CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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The nullity spaces of the curvature operator

Cahiers de topologie et géométrie différentielle catégoriques, tome 8 (1966), exp. nº 4, p. 1-20

<http://www.numdam.org/item?id=CTGDC_1966__8__A4_0>

CAHIERS DE TOPOLOGIE *Vol. VIII*
ET GEOMETRIE DIFFERENTIELLE

THE NULLITY SPACES OF THE CURVATURE OPERATOR

by Robert MALTZ (*)

ACKNOWLEDGMENTS.

It is a pleasure to acknowledge my debt to Professor Barrett O'Neill, as teacher and thesis supervisor. Also special thanks are due to Professor Charles Ehresmann for several helpful conversations and much encouragement while the author was in Paris. And finally, I would like to thank Professor Yeaton H. Clifton for his enthusiasm and interest in my work, and for his many suggestions and helpful criticism along the way.

This work was supported in part by funds from the National Science Foundation; and a major portion was completed while the author was an exchange student in Paris under the Fulbright-Hays Act.

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(*) Dissertation submitted for the degree Doctor of Philosophy in Mathematics (University of California, Los Angeles, 1965).

Introduction.

Let M be a C^∞ Riemannian manifold, R the curvature operator, and M_m the tangent space at the point m. Then let

$$N(m) = \{ x \in M_m \mid R_{xy} = 0 \text{ for all } y \in M_m \}$$

be the *nullity space* at m. Set $\mu(m) = dim\, N(m)$. μ is the Index of Nullity. Chern and Kuiper showed that if μ is constant in a neighborhood then N constitutes a completely integrable field of planes, and that the leaves of the resulting foliation are locally flat. In this paper the following results are established : (1) The leaves are totally geodesic submanifolds of M (this implies they are locally flat). Let G be the open set on which μ takes its minimum value μ_o (assumed > 0). (2) Assuming M is complete, the leaves of the nullity foliation of G are also complete. (3) If μ is constant in a deleted neighborhood of a point p, then it has that same value at p also. (4) The boundary of G is the union of geodesics tangent to N.

1. Intrinsic Riemannian Geometry.

Let M be a d-dimensional C^∞ Riemannian manifold, and $\langle\, ,\rangle$ its Riemannian inner product (metric). Let M_m denote the tangent space to M at the point m, $\mathfrak{F}(M)$ the algebra of C^∞-differentiable real-valued functions on M and $\mathfrak{X}(M)$ the algebra of vector fields on M. $\mathfrak{X}(M)$ forms a Lie algebra under the bracket product

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

The bracket operator is bilinear over R, anti-commutative, and satisfies the Jacobi Identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

Associated with the Riemannian metric there is the unique Riemannian (symmetric) connection, which essentially defines the parallel translation of tangent vectors. That is, given any (smooth) curve $\alpha : [0, 1] \to M$ and a vector $x \in M_{\alpha(0)}$, x can be extended to a uniquely defined *parallel vector field* X along α. A *frame* at $m \in M$ is an ordered orthonormal basis for the tangent space M_m. Parallel translation of each of the basis vectors of a frame along a curve α gives rise to a *parallel frame field* along α, said to be obtained by parallel translation of the frame. If $E = (E_1, \ldots, E_d)$ is a parallel frame field along α, so that $E(t) = (E_1(t), \ldots, E_d(t))$ is a frame at $\alpha(t)$, and $X(t)$ is a vector field along α such that $X(t) = \Sigma(x^i(t))E_i(t)$, then the *covariant derivative* $\nabla_{\alpha'(t)} X(t)$ is the vector field on α defined by the expression $\Sigma\, d/dt\, \{x^i(t)\}\, E_i(t)$. More generally, for Y in $\mathfrak{X}(M)$, we define $\nabla_Y X$ by foliating M (locally) by integral curves of Y, i.e. by curves α such that $\alpha'(t) = Y(\alpha(t))$ (This can always be done, by the Existence Theorem for solutions of ordinary differential equations). Then $\nabla_Y X = \nabla_{\alpha'} X$ along any particular integral curve α of Y. It follows from this definition that a vector field X on a curve α is parallel if and only if $\nabla_{\alpha'} X = 0$. By convention we extend ∇ to $\mathfrak{F}(M)$ by setting $\nabla_Y f = Y(f)$ for f in $\mathfrak{F}(M)$.

PROPOSITION 1. 1. ∇ *has the following properties (see* [4]*)*:

(i) $\nabla_{fX+gY}(Z) = f\nabla_X(Z) + g\nabla_Y(Z)$

(ii) $\nabla_Z(X+Y) = \nabla_Z(X) + \nabla_Z(Y)$

(iii) $\nabla_Z(fX) = f\nabla_Z(X) + Z(f)X$

(iv) $X\langle Y, Z\rangle = \langle \nabla_X(Y), Z\rangle + \langle Y, \nabla_X(Z)\rangle$

(v) $\nabla_X(Y) - \nabla_Y(X) = [X, Y]$ *where* $X, Y, Z \in \mathfrak{X}(M)$ *and* $f, g \in \mathfrak{F}(M)$.

A *tensor field* T^a_b of degree (a, b) is a differentiable $\mathfrak{F}(M)$-multilinear real-valued map defined on $\mathfrak{X}^*(M)\times \ldots \mathfrak{X}^*(M)\times\mathfrak{X}(M)\times\ldots\mathfrak{X}(M)$, where $\mathfrak{X}^*(M)$ is the dual space to $\mathfrak{X}(M)$ and there are a copies of $\mathfrak{X}^*(M)$ and b factors $\mathfrak{X}(M)$ in the product. If $X^1, \ldots, X^d$ are linearly independent elements of $\mathfrak{X}^*(M)$ and $X_1, \ldots, X_d$ are linearly independent in $\mathfrak{X}(M)$, the *components* $T^{i_1 \ldots i_b}_{j_1 \ldots j_a}$ of T^a_b with respect to this basis are defined to be

$$T^a_b(X^{j_1}, \ldots, X^{j_a}, X_{i_1}, \ldots, X_{i_b}),$$

where the indices take on all possible values from 1 to d.

