : H^{2*}(B_{U(n)}, \Lambda) \to H^{2*}(B_{T_n}, \Lambda),$$

is injective (i.e. verifies the condition *(1)* of Corollary 1) and its image consists exactly of the symmetric power series [6].

(D) If Λ contains $\frac{1}{2}$, we have the monomorphism [6]

$$j^* : H^{2*}(B_{SO(2n)}, \Lambda) \to H^{2*}(B_{T_n}, \Lambda)$$

(i.e. *1* of Corollary 1 is verified), and the image consists of those symmetric power series which are invariant under an even number of changes of signs of the α_i (of x, y under the identification of *(A)*).

(E) There is an isomorphism of rings [3]

$$\hat{\Phi} : \mathbf{Z}[[t_1, \ldots, t_n]] \underset{\approx}{\to} \mathbb{K}(B_{T_n}),$$

defined by $\hat{\Phi}(t_i) = \xi^i_1 - 1$, where $\xi^i_1 = p^*_i(\xi^o_1)$ is the pull back by p_i of the universal complex line bundle ξ^o_1 over B_T, and where 1 denotes the trivial complex line bundle over B_{T_n}. When $n = 1, 2$, we write simply

$$\Lambda[[t]] \underset{\approx}{\overset{\hat{\Phi}}{\to}} K(B_T), \qquad \Lambda[[x,y]] \underset{\approx}{\overset{\hat{\Phi}}{\to}} K(B_{T_2}).$$

(F) Under the identification $\hat{\Phi}$, the homomorphism induced by the ρ's (c.f. *(B)*) on the K is given by

$$\rho^*_+(t) = (x+1)(y+1) - 1, \qquad \rho^*_-(t) = \frac{x+1}{y+1} - 1, \qquad \rho^*_o(t) = \frac{1}{1+t} - 1 .$$

(G) The homomorphism induced by the inclusion of the maximal torus T_n into $U(n)$,

$$j^* : K(B_{U(n)}) \to K(B_{T_n}).$$

is injective [3] (i.e. verifies the condition (1) of Corollary 1) and its image consists exactly of the symmetric formal power series (c.f. *(E)*) under the identification $\hat{\Phi}$.

(H) The homomorphism induced by the inclusion of the maximal torus T_n into $SO(2n)$,

$$j^* : K(B_{SO(2n)}) \to K(B_{T_n}),$$

is injective [3]. And its image consists exactly of those symmetric formal power series (c.f. *(E)*) $f(t_1, t_2, \ldots, t_n)$ which are invariant under an even number of substitutions of the form $t \to \frac{1}{1+t} - 1$. (E.g. for $n = 2$, then

$$f(x,y) = f\left(\frac{1}{1+x} - 1, \frac{1}{1+y} - 1\right)).$$

REMARK 9.1. The facts in *(F)*, *(G)*, *(H)* are just an explanation of the invariants of the Weyl group operation. And $\rho^*_o(\xi^o_1) = \bar{\xi}^o_1$ is the conjugation of ξ^o_1 .

Now we are ready to give the proofs. We shall conserve the number and notations of the original announcement without repeating it.

(Lemma 2.1). This is a consequence of Lemmas 9.1 and 9.2.

(Lemma 2.2). This follows from the universal property of the canonical homomorphism from $\mathcal{E}_{\mathbf{C}}(X)$ into $K(X)$, because H^{2*} and K are abelian groups with respect to addition (c.f. last paragraph in § 1).

(Theorem 3.1). Define φ (resp. Ψ) as follows : For each $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$ (resp. $+,+$), let $\tilde{f}(t) = \varphi(f) \in \Lambda[[t]]$ (resp. $\Psi(f)$) be the formal power series such that

$$\tilde{f}(t) = \tilde{f}(\alpha_o) = f_{B_T}(\xi_1^o)$$

under the identification in (A). It is clear that φ is a homomorphism from $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{\mu}$ into $\Lambda[[t]]_{\mu}$. Now construct another homomorphism

$$\varphi' : \Lambda[[t]]_{\times} \to \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$$

(resp. Ψ') in the following way : First (C) implies that there is a unique element $\tilde{\alpha} \in H^{2*}(B_{U(n)})$ such that

$$j^*(\tilde{\alpha}) = \tilde{f}(\alpha_1) \dots \tilde{f}(\alpha_n), \quad \tilde{f} \in \Lambda[[t]],$$

where $j^* : H^{2*}(B_{U(n)}) \to H^{2*}(B_{T_n})$ is induced by the inclusion $T_n \subset U(n)$ (resp.

$$j^*(\tilde{\alpha}) = \tilde{f}(\alpha_1) + \dots + \tilde{f}(\alpha_n)).$$

Then Lemma 9.2 implies there is a unique element $\varphi'(\tilde{f}) \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})$ (resp. $\Psi'(\tilde{f})$) such that

$$\varphi'(\tilde{f})_{B_{U(n)}}(\xi_n^o) = \tilde{\alpha} \quad (\text{resp. } \Psi'(\tilde{f})_{B_{U(n)}}(\xi_n^o) = \tilde{\alpha}).$$

Moreover $\varphi'(\tilde{f})$ is contained in $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$; in fact this follows from Lemma 9.3 and from the commutative diagram

$$\begin{array}{ccc} B_{T_n} \times B_{T_m} & \longrightarrow & B_{T_{n+m}} \\ \downarrow & & \downarrow \\ B_{U(n)} \times B_{U(m)} & \longrightarrow & B_{U(n+m)}. \end{array}$$

