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FILTRATIONS

by Peter HILTON

We extract from the theory of spectral sequences the following problem. Let \mathcal{A} be an abelian category, let X be an object of \mathcal{A} , and let

$$\dots \subseteq X^{p-1} \subseteq X^p \subseteq \dots \subseteq X$$

be a filtration of X , $-\infty < p < \infty$. Similarly let

$$\dots \subseteq \tilde{X}^{p-1} \subseteq \tilde{X}^p \subseteq \dots \subseteq \tilde{X}$$

be a filtration of $\tilde{X} \in \mathcal{A}$. Let $\varphi: X \rightarrow \tilde{X}$ be a morphism of filtered objects, so that φ induces

$$\varphi^p: X^p \rightarrow \tilde{X}^p, \quad -\infty < p < \infty.$$

Associated with the given filtrations of X and \tilde{X} we have the graded objects GX , $G\tilde{X}$ given by

$$(GX)^p = X^p / X^{p-1}, \quad (G\tilde{X})^p = \tilde{X}^p / \tilde{X}^{p-1}$$

and φ obviously induces the morphism

$$G\varphi: GX \rightarrow G\tilde{X}$$

of graded objects. We then ask under what circumstances G reflects isomorphisms. Further we are interested in devising a functorial process which will convert the given filtered objects X and \tilde{X} into new filtered objects \bar{X} and $\tilde{\bar{X}}$ in such a way that the associated graded objects are unchanged and that $\bar{\varphi}: \bar{X} \rightarrow \tilde{\bar{X}}$ if $G\varphi: GX \rightarrow G\tilde{X}$. Such a process was described in [2] but under extra hypotheses on \mathcal{A} which do not appear in this presentation. Moreover, we also describe here a simplification of the process which does not enjoy the strong property of the completion \bar{X} above but has the good features (a) that it does far less violence to the filtration than the completion process, and (b) that, after carrying it out, every non-zero element

of the filtered object is represented by a unique non-zero element of the associated graded object. We describe the simplified process as *quasi-completion*.

Details of this work, which is joint work with B. Eckmann, will appear elsewhere [1].

1. Definitions and notations.

Let \mathbf{N} be the ordered set of integers regarded as a category and let $\mathcal{U}^{\mathbf{N}}$ be the functor category. Thus an object of $\mathcal{U}^{\mathbf{N}}$ is a sequence $(D^{\bullet}, \alpha^{\bullet})$,

$$\dots \longrightarrow D^p \xrightarrow{\alpha^p} D^{p+1} \longrightarrow \dots,$$

and a morphism $\varphi^{\bullet}: (D^{\bullet}, \alpha^{\bullet}) \rightarrow (E^{\bullet}, \beta^{\bullet})$ is a sequence of morphisms $\varphi^p: D^p \rightarrow E^p$ such that $\beta^p \varphi^p = \varphi^{p+1} \alpha^p$. Note that we may also regard an object of $\mathcal{U}^{\mathbf{N}}$ as an object D^{\bullet} of the graded category $\mathcal{U}^{\mathbf{Z}}$ together with an endomorphism $\alpha^{\bullet}: D^{\bullet} \rightarrow D^{\bullet}$ of degree +1.

Let $P: \mathcal{U} \rightarrow \mathcal{U}^{\mathbf{N}}$ be the embedding functor which associates with $X \in \mathcal{U}$ the sequence

$$\dots \xrightarrow{1} X \xrightarrow{1} X \longrightarrow \dots$$

PROPOSITION 1.1. P is a full embedding, and preserves monics and epics.

A *filtration* of X is a monic $\mu^{\bullet}: (X^{\bullet}, \xi^{\bullet}) \rightarrow PX$; note that each ξ^p is monic.

A *cofiltration* of X is an epic $\varepsilon_{\bullet}: PX \rightarrow (X_{\bullet}, \xi_{\bullet})$; note that each ξ_p is epic.

The graded object associated with the filtration μ^{\bullet} is $\mathcal{G}(\mu^{\bullet})$ given by

$$\mathcal{G}(\mu^{\bullet})^p = \text{coker } \xi^{p-1}.$$

The graded object associated with the cofiltration ε_{\bullet} is $\mathcal{G}(\varepsilon_{\bullet})$ given by

$$\mathcal{G}(\varepsilon_{\bullet})^p = \text{ker } \xi_p.$$

Note that $\mathcal{G}(\mu^{\bullet})$ depends only on $(X^{\bullet}, \xi^{\bullet})$ and $\mathcal{G}(\varepsilon_{\bullet})$ depends only on $(X_{\bullet}, \xi_{\bullet})$. We suppose henceforth that $\mu^{\bullet}, \varepsilon_{\bullet}$ are *mutual annihilators* so that

$$(1.2) \quad (X^\bullet, \xi^\bullet) \xrightarrow{\mu^\bullet} PX \xrightarrow{\varepsilon} (X_\bullet, \xi_\bullet)$$

is exact in $\mathcal{U}^{\mathbb{N}}$. We may then think of X_p as the quotient X/X^p .