Now ∇ can be extended to tensor fields as follows. Given any tensor field T^a_b and a curve α, let E be a parallel frame field on α. Then if $T^{i_1\cdots}_{j_1\cdots}(t)$ are the components of T^a_b with respect to the basis $E(t)$ and its dual $E^*(t)$, then $\nabla_{\alpha'}T^a_b$ is the tensor whose components are $d/dt(T^{i_1\cdots}_{j_1\cdots}(t))$. By proceeding as in the vector field case we can define $\nabla_Y T^a_b$ for any Y in $\mathfrak{X}(M)$.

PROPOSITION 1.2. *Let T^a_b be a tensor of degree (a, b), and let $X^1, \dots X^a$ be in $\mathfrak{X}^*(M)$, $X_1, \dots, X_b$ in $\mathfrak{X}(M)$. Then*

$$\nabla_Y\{T^a_b(X^1, \dots, X^a, X_1, \dots, X_b)\} = (\nabla_Y T^a_b)(X^1, \dots, X^a, X_1, \dots, X_b) +$$
$$+ \sum_j T^a_b(X^1, \dots, \nabla_Y X^j, \dots, X_1, \dots, X_b) +$$
$$+ \sum_i T^a_b(X^1, \dots, X^a, \dots, \nabla_Y X_i, \dots, X_b).$$

PROOF. This proposition is easily checked by writing out the X^i and the X_j in terms of a parallel frame field along an integral curve of Y.

Now we can note that by Proposition 1.1, *(i)*, $\nabla_Y T^a_b$ is linear in Y, so that T^a_b can be considered a tensor of degree $(a, b+1)$. Also it should be noted that by fixing a certain number of variables in a tensor T^a_b the resulting operator is still multilinear in the remaining variables, and hence defines a new tensor of lower degree. In computing the covariant derivative of the new tensor the appropriate generalization to 1.2 must be used.

The *curvature tensor* of a Riemannian manifold M is a $(1, 3)$ tensor, which for $X, Y \in \mathfrak{X}(M)$ can be defined as the operator $R_{XY} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$R_{XY} = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y],$$

where

$$[\nabla_X, \nabla_Y] \equiv \nabla_X \nabla_Y - \nabla_Y \nabla_X.$$

The curvature has the following properties :

PROPOSITION 1.3.

(i) $R_{XY} = -R_{YX}$

(ii) $\langle R_{XY}(Z), W \rangle = -\langle R_{XY}(W), Z \rangle$

(iii) $R_{XY}(Z) + R_{ZX}(Y) + R_{YZ}(X) = 0$

(iv) $\langle R_{XY}(Z), W\rangle = \langle R_{ZW}(X), Y\rangle$.

R_{XY} *is an* $\mathfrak{F}(M)$*-linear operator, and is* $\mathfrak{F}(M)$*-linear in* X *and* Y. *It follows from this that we can define the operation of* R *on* M_m, *as follows:*

$$\{R_{XY}(Z)\}(m) = R_{xy}(z),$$

where $X, Y, Z \in \mathfrak{X}(M)$ *and*

$$X(m) = x, \quad Y(m) = y, \quad Z(m) = z.$$

If $\xi = (x^1, \ldots, x^d)$ is a local coordinate system, then

$$\langle R_{\partial/\partial x^i\, \partial/\partial x^j}(\partial/\partial x^k), \partial/\partial x^l\rangle = R_{ijkl},$$

one of the classical forms of the curvature tensor.

The covariant derivative of R is subject to the following condition, known as *Bianchi's Identity*:

$$(\nabla_X R)_{YZ} + (\nabla_Z R)_{XY} + (\nabla_Y R)_{ZX} = 0,$$

for $X, Y, Z \in \mathfrak{X}(M)$. This will be abbreviated to

$$\mathfrak{S}_{X,Y,Z}(\nabla_X R)_{YZ} = 0,$$

by using the cyclic summation symbol $\mathfrak{S}$.

It is vital to note the position of the parentheses in this identity. We do not have $\mathfrak{S}\nabla_X(R_{YZ}) = 0$. It is interesting to note, though, that if $[X, Z]$, $[X, Y]$, $[Y, Z]$ all vanish then the last equality holds. This is the case when $X = \partial/\partial x^i$, $Y = \partial/\partial x^j$, $Z = \partial/\partial x^k$ for some coordinate system $\xi = (x^1, x^2, \ldots, x^d)$. The classical coordinate version of Bianchi's Identity is actually

$$\mathfrak{S}_{i,j,k}\nabla_{\partial/\partial x^i}(R_{\partial/\partial x^j \partial/\partial x^k}) = 0.$$

LEMMA 1. *If* $[X, Y] = [X, Z] = [Y, Z] = 0$, *then* $\mathfrak{S}\Delta_X(R_{YZ}) = 0$.

PROOF. These remarks can be verified by expanding

$$\nabla_X(R_{YZ}) = (\nabla_X R)_{YZ} + R_{\nabla_X Y, Z} + R_{Y, \nabla_X Z}$$

according to Proposition 1.2, taking the cyclic sum, and cancelling by using

$$\nabla_X Y - \nabla_Y X = [X, Y] = 0.$$

Now let Π be a map assigning to each $m \in M$ a b-dimensional linear subspace $\Pi(m) \subseteq M_m$, for some fixed $b \leq d$. We write $X \in \Pi$ for a vector field X if $X(m) \in \Pi(m)$ for all m. If there are b linearly independent vector fields $X_1, \dots, X_b \in \Pi$ in a neighborhood O_p of every point $p \in M$, Π is said to be a *(differentiable) field of b-planes*. The Theorem of Frobenius states (see Bishop and Crittendon, [1]): If $X, Y \in \Pi$ implies that $[X, Y] \in \Pi$ also, then there exists a foliation of M by b-dimensional maximal connected submanifolds, the *leaves*, such that $\Pi(m)$ is the tangent plane of the leaf through m. Π is said to be *completely integrable* if it has this property.

A curve γ in M is called a *geodesic* if γ' is parallel along γ, i.e. $\gamma'' = \nabla_{\gamma'}\gamma' = 0$.

In order to get a useful characterization of geodesics, we now define the *frame bundle* $F(M)$. $F(M)$ is the set of all orthonormal frames on M, given a natural differentiable structure so that the projection map π, which assigns to each frame f its base point in M, is differentiable (see Bishop and Crittendon, [1]).