Hence φ' is well defined and it is clear that $\varphi \circ \varphi'$ is the identity. On the other hand, Corollary 9.1 implies that $\varphi' \circ \varphi$ is also the identity, and the isomorphism property of φ is proved (resp. Ψ). Now suppose

$$f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{ring} \subseteq \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,+};$$

then (notation of corollary 9.2), from

$$f_{B_{T_2}}(\xi_1^1 \otimes \xi_1^2) = f_{B_{T_2}}(\xi_1^1) \cdot f_{B_{T_2}}(\xi_1^2)$$

it follows that the relation $\tilde{f} = \Psi(f)$ should be verified (c.f. (B)) :

$$\tilde{f}(x+y) = \tilde{f}(x) \cdot \tilde{f}(y),$$

because $\xi_1^1 \otimes \xi_1^2 = \rho_+^*(\xi_1^o)$. Hence $\tilde{f}(t) = e^{at}$. Inversely the corollary 9.2 proves that in fact $\Psi'(e^{at})$ is an element of $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, H^{2*})_{ring}$.

(Theorem 3.2). Let $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^+, H^{2*})_{+,\times}$ and define $\tilde{f} = \varphi(f)$ by $f(\alpha_o) = f_{B_T}(\eta_2^o)$, where η_2^o is the universal oriented plane bundle over B_T. We must show first that φ is well defined, i.e. $\tilde{f}(t) \in \Lambda^{\pm}[[t]]$. If $\tilde{f}$ is even, i.e. $\tilde{f}(t) = \tilde{f}(-t)$, then it is already in $\Lambda^{\pm}[[t]]$ (notation § 1, (ii)); hence suppose this is not the case; so we can write

$$\tilde{f}(t) = \tilde{f}_1(t) + t \cdot \tilde{f}_2(t),$$

where $\tilde{f}_1, \tilde{f}_2$ are both even and $\tilde{f}_2 \neq 0$. We shall prove that $\tilde{f}_1(t) = 0$, which implies $\tilde{f} \in \Lambda^{\pm}[[t]]$. Suppose $\tilde{f}_1 \neq 0$, and let a (resp. b) be the first non vanishing coefficient of $\tilde{f}_1$ (resp. $\tilde{f}_2$); we shall obtain a contradiction. In fact, consider the value of f on $B_{SO(4)}$; it follows from Lemma 9.3 that

$$j^*(\eta_4^o) = \eta_2^1 + \eta_2^2 ;$$

hence in $H^{2*}(B_{T_2})$, we have

$$j^* f(\eta_4^o) = f(\eta_2^1) \cdot f(\eta_2^2) = p_1^*(f(\eta_2^o)) \cdot p_2^*(f(\eta_2^o)) =$$
$$= (\tilde{f}_1(x) + x\tilde{f}_2(x)) \cdot (\tilde{f}_1(y) + y\tilde{f}_2(y))$$

under the identification in (A). Then (D) implies

$$(\tilde{f}_1(x) + x\tilde{f}_2(x)) \cdot (\tilde{f}_1(y) + y\tilde{f}_2(y)) =$$
$$= (\tilde{f}_1(x) - x\tilde{f}_2(x)) \cdot (\tilde{f}_1(y) - y\tilde{f}_2(y)),$$

since $\tilde{f}_1, \tilde{f}_2$ are both even. This implies

$$x\tilde{f}_2(x) \cdot \tilde{f}_1(y) + y\tilde{f}_2(y) \cdot \tilde{f}_1(x) = 0,$$

because Λ contains $\frac{1}{2}$. But this gives the contradiction $a \cdot b = 0$; therefore $\tilde{f}_1 = 0$ and $\tilde{f} \in \tilde{\Lambda}^{\pm}[[t]]$. The construction of φ',

$$\varphi' : \Lambda^{\pm}[[t]] \to \mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, H^{2*})_{+,\times},$$

is exactly the same as the proof of Theorem 3.1. It is a well defined homomorphism by (D), and we prove by Corollary 9.1 that φ' is the inverse of φ.

Now let $f \in \mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, H^{2*})_{+,+}$ and define $\Psi(f) = \tilde{f}$ by $\tilde{f}(\alpha_o) = = f_{B_T}(\eta^o)$; as before we must show that $\tilde{f} \in \Lambda^{+}[[t]]$. Write $\tilde{f} = \tilde{f}_1 + t\tilde{f}_2$, where $\tilde{f}_1, \tilde{f}_2$ are even $\tilde{f}_i(t) = \tilde{f}_i(-t)$; we shall prove that $\tilde{f}_2 = 0$. In fact, consider the value of f on $B_{SO(4)}$. From the fact $j^*(\eta^o_4) = \eta^1_2 + \eta^2_2$ (c.f. lemma 9.3), we deduce in $H^{2*}(B_{T_2})$:

$$j^* f(\eta^o_4) = p_1(f(\eta^o_2)) + p_2(f(\eta^o_2)) = \tilde{f}(x) + \tilde{f}(y) = \tilde{f}(-x) + \tilde{f}(-y),$$

where the last equality follows from (D). This implies (because Λ contains $\frac{1}{2}$)

$$x\tilde{f}_2(x) + y\tilde{f}_2(y) = 0;$$

hence $\tilde{f}_2 = 0$, and φ is well defined. On the other hand, (D) and Lemma 9.2 imply the existence of

$$\Psi' : \Lambda^{+}[[t]] \to \mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, H^{2*})_{+,+},$$

determined by the condition

$$j^*(\Psi'(\tilde{f})_{B_{SO(2n)}}(\eta^o_{2n})) = \tilde{f}(\alpha_1) + \ldots + \tilde{f}(\alpha_n), \quad \tilde{f} \in \Lambda^{+}[[t]];$$

and it follows from Corollary 9.2 that Ψ' is the inverse of Ψ.

