CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 8 (1966), exp. nº 2, p. 1-5

<http://www.numdam.org/item?id=CTGDC_1966__8__A2_0>

CAHIERS DE TOPOLOGIE Vol. VIII
ET GEOMETRIE DIFFERENTIELLE
Novembre 1965

SUCCESSIVE HOMOLOGY OPERATIONS AND THEIR APPLICATIONS

by Su-Cheng CHANG

Successive homology operations are introduced in terms of exact couples. They are explicitly expressed. Applications to the theory of cohomotopy groups and the realization theorems between homotopy types of certain polyhedra and equivalence classes of certain algebraic systems are indicated here.

1. Let K be a CW-complex. In the following diagram

$$\begin{array}{ccccc}
\pi_r(K^s, K^{s-1}) & \xrightarrow{\beta} & \pi_{r-1}(K^{s-1}) & \xrightarrow{j} & \pi_{r-1}(K^{s-1}, K^{s-2}) \\
 & & \uparrow i & & \\
\pi_r(K^{s-1}, K^{s-2}) & \xrightarrow{\beta} & \pi_{r-1}(K^{s-2}) & \xrightarrow{j} & \pi_{r-1}(K^{s-2}, K^{s-3}) \\
 & & \uparrow i & & \\
\pi_r(K^{s-2}, K^{s-3}) & \xrightarrow{\beta} & \pi_{r-1}(K^{s-3}) & \xrightarrow{j} & \pi_{r-1}(K^{s-3}, K^{s-4}) \\
 & & \uparrow i & & \\
\pi_r(K^{s-3}, K^{s-4}) & \xrightarrow{\beta} & \pi_{r-1}(K^{s-4}) & \xrightarrow{j} & \pi_{r-1}(K^{s-4}, K^{s-5})
\end{array}$$

the homomorphisms i, j and β are used with the assumption that no confusion will be aroused. Define $\partial = j\beta$ and $\mathcal{H}_{r,s} = \partial^{-1}(0)/\,\partial\pi_{r+1}(K^{s+1}, K^s)$, where $\partial^{-1}(0)$ is the kernel of

$$\partial : \pi_r(K^s, K^{s-1}) \to \pi_{r-1}(K^{s-1}, K^{s-2}).$$

If Z^r represents an element of $\mathcal{H}_{r,s}$, then $ji^{-1}\beta Z^r$ evidently determines an element of $\mathcal{H}_{r-1,s-2}$. This gives rise to a homomorphism

$$\Gamma_1 : \mathcal{H}_{r,s} \to \mathcal{H}_{r-1,s-2}.$$

If $\{Z^r\} \in \Gamma_1^{-1}(0)$, there exists an element $\lambda_1 \in \pi_r(K^{s-1}, K^{s-2})$ so that

$$(1) \qquad ji^{-1}\beta Z^r - j\beta\lambda_1 = 0.$$

Now we may consider $i^{-1}(i^{-1}\beta Z^r - \beta\lambda_1)$. Before going on it worths noting that i^{-1} is many valued. However, for two values of $i^{-1}\beta Z^r$, their difference belongs to $\beta\pi_r(K^{s-1}, K^{s-2})$. As a result the existence of λ_1 in (1) is or is not true for all possible values of i^{-1}. Moreover $i^{-1}\beta Z^r - \beta\lambda_1$, is uniquely determined except an element of

$$\beta\partial^{-1}(0)\,(\partial^{-1}(0) \subset \pi_r(K^{s-1}, K^{s-2})).$$

By $ji^{-1}(i^{-1}\beta Z^r - \beta\lambda_1)$ we may determine a coset $\mathcal{H}_{r-1,s-3}/\Gamma_1\mathcal{H}_{r,s-1}$.

In general, we define a sequence of subgroups of $\mathcal{H}_{r,s}$ as follows :

$$\mathcal{H}_{r,s} \supset \Gamma_1^{-1}(0) \supset \Gamma_2^{-1}(0) \supset \dots \supset \Gamma_l^{-1}(0) \supset \dots,$$

where $\Gamma_l^{-1}(0)$ consists of those elements $\{Z^r\}$ of $\mathcal{H}_{r,s}$, for which there exist $\lambda_u \in \pi_r(K^{s-u}, K^{s-u-1})$ such that [2]

$$
\begin{aligned}
& j(i^{-1}\beta Z^r - \beta\lambda_1) = 0, \\
(2) \quad & j(i^{-1}(i^{-1}\beta Z^r - \beta\lambda_1) - \beta\lambda_2) = 0, \\
& \cdots\cdots \\
& j[(i^{-1})^l\beta Z^r - (i^{-1})^{l-1}\beta\lambda_1 - \dots - (i^{-1})\beta\lambda_{l-1} - \beta\lambda_l] = 0,
\end{aligned}
$$

here $[(i^{-1})^l\beta Z^r - (i^{-1})^{l-1}\beta\lambda_1 - \dots - (i^{-1})\beta\lambda_{l-1} - \beta\lambda_l]$ means

$$i^{-1}[(i^{-1})^{l-1}\beta Z^r - (i^{-1})^{l-2}\beta\lambda_1 - \dots - \beta\lambda_{l-1}]\,\beta\lambda_l.$$

