CAHIERS DU SÉMINAIRE D'HISTOIRE DES MATHÉMATIQUES

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Semialgebraic and semianalytic sets

Cahiers du séminaire d'histoire des mathématiques 2e série, tome 1 (1991), p. 59-70
<http://www.numdam.org/item?id=CSHM_1991_2_1__59_0>

SEMIALGEBRAIC AND SEMIANALYTIC SETS

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In this talk I shall discuss the notion and some basic features of semialgebraic and semianalytic sets, which are one main concern of Real Geometry.

1. Algebraic Geometry and systems of polynomial equations.

As is well known, Algebraic Geometry studies polynomial equations or, more geometrically, the sets defined by systems of the type

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ f_k(x_1, \dots, x_n) = 0 \end{cases}$$

where the f_i's are polynomials. The classical ground field for this study is the filed $\mathbb{C}$ of complex numbers (or, in general, an algebraically closed field). The reason for this choice was that in $\mathbb{C}$ "all equations have enough solutions", while in the field $\mathbb{R}$ of real numbers one encouters many surprises. For instance, the polynomial

$$f(t) = t^6 - 6\varepsilon t^4 + 11\varepsilon^2 t^2 - 6\varepsilon^3$$

has six complex roots $\pm\sqrt{\varepsilon}$, $\pm\sqrt{2\varepsilon}$, $\pm\sqrt{3\varepsilon}$ for any ε (a uniform behaviour) but none of them is real for negative ε. Another example : the equation

$$(x^2y-2)^2 + x^4y^2 = 0$$

represents nothing over the reals (check it !), but many points over the complex, e.g.

$$(x,y) = (\lambda, \frac{1\pm i}{\lambda^2}) \text{ for } \lambda \neq 0 .$$

Nonetheless, it is apparent the interest of understanding the qualities of real solutions, merely their existence. In the end, most problems arising in nature involve real parameters, i.e lead to systems with *real* coefficients, and ask for *real* solutions. It is at least remarkable that Real Algebraic Geometry became a specific research field only after 1970[(1)], but we shall not analyse the history of that delay here.

2. Classical Algebraic Geometry and Real Algebraic Geometry.

Hence, we have two different settings for polynomial systems :

$$\text{algebraically closed fields (think of } \mathbb{C}) \leftrightarrow$$
$$\leftrightarrow \text{ Classical Algebraic Geometry}$$

and

$$\text{real closed fields (think of } \mathbb{R}) \leftrightarrow$$
$$\leftrightarrow \text{ Real Algebraic Geometry .}$$

Of course this scheme is highly simplifying : as H. Whitney put it once, the complex case "is a prerequisite for a full study of the real case" !

However, at the basis of the theory, real objects need a specific treatment as they present difficulties that make no sense in the complex case. For instance, any equation

$$f(x_1, \ldots, x_n) = 0$$

defines in complex affine n-espace a set of the maximal predictable dimension $n-1$, while in real affine n-espace it may be even empty (remember the example $(x^2y-2)^2 + x^4y^2 = 0$). Hence, since we are to discuss very basic facts, we separate the real case from the complex.

3. Real Algebraic Geometry and systems of inequations.

We have seen that some equations do not have real solutions. This is a consequence of the order structure of the real line. Because of this order, a square is never negative, and a sum of squares vanishes only when all squares vanish. Consequently, the typical system

$$f_1 = 0, \ldots, f_k = 0$$

can be substituted by a single equation

$$f_1^2 + \ldots + f_k^2 = 0 ,$$

and an equation like

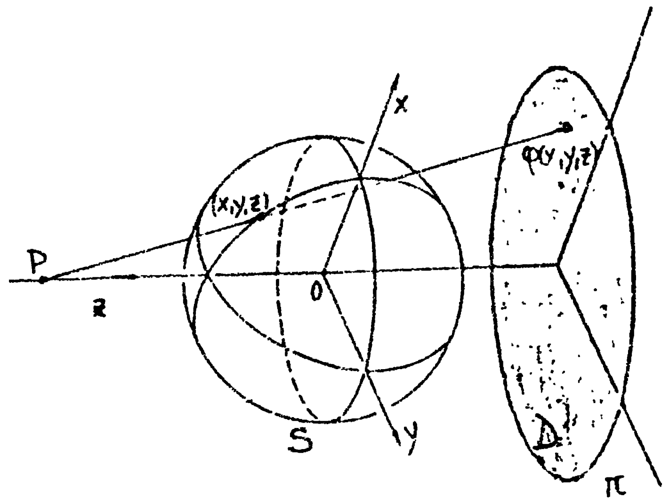
$$f_1^2 + \ldots + f_k^2 + 1 = 0$$

(1) Once Dubois and Risler rediscovered the Real Nullstellensatz proved by Krivine in 1964.