Now suppose

$$f \in \mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, H^{2*})_{ring} \subseteq \mathcal{H}om(\mathcal{E}^{+}_{\mathbf{R}}, H^{2*})_{+,+};$$

then by definition we have

$$f_{B_{T_2}}(\eta^1_2 \otimes \eta^2_2) = f_{B_{T_2}}(\eta^1_2) \cdot f_{B_{T_2}}(\eta^2_2).$$

On the other hand in $\mathcal{E}^{+}_{\mathbf{R}}(B_{T_2})$ we have

$$\eta^1_2 \otimes \eta^2_2 = \rho^*_{+}(\eta^o_2) + \rho^*_{-}(\eta^o_2)$$

(notation of (B)). In fact, this is a consequence of the equality

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \otimes \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix} = \frac{1}{2}\mu \cdot A \cdot \mu^{-1},$$

$$\mu = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0 & 0 \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & 0 \\ 0 & 0 & \cos(-\theta_1+\theta_2) & \sin(-\theta_1+\theta_2) \\ 0 & 0 & -\sin(-\theta_1+\theta_2) & \cos(-\theta_1+\theta_2) \end{pmatrix}$$

Hence we deduce, from the fact that f is also additive :

$$\rho_+^*(f(\eta_2^o)) + \rho_-^*(f(\eta_2^o)) = p_1^*(f(\eta_2^o)) + p_2^*(f(\eta_2^o)).$$

Then (B) implies

$$\tilde{f}(x+y) + \tilde{f}(x-y) = \tilde{f}(x).\tilde{f}(y),$$

where $\tilde{f} = \Psi(f)$, and Lemma 9.4 implies

$$\tilde{f}(t) = e^{\alpha t} + e^{-\alpha t}, \quad \alpha \in \mathbf{Q}^+.$$

Inversely it is clear that $f = \Psi'(e^{\alpha t} + e^{-\alpha t})$ satisfies

$$f_{B_{T_2}}(\eta_2^1 \otimes \eta_2^2) = f_{B_{T_2}}(\eta_2^1).f_{B_{T_2}}(\eta_2^2);$$

hence Corollary 9.2 implies that $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^+, H^{2*})_{ring}$.

Q.e.d.

(Corollary 3.1). If f is stable, then

$$f(\xi).f(1) = f(\xi+1) = f(\xi),$$

where 1 is the trivial complex line bundle , but this implies $f(1) = 1$; in other words, $\tilde{f} = \varphi(f)$ satisfies $\tilde{f}(0) = 1$. The fact «$\tilde{f}(0) = 1$ implies $\varphi'(\tilde{f})$ is stable» follows immediatly from the definition of φ'. Similarly we obtain the case of invertible elements, and the corollary 9.1 gives the relation with Hirzebruch's multiplicative sequences.

Q.e.d.

(Corollary 3.2). It is sufficient to remark that the conjugation of the complex line bundle ξ_1^o over B_T is given by $\bar{\xi}_1^o = \rho_o(\xi_1)$, for (B) implies

$$\varphi(\bar{f}) = f(\bar{\xi}_1) = \rho_o^*(f(\xi_1^o)) = \tilde{f}(-t).$$

Q.e.d.

(Corollary 3.3). It is sufficient to remark that, in $\mathcal{E}_{\mathbf{C}}(B_T)$, we have

$$\eta_2^o \otimes \mathbf{C} = \xi_1^o + \bar{\xi}_1^o.$$

Then the condition which characterizes the uniqueness of φ (resp. Ψ) in Theorem 3.2 and Corollary 3.2 implies the result.

Q.e.d.

(Corollary 3.4). Remark first that, if $f \in \mathcal{H}om(\mathcal{E}, H^{2*})_\mu$, $\beta \in \Lambda$, then the map defined by

$$f^\beta : \xi \to f_o(\xi) + \beta f_1(\xi) + \ldots + \beta^m f_m(\xi) + \ldots$$

for every $\xi \in \mathcal{E}$ is clearly another element of $\mathcal{H}om(\mathcal{E}, H^{2*})_\mu$, where we write

$$f(\xi) = f_o(\xi) + f_1(\xi) + \ldots + f_m(\xi) + \ldots$$

with $f_m(\xi) \in H^{2m}$. Now this corollary follows from the condition of uniqueness of φ (resp. Ψ) in Theorem 3.1, because on ξ_1^o (resp. η_2^o) f^β and $\varphi'(\varphi(f)(\beta t))$ are evidently equal.