A curve $\bar{\alpha}$ in $F(M)$ will be called *horizontal* if it is a *horizontal lifting* of a curve α in M, i.e. if it is a parallel frame field on α. A vector in $F(M)_f$ is called *horizontal* if it is tangent to a horizontal curve through f. It follows that for each vector $x \in M_m$ and frame f at m, there is a unique horizontal vector $\bar{x} \in F(M)_f$ such that $d\pi(\bar{x}) = x$.

The *basic vector field* B_c on $F(M)$ can now be defined, for each d-tuple of real numbers $c = (c_1, c_2, \dots, c_d)$. If $f = (f_1, f_2, \dots, f_d) \in F(M)$, then $B_c(f)$ is the unique horizontal vector in $F(M)_f$ such that

$$d\pi(B_c(f)) = \sum_i c_i f_i .$$

PROPOSITION 1.4. *A curve γ in M is a geodesic if and only if it has a horizontal lift $\bar{\gamma}$ in $F(M)$ which is an integral curve of a basic vector field.*

PROOF. Let f be an arbitrary frame at some point $\gamma(t_0)$ on γ. Parallel translate f along γ to define a parallel frame field $F(t)=(f_1(t),\ldots, f_d(t))$ and hence a horizontal lifting $\bar{\gamma}$ of γ into $F(M)$. Now if $\gamma'(t_0) = \Sigma c_i f_i$, the fact that $F(t)$ and γ' are both parallel along γ assures that $\gamma'(t) = \Sigma c_i f_i(t)$. Now

$$d\pi \bar{\gamma}'(t) = \gamma'(t) = \Sigma c_i f_i(t),$$

so $\bar{\gamma}'(t)$ must be the unique horizontal vector in $F(M)_{f(t)}$ projecting to $\Sigma c_i f_i(t)$. But that means

$$\bar{\gamma}'(t) = B_c(f(t)) = B_c \circ \bar{\gamma}(t),$$

or $\bar{\gamma}$ is an integral curve of B_c

Reversing the steps proves the converse.

2. Immersions.

Let M and $\bar{M}$ be Riemannian manifolds with inner products $\langle\, , \rangle$ and $\overline{\langle\, , \rangle}$ respectively, and curvature operators R and $\bar{R}$. A differentiable map $j : M \to \bar{M}$ is said to be an *isometric immersion* if

$$\overline{\langle dj(x), dj(y)\rangle} = \langle x, y\rangle$$

for any vectors $x, y \in M_m$, all $m \in M$. (Here dj denotes the (linear) differential map induced on the tangent spaces of M by j). From now on we will suppress j in the notation and consider M to be a subset of $\bar{M}$, and identify $\langle\, , \rangle$ and $\overline{\langle\, , \rangle}$. Now let $\mathfrak{F}(M)$ be the algebra of real-valued C^∞ functions on M, $\mathfrak{X}(M)$ the Lie algebra of vector fields on M, $\bar{\mathfrak{X}}(M)$ the algebra of restrictions to M of vector fields on $\bar{M}$. Then we have $\bar{\mathfrak{X}}(M) = \mathfrak{X}(M) \oplus \mathfrak{X}(M)^\perp$ where $\mathfrak{X}(M)^\perp$ denotes the set of vector fields perpendicular to M. Let $P : \bar{\mathfrak{X}}(M) \to \mathfrak{X}(M)$ be the orthogonal projection. Let ∇ be the Riemannian connection (covariant differentiation operator) of M and $\bar{\nabla}$ the Riemannian connection of $\bar{M}$ restricted to $\bar{\mathfrak{X}}(M)$. The *difference operator* $T : \mathfrak{X}(M) \times \bar{\mathfrak{X}}(M) \to \bar{\mathfrak{X}}(M)$ is defined as follows :

(2.1) $\qquad T_X(Y) = \bar{\nabla}_X(Y) - \nabla_X(Y)$ for $X, Y \in \mathfrak{X}(M)$,

(2.2) $\qquad T_X(Z) = P\bar{\nabla}_X(Z)$ for $X \in \mathfrak{X}(M)$, $Z \in \mathfrak{X}(M)^\perp$.

PROPOSITION 2.1. *T has the following properties :*

(i) T *is bilinear over* $\mathfrak{F}(M)$.

(ii) $T_X(Y) = T_Y(X)$ *for* $X, Y \in \mathfrak{X}(M)$.

(iii) $\langle T_X(Y), Z \rangle = - \langle T_X(Z), Y \rangle$ *for* $X \in \mathfrak{X}(M)$, $Y, Z \in \overline{\mathfrak{X}}(M)$.

(iv) $T_X(\mathfrak{X}(M)) \subseteq \mathfrak{X}(M)^{\perp}$; $T_X(\mathfrak{X}(M)^{\perp}) \subseteq \mathfrak{X}(M)$ *for* $X \in \mathfrak{X}(M)$.

Note that from *(iii)* it follows that T_X is determined by its effect on $\mathfrak{X}(M)$.

PROPOSITION 2.2. *Let* $X, Y \in \mathfrak{X}(M)$. *Then on* $\mathfrak{X}(M)$ *the Gauss Equation holds :*

$$P\overline{R}_{XY} = R_{XY} - [T_X, T_Y].$$

PROOF. Use $\overline{R}_{XY} = \overline{\nabla}_{[X,Y]} - [\overline{\nabla}_X, \overline{\nabla}_Y]$, apply P.

T is related to the classical second fundamental form as follows : let $\xi = (x^1, \ldots, x^{n+k})$ be a coordinate system in a neighborhood of $p \in M$ such that the $\partial/\partial x^i$ are tangent to M for $1 \leq i \leq n$ and the $\partial/\partial x^\alpha$ are perpendicular to M for $n+1 \leq \alpha \leq n+k$. The second fundamental form $b_{ij\alpha}$ is then related to T by

$$T_{\partial/\partial x^i}(\partial/\partial x^j) = \sum_{\alpha=n+1}^{n+k} b_{ij\alpha}\, \partial/\partial x^\alpha.$$

By Proposition 2.1, *(iii)*, T and $b_{ij\alpha}$ contain the same information.

NOTE. The T operator was originally defined by Ambrose and Singer using a frame bundle approach. I am following Alfred Gray [6] in defining T in terms of ∇ and $\overline{\nabla}$.

M is said to be *totally geodesic* in $\overline{M}$ if for any geodesic $\gamma \in M$, $j \circ \gamma$ is a geodesic of $\overline{M}$.

