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TRENDS TOWARD UNITY IN MATHEMATICS

by Charles EHRESMANN(*)

Though mathematical results are commonly considered as being immutable truths, Mathematics is not a rigid body of theorems, or perhaps a somewhat expanding collection of theorems, giving rise to more or less complicated exercises as well as to numerous applications in other sciences, but really it is a living science, actually in rapid evolution. And this is a time of proliferation of mathematics; however we can recognize also signifiant trends toward unity.

The same development which leads to a new literature where novels do not need to have a plot, to an abstract music, sometimes written by a computer, to abstract sculpture and painting, which do not intend to give an ordinary representation of real objects, this same development toward abstraction leads to a kind of Mathematics much less motivated by possible applications than by a profound desire to find in each problem the very essence of it, the general structure on which it depends. This is not surprising, for Mathematics is very akin to Art; a mathematical theory not only must be rigourous, but it must also satisfy our mind in quest of simplicity, of harmony, of beauty; and a beautiful theory is an inspired creation like a piece of Art.

For the Platonists among the mathematicians, the motivation of their work lies in this search for the true structure in a given situation and in the study of such an abstract structure for itself. For the more pragmatic mathematician, the purpose of his efforts is to solve a preassigned problem arising in pure or applied Mathematics with any means at his disposal, avoiding as much as possible the introduction of new general concepts. But

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all mathematicians agree that the value of a work in mathematics is best proved if it stimulates new research, and the main range of applications of mathematics is Mathematics itself.

Until recently most philosophers, even Bergson, talked about Mathematics as a science concerned with numbers and with quantities in ordinary space, but this is no longer adequate and corresponds more or less to the mathematics of the classical Greeks.

For the Greeks, Mathematics was Arithmetics, i.e. the science of natural numbers, and Geometry, i.e. the study of figures and of ratios of geometrical quantities in ordinary space. Their Geometry was really an axiomatic theory, but they thought the axioms were imposed by « evidence » and in fact they implicitly assumed more axioms than they explicitly stated. It may surprise that they never introduced the notion of a real number, though Eudoxe's theory of ratios of quantities was not essentially different from the definition of real numbers given by Dedekind more than 20 centuries later. This abstraction which consists in considering as a new object a class of previously known objects, in this case a class of rational numbers, was entirely foreign to their mind. Even Archimedes, who invented new domains like Statics and Hydrodynamics, and opened the way for integration theory, was not willing to define abstractly real numbers. After him the urge for invention seemed to be exhausted and Mathematics slumbered throughout the Middle Ages.

The revival came through the introduction of new notions of numbers, the negative numbers and the imaginary numbers by the italian mathematicians of the 16th century, and the introduction of algebraic notations by Viete. The Greeks had a kind of geometrical algebra, but they had no algebraic notations, so that their works are very difficult to read.

A new impulse came from Descartes and Fermat who unified Algebra and Geometry in Analytical Geometry. The problem of the definition and of the determination of the tangent to a curve, already solved in very special cases (for example the spiral by Archimedes) could now be studied in an efficient way and led to the discovery of differential Calculus by Newton and Leibniz. It seems that Leibniz had guessed many of the future developments of Mathematics. Not only did he introduce clearly the notion of a function as a mathematical object, and so prepared the path to functional Analysis, but in his unachieved theory of universal characteristics he dreamed to uncover the algebraic structure of all things and to introduce an universal algorithm to express it. So he was not satisfied with Descartes' analytic Geometry which uses arbitrarily a coordinate system; confusely he foresaw an intrinsic algorithm for Geometry, dream which may be considered as partially realized in linear algebra and Grassmann algebra. Unfortunately his ideas were too advanced for his time and he had not followers enough to persevere in this trail. However his work on differential and integral Calculus was adopted, especially with his notations, and Calculus became for a long period the principal domain of Mathematics.

A further advance came from the discovery in the 19th century of non-euclidean geometries (Lobatchewsky, Bolyai). Now all the ancient bounds of Mathematics were broken : Euclidean Geometry was no longer imposed by perception, but it was a human creation based on axioms; and many other systems of axioms could be devised. Kant's «a priori» of our conception of space becomes hereby obsolete. What was then the essence of Geometry ? The unifying and generalizing notion for the geometries of that time was discovered in the notion of a space with a transitive group of transformations, the group of the euclidean geometry being the group of euclidean displacements. Thus Geometry becomes the theory of invariants and covariants of a group of transformations. In fact this definition applies only to geometries of homogeneous spaces, and already other geometries were discovered and the need for different generalizations was felt. This led ultimately to the definition of topological spaces, which is the proper setting for all questions concerning continuity, limits and approximations, stressing the common structure underlying most problems of Analysis and Geometry.

At the same period Cantor's theory of sets appeared and became more and more the unifying basis of all Mathematics. It was a new abstraction. From now on «Mathematics is entirely free in its developments», as says Cantor, «and its concepts have only to be non contradictory and linked with concepts previously introduced by precise definitions». Though paradoxes were discovered soon after, endangering the whole theory of sets and hence the whole edifice of Mathematics, Cantor's masterpiece opened the road to modern mathematical thinking.