(Theorem 4.1). The lemma 9.2 and the fact that «the isomorphism $\mathbf{Z}[[t_1, \ldots, t_n]] \approx \mathcal{K}(B_{T_n})$ (c.f. (E)) implies that

$$\Lambda[[t_1, \ldots, t_n]] \approx \mathcal{K}(B_{T_n}) \otimes_{\mathbf{Z}} \Lambda,$$

which is compatible with the inclusion $\mathcal{K}(B_{U(n)}) \to \mathcal{K}(B_{T_n})$ (resp. $B_{SO(2n)}$)» show that it is sufficient to study the case where $\Lambda = \mathbf{Z}$. Let $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times}$ (resp. $+,+$) and define $\tilde{f}(t) = \varphi(f)$ (resp. $\Psi(f)$) by

$$f_{B_T}(\xi_1^o) = \tilde{f}(\xi_1^o - 1),$$

where 1 denotes the trivial complex line bundle and $\xi_1^o - 1 \in \mathcal{K}(B_T)$. In other words, $f_{B_T}(\xi_1^o)$ corresponds to $\tilde{f}(t)$ under the isomorphism $\hat{\Phi}$ given in (E) (c.f. lemma 9.1) $\mathcal{K}(B_T) \approx \mathbf{Z}[[t]]$. According to Lemma 9.2 and (G), we obtain a well defined homomorphism

$$\varphi' : \mathbf{Z}[[t]] \to \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)$$

by the condition

$$j^*(\varphi'(\tilde{f})(\xi_n^o)) = \tilde{f}(\xi_1^1 - 1) \ \ldots \ \tilde{f}(\xi_1^n - 1), \qquad \tilde{f} \in \mathbf{Z}[[t]]$$

(resp. $j^*(\Psi'(\tilde{f})(\xi_n^o)) = \tilde{f}(\xi_1^1 - 1) + \ldots + \tilde{f}(\xi_1^n - 1))$.

The same reasoning as in the proof of Theorem 3.1 shows that the image of φ' is in $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times}$, and Corollary 9.1 proves that φ' is the inverse of φ (similarly for Ψ).

Now let

$$f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{ring} \subseteq \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,+} ;$$

then from

$$f_{B_{T_2}}(\xi_1^1 \otimes \xi_1^2) = f_{B_{T_2}}(\xi_1^1) \cdot f_{B_{T_2}}(\xi_1^2) \text{ and } \rho_+^*(\xi_1^o) = \xi_1^1 \otimes \xi_1^2,$$

we deduce (c.f. (F))

$$f_{B_{T_2}}(\xi_1^1 \otimes \xi_1^2) = \rho_+^*(f(\xi_1^o)) = \tilde{f}((x+1).(y+1)-1)$$

under the identification of (E); hence

$$\tilde{f}((x+1)(y+1)-1) = \tilde{f}(x) . \tilde{f}(y).$$

It follows from Lemma 9.4 (ii) that $\tilde{f}(t) = 0$ or $(1+t)^k$, $k \in \mathbf{Z}$. Inversely, it follows from Corollary 9.2 that every $\tilde{f}(t)$ of this form defines by φ' a unique element of $\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{ring}$.

Q.e.d.

(Theorem 4.2). Let $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^+, K)_{+,\times}$ and defines $\tilde{f}(t) = \varphi(f)$ by

$$f_{B_T}(\eta_2^o) = \tilde{f}(\xi_o - 1),$$

i.e. under the identification $\hat{\Phi}$ of (E) to $f_{B_T}(\eta_2^o)$ corresponds $\tilde{f}(t)$. We must first show that $\tilde{f}(t)$ is in $\mathbf{Z}_e[[t]]$ (c.f. § 1 (ii)). Indeed consider the value of f on $B_{SO(4)}$; then Lemma 9.3 implies

$$j^*(f_{B_{SO(4)}}(\eta_4^o)) = f_{B_{T_2}}(\eta_2^1 + \eta_2^2) = p_1^*(f(\eta_2^o)) . p_2^*(f(\eta_2^o)) = \\ = \tilde{f}(x) . \tilde{f}(y)$$

under the identification of (E). As this element is contained in the image of j^*, (H) implies

$$\tilde{f}(x) . \tilde{f}(y) = \tilde{f}(\frac{1}{1+x} - 1) . \tilde{f}(\frac{1}{1+y} - y).$$

Putting $t = x = y$, we obtain that $\tilde{f}(t)$ satisfies

$$\tilde{f}(t) = \pm \tilde{f}(\frac{1}{1+t} - 1), \quad \text{i.e. } \tilde{f}(t) \in \mathbf{Z}_e^{\pm}[[t]]$$

and φ is well defined. Using again (H) and lemma 9.2, we obtain a unique homomorphism

$$\varphi' : \mathbf{Z}_e^{\pm}[[t]] \to \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K)$$

which is determined by the condition

$$j^*(\varphi'(\tilde{f})(\eta_{2n}^{o})) = \tilde{f}(\xi_1^1 - 1), \ldots, \tilde{f}(\xi_1^n - 1), \quad \tilde{f} \in \mathbf{Z}_e^{\pm}[[t]],$$

where j^* is the inclusion in (H) and ξ_1^1 is defined in Lemma 9.3. By the same reasoning as before (c.f. proof of Theorem 3.1) we find that the image of φ' is contained in $\mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K)_{+,\times}$ and that it is the inverse of φ.