has no solution. This leads to include *sign conditions* in the systems under consideration :

$$\begin{cases} f_1(x_1,\dots,x_n)=0 \\ f_k(x_1,\dots,x_n)=0 \\ g_1(x_1,\dots,x_n)>0 \\ g_l(x_1,\dots,x_n)>0 \\ h_1(x_1,\dots,x_n)\geq 0 \\ h_p(x_1,\dots,x_n)\geq 0 \end{cases}$$

There are geometric operations behind this. For instance, let us project the sphere $S : x^2+y^2+z^2 = 1$ from the exterior point $P = (0,0,2)$ into the plane $\pi : z = 2$.



Then the image of the projection is the disk $D = \{x^2+y^2 \leq 2\}$. Of course, allowing *complex* points would produce additional points, even additonal *real* points :

$$Q = (\,0, 1-\frac{1}{\sqrt{2}}\,i,\ 1+\frac{1}{\sqrt{2}\,i}\,) \in \mathrm{S}$$

$$\varphi(Q) = (0,4) \notin \mathrm{D}.$$

4. Semialgebraic sets.

After these remarks and examples we are ready for a

Definition. *A semialgebraic set is a subset S of the affine space* $\mathbb{R}^n$ *of the form* :

$$S = \bigcup_{i=1}^{s} \{x \in \mathbb{R}^n : g_{i1}(x) > 0, \dots, g_{ir}(x) > 0, f_i(x) = 0\} = \bigcup_{i=1}^{s} \{ g_{i1} > 0, \dots, g_{ir} > 0,\ f_i = 0\},$$

where the g_i , f_i *s are polynomials. In other words, S is defined by finitely many systems of equations and/or inequations of polynomials.*

Note here that :

1) Starting from pieces $\{f > 0\}$ and making boolean operations, we obtain exactly all the semialgebraic sets. For instance :

$$\mathbb{R}^n \setminus \{f>0\} = \{f \leq 0\} = \{f<0\} \cup \{f=0\} = \{-f>0\} \cup \{f=0\}$$

2) Using tricks licke :

$$f_1 = 0, \dots, f_r = 0 \ \Leftrightarrow f_1^2 + \dots + f_r^2 = 0 ,$$
$$f = 0 \ \Leftrightarrow\ -f^2 \geq 0$$

a given S can be described by different systems.

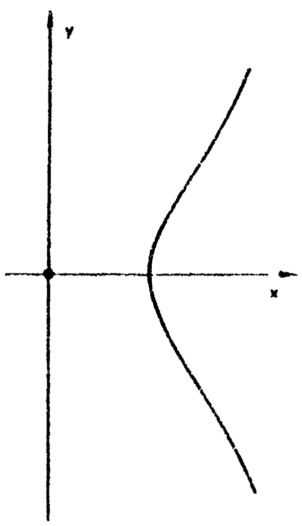
Thus we come to the core of the matter : the relationship between a semialgebraic set S and its various defining systems. In order to make this more explicit, let me state two quite natural questions :

a) Which operations preserve the semialgebraic nature of S ?

b) Which are the cheapest defining systems of S ?