PROPOSITION 1.3. *If (1.2) is exact then $\mathcal{G}(\mu^\bullet) = \mathcal{G}(\varepsilon_\bullet)$.*

If \mathcal{U} has countable products (sums) then P has a right (left) adjoint. So we suppose \mathcal{U} has countable products and sums. Let

$$R, L: \mathcal{U}^{\mathbb{N}} \rightarrow \mathcal{U}$$

be such that

$$(1.4) \quad P \dashv R, \quad L \dashv P.$$

Then $RP = Id$, and there is a natural transformation $\tau: PR \rightarrow Id$ such that $\tau P = 1, R\tau = 1$. Also $LP = Id$, and there is a natural transformation $\pi: Id \rightarrow PL$ such that $\pi P = 1, L\pi = 1$. The *limit* (= *limite projective*) of $(D^\bullet, \alpha^\bullet)$ is the object $R(D^\bullet, \alpha^\bullet)$ together with the morphism

$$\tau(D^\bullet, \alpha^\bullet): PR(D^\bullet, \alpha^\bullet) \rightarrow (D^\bullet, \alpha^\bullet).$$

The *colimit* (= *limite inductive*) of $(D^\bullet, \alpha^\bullet)$ is the object $L(D^\bullet, \alpha^\bullet)$ together with the morphism $\pi(D^\bullet, \alpha^\bullet): (D^\bullet, \alpha^\bullet) \rightarrow PL(D^\bullet, \alpha^\bullet)$. We will write

$$X^\infty = L(X^\bullet, \xi^\bullet), \quad X^{-\infty} = R(X^\bullet, \xi^\bullet), \quad X_\infty = L(X_\bullet, \xi_\bullet), \quad X_{-\infty} = R(X_\bullet, \xi_\bullet).$$

PROPOSITION 1.4. (i) *For any $\varphi^\bullet: (D^\bullet, \alpha^\bullet) \rightarrow PX, PL \varphi^\bullet \circ \pi(D^\bullet, \alpha^\bullet) = \varphi^\bullet$.*

(ii) *For any $\psi_\bullet: PX \rightarrow (D_\bullet, \alpha_\bullet), \tau(D_\bullet, \alpha_\bullet) \circ PR \psi_\bullet = \psi_\bullet$.*

COROLLARY 1.5. (i) *If μ^\bullet is a filtration, $\pi(X^\bullet, \xi^\bullet): (X^\bullet, \xi^\bullet) \rightarrow PX^\infty$.*

(ii) *If ε_\bullet is a cofiltration, $\tau(X_\bullet, \xi_\bullet): PX_{-\infty} \rightarrow (X_\bullet, \xi_\bullet)$.*

We say that μ^\bullet (or its annihilator ε_\bullet) *generates* X if $L\mu^\bullet$ is epic; μ^\bullet (or ε_\bullet) *cogenerates* X if $R\varepsilon_\bullet$ is monic; μ^\bullet (or ε_\bullet) is *quasi-complete* if it generates and cogenerates X .

We say that μ^\bullet (or ε_\bullet) is *L-complete* if $L\mu^\bullet$ is an isomorphism; μ^\bullet (or ε_\bullet) is *R-complete* if $R\varepsilon_\bullet$ is an isomorphism; μ^\bullet (or ε_\bullet) is *complete* if $L\mu^\bullet$ and $R\varepsilon_\bullet$ are isomorphisms.