Now, let $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K)_{+,+}$; we define $\tilde{f}(t) = \Psi(f)$ exactly like φ, and then $f(t) \in \mathbf{Z}_e^{+}[[t]]$. Indeed, from the additivity of f it follows (c.f. lemma 9.3 and (E))

$$\tilde{f}(x) + \tilde{f}(y) = \tilde{f}\left(\frac{1}{1+x} - 1\right) + \tilde{f}\left(\frac{1}{1+y} - 1\right).$$

Putting $y = 0$, $x = t$, we obtain

$$\tilde{f}(t) = \tilde{f}\left(\frac{1}{1+t} - 1\right), \text{ i.e. } \tilde{f}(t) \in \mathbf{Z}_e^{+}[[t]]$$

and Ψ is well defined. Using (H) and Lemma 9.2, we obtain a unique homomorphism

$$\Psi' : \mathbf{Z}_e^{+}[[t]] \to \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K)_{+,+}$$

determined by the condition

$$j^*(\Psi'(\tilde{f})(\eta_{2n}^{o})) = \tilde{f}(\xi_1^1 - 1) + \ldots + f(\xi_1^n - 1), \quad \tilde{f} \in \mathbf{Z}_e^{+}[[t]],$$

where j^* (resp. ξ_1^i) is defined in (H) (resp. lemma 9.3); and Corollary 9.2 implies that Ψ' is the inverse of Ψ.

Finally let

$$f \in \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K)_{ring} \subseteq \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K)_{+,+}.$$

Then it follows from the equality (c.f. (B) and the proof of Theorem 3.2)

$$f_{B_{T_2}}(\eta_2^1 \otimes \eta_2^2) = f_{B_{T_2}}(\eta_2^1) . f_{B_{T_2}}(\eta_2^2),$$

$$\eta_2^1 \otimes \eta_2^2 = \rho_+^*(\eta_2^o) + \rho_-^*(\eta_2^o)$$

that we deduce

$$\rho_+^*(f(\eta_2^o)) + \rho_-^*(f(\eta_2^o)) = p_1^*(f(\eta_2^o)).p_2^*(f(\eta_2^o)),$$
$$\rho_+^*(\tilde{f}(t)) + \rho_-^*(\tilde{f}(t)) = p_1^*(\tilde{f}(t)).p_2^*(\tilde{f}(t)).$$

Hence it follows from (F) that

$$\tilde{f}((x+1)(y+1)-1) + \tilde{f}(\frac{x+1}{y+1}-1) = \tilde{f}(x).\tilde{f}(y).$$

Then Lemma 9.4 (iii) gives

$$\tilde{f}(t) = (1+t)^n + (1+t)^{-n}, \quad n \in \mathbf{Z}^+ .$$

Inversely the corollary 9.2 implies that any $\tilde{f}$ satisfying the above relation defines by Ψ' a unique element of $\mathcal{H}om(\mathcal{E}_{\mathbf{R}}^+, K)_{ring}$.

Q.e.d.

(Corollary 4.1). This is an immediate consequence of Theorem 4.1 and of the fact

$$\mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{ring} \approx \mathcal{H}om(K, K)_{ring},$$

as indicated in Lemma 2.2.

Q.e.d.

(Corollary 4.2). It is sufficient to check the value of $\bar{f}$ on ξ_1^o or, what is the same, the value of f on $\bar{\xi}_1^o = \rho_o^*(\xi_1^o)$; hence it follows from (F)

$$\varphi(\bar{f}) = f(\bar{\xi}_1) = \rho_o^*(f(\xi_1^o)) = \tilde{f}(\frac{1}{1+t} - 1).$$

Q.e.d.

(Corollary 4.3). The proof is similar to that of Corollary 4.2.

(Corollary 4.4). It is sufficient to check the value of $f \otimes \mathbf{C}$ on η_2^o or, what is the same, the value of f on $\eta_2^o \otimes \mathbf{C} = \xi_1^o + \bar{\xi}_1^o$. Hence, it follows that (F) implies

$$\widetilde{(f \otimes \mathbf{C})}(t) = f(\xi_1^o + \bar{\xi}_1^o) = f(\xi_1^o).f(\bar{\xi}_1^o) = \tilde{f}(t).\tilde{f}(\frac{1}{1+t} - 1)$$

(under the identification $\hat{\Phi}$ of (E)). Similarly, we obtain the case $(+,+)$.

Q.e.d.

(Corollary 4.5). It is sufficient to verify that the value Λ' (resp. Λ) on ξ_1^o is equal to the value of $\varphi'(-t)$ (resp. $\varphi'(2+t)$ on ξ_1^o). But this is immediate :

$$\Lambda'(\xi_1^o) = 1 - \xi_1^o = -(\xi_1^o - 1) = \varphi'(-t)(\xi_1^o),$$

because ξ_1^o is a one dimensional complex bundle (resp. $\Lambda = \varphi'(2+t)$).

Q.e.d.

(Corollary 4.6). It is sufficient to verify that the value of

$\varphi'((1+t)-(1+t)^{-1})$ (resp. $\varphi'((1+t)+(1+t)^{-1})$) on ξ_1^o

is equal to the virtual class $\xi_1^o - \bar{\xi}_1^o$ (resp. $\xi_1^o - \bar{\xi}_1^o$). But this is immediate from the definition of φ' given in the proof of Theorem 4.1.

Q.e.d.

(Theorem 5.1). Let $g = ch^{\beta} \in \mathcal{H}om(K, H^{2*})_{ring}$ (c.f. convention 3.1) and $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times}$; we have to compute the value of $g \circ f$ on ξ_1^o:

$$(g \circ f)(\xi_1^o) = g(f(\xi_1^o)) = g(a_o + a_1(\xi_1^o - 1) + \ldots + a_m(\xi_1^o - 1)^m + \ldots),$$

where $\varphi(f) = a_o + a_1 t + \ldots + a_m t^m + \ldots$ and where 1 is the trivial complex line bundle. As g is a ring homomorphism, we have

$$(g \circ f)(\xi_1^o) = a_o + a_1 g(\xi_1^o - 1) + \ldots + a_m (g(\xi_1^o - 1))^m + \ldots$$
$$= a_o + a_1(g(\xi_1^o) - 1) + \ldots + a_m(g(\xi_1^o) - 1)^m + \ldots,$$

because $g(1) = 1$; i.e., using the identification in (A),

$$= a_o + a_1(g(t) - 1) + \ldots + a_m(g(t) - 1)^m + \ldots .$$

Hence, it follows from Theorem 3.1 that

$$\varphi(g \circ f) = f(g(t) - 1) = f(e^{\beta t} - 1).$$

Similarly we obtain the other cases.