Suppose we have defined the homomorphisms $\Gamma_1, \dots, \Gamma_l$ on the groups $\mathcal{H}_{r,s}, \Gamma_1^{-1}(0), \dots, \Gamma_{l-1}^{-1}(0)$ respectively, then a homomorphism Γ_{l+1} is defined on $\Gamma_l^{-1}(0)$ by virtue of the last line in (2) such that to $\{Z^r\} \subset \Gamma_l^{-1}(0)$ we attach an element of $\mathcal{H}_{r-1,s-l-2}/G_{l+1}$ determined by (2)

$$(3) \qquad ji^{-1}[(i^{-1})^l\beta\lambda^r - (i^{-1})^{l-1}\beta\lambda_1 - \dots - (i^{-1})\beta\lambda_{l-1} - \beta\lambda_l],$$

where G_{l+1} is the subgroup of $\mathcal{H}_{r-1,s-l-2}$ generated by representatives in $\mathcal{H}_{r-1,s-l-2}$ of images of the homomorphisms $\Gamma_1, \Gamma_2, \dots, \Gamma_l$. No doubt

$$f_*\Gamma = \Gamma f_*.$$

We call Γ_l as successive homology operations, which are actually another edition of Massey's theory [1] of exact couples. But explicit definitions (2) and (3) of $\Gamma_l^{-1}(0)$ and Γ_{l+1} are useful.

If the $(n+r-1)$- skeleton of K^{n+r} is S^n, then

$$\pi_m(K^{n+r-u}, K^{n+r-u-1}) = 0, \qquad u = 1, \dots, r-1,$$

whence

$$j : \pi_{n+r-1}(K^{n+r-u}) \to \pi_{n+r-1}(K^{n+r-u}, K^{n+r-u-1}), \qquad u = 1, \dots, r-1$$

are trivial, consequently $\Gamma_{r-2}^{-1}(0) = H_{n+r}(K^{n+r})$. Now

$$\mathcal{H}_{n+r,n+r-1} = \dots = \mathcal{H}_{n+r,n+2} = 0,$$

hence $G_{r-1} = 0$. It reduces Γ_{r-1} to

$$(4) \qquad \Gamma_{r-1} : H_{n+r}(K^{n+r}) \to \mathcal{H}_{n+r-1,n} \approx \pi_{n+r-1}(S^n) \approx H_n(S^n, \pi_{n+r-1}(S^n)).$$

Let J^{n+r} be a cellular complex. If we have a continuous mapping $f : J^{n+r-1} \to S^n$, then we may identify the points of J^{n+r-1} and their image under f to construct a new space $\tilde{J}^{n+r}$. Each cell σ^{n+r} of J becomes a generator, $\tilde{\sigma}^{n+r}$ of $H_{n+r}(\tilde{J}^{n+r})$. Now (3) leads to

$$\Gamma_{r-1}\tilde{\sigma}^{n+r} = (C^{n+r}(f).\sigma^{n+r})S^n,$$

where $C^{n+r}(f)$ is the obstruction cocycle. Here Γ_{r-1} is dual to $C^{n+r}(f)$.

Suppose two mappings $f, g : K^{n+r-1} \to S^n$ are identical in the $(n+r-2)$-dimensional skeleton of the complex K. Then we have a map

$$F : K^{n+r-1} \times 0 \cup K^{n+r-2} \times I \cup K^{n+r-1} \times I \to S^n$$

such that $F \mid K^{n+r-1} \times 0 = f$, $F \mid K^{n+r-1} \times 1 = g$, $F \mid K^{n+r-2} \times t = f$ for $0 \leq t \leq 1$. Identify the points of the $(n+r-1)$-dimensional skeleton of $K^{n+r-1} \times I$ with their image under F to construct a new space $\tilde{K}^{n+r}$. Now $f \sim g$ if, and only if, Γ_{r-1} is trivial in $\tilde{K}^{n+r}$, whose $n+r-1$ skeleton is now S^n only.

2. This sort of homology operations may be used to construct «homology systems» whose equivalence classes are $1-1$ corresponded with homotopy types of certain polyhedra as was initialed by J.H.C. Whitehead. We give an example. Let $n > 3$. Let $H_n, H_{n+1}, H_{n+2}, H_{n+3}$ be abelian groups, H_{n+3} being free. By ${}_2G$ we mean a subgroup of the group G such that $2x = 0$ if $x \in {}_2G$. Define

$$H_{n+r}(2) = H_{n+r}/2H_{n+r} + \Delta_{*\,2}H_{n+r-1}, \qquad r = 1, 2, 3,$$

where + means direct sum and Δ_* is an isomorphism which is unique up to an arbitrary homomorphism of ${}_2H_{n+r-1}$ into $H_{n+r}/2H_{n+r}$. Let

$$\mu : H_{n+r} \to H_{n+r}(2)$$

be the natural map of H_{n+r} into $H_{n+r}/2H_{n+r} \subset H_{n+r}(2)$ and let

$$\Delta : H_{n+r}(2) \to H_{n+r-1}$$

be defined by $\Delta(x+\Delta_* y)=y$ where $x \in H_{n+r}/2H_{n+r}$ and $y \in {}_2H_{n+r-1}$.
We may arbitrarily assign the homomorphisms

$$\Gamma_{1,1} : H_{n+3}(2) \to H_{n+1}(2),$$
$$\Gamma_{1,2} : H_{n+2}(2) \to H_n(2),$$
$$\Gamma_2 \quad : \Gamma^{-1}_{1,1}(0) \to H_n(2)/\Gamma_{1,2}H_{n+2}(2).$$