2. The completion procedure.

Consider the exact sequence (1.2). We write $\varepsilon \square \mu^*$ to show they are mutual annihilators. We construct filtrations and cofiltrations as follows :

$$\begin{aligned}
 (2.1) \quad & \pi^* : (X^*, \xi^*) \twoheadrightarrow P X^\infty, & \tau_* : P X_{-\infty} &\twoheadrightarrow (X_*, \xi_*), \\
 & \zeta^* : (X_{-\infty}^*, \xi_{-\infty}^*) \twoheadrightarrow P X_{-\infty}, & \eta_* : P X^\infty &\twoheadrightarrow (X_*^\infty, \xi_*^\infty), \\
 & \pi_{-\infty}^* : (X_{-\infty}^*, \xi_{-\infty}^*) \twoheadrightarrow P (X_{-\infty}^*)^\infty, & \tau_*^\infty : P (X^\infty)_{-\infty} &\twoheadrightarrow (X_*^\infty, \xi_*^\infty).
 \end{aligned}$$

Explicitly, $\pi^* = \pi(X^*, \xi^*)$, so π^* is certainly L -complete; $\tau_* = \tau(X_*, \xi_*)$ so τ_* is R -complete : then $\eta_* \square \pi^*$, $\tau_* \square \zeta^*$, so ζ^* is R -complete, η_* is L -complete; finally, $\pi_{-\infty}^* = \pi(X_{-\infty}^*, \xi_{-\infty}^*)$, $\tau_*^\infty = \tau(X_*^\infty, \xi_*^\infty)$, so $\pi_{-\infty}^*$ is L -complete, τ_*^∞ is R -complete.

From Proposition 1.3 we immediately deduce

PROPOSITION 2.2. *All the filtrations and cofiltrations of (2.1) determine the same associated graded object.*

We now explain precisely in what sense we may regard the generation of the morphisms of (2.1) as a completion procedure. We state the result as a comprehensive theorem.

THEOREM 2.3. (i) *There is a natural isomorphism $(X_{-\infty}^*)^\infty = (X^\infty)_{-\infty}$ under which the two objects may be identified to a single object $X_{-\infty}^\infty$.*

(ii) $\tau_*^\infty \square \pi_{-\infty}^*$, so that $\pi_{-\infty}^*$ and τ_*^∞ are complete.

(iii) *The square*

$$(2.4) \quad \begin{array}{ccc}
 X^\infty & \xrightarrow{L\mu^*} & X \\
 \downarrow R\eta_* & & \downarrow R\varepsilon_* \\
 X_{-\infty}^\infty & \xrightarrow{L\zeta^*} & X_{-\infty}
 \end{array}$$

is bicartesian (pull-back and push-out).

Thus the completion procedure is natural with respect to morphisms of filtrations or cofiltrations and self-dual.

COROLLARY 2.5. *The filtration μ^\cdot is complete if and only if (2.4) is a diagram of isomorphisms. Moreover, the completion procedure leaves a complete filtration unchanged.*

Given (1.2) there are objects $X_q^p = \text{coker } X^q \twoheadrightarrow X^p = \text{ker } X_q \twoheadrightarrow X_p$ $q \leq p$. There are also morphisms

$$\begin{aligned} \varepsilon_q^p : X_q^p &\twoheadrightarrow X_{q+1}^p, & q+1 \leq p, \\ \mu_q^p : X_q^p &\twoheadrightarrow X_q^{p+1}, & q \leq p, \end{aligned}$$

such that the square

$$(2.6) \quad \begin{array}{ccc} X_q^p & \xrightarrow{\varepsilon_q^p} & X_{q+1}^p \\ \downarrow \mu_q^p & & \downarrow \mu_{q+1}^p \\ X_q^{p+1} & \xrightarrow{\varepsilon_q^{p+1}} & X_{q+1}^{p+1} \end{array}$$

is bicartesian, $q+1 \leq p$.

THEOREM 2.7. *The natural map*

$$\lim_{\substack{\rightarrow \\ p}} \lim_{\leftarrow \\ q} (X_q^p; \varepsilon_q^p; \mu_q^p) \rightarrow \lim_{\leftarrow \\ q} \lim_{\rightarrow \\ p} (X_q^p; \varepsilon_q^p; \mu_q^p)$$

is an isomorphism and the common double limit is $X_{-\infty}^\infty$.

Moreover $\lim_{\leftarrow \\ q} (X_q^p; \varepsilon_q^p) = X_{-\infty}^p$ and μ_q^p induces $\xi_{-\infty}^p : X_{-\infty}^p \twoheadrightarrow X_{-\infty}^{p+1}$;

and dually,

$$\lim_{\rightarrow \\ p} (X_q^p; \mu_q^p) = X_q^\infty \quad \text{and} \quad \varepsilon_q^p \text{ induces } \xi_q^\infty : X_q^\infty \twoheadrightarrow X_{q+1}^\infty.$$

COROLLARY 2.8. *Let $\varphi : \mu^\cdot \rightarrow \tilde{\mu}^\cdot$ be a morphism of filtrations, inducing*

$$(2.9) \quad \mathcal{G}\varphi : \mathcal{G}(\mu^\cdot) \rightarrow \mathcal{G}(\tilde{\mu}^\cdot).$$

Suppose $\mathcal{G}\varphi$ is an isomorphism and $\mu^\cdot, \tilde{\mu}^\cdot$ are complete. Then $\varphi : \mu^\cdot \cong \tilde{\mu}^\cdot$.