Q.e.d.

(Corollary 5.1). Let $g = ch^{\beta} \in \mathcal{H}om(K, H^{2*})_{ring}$ and $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K)_{+,\times}$, then it follows from Corollary 3.3 and Theorem 5.1 that

$$\varphi((g \circ f) \otimes \mathbf{C}) = \varphi(g \circ f)(t) \,.\, \varphi(g \circ f)(-t) = f(e^{\beta t} - 1) \,.\, f(e^{-\beta t} - 1),$$

(where φ is the isomorphism in Theorem 3.2). On the other hand, it follows from Corollary 4.4 that

$$\varphi(g \circ (f \otimes \mathbf{C})) = \varphi(f \otimes \mathbf{C})(e^{\beta t} - 1) = f(e^{\beta t} - 1) \,.\, f\left(\frac{1}{1 + e^{\beta t} - 1} - 1\right) =$$
$$= f(e^{\beta t} - 1) \,.\, f(e^{-\beta t} - 1),$$

hence $(g \circ f) \otimes \mathbf{C} = g \circ (f \otimes \mathbf{C})$.

Q.e.d.

(Corollary 5.2). Let $g = ch^{\beta} \in \mathcal{H}om(K, H^{2*})_{ring}$, $f \in \mathcal{H}om(\mathcal{E}_{\mathbf{R}}^{+}, K)_{+,\times}$; then it follows from Corollary 3.2 that

$$\varphi(\overline{g \circ f}) = \overline{f(g(t)-1)} = f(g(-t)-1) = f(e^{-\beta t}-1).$$

On the other hand, Corollary 4.2 implies

$$\varphi(g \circ \overline{f}) = f(\frac{1}{1+\overline{g(t)-1}} - 1) = f(e^{-\beta t}-1),$$

hence $\overline{g \circ f} = g \circ \overline{f}$. Similarly we obtain the other cases.

Q.e.d.

(Corollary 5.3). This is immediate consequence of Theorem 5.1, i.e.

$$\varphi(ch^{\beta} \circ f) = f(e^{\beta t}-1) = f(e^{t}-1)(\beta t) = \varphi((ch \circ f)^{\beta})$$

(where φ is the isomorphism in Theorem 3.1).

Q.e.d.

(Lemma 6.1). This follows immediately from the isomorphism

$$\varphi: \mathcal{H}om(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times} \approx \Lambda[[t]],$$

because it maps $\mathcal{H}om^{\#}(\mathcal{E}_{\mathbf{C}}, K \otimes_{\mathbf{Z}} \Lambda)_{+,\times}$ onto the sub-semi-group $\Lambda^{\#}[[t]]$ of those formal power series whose first non vanishing coefficient is an invertible element of Λ, and because the homomorphism τ associates with each such f its inverse modulo the order $\omega(f)$, i.e. $f.\tau f = t^{\omega(f)}$ (c.f. §1 (ii)). Hence it is evident that τf is invertible. As every invertible element g has order zero, we have $g.\tau g = 1$ and in particular $\tau^2 f.\tau f = 1$.

Q.e.d.

(Corollary 6.1). The first assertion follows immediately from the definition of τ and that of the Euler class χ (c.f. definition 4.1). For the last assertion, write

$$\varphi(f) = a_k t^k + a_{k+1} t^{k+1} + \dots, \quad a_k \text{ invertible in } \Lambda;$$

then it is clear that

$$\varphi(\tau^2 f) = \tau(\frac{1}{a_k + a_{k+1} t + \dots}) = a_k + a_{k+1} t + \dots .$$

Q.e.d.

(Corollary 6.2). Let $f \in \mathcal{H}om^{\#}(\mathcal{E}_{\mathbf{C}}, H^{2*})_{+,\times}$ and write

$$\varphi(f) = a_k t^k + a_{k+1} t^{k+1} + \dots, \quad a_k \text{ invertible in } \Lambda .$$

Then it follows from Corollary 3.3 that

$$\varphi(\tau(f \otimes \mathbf{C})) = \tau(f(t).f(-t)) = \frac{(-1)^k}{a_k + a_{k+1} t + \dots} \cdot \frac{1}{a_k - a_{k+1} t + \dots} .$$

On the other hand, we have

$$\varphi(\tau f\otimes \mathbf{C}) = \tau f(t).\tau f(-t) = \frac{1}{a_k + a_{k+1}t + \dots}\cdot\frac{1}{a_k - a_{k+1}t+\dots};$$

but the degree of $f\otimes\mathbf{C}$ is equal to $2k$, and the result follows.

Q.e.d.