5. Topological operations.

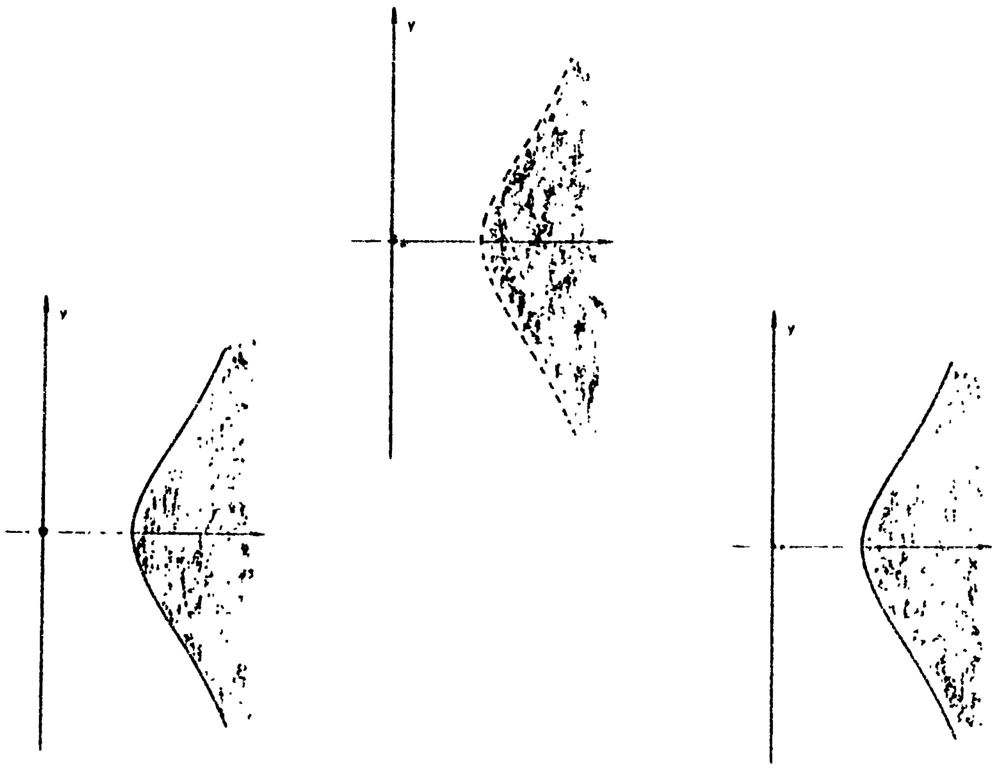
To explain question a) consider the curve $C : y^2+x^2-x^3 = 0$



It consists of a smooth brach C' plus an isolated point at the origin. Each of these two components is semialgebraic, but to define them we need an extra equation. For instance,

$$C' : y^2+x^2-x^3 = 0, \ x \geq \frac{1}{2}.$$

Now take $S : y^2+x^2-x^3 < 0$

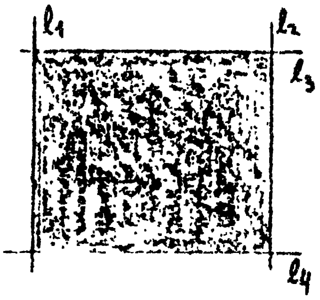


In this case, the limit point of S are those in the branch C', and not the origin. Hence, the closure $\overline{S}$ of S is not obtained by relaxing inequalities, and we need again the extra equation $x \geq \frac{1}{2}$. Note here that also $x > \frac{1}{2}$ does the job, but we prefer the non strict inequation because it reflects closedness.

Thus, these natural topological operations furnish new semialgebraic sets, but we have to find additional equations to show it. Furthermore, we succeeded by looking at the pictures, which cannot be done in general.

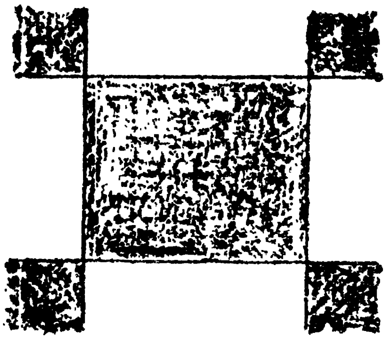
6. The number of inequations needed.

For question b) concerning the change of the defining system let us consider planar polygons like

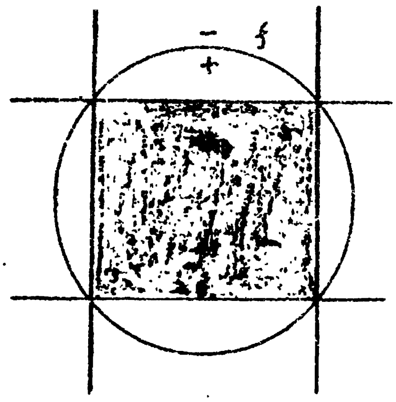


$$S : l_1 \geq 0,\ l_2 \geq 0,\ l_3 \geq 0,\ l_4 \geq 0 .$$

This semialgebraic set can be defined in a more economical way.



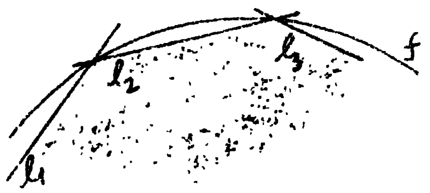
1[st] step : $l_1 l_2 l_3 l_4 \geq 0$ (chessboard coloring)



2[st] step : $l_1 l_2 l_3 l_4 \geq 0,\ f \geq 0$ (drawing a circumference to get rid of the extra pieces)

Thus we need only two inequations instead of four (but notice that the degrees increase : the l_i' s are lines, so degree 1, while $l_1 ... l_4$ has degree 4 and f degree 2).