The notation means that φ gives a commutative diagram

$$(2.10) \begin{array}{ccccccccc} X^{p-1} & \xrightarrow{\xi^{p-1}} & X^p & \xrightarrow{\mu^p} & X & \xrightarrow{\varepsilon_p} & X_p & \xrightarrow{\xi_p} & X_{p+1} \\ \downarrow \varphi^{p-1} & & \downarrow \varphi^p & & \downarrow \varphi & & \downarrow \varphi_p & & \downarrow \varphi_{p+1} \\ \tilde{X}^{p-1} & \xrightarrow{\tilde{\xi}^{p-1}} & \tilde{X}^p & \xrightarrow{\tilde{\mu}^p} & \tilde{X} & \xrightarrow{\tilde{\varepsilon}_p} & \tilde{X}_p & \xrightarrow{\tilde{\xi}_p} & \tilde{X}_{p+1} \end{array}$$

for each p . Then (2.9) coincides with the morphism $G\varphi: GX \rightarrow G\tilde{X}$ of the introduction and the theorem asserts that if $\mathcal{G}\varphi$ is an isomorphism and μ^* , $\tilde{\mu}^*$ are complete then all vertical maps in (2.10) are isomorphisms. To see this we invoke the remarks following theorem 2.7 to deduce that if μ^* , $\tilde{\mu}^*$ are R -complete and $\mathcal{G}\varphi$ is an isomorphism, then

$$\varphi^p : X^p \cong \tilde{X}^p, \text{ all } p.$$

Similarly if μ^* , $\tilde{\mu}^*$ are L -complete and $\mathcal{G}\varphi$ is an isomorphism, then

$$\varphi_p : X_p \cong \tilde{X}_p, \text{ all } p.$$

The corollary now follows from these observations and (2.10). We note that, assuming only that $\mathcal{G}\varphi$ is an isomorphism, it follows that φ induces

$$\varphi_{-\infty}^\omega : X_{-\infty}^\omega \cong \tilde{X}_{-\infty}^\omega ;$$

we note also that we have achieved the objective of the introduction since the completion procedure is functorial and does not change the associated graded object.

3. The quasi-completion procedure.

We again consider the exact sequence (1.2) and describe the quasi-completion procedure. Set

$$\alpha^* = \tau(X^*, \xi^*) : PX^{-\omega} \rightarrow (X^*, \xi^*). \text{ Then } \mu^* \alpha^* = PR\mu^* \text{ and } PR\mu^*$$

is monic since R has a left adjoint. Thus we have

$$\begin{array}{ccc} PX^{-\omega} & = & PX^{-\omega} \\ \downarrow \alpha^* & & \downarrow PR\mu^* \\ (X^*, \xi^*) & \longrightarrow & PX \end{array}$$

and passing to quotients we obtain the filtration

$$(3.1) \quad \mu_R^\bullet : (X_R^\bullet, \xi_R^\bullet) \twoheadrightarrow PX_R;$$

its annihilator is

$$(3.2) \quad \varepsilon_{R\bullet} : PX_R \twoheadrightarrow (X_\bullet, \xi_\bullet).$$

We say that μ_R^\bullet is obtained from μ^\bullet by *killing* $X^{-\infty}$. Dually we may obtain a new cofiltration from ε_\bullet or $\varepsilon_{R\bullet}$ by *killing* X_∞ . We call these cofiltrations ε_\bullet^L and $\varepsilon_{R\bullet}^L$ respectively,

$$(3.3) \quad \varepsilon_\bullet^L : PX^L \twoheadrightarrow (X_\bullet^L, \xi_\bullet^L),$$

$$(3.4) \quad \varepsilon_{R\bullet}^L : P(X_R)^L \twoheadrightarrow (X_R^L, \xi_R^L),$$

with annihilators

$$(3.5) \quad \mu^{L\bullet} : (X^\bullet, \xi^\bullet) \twoheadrightarrow PX^L$$

$$(3.6) \quad \mu_{R\bullet}^L : (X_R^\bullet, \xi_R^\bullet) \twoheadrightarrow P(X_R)^L.$$

THEOREM 3.7. (i) $\mu_{R\bullet}^L$ cogenerates X_R and ε_\bullet^L generates X^L .