PROPOSITION 2.3. *M is totally geodesic in $\overline{M}$ if and only if $T = 0$.*

PROOF. $T_X(X) = 0$ if and only if $\nabla_X(X) = \overline{\nabla}_X(X)$. This is equivalent to

$$\nabla_{\gamma'}(\gamma') = \overline{\nabla}_{\gamma'}\gamma' = 0,$$

γ' is a geodesic in M. $T_X(X) = 0$ for all X if and only if $T = 0$.

PROPOSITION 2.4. *If* M *is totally geodesic in* $\overline{M}$ *then* $\overline{M}$*-parallel translation along a curve* α *in* M *preserves tangency and orthogonality of vectors with respect to* M.

PROOF. Since $\overline{\nabla}_X - \nabla_X = T_X = 0$ for $X \in \mathfrak{X}(M)$, we have $\overline{\nabla}_{\alpha'} = \nabla_{\alpha'}$. Hence $\overline{M}$-parallelism and M-parallelism coincide along α. But M-parallel translation preserves tangency of vectors on M; hence the same is true for $\overline{M}$-parallelism along α. But orthogonality must also be preserved since, if x is tangent to M at $\alpha(t_o)$ and y is orthogonal, we have $\langle x, y \rangle = 0$. Now if X and Y are the parallel vector fields on α generated by x and y, we have

$$\overline{\nabla}_{\alpha'} \langle X, Y \rangle = \langle \overline{\nabla}_{\alpha} X, Y \rangle + \langle X, \overline{\nabla}_{\alpha'} Y \rangle = 0.$$

Hence $\langle X, Y \rangle$ is constant along α. But

$$\langle X, Y \rangle (\alpha(t_o)) = \langle x, y \rangle = 0.$$

So Y is orthogonal to M along α.

3. The Index of Nullity.

The *index of nullity* μ is a non-negative integer - valued function defined on M^d as follows : at each point $m \in M^d$, $\mu(m)$ is the dimension of the vector subspace $N(m)$ of M_m spanned by tangent vectors x such that $R_{xy} = 0$ for all $y \in M_m$. $N(m)$ will be called the *nullity space* at m, while N will denote the field of nullity planes. If Y is a vector field, $Y \in N$ will mean Y is a *nullity vector field,* i.e. $Y(m) \in N(m)$ for all m in question. In the sequel we assume $\mu \neq 0$, $\mu \neq d$ unless otherwise specified.

We now state explicitly some simple algebraic consequences of this definition. Let $x \in N(m), y, z, w, u \in M_m$. Then $R_{xy}(z) = R_{yx}(z) = 0$. Futhermore

$$-\langle R_{yz}(x), w \rangle = \langle R_{yz}(w), x \rangle = \langle R_{wx}(y), z \rangle = 0.$$

Since y, z and w were chosen arbitrarily in M_m, it follows that $R_{yz}(x) = 0$ also. *Hence the* R*-operator vanishes if any of its entries are nullity vectors.* Finally $\langle R_{yx}(w), x \rangle = 0$ implies that $R_{yz}(w)$ is always in

$N^{\perp}(m)$, the orthogonal complement of $N(m)$ in M_m. And conversely, if $\langle R_{yz}(w), u\rangle = 0$ for all $y, z, w \in M_m$, then $u \in N(m)$. So we have the following alternative definition of μ : *$d - \mu(m)$ is the rank of the subspace $N^{-}(m)$ of M_m spanned by all vectors of the form $R_{zy}(w)$, $(y, z, w \in M_m)$.*

Now we can see that if $\mu \neq d$, then $d - \mu \geq 2$. This is true because R_{xy} is an anti-symmetric linear operator on M_m and hence has even rank.

In classical notation $d - \mu(m)$ is the number of linearly independent vectors at m of the form $\sum_l R_{ijkl}\partial/\partial x^l$, $\xi = (x^1, x^2, \ldots, x^d)$ a coordinate system at m. Or once again, the smallest number of linearly independent differential forms $\omega^1, \omega^2, \ldots$ in a neighborhood of m needed to express the curvature form

$$\Omega_{ij} = \sum_{k,l} R_{ijkl}\, \omega^k \wedge \omega^l .$$

Chern and Kuiper [2] showed that if μ is constant in an open set, then the nullity spaces N constitute a completely integrable field of μ-planes. We now reestablish this result using the covariant differentiation operator ∇. We further show that the resulting leaves are totally geodesic. It follows as a corollary that the leaves are locally flat in the induced metric, also established in [2].

THEOREM 3.1. *If μ is constant on an open submanifold $\tilde{G}$ then the nullity field of planes N is completely integrable on $\tilde{G}$.*

PROOF. We suppose U, V are vector fields in N, and Z is an arbitrary vector field. We show $[U, V] \in N$ also, i.e. $R_{[U,V],Z} = 0$.

We start by expanding $\nabla_Z(R_{UV})$ by Proposition 1.2, and then summing cyclically over U, V and Z. R_{UV}, R_{VZ}, etc., vanish identically, so we have :

$$0 = \mathop{\mathfrak{S}}_{U,V,Z} \nabla_Z(R_{UV}) = \mathop{\mathfrak{S}}_{U,V,Z} \{(\nabla_Z R)_{UV} + R_{\nabla_Z U, V} + R_{U, \nabla_Z V}\}$$

But $\mathop{\mathfrak{S}}_{U,V,Z} (\nabla_Z R)_{UV} = 0$ by Bianchi's Identity. Most of the remaining terms on the right are zero since U and V are nullity, but we find after summing that

$$0 = R_{Z,\nabla_V U} + R_{\nabla_U V, Z} = R_{\nabla_U V - \nabla_V U, Z}\,.$$

But $\nabla_U V - \nabla_V U = [U, V]$, the symmetry condition on ∇. So we have $R_{[U,V],Z} = 0$ as required.

THEOREM 3.2. *Let L be a leaf of the nullity foliation. Then L is a totally geodesic submanifold of M.*

PROOF. We have an immersion $j : L \to M$ so we use the terminology for describing immersions as developed in §2. However we continue to use R for the curvature of M; let ρ denote the curvature of L. $N(m)$ is identified with L_m, and $N^{\perp}(m)$ with $L_m^{\perp}$, $m \in L$. Our task is to show that $T_X = 0$ for all $X \in \mathfrak{X}(L)$.