In general, an n-gnon will be :

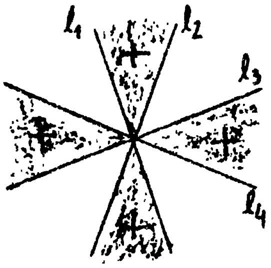


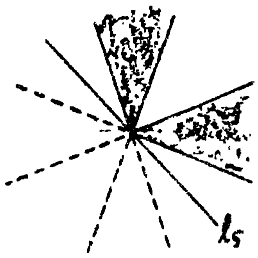
$$S : l_1 l_2 ... l_n \geq 0,\ \ f \geq 0$$

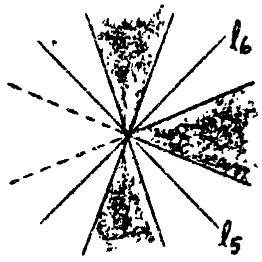
Hence, despite the number of edges, we are done with two equations.

7. The number of systems needed.

Let us turn to a more elaborated situation. Look at the following pictures, focusing the attention around the origin :







$S_1 : l_1 l_2 l_3 l_4 \geq 0,$ $\qquad S_2 : \begin{cases} l_1 l_2 l_3 l_4 \geq 0 \\ l_5 \geq 0 \end{cases}$ $\qquad S_3 : \{ l_1 l_2 \geq 0 \quad \text{or} \quad \begin{cases} l_3 l_4 \geq 0 \\ l_5 \geq 0 \end{cases}$

These semialgebraic sets are examples and counterexamples of two impotant notions :

S_1 is principal, i.e. can be defined with one single inequality,

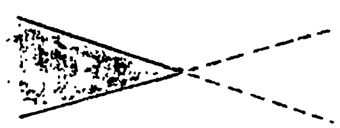
S_2 is not principal, but it is basic, i.e. can be defined with one single system,

S_3 is not basic.

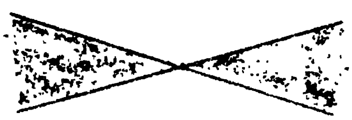
To understand why S_2 is not principal, note that in case $S_2 = \{ f \geq 0 \}$, then f would change sign in passing the boundary of S_2 and so it would vanish on the upper half of the line l_1. But then it would vanish on the lower half too, and this lower half would be contained in S_2.

More difficult is to explain why S_3 is not basic. But imagine it were : $S_3 = \{ g_1 \geq 0, \ldots, g_r \geq 0 \}$. Then some g_i would be negative on the left half of $l_3 l_4$ and positive on S_3. Thus the curve $g_i = 0$ would converge to the origine with slopes like l_5 or l_6 in the picture. Then the sign changing along those slopes would necessarily exclude some of the halves of $l_1 l_2$, which is not possible.

A complete analysis of this example would show that the basic sets of this form are either a union of separated halves or a union of full sectors.



a separate half



a full sector

Furthermore, two equations suffice in that case.

8. Is it possible to get rid of a corner ?

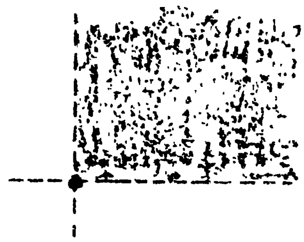
These pictures tell us that there is something special about corners, and I cannot help posing a curious open problem : is there any polynomial mapping

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2 : (x,y) \longmapsto (P(x,y),Q(x,y)), \quad P, Q \text{ polynomials}$$

whose image is exactly the open quadrant

$x > 0, \ y > 0$

Remark : The image of : $(x,y) \longmapsto (x^2y, x^2y^4)$ is "almost" a solution, as it maps $\mathbb{R}^2$ onto the open quadrant plus the corner !



9. Lojasiewicz's and Bröcker's theorems.

The previous examples are all very well, but as far, I have not stated any theorem, something improper for a mathematician. Hence, to respect tradition, I shall formulate two that give answers to our initial questions a) and b).

Theorem 1 (S. Lojasiewicz). *Let S semialgebraic. Then*

(1) *S has finitely many connected components, which are all semialgebraic.*

(2) *The closure of S is semialgebraic, and it can be described using only non strict inequalities.*

Note that the finiteness in (1) fails if we do not restrict to polynomials. Consider for instance the set

$$T = \{y - \cos x \leq 0, y \geq 0\} \subset \mathbb{R}^2 .$$

Also, the use of $\geq$ usually requires additional equations as in the example $y^2+x^2-x^3$ considered before.