(ii) The processes of killing $X^{-\infty}$ and X_∞ commute.

Precisely there is a natural isomorphism $(X_R)^L \cong (X^L)_R$ under which the two objects may be identified to X_R^L and then (3.6) is obtained from (3.5) by killing $X^{-\infty}$. Thus $\mu_{R\bullet}^L$ (or $\varepsilon_{R\bullet}^L$) is quasi-complete.

Thus we may describe the quasi-completion procedure as killing X_∞ and $X^{-\infty}$. As a result we replace the original (X^\bullet, ξ^\bullet) by its quotient by $X^{-\infty}$ and (X_\bullet, ξ_\bullet) by its subobject by X_∞ and X is replaced by its subquotient X_R^L . A filtration is quasi-complete if and only if $X_\infty = X^{-\infty} = 0$ and the procedure leaves such a filtration unchanged. Moreover the relation of the completion and quasi-completion procedures is described precisely in the following theorem.

THEOREM 3.8. Let each morphism in (2.4) be factorized as epic followed by monic. This divides (2.4) into four bicartesian squares of which the top right hand square is

(3.9)

$$\begin{array}{ccc}
 X^L & \xrightarrow{\lambda} & X \\
 \downarrow \kappa^L & & \downarrow \kappa \\
 X_R^L & \xrightarrow{\lambda_R} & X_R
 \end{array}$$

where $\kappa \square R \mu^*$, $L \varepsilon \square \lambda$, $\kappa^L \square R \mu^{L*}$, $L \varepsilon_R \square \lambda_R$. In particular (3.9) gives a self-dual description of X_R^L as $Im \kappa \lambda$, or equivalently $Im R \varepsilon \circ L \mu^*$.

4. Remarks.

We remark that the completion procedure depends only on (X^*, ξ^*) (or (X, ξ)) and not on μ^* (or ε). That is, provided (X^*, ξ^*) is the domain of some filtration μ^* , then the completion is independent of the choice of μ^* . On the other hand the quasi-completion procedure does depend in general on μ^* and not simply on (X^*, ξ^*) . For let μ^* generate X without being L -complete; that is $L\mu^*$ is epic with non-zero kernel K ,

$$K \twoheadrightarrow X^\infty \xrightarrow{L\mu^*} X.$$

Then μ^* and π^* both generate; but the object we get in quasi-completing μ^* is $X/X^{-\infty}$, while the object we get in quasi-completing π^* is $X^\infty/X^{-\infty}$ and the kernel of $X^\infty/X^{-\infty} \twoheadrightarrow X/X^{-\infty}$ is again K .

In a category of modules $L\mu^*$ is always monic (but $R\varepsilon$ is not always epic!). Thus we have

PROPOSITION 4.1. *If \mathcal{U} is a category of modules then*

- (i) μ^* is L -complete if it generates X ;
- (ii) the quasi-completion of μ^* depends only on (X^*, ξ^*) .

Reverting to the general case, we may consider criteria of completeness (or quasi-completeness) instead of the procedures. Plainly the question whether $\mu^* : (X^*, \xi^*) \twoheadrightarrow PX$ is complete depends on μ^* itself; but, given that (X^*, ξ^*) is the domain of some filtration, one may give necessary and sufficient conditions for it to be the domain of a complete filtration. Let us call (X^*, ξ^*) a *prefiltration* if it is the domain of some filtration (i.e., if $\pi^* : (X^*, \xi^*) \rightarrow PX^\infty$ is monic), and a *complete prefil-*

tration if it is the domain of some complete filtration.

THEOREM 4.2. *If (X^\bullet, ξ^\bullet) is a prefiltration, then it is complete if and only if $R\eta_\bullet$ is an isomorphism. If $R\eta_\bullet$ is an isomorphism then π^\bullet is the unique complete filtration with domain (X^\bullet, ξ^\bullet) .*

Here $\eta_\bullet \square \pi^\bullet$, $\eta_\bullet : PX^\infty \rightarrow (X^\bullet, \xi^\bullet)$.

Insofar as quasi-completeness is concerned we have the following result.