We first show

$$T_X . R_{YZ} = 0 \text{ for } X \in \mathfrak{X}(L),\ Y, Z \in \mathfrak{X}^{\perp}(L)$$

(the product here is composition of linear operators, of course). Note that we are using the fact that $d - \mu \geq 2$. Since $R_{YZ}(U) \in \mathfrak{X}^{\perp}(L)$ we have

$$T_X . R_{YZ}(U) = P . \overline{\nabla}_X (R_{YZ}(U)),$$

where $U \in \mathfrak{X}(M)$. Taking the cyclic sum $\mathfrak{S}_{X,Y,Z}$ we get

$$\mathfrak{S}_{X,Y,Z}\, T_X . R_{YZ}(U) = T_X . R_{YZ}(U)$$

by nullity of X. Hence

$$T_X . R_{YZ}(U) = \mathfrak{S} P . \overline{\nabla}_X (R_{YZ}(U) = \underset{X,Y,Z}{\mathfrak{S}} \{ P . (\overline{\nabla}_X R)_{YZ}(U) \} +$$
$$+ P . R_{\overline{\nabla}_X Y, Z}(U) + \overline{P} . R_{Y, \overline{\nabla}_X Z}(U) +$$
$$+ P . R_{XY}(\nabla_X U).$$

But

$$\underset{X,Y,Z}{\mathfrak{S}} \{ P . (\nabla_X R)_{YZ}(U) \} = 0$$

by Bianchi's Identity, and the remaining terms are zero since the image space of the curvature operator is precisely the non-nullity vector fields, which is just $\mathfrak{X}^{\perp}(L)$, the kernel of P.

Hence $T_X . R_{YZ} = 0$. But $T_X . R_{YZ} = R_{YZ} . T_X = 0$ since T_X and

R_{YZ} are both antisymmetric linear operators. So $R_{YZ} \cdot T_X = 0$ for all $Y, Z \in \mathfrak{X}^{\perp}(L)$.

Now the images of $\mathfrak{X}(L)$ under T_X are in $\mathfrak{X}^{\perp}(L)$. But given any non-zero vector field $W \in \mathfrak{X}^{\perp}(L)$ there must be some $Y, Z \in \mathfrak{X}^{\perp}(L)$ for which $R_{YZ}(W) \neq 0$, since $\langle R_{YZ}(U), W \rangle \neq 0$ for some U, Y, Z; and

$$\langle R_{YZ}(U), W \rangle = - \langle R_{YZ}(W), U \rangle .$$

So all images under T_X must vanish, or L is totally geodesic.

COROLLARY 3.3. *L is locally flat in the induced metric.*

PROOF. We use the Gauss Equation

$$P \cdot R_{XY} = \rho_{XY} + [\, T_X, T_Y \,] .$$

For any $X, Y \in \mathfrak{X}(L)$ we get immediately $\rho_{XY} = 0$, since T_X and T_Y vanish.

4. The set G of minimal nullity.

In this section we prove some theorems about the set G on which μ attains its minimal value $\mu_o > 0$.

LEMMA 4.1. *Given any $p \in M$, there exists a certain neighborhood O of p such that $\mu(m) \leq \mu(p)$ for all m in O.*

PROOF. Choose a coordinate system $\xi = (x^1, x^2, \dots, x^d)$ on a neighborhood of m. Then there are $d - \mu(m)$ vector fields $Y_1, Y_2, \dots, Y_{d-\mu(m)}$ all of form $\sum_l R_{ijkl}\, \partial / \partial x^l$ which are linearly independent at m. But then $Y_1 \wedge Y_2 \wedge \dots \wedge Y_{d-\mu(m)}$ must be non-zero at m, and hence by continuity non-zero in a neighborhood of m. But that means $d - \mu(m) \geq d - \mu(p)$ everywhere on O, or $\mu(p) \geq \mu(m)$ on O.

THEOREM 4.2. *The set G on which μ takes on its minimum value μ_o is an open submanifold of M.*

PROOF. Let $p \in G$. Then by Lemma 4.1 $\mu(p) = \mu_o \geq \mu(m)$ on some nbd. O of p. But μ_o was assumed minimal, so $\mu_o = \mu(m)$ on O. But then $p \in O \subset G$, so G is open.

THEOREM 4.3. *Assume M is complete, and let G be the open set on which μ takes its minimum value μ_o. Then the leaves L of the nullity foliation induced on G are complete.*

Before proving the theorem we recall a few definitions and facts from the calculus of variations needed in the proof to the theorem.

A *rectangle* or *1-parameter family of curves* is a C^{∞} map $Q : R^2 \to M$. Let u^1 and u^2 denote the natural coordinate functions in R^2. The *longitudinal curves* of the rectangle are defined by restricting Q to the lines u^2 = constant in R^2, while the *transverse curves* arise by restricting Q to the lines u^1 = constant.

The *associated vector field* to Q, denoted by X, is defined by the velocity vector fields of the transverse curves. If the longitudinal curves are all geodesics, then Q is called a *1-parameter family of geodesics*, and X is called a *Jacobi vector field.* Now we have the following well-known

LEMMA. *If Q is a 1-parameter family of geodesics, X satisfies the Jacobi Equation $X'' = \nabla_{\sigma'}(\nabla_{\sigma'} X) = R_{X\sigma'}(\sigma')$ along any longitudinal curve σ.*

PROOF. $X = dQ(\partial/\partial u^1)$, $\sigma' = dQ(\partial/\partial u^2)$. But $[\partial/\partial u^1, \partial/\partial u^2] = 0$, so $[x, \sigma'] = dQ[\partial/\partial u^1, \partial/\partial u^2] = 0$.

Hence $R_{X\sigma'}(\sigma') = \nabla_{[X,\sigma']}(\sigma') - [\nabla_X \nabla_{\sigma'} - \nabla_{\sigma'}\nabla_X](\sigma') =$

$$= -\nabla_X\nabla_{\sigma'}(\sigma') + \nabla_{\sigma'}\nabla_X(\sigma') = \nabla_{\sigma'}\nabla_X(\sigma')$$

since $\nabla_{\sigma'}(\sigma') = 0$. But $\nabla_X(\sigma') - \nabla_{\sigma'}(X) = [X, \sigma'] = 0$, so we have $R_{X\sigma'}(\sigma') = \nabla_{\sigma'}(\nabla_{\sigma'} X)$.