The freedom in the creation of mathematical theories has led since the beginning of our century to a multitude of new kinds of structures considered on sets : Besides the various types of algebraic structures (like groups, rings, fields,semi-groups, modules, algebras, Lie algebras, etc...), there are the structures of measure and of probability theory and the numerous refinements of topological structures: uniform structures, metric spaces, topological manifolds, differentiable or analytic manifolds with all sorts of infinitesimal structures like riemannian structures and connections, algebraic manifolds, etc... By association of different structures on the same set, new structures are created such as Lie groups, topological vector spaces, Banach spaces, Hilbert spaces, normed algebras, etc... These structures have mostly been introduced for the needs of pure Mathematics, but naturally they will have ever more applications in other fields as soon as they will be more generally known, and the utilizers of Mathematics will be more and more numerous.

After the introduction of all these different kinds of structures, the necessity of unification was deeply felt; without some unifying theory following a period of rapid expansion, the mathematicians would fatally tend to use divergent, incompatible languages, like the builders of the tower of Babel.

Considering the similarities of all theories, a unification is obtained by giving a general definition of the notion of a structure, or more precisely of a species of structures over sets. This idea is developed by Bourbaki and is the basis of the order and contents of his treatise «Eléments de Mathématiques». The two initial structures considered in Mathematics, the set of integers and the euclidean space, once axiomatically defined, correspond to rigid species of structures over sets, i.e. the structures of such a species are all isomorphic. The species of structures over sets introduced more recently (for example groups or topologies) do not have this rigidity.

The theory of structures over sets admits a more general and axiomatic form within the theory of categories and functors, and this theory of categories seems to be the most characteristic unifying trend in present day Mathematics; for that reason I think it will soon have to be taught at the University level like other fundamentals, as early as linear Algebra or Topology.

A category is a class together with a partially defined law of composition satisfying some axioms. A group is a particular category, with only inversible elements and one unit; but the most typical categories are the categories of mappings, the elements of which are mappings of a set into a set with the usual composition of mappings. The axioms of an abstract category are suggested by these categories of mappings. An element of a category, instead of mapping, is called a morphism and pictured as an arrow from one unit, its source, to another unit, its target. So this general notion of a morphism generalizes the notion of a mapping, which was considered by Dedekind as the basic tool of Mathematics.

Functors are mappings between categories compatible with the laws of composition. They are again morphisms of a category, the category of functors. The usual homomorphisms between the structures of a given species of structures over sets are the morphisms of a category, and this category admits a canonical functor toward a category of mappings; it is the forgetting functor, i.e. forgetting the structures and remembering only the underlying sets. For example, we have the forgetting functor from the category of continuous mappings between topological spaces, or the forgetting functor from the category of homomorphisms between groups.

Now more abstractly we may consider a functor p from a category H toward a category C. A unit (or object) S of the category H will then be called a structure relative to the functor p, or a p-structure, on the unit $p(S)$ of the category C. So H is considered as a category of morphisms between p-structures. Surprisingly, most of the results obtained for specific species of structures over sets can be integrated in a general theory of p-structures, where one defines and studies sub-structures, quotient structures, free structures, products and sums of families of structures, inductive and projective limits of a functor, etc... At present, given this definition of a p-structure as a precise mathematical object,mathematical research, I believe, will be less concerned with the study of a given p-structure or even of a given functor p; instead its aim will be to define classes of functors p such that, for the corresponding p-structures, a certain theorem previously established for a particular functor p be valid. Once the true reasons for the validity of this theorem are understood, it will generally be seen that only a few of the assumptions are really necessary, and so the class of functors p to which the idea of the theorem may be extended contains many functors besides the original one. In particular, it may contain well known functors to which the initial theorem did not seem to apply. For example, the theorems of compactification of a topological space, of completion of a uniform space, the construction of a free group, a free module, or generally a free algebraic structure generated by a given set, are all special cases of an abstract existence theorem of free p-structures.

Naturally the scheme just described is merely a rough scheme. In fact it is only the creative power of a mathematician which will enable him to discover interesting new classes of functors. As we have seen, one characteristic creative process in Mathematics consists in recognizing as a new object a class of previously defined objects. Have we just reached a stage of the same kind but of superior order when we begin to study classes of functors and, once this new theory will be again sufficiently entangled and sophisticated, will it be necessary to discover a higher degree of unification ? We will not try to answer this question. Yet we realize more and more that Mathematics is a never finished creation, which has not to justify its existence by the importance and the expanding number of its applications; it is not just the «bulldozer of Physics».It is the key for the understanding of the whole Universe, unifying all human thinking, from Sciences to Philosophy and Metaphysics. So the great ideal of Plato and Leibniz, the ideal of Mathematics as the essence of all knowledge, might at last be attained.