THEOREM 4.3. *The prefiltration (X^\bullet, ξ^\bullet) is quasi-complete if and only if $R(X^\bullet, \xi^\bullet) = 0$. If $R(X^\bullet, \xi^\bullet) = 0$, then $\mu^\bullet : (X^\bullet, \xi^\bullet) \rightarrow PX$ is quasi-complete if and only if $L\mu^\bullet$ is epic. If also \mathcal{A} is a category of modules then π^\bullet is the unique quasi-complete filtration with domain (X^\bullet, ξ^\bullet) .*

5. Examples.

Since we wish to exhibit the difference between completion and quasi-completion by examples within a category of modules, we are content (see Proposition 4.1 (i)) to consider L -complete filtrations.

EXAMPLE 1. Let $\theta : D \rightarrow D$ be an endomorphism of the module D . We consider the filtration $\dots \subseteq \theta^n D \subseteq \theta^{n-1} D \subseteq \dots \subseteq \theta D \subseteq D$ of D . Since the filtration is right-finite it is evidently L -complete. We render it quasi-complete by factoring out $\lim_{\leftarrow n} \theta^n D = \bigcap_n \theta^n D$; we pass to the cofiltration and we have an exact sequence

$$\bigcap_n \theta^n D \longrightarrow D \xrightarrow{\omega} \lim_{\leftarrow n} D / \theta^n D .$$

Then the filtration obtained by factoring out $\bigcap_n \theta^n D$ is complete if and only if ω is epic. Plainly ω is not always epic. For example, let D be the graded abelian group $D = \bigoplus_{k \geq 0} D_k$, with

$$D_k = (a_k, a_{k+1}, \dots), \quad k \geq 0,$$

the free abelian group on generators a_k, a_{k+1}, \dots . Further let θ be of degree -1 , $\theta |_{D_{k+1}} : D_{k+1} \rightarrow D_k$ being the obvious embedding. Then

$\bigcap_n \theta^n D = 0$, so $\omega : D \twoheadrightarrow \lim_{\leftarrow n} D / \theta^n D$. Moreover,

$$(D / \theta^n D)_k = (a_k, a_{k+1}, \dots, a_{k+n-1}), \quad k \geq 0,$$

so $(\lim_{\leftarrow n} D / \theta^n D)_k = \prod (a_k, a_{k+1}, \dots)$,

the direct product of cyclic groups generated by a_k, a_{k+1}, \dots . On the other hand D_k is the direct sum, so ω is not epic, and the original filtration, although quasi-complete, is not complete. To complete it we must replace D by $D_{-\infty} = \lim_{\leftarrow} D / \theta^n D$ and annihilate the cofiltration

$$D_{-\infty} \twoheadrightarrow \dots \twoheadrightarrow D / \theta^n D \twoheadrightarrow D / \theta^{n-1} D \twoheadrightarrow \dots \twoheadrightarrow D / \theta D \twoheadrightarrow 0.$$

This amounts to replacing D by $D_{-\infty}$ and extending θ to $\theta_{-\infty} : D_{-\infty} \rightarrow D_{-\infty}$ in the obvious way.

EXAMPLE 2. Let b be a cohomology theory defined on the category of CW-complexes. Let $\{K_n\}$ be the skeleton decomposition of the complex K and let $X^n = \ker b(K) \rightarrow b(K_n)$, $X = b(K)$. Then we have the filtration

$$(5.1) \quad \dots \subseteq X^n \subseteq X^{n-1} \subseteq \dots \subseteq X^0 \subseteq X$$

which is again evidently L -complete. We make it quasi-complete by factoring out $\bigcap X^n$; that is, the subgroup of X consisting of cohomology classes which vanish on every skeleton. Call this subgroup $b'(K)$. Moreover let $b_n(K)$ be the image of $b(K)$ in $b(K_n)$. Then we pass to the annihilating cofiltration of (5.1) and obtain the exact sequence

$$b'(K) \twoheadrightarrow b(K) \xrightarrow{\omega} \lim_{\leftarrow n} b_n(K)$$

and the question at issue is whether ω is epic. This is certainly the case if b is representable (by an Ω -spectrum). Thus in the case of the application of a representable cohomology theory to a skeleton decomposition the quasi-completion of the resulting filtration coincides with the completion ¹⁾, and consists of factoring out $b'(K)$. However it is easy to construct examples of cohomology theories wherein ω is not in general epic.

¹⁾ We need not confine attention to a skeleton decomposition; we could take any filtration of K by subcomplexes K_n such that $K_n = 0$, $n < 0$, and $\bigcup_n K_n = K$.

References.

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- [2] S. EILENBERG and J.C. MOORE. *Limits and spectral sequences*, *Topology I* (1962), 1-24.