PROOF OF THEOREM.

Let $\gamma : [0, c) \to L$ be a geodesic segment in L. It suffices to show that γ can be extended, as a geodesic of L, over the half-line $[0, \infty)$. Suppose this cannot be done, and that γ as given is maximal. Since M is complete, γ can be extended as a geodesic $\tilde{\gamma}$ of $M (\gamma = \tilde{\gamma} \cap L)$. Since L is totally geodesic in M, it follows that $\tilde{\gamma}(c)$ is not in G. But that means that $\mu(\tilde{\gamma}(c)) > \mu_o$. We now show that is impossible.

First let $p = \gamma(0), \tilde{p} = \tilde{\gamma}(c)$, and let us make the convention that $1 \leq i, j, k \leq \mu_o$ are «nullity» indices, $\mu_o + 1 \leq \alpha, \beta, \gamma \leq d$ are «non-nullity» indices, while $1 \leq I, J, K \leq d$ are unrestricted indices.

Now we note that if we have a coordinate system $\xi = (x^1, \ldots, x^d)$ in a neighborhood U of $\tilde{p}$, with $\partial/\partial x^1 = \gamma'$ along γ and $\partial/\partial x^i$ nullity on $U \cap G$, then by Lemma 1 of paragraph 1, we have

$$\mathfrak{S}\nabla_{\partial/\partial x_1}(R_{\partial/\partial x^\alpha \partial/\partial x^\beta}) = 0.$$

$\nabla_{\partial/\partial x_1}(R_{\partial/\partial x^\alpha \partial/\partial x^\beta}) = 0$ then also, using the fact that the tensors $R_{\partial/\partial x^1 \partial/\partial x^\alpha}, R_{\partial/\partial x^1 \partial/\partial x^\beta}$ vanish identically in $U \cap G$, by nullity of $\partial/\partial x^1$. But this means that $R_{\partial/\partial x^\alpha \partial/\partial x^\beta}$ is parallel along γ in $U \cap G$. Now let $E = (E_1, \ldots, E_{\mu_o}, \ldots, E_d)$ be a parallel frame field along $\tilde{\gamma}$, adapted to N on G, i.e. $E_i \in N$, $E_\alpha \notin N$. (This is possible since L is totally geodesic. Cf. Prop. 2.4). Now if E_I is nullity at $\tilde{p}$, for some I, we have: $R_{\partial/\partial x^\alpha \partial/\partial x^\beta}(E_I)$ is a parallel vector field along $\tilde{\gamma} \mid U \cap G$ vanishing at $\tilde{\gamma}(c)$ by assumption, so it must vanish identically on $\tilde{\gamma} \mid U \cap G$. Hence $E_I \in N$ on $\tilde{\gamma} \mid U \cap G$. This proves that μ cannot increase at $\tilde{p}$.

We now establish the existence of a coordinate system ξ as above, starting with a Frobenius coordinate system $\eta = (y^1, \ldots, y^d)$ on a neighborhood V of $\gamma(0) = p$. We can further assume that $\eta(p) = (0, \ldots, 0)$ the origin in R^d, and that $(\partial/\partial y^1)_p = \gamma'(0)$, $\partial/\partial y^i \in N$ on V. (If η can be extended to $\tilde{p}$ then the proof can be finished as above, but in general this cannot be done).

Now let Σ be the slice of V determined by $y^i = 0$, and let

$$E = (E_1, \ldots, E_{\mu_o}, \ldots, E_d)$$

be a C^∞-frame field on Σ adapted to the nullity field $(E_i \in N)$, and such that $E_1(p) = \gamma'(0)$. $\eta_2 = (y^{\mu_o+1}, \ldots, y^d)$ defines a coordinate system on Σ; set $\eta_2(\Sigma) = W \subset R^{d-\mu_o}$. Now define $F : R^{\mu_o} \times W \to M$ by

$$F(x^1, \ldots, x^{\mu_o}, \eta_2(s)) = exp_s(\overline{x}),$$

where $s \in \Sigma$ and $\overline{x} = \Sigma x^i E_i(s)$. Since M is complete, F is defined for all values in R^{μ_o}.

We now prove F is regular along $\tilde{\gamma}$. First we identify $R^{\mu_o} \times W$ with a subset U of R^d, and let $u^1, \ldots, u^d$ be the natural Euclidean coordinate functions on U. Fixing $u^I = 0$ for all $I \neq 1$, $I \neq \alpha$, and restricting F to the plane so defined in U, we obtain an induced mapping $F_\alpha : R^2 \to M$, which is just a rectangle. Furthermore the longitudinal curves of F_α are the geodesics $exp_s(tE_1(s))$, where s is a point in the slice Σ_α of Σ defined by $u^\beta = 0$ for $\beta \neq \alpha$. It follows that the associated vector field X_α to F_α is a Jacobi vector field, satisfying the Jacobi equation $X''_\alpha = R_{X_\alpha \tilde{\gamma}'}(\tilde{\gamma}')$ along the geodesic $\tilde{\gamma} = exp_p(tE_1(p))$ in particular. But $R_{X_\alpha \tilde{\gamma}'}(\tilde{\gamma}') = 0$ in G since $\gamma' \in N$, so we have $X''_\alpha = 0$ along γ, or

$$X_\alpha(t) = A_\alpha(t) + tB_\alpha(t),$$

where A_α and B_α are parallel vector fields along γ. Hence X_α is well-defined, bounded and continuous on $\tilde{\gamma}([0, c))$. (We are setting $X_\alpha(t) = X_\alpha(\gamma(t))$ along γ here, of course). Also note that $X_\alpha = dF_\alpha(\partial/\partial u^\alpha)$ since X_α is the associated vector field of the rectangle F_α. Writing out the components of $X_\alpha(t)$ with respect to the parallel adapted frame field $E(t)$, we have $X_\alpha(t) = A_\alpha(t) + tB_\alpha(t) = \Sigma_I A^I_\alpha E_I(t) + \Sigma t B^I_\alpha E_I(t)$ where the components A^I_α and B^I_α are constants since A_α and B_α are parallel along γ. Set $X^\perp_\alpha(t) = \Sigma_\beta A^\beta_\alpha E_\beta(t) + \Sigma t B^\beta_\alpha E_\beta(t)$, the «late» components of $X_\alpha(t)$. (Note that at $\tilde{p}$ the «early» vector fields $E_i(t)$ remain nullity by continuity, so that $X_\alpha - X^\perp_\alpha \in N$ on $\tilde{\gamma}([0, c])$.