(Theorem 6.1). It follows from Corollary 4.5 and Theorem 5.1 that, for any $ch^\beta \in \mathcal{H}om(K, H^{2*})_{ring}$, $\beta \neq 0$, we have

$$\varphi(\tau(ch^\beta \circ \Lambda')) = \tau(-(e^{\beta t}-1)) = \frac{t}{1-e^{\beta t}}.$$

Hence, if $\beta = -1$, we obtain $\tau(ch^{-1}\circ\Lambda') = \mathfrak{T}$, and, if $\beta = 1$, Corollary 3.2 implies $\tau(ch\circ\Lambda') = \overline{\mathfrak{T}}$. Similarly, it follows from Corollary 4.6 that

$$\varphi(\tau(ch^\beta\circ\tilde{\Lambda}^-.\tau\tilde{\Lambda}^+)) = \tau(\frac{e^{\beta t}-e^{-\beta t}}{e^{\beta t}+e^{-\beta t}}) = \frac{t(e^{\beta t}+e^{-\beta t})}{e^{\beta t}-e^{-\beta t}} = \frac{1}{\beta}\cdot\frac{\beta t}{\tanh \beta t};$$

in particular if $\beta = 1$, we obtain

$$\tau(ch\circ\tilde{\Lambda}^-.t\tilde{\Lambda}^+) = L.$$

Finally we have

$$\varphi(\tau(ch^\beta\circ\tilde{\Lambda}^{-1})) = \tau(e^{\beta t}-e^{-\beta t}) = \frac{t}{e^{\beta t}-e^{-\beta t}} = \frac{1}{2\beta}\cdot\frac{\beta t}{\sinh\beta t};$$

when $\beta = 1$, we obtain

$$\tau(ch\circ\tilde{\Lambda}^-) = \frac{1}{2}A.$$

(One may remark that $\tau(ch^\beta\circ\tilde{\Lambda}^-)$ is defined for $\tilde{\Lambda}^-\in\mathcal{H}om(\mathfrak{S}_{\mathbf{R}}, K)_{+,\times}$, but $\tau(ch^\beta\circ\tilde{\Lambda}^-.\tau\tilde{\Lambda}^+)$ is defined for $\tilde{\Lambda}^-.\tau\tilde{\Lambda}^+ \in \mathcal{H}om(\mathfrak{S}_{\mathbf{R}}, K\otimes_{\mathbf{Z}}\mathbf{Q})_{+,\times}$).

Q.e.d.

(Corollary 6.3). This follows from the definition of τ and from the fact that the order of $ch^{-1}\circ\Lambda'$ is 1 (c.f. Corollary 6.1).

Q.e.d.

(Lemma 6.2). Let $ch^\beta\in\mathcal{H}om(K, H^{2*})_{ring}$, $f\in\mathcal{H}om^{\#}(\mathfrak{S}_{\mathbf{C}}, K)_{+,\times}$, and write

$$\varphi(f) = a_k t^k + a_{k+1}t^{k+1}+\dots,\quad a_k = \pm 1,$$

where φ is the isomorphism in Theorem 4.1. Then Theorem 5.1 implies

$$\varphi(\tau(ch^\beta\circ f)) = \tau(a_k(e^{\beta t}-1)^k + a_{k+1}(e^{\beta t}-1)^{k+1}+\dots)$$

$$= \frac{t^k}{(e^{\beta t}-1)^k}\cdot\frac{1}{a_k + a_{k+1}(e^{\beta t}-1)+\dots},$$

where φ is the isomorphism in Theorem 3.1 . Similarly we have

$$\varphi(\tau(ch^{\beta}\circ\chi)^{k}.(ch^{\beta}\circ\tau(f))) = \tau(e^{\beta t}-1)^{k}.\frac{1}{a_{k}+a_{k+1}(e^{\beta t}-1)+\dots}$$

$$= \frac{t^{k}}{(e^{\beta t}-1)}\cdot\frac{1}{a_{k}+a_{k+1}(e^{\beta t}-1)+\dots}$$

Q.e.d.

(Corollary 6.4). It follows from Remark 4.1 that it is sufficient to prove

$$\tau(ch^{\beta/2}\circ\tilde{\Lambda}^{-}.\tau\tilde{\Lambda}^{+}) = -\tau^{2}(ch^{\beta}\circ\tilde{\Lambda}^{-}).\tau(ch^{-\beta}\circ\Lambda'\otimes\mathbf{C}).$$

We have already, in the proof of Theorem 6.1, the equalities

$$\varphi(\tau(ch^{\beta/2}\circ\tilde{\Lambda}^{-}.\tau\tilde{\Lambda}^{+})) = \frac{t(e^{\beta t/2}+e^{-\beta t/2})}{e^{\beta t/2}-e^{-\beta t/2}},$$

$$\varphi(-\tau^{e}(ch^{\beta}\circ\tilde{\Lambda}^{-})) = -\tau(\frac{t}{e^{\beta t}-e^{-\beta t}}) = -\frac{e^{\beta t}-e^{-\beta t}}{t}.$$

Now Corollary 4.4 and Theorem 5.1 imply

$$\varphi(\tau(ch^{-\beta}\circ\Lambda'\otimes\mathbf{C})) = \tau((e^{-\beta t}-1)(e^{\beta t}-1)) = \frac{t^{2}}{(e^{-\beta t}-1)(e^{\beta t}-1)}.$$

Then the result follows immediately from the trivial identity

$$(x^{-1}-1)(x-1) = -(x^{-1/2}-x^{1/2})^{2}$$

and Remark 6.3.

Q.e.d.