Then
$\{H_n, H_{n+1}, H_{n+2}, H_{n+3}, H_n(2), H_{n+1}(2), H_{n+2}(2), H_{n+3}(2), \mu, \Delta, \Gamma_{1,1}, \Gamma_{1,2}, \Gamma_2\}$
contribute an algebraic homology A^3_n- system.

THEOREM. *Properly equivalent classes of homology A^3_n-systems are 1–1 corresponded with the homotopy types of A^3_n-polyhedra if $n > 3$.*

If an A^3_n-polyhedron, K^{n+3}, is given, assume its $(n+2)$-skeleton K^{n+2}, is normalized. Let $B_4 = S^n \cup e^{n+2}$, where e^{n+2} is attached to S^n by the essential element of $\pi_{n+1}(S^n)$. I remark

$$\omega : \pi_{n+2}(B_4) \to Z(B_4) \approx H_4(B_4)$$

is not onto, where ω denotes the Hurewicz homomorphism. By W^{n+2} or W^{n+1} we mean the set of $n+2$ or $n+1$ dimensional cells of K^{n+2}, each of which is a cycle. If α represents an element of $H_{n+3}(2)$, then

$$\alpha \in \pi_{n+3}(K^{n+3}, K^{n+2}).$$

We observe the following diagram $(*)$

$$\pi_{n+3}(K^{n+3}, K^{n+2}) \xrightarrow{\beta} \pi_{n+2}(K^{n+2}) \to \pi_{n+2}(K^{n+2}, K^{n+1} \cup W^{n+2})$$
$$\uparrow i$$
$$\pi_{n+3}(K^{n+2}, K^{n+1} \cup W^{n+2}) \xrightarrow{\beta} \pi_{n+2}(K^{n+1} \cup W^{n+2}) \xrightarrow{j} \pi_{n+2}(K^{n+1} \cup W^{n+2}, K^n \cup W^{n+2})$$
$$\uparrow i$$
$$\pi_{n+3}(K^{n+1} \cup W^{n+2}, K^n \cup W^{n+2}) \xrightarrow{\beta} \pi_{n+2}(K^n \cup W^{n+2}).$$

Since each cell of W^{n+2}, being a cycle, is attached to K^n, the complex $K^n \cup W^{n+2}$ in the diagram is well defined. Futhermore

$$\pi_{n+2}(K^{n+1} \cup W^{n+2}, K^n \cup W^{n+2})$$

is actually the $(n+1)$ dimensional chain group with coefficients in Z_2. Here $j\beta\alpha = 0$, whence we have $ji^{-1}\beta\alpha$, which offers an element of $H_{n+1}(K^{n+3}, 2)$. This defines $\Gamma_{1,1} : H_{n+3}(2) \to H_{n+1}(2)$.

The homomorphism $\Gamma_{1,2} : H_{n+2}(2) \to H_n(2)$ may be defined in a similar way with W^{n+1} in place of W^{n+2}. In fact, W^{n+1} consists of a set of $(n+1)$-dimensional spheres touching at a point.

In the diagram $(*)$ if $\alpha \in \Gamma^{-1}_{1,1}(0)$, then there is an element

$$b \in \pi_{n+3}(K^{n+2}, K^{n+1} \cup W^{n+2})$$

such that $j(i^{-1}\beta\alpha - \beta b) = 0$. Let $K^n \cup W^{n+2} = \sum_1^{N_1} S^n_j \cup \sum_{k=1}^{N_2} e^{n+2}_k$. Evidently

$$i^{-1}(i^{-1}\beta\alpha - \beta b) = \sum_{j=1}^{N_1} \varepsilon_j \eta(s^n_j) + \sum_{k=1}^{N_2} l_k g_k(e^{n+2}_k),$$

where $\varepsilon_j = 0$ or 1, $\eta(s^n_j)$ is the generator of $\pi_{n+2}(S^n_j)$, l_k is an integer and $g_k(e^{n+2}_k)$ is the free generator of the group $\pi_{n+2}(S^n_k \cup e^{n+2}(\eta))$. Now

$$\sum_{j=1}^{N} \varepsilon_j \eta(s^n_j)$$

determines the image of $\Gamma_2(\alpha)$ in $H_n(2)/\Gamma_{1,2} H_{n+2}(2)$. This completes the definition of $\Gamma_{1,1}, \Gamma_{1,2}$ and Γ_2 required in the theorem.

References.

[1] MASSEY, W.S., Annals of Math, 56 (1952), 363-396.

[2] CHANG, S.C., Acta Math. Sinica, 13 (1963), 231-237.