We will now show the $X^\perp_\alpha$ remain linearly independent on $\tilde{\gamma}([0, c])$. First of all, the $X^\perp_\alpha$ are linearly independent at p since

$$X_\alpha(0) = dF(\partial/\partial u^\alpha)_p = d\eta_2^{-1}(\partial/\partial u^\alpha) = (\partial/\partial y^\alpha)_p.$$

Hence the $X_\alpha(0)$ form a basis for the non-nullity space $N^\perp(p)$, which has dimension $d - \mu_o$. But the $X^\perp_\alpha(0)$ also span $N^\perp(p)$. Since there are exactly $d - \mu_o$ $X^\perp_\alpha(0)$, they are linearly independent. Now suppose there is some linear combination $X = \Sigma c^\alpha X^\perp_\alpha$ such that $X(t_o) = \Sigma c^\alpha X^\perp_\alpha(t_o) = 0$ for some $t_o \leq c$. Now $\mathfrak{S}\nabla_{\gamma'}(R_{X_\alpha X_\beta}) = \nabla_{\gamma'}(R_{X_\alpha X_\beta}) - 0$ along γ, since

$$[X_\alpha, X_\beta] = dF([\partial/\partial u^\alpha, \partial/\partial u^\beta]) = 0,$$

$$[\gamma', X_\alpha] = dF[\partial/\partial u^1, \partial/\partial u^\alpha]) = 0, \qquad [\gamma', X_\beta] = 0,$$

so we can use the Lemma 1 of paragraph 1 again. $R_{X_\alpha^\perp X_\beta} = R_{X_\alpha X_\beta}$ on $\tilde{\gamma}([0, c])$ since R vanishes on the nullity components of X_α. Hence it follows from $\nabla_{\gamma'}(R_{X_\alpha X_\beta}) = 0$ that the components of $R_{X_\alpha^\perp X_\beta}$ with respect to the parallel frame field $E(t)$ are constants, and the same is true of the components of R_{XX_β}. But $R_{XX_\beta} = 0$ at t_o since $X(t_o) = 0$. Hence $R_{XX_\beta} = 0$ everywhere on γ. In particular this must be true at p, and for all $\beta \geq \mu_o + 1$. But the X_β span $N^\perp$ at p, so $R_{XX_\beta} = 0$ implies $X(0) \in N(p)$. On the other hand $X(0) = \Sigma\, c^\alpha X_\alpha^\perp(0) \in N^\perp(p)$, so this is possible only if all $c^\alpha = 0$. Therefore the $X_\alpha^\perp$ must remain linearly independent on $\tilde{\gamma}([0, c])$.

Now define the map F_1 by

$$F_1(x^1, \ldots, x^{\mu_o}) = F(x^1, \ldots, x^{\mu_o}, 0, \ldots, 0).$$

Then F_1 defines a regular mapping onto L, since

$$F_1(x^1, \ldots, x^{\mu_o}) = exp_p(\Sigma\, x^i E_i(p)) \in L,$$

and since L is locally flat, exp_p is a local isometry. Hence dF_1 is an orthogonal linear transformation, and $dF_1(\partial/\partial u^i)$ are orthonormal at each point of L. Hence by continuity $dF(\partial/\partial u^i)$ are orthonormal on the boundary of L as well; in particular at $\tilde{p}$. But $dF_1(\partial/\partial u^i) = dF(\partial/\partial u^i)$. So $dF(\partial/\partial u^i)$ are orthonormal at $\tilde{p}$. Furthermore $dF(\partial/\partial u^i) \in N$ on L, hence by continuity $dF(\partial/\partial u^i)_{\tilde{p}} \in N(\tilde{p})$.

Now we can see that F must be regular on $\tilde{\gamma}([0, c])$. First let $\tilde{N}(t)$ be the μ_o- plane at $\tilde{\gamma}(t)$ spanned by the «early» vectors $E_i(t)$, and $\tilde{N}^\perp(t)$ be the orthogonal complement spanned by the $E_\alpha(t)$ ($N(\gamma(t)) = \tilde{N}(t)$ on L, of course). Then the $dF(\partial/\partial u^i)$ are linearly independent on $\tilde{\gamma}([0, c])$ and span $\tilde{N}(t)$, $0 \leq t \leq c$. Furthermore the $dF(\partial/\partial u^\alpha) = X_\alpha$ are linearly independent, and their late components $X_\alpha^\perp$ span $\tilde{N}^\perp(t)$, $0 \leq t \leq c$. Hence the rank of dF is exactly d everywhere on $\tilde{\gamma}([0, c])$.

In particular F is regular at $\tilde{p} = \tilde{\gamma}(c)$, so F^{-1} defines a coordinate system $\xi = (x^1, \ldots, x^d)$ on a neighborhood U of F. Also $\partial/\partial x^i \in N$ on $U \cap G$, $\partial/\partial x^1 = \tilde{\gamma}'$ along $\tilde{\gamma}$. Hence ξ is the required coordinate system, and theTheorem is established.

It is a pleasure to acknowledge essential aid given by Professor Y.H. Clifton in constructing this proof.

THEOREM 4.4. *Suppose the nullity index* μ *has the constant value* μ_1 *everywhere in the deleted neighborhood* O *of a point* $p \in M$. *Then* μ *has the same value* μ_1 *at* p *as well.* [NOTE. By Lemma 3.3 we know that $\mu(p) \geq \mu_1$. The Theorem claims that $\mu(p) = \mu_1$] .