(Proposition 7.1). Consider the case of a $\mathfrak{S}^{+}_{\mathbf{R}}$-pair (X,η). Let $\theta\in\mathcal{H}om(\mathfrak{S}^{+}_{\mathbf{R}},K)$ and $x\in U_{\theta}(t(\eta))$; then by definition there is a representation

$$\rho: G\to SO(2n), \qquad \rho: B_{G}\to B_{SO(2n)},$$

verifying the conditions (1) and (2) of § 7, and there is an element $\bar{x}\in K(t(\rho^{*}\eta^{o}_{2n}))$ such that

$$h^{*}_{\rho}(\bar{x}) = x \text{ and } p(\bar{x}) = \theta(\rho^{*}\eta^{o}_{2n})$$

(where we use the same notation as in (2) §7). Then in $H^{2*}(B_{H},\mathbf{Q})$ we have

$$\Phi \circ f(\bar{x}) = f^{(\theta)}(\rho^* \eta^o_{2n})$$

(where $f^{(\theta)} = \chi^{\omega-1}\tau^2(f \circ \theta)$ and where $\omega = \omega(f \circ \theta)$ is the order of $f \circ \theta$ as defined in the proposition). Indeed, let $\bar{\chi} \in H^{2*}(B_G, \mathbf{Q})$ be the Euler class of the real oriented bundle $\rho^* \eta^o_{2n}$. Then the condition *(1)* of §7 implies $\bar{\chi} \neq 0$, because, from Lemma 9.5, Lemma 9.6 and *(D)*,

$$\rho^* : H^{2*}(B_{SO(2n)}, \mathbf{Q}) \to H^{2*}(B_G, \mathbf{Q})$$

is a monomorphism, hence $\bar{\chi} = \rho^*(\chi_{\eta^o_{2n}}) \neq 0$. Now from the fact that the ring $H^{2*}(B_G, \mathbf{Q}) \subseteq H^{2*}(B_T, \mathbf{Q})$ has a non zero divisor, the following equality

$$(ii) \qquad \bar{\chi} \cup \Phi \circ f(\bar{x}) = \bar{\chi} \cup f^{(\theta)}(\rho^* \eta^o_{2n}) = (\chi \cdot f^{(\theta)})(\rho^* \eta^o_{2n})$$

will imply *(i)*. (Here we use the « ∪ » product in H^{2*} to distinguish it from that of $\mathcal{H}om$). To prove *(ii)*, recall that the following diagram is commutative

$$\begin{array}{ccc} H^*(t(\rho^* \eta^o_{2n})) & \xrightarrow{\ \Phi\ } & H^*(B_G) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \bar{\chi}\cup} & \\ & H^*(B_G) & \end{array}$$

where $t(\rho^* \eta^o_{2n})$ is the Thom space of $\rho^* \eta^o_{2n}$; hence *(ii)* is equivalent to

$$f \circ \theta(\rho^* \eta^o_{2n}) = (\chi \cdot f^{(\theta)})(\rho^* \eta^o_{2n}),$$

because

$$\bar{\chi} \cup \Phi \circ f(\bar{x}) = p \circ f(\bar{x}) = f(p\bar{x}) = f(\theta(\rho^* \eta^o_{2n})).$$

But this is a trivial consequence of the definition of $f^{(\theta)}$ and Corollary 6.1

$$\chi \cdot f^{(\theta)} = \chi \cdot \chi^{\omega-1} \cdot \tau^2(f \circ \theta) = \chi^{\omega} \cdot \tau^2(f \circ \theta) = f \circ \theta,$$

where $\omega = \omega(f \circ \theta)$. Hence *(i)* is proved. On the other hand *(i)* implies the condition *(3)* of §7 : $\Phi \circ f(x) = f^{(\theta)}(\eta)$. Indeed, the condition *(1)* of §7 says precisely $\eta = h^*_\rho(\rho^* \eta^o_{2n})$, hence (since $h^*_\rho(\bar{x}) = x$ by hypothesis)

$$\Phi \circ f(x) = \Phi \circ f(h^*_\rho(\bar{x})) = h^*_\rho(\Phi \circ f(\bar{x})) =$$

$$= h_\rho^*(f^{(\theta)}(\rho^*\eta_{2n}^{o})) = f^{(\theta)}(h_\rho^*(\rho^*\eta_{2n}^{o})) = f^{(\theta)}(\eta).$$

Q.e.d.

(Lemma 7.1). This follows from the fact that $f = ch^\beta$ commutes with the product.

$$\begin{array}{ccc} K(\eta)\otimes K(t(\eta)) & \longrightarrow & K(t(\eta)) \\ \downarrow{\scriptstyle f\otimes f} & & \downarrow{\scriptstyle f} \\ H^{2*}(\eta)\otimes H^{2*}(t(\eta)) & \xrightarrow{\ \cup\ } & H^{2*}(t(\eta)), \end{array}$$

and the property of the Thom isomorphism Φ, i.e. the following diagram is commutative

$$\begin{array}{ccc} H^*(\eta)\otimes H^*(t(\eta)) & \xrightarrow{\ \cup\ } & H^*(t(\eta)) \\ \downarrow{\scriptstyle p\otimes\Phi} & & \downarrow{\scriptstyle \Phi} \\ H^*(X)\otimes H^*(X) & \xrightarrow{\ \cup\ } & H^*(X). \end{array}$$

This is an immediate consequence of the associativity of the cup-product and of the definition of Φ by the cup-product of the Thom class (where p is the isomorphism induced by the bundle map). Indeed the naturality of f then implies the result.

Q.e.d.

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