PROOF. If γ is any nullity geodesic in O (i.e. $\gamma' \in N$ in O), and p lies on γ, then p lies in the closure of a leaf of the nullity foliation. In that case the proof of Theorem 4.3 can be applied to show $\mu(p) = \mu_1$. To show the existence of such a geodesic, we consider a segment of an arbitrary geodesic $\alpha : (0, 1) \to O$ starting at p. Let $t_1, t_2, \ldots$ be an infinite convergent sequence of real numbers in $(0, 1)$ such that $\lim t_i = 0$. At each point $\alpha(t_i)$ we pick a (unit-speed) geodesic γ_i starting in a nullity direction at $\alpha(t_i)$. Then the γ_i lie in leaves of the nullity foliation and are nullity geodesics in O. Now consider the sequence of tangent vectors $\gamma_i'(0)$. This sequence defines a sequence of points $\tilde{\gamma}_i'(0)$ in the sphere-bundle B over the closed segment $\alpha : [0, 1] \to M$, and this bundle is a compact set. Hence we can extract a convergent subsequence $\tilde{\gamma}_j'(0)$. Now the limit point $\tilde{\gamma}'(0)$ of the sequence $\tilde{\gamma}_j'(0)$ must lie over $p = \alpha(0)$, since the bundle projection π is a continuous function, so $\pi(\tilde{\gamma}'(0))$ must be a limit point of $\pi(\tilde{\gamma}_j'(0)) = \alpha(t_j)$; but $\alpha(0)$ is the only such limit point. Hence $\tilde{\gamma}'(0)$ defines a unique tangent vector $\gamma'(0)$ at $\alpha(0)$.

Now let γ be the geodesic starting at p in the $\gamma'(0)$ direction. We will show γ is a nullity geodesic in O. To do so choose an $\varepsilon_o > 0$ small enough so that all the segments $\gamma_j([0, \varepsilon_o])$ are in O. We will show that for $0 \leq \varepsilon \leq \varepsilon_o$ the points $\gamma_j(\varepsilon)$ converge to $\gamma(\varepsilon)$, and hence that the tangent vectors $\gamma_j'(\varepsilon)$ converge to $\gamma'(\varepsilon)$ (these assertions are actually true for all ε). This would prove that $\gamma'(\varepsilon)$ is a nullity vector, since the $\gamma_j'(\varepsilon)$ all are nullity vectors when ε is properly restricted. [PROOF. Given any $R_{\gamma' y}$, we can set $y = \lim y_j$, $y_j \in M_{\alpha(t_j)}$. Then $R_{\gamma' y} = \lim R_{\gamma_j' y_j}$, while the terms of the sequence all vanish. Hence $R_{\gamma' y} = \|0\|$ also. Hence the limit of a sequence of nullity vectors is itself a nullity vector.]

To do this we introduce a sequence of frames

$$E(t_j) = (e_1(t_j), e_2(t_j), \ldots, e_\alpha(t_j))$$

such that $e_1(t_j) = \gamma_j'(0)$. We may assume that the $E(t_j)$ converge to a definite limit frame $E(0)$ at $\alpha(0)$, by repeating the sphere-bundle argument above, substituting $F(M)$ for B everywhere, $E(t_j)$ for $\tilde{\gamma}_j'(0)$ [or else by using the sphere-bundle argument iteratively on the vector sequences $e_i(t_j)$]. In this process

$$E_1(0) = \lim \gamma_j'(0) = \gamma'(0)$$

also. Now we parallel translate $E(t_j)$ along γ_j, thus defining a horizontal lifting $\overline{\gamma}_j$ of γ_j into $F(M)$, with initial value $E(t_j)$. Now the $\overline{\gamma}_j$ are integral curves of the basic vector field $B_{(1,0,\ldots,0)}$. Hence the $\overline{\gamma}_j$ are essentially solutions to an ordinary differential equation

$$\overline{\beta}'(f) = B_{(1,0,\ldots,0)}(f)$$

in $F(M)$; these solutions are hence continuous functions of the initial values $E(t_j)$. Hence $\overline{\gamma}_j(\varepsilon) \to \overline{\gamma}(\varepsilon)$ as $E(t_j) \to E(0)$. Since the bundle projection π is continuous, we find $\gamma_j(\varepsilon) \to \gamma(\varepsilon)$ as required.

THEOREM 4.5. *The boundary set of G (the set on which μ has its minimum value μ_0) is the union of nullity geodesics, which are limits of nullity geodesics in G.*

PROOF. Let p be a boundary point of G. By repeating the argument of the preceding Theorem *, we find a nullity geodesic γ going through p. γ is the limiting geodesic of a sequence of nullity geodesics γ_j in G, and γ is nullity throughout its length since the γ_j all have that property. Hence γ cannot be in G anywhere, for then it would lie in a leaf of the nullity foliation in G, and would have to stay in G throughout its length, contradicting $p \notin G$. But γ is arbitrarily close to geodesics γ_j in G, so γ is in the boundary of G.

EXAMPLE. In R^3, define differential forms $\omega^1, \omega^2, \omega^3$, etc... as follows:

a) when $x > 0$: $\omega^1 = dz - e^x dy;\ \omega^2 = e^x dx + z dy;\ \omega^3 = (e^x + e^{-1/x})dy;$

$$\omega_2^3 = (1 + x^{-2} e^{-1/x - x})dy;\ \omega_1^2 = dy;\ \omega_1^3 = 0;$$

$$\Omega_2^3 = (x^{-4} - x^{-2} - 2x^{-3}) e^{-1/x - x}\, dx\, dy.$$

b) when $x \leq 0$: $\omega^1 = dz - e^x dy;\ \omega^2 = e^x dx + z dy;\ \omega^3 = e^x dy;$

$$\omega_2^3 = dy;\ \omega_1^2 = dy;\ \omega_1^3 = 0;\ \Omega_j^i = 0.$$

c) define coordinate transformations

$$\xi = e^x \cos \sqrt{2}\, y + \frac{1}{\sqrt{2}} z \sin \sqrt{2}\, y;\ \eta = e^x \sin \sqrt{2}\, y - \frac{1}{\sqrt{2}} z \cos \sqrt{2}\, y;\ \zeta = \frac{z}{2}.$$

This maps (x, y, z)-space one-to-one into (ξ, η, ζ)-space. $x > 0$ goes into the exterior of the ruled hyperboloid $\xi^2 + \eta^2 = 1 + 2\zeta^2$. Inside this surface

$$ds^2 = d\xi^2 + d\eta^2 + d\zeta^2 \quad \text{(i. e. } \mu = 3).$$

Outside this surface $\mu = \mu_o = 1$. The nullity geodesics are straight lines lying on the hyperboloids

$$\xi^2 + \eta^2 - 2\zeta^2 = constant.$$

In this case(*), the boundary set of G is a hyperboloid of revolution.

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* We cannot assume the existence of a curve in G leading into p. But all we need is a sequence of geodesics in G arbitrarily close to p in order to carry out the argument of 4.4.

(*) This example is due to Prof. Clifton.

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