The second important result is

Theorem 2 (L. Bröcker, C. Scheiderer). *Let S be semialgebraic of the form*

$$\{f_1 > 0, \ldots, f_s > 0\} \text{) (resp. } \{f_1 \geq 0, \ldots, f_s \geq 0\}) .$$

Then, after replacing the polynomial f_i by suitable new ones, we have

$$s \leq n \text{ (resp. } s \leq \frac{1}{2} n(n+1)) .$$

In other words, no matter how many inequations there are in the defining system at the begining, we can find another system with no more than n (resp. $\frac{1}{2} n(n+1)$) inequations !

10. The variety of methods involved.

The methods used in the proofs of the previous results are very different, running from Quadratic Forms to Model Theory or from Commutative Algebra to Valuation Theory. And this does not mean that there are different proofs using different tools, but that any proof needs all of the tools. This interdisciplinary nature is one appealing characteristic of Real Algebraic Geometry and has the good consequence that very often the ideas developed in a concrete context are useful in others. This is indeed the case with our problems, which can be posed and attacked also in the analytic case.

11. From Real Algebraic to Real Analytic Geometry.

Like Real Algebraic Geometry, Real Analytic Geometry deals with systems of equations and inequations. The difference comes from the use of analytic functions (like sin, cos, exp, etc.) instead of polynomials. An analytic function f is characterized by the fact that close enough to any fixed point x, f can be substituted by its Taylor expansion $T_x f$. In some sense, f can be replaced by the family $\{T_x f\}_x$ of all its Taylor expansions at the points where f is defined. Thus our viewpoint splits into a local view (fix x and take $T_x f$) and a global one (take the whole family $\{T_x f\}_x$).

With this idea in mind one can imagine the framework for real geometry as follows

Algebraic Geometry $\leftrightarrow$ polynomials
1st ↙ ↘ completion
Local Analytic Geometry $\leftrightarrow$ power series
2nd ↙ ↖ Taylor expansion
Global Analytic Geometry $\leftrightarrow$ analytic functions

Then an often walked path from the algebraic context to the analytic is

1^{st} : Mimic for power series the arguments that work for polynomials. Usually they go by induction on the number n of variables and Weierstrass' theorems ease the translation. It is remarkable that sometimes things become easier for power series (according to S. Abhyankar this is by no means surprising, because writting a polynomial

$$a_0 + a_1 x + \ldots + a_d x^d$$

takes more time than writting a series

$$a_0 + a_1 x + \ldots) .$$

This way one obtains information locally around every fixed point.

2^{nd} : Take as local data what has been found for power series and try to glue them together. This is by far the hard part of the whole affair.

12. Semianalytic sets.

The scheme proposed above works well for the problems I am discussing here and produces the analytic counterparts of Theorems. 1 and 2. To state them, let us fix a compact real analytic manifold M of dimension n (i.e. a compact space which is locally analytically diffeomorphic to $\mathbb{R}^n$, think of a sphere, a torus, a projective space). Then

Definition. *A semianalytic set is a subset S of M of the form*

$$S = \bigcup_{i=1}^{s} \{x \in M : g_{i1}(x) > 0, \ldots, g_{ir}(x) > 0, f_i = 0\} =: \bigcup_{i=1}^{s} \{ g_{i1} > 0, \ldots, g_{ir} > 0, f_i = 0\} ,$$

where the g_{ij}, f_i' s are analytic functions $M \longrightarrow R$.

With this notion, patterned upon the semialgebraic one, the following holds true :

Theorem 1 (analytic). *Let $S \subset M$ be semianalytic. Then :*

(1) *S has finitely many connected components, which are all semianalytic.*

(2) *The closure of S is semianalytic, and it can be described using only non strict inequalities.*

Theorem 2 (analytic). *Let $S \subset M$ be semianalytic of the form*

$$\{f_1 > 0, \ldots, f_s > 0\} \quad (\text{resp. } \{f_1 \geq 0, \ldots, f_s \geq 0\}) .$$

Then after replacing the analytic functions f_i by suitable new ones, we have :

$$s \leq n \; (\text{resp. } s \leq \tfrac{1}{2} n(n+1)) .$$

In these results we assume M to be compact. This excludes the example $\{y - \cos x \leq 0, y \geq 0\}$ and implies finiteness in (1). This compactness assumption is used only at the very end of the globalization process and it can be somewhat weakened. However, the answers to questions a) and b) in full generality for a non-compact analytic manifold like $\mathbb{R}^n$ remain today a mos interesting open problem.

Bibliography

The oldest reference for the topics we have discussed is

[L] Lojasiewicz S. : *Ensembles semianalytiques*. I.H.E.S. (prépublication) (1964).

Much more recent is the book

[B C R] Bochnak J., Coste M., Roy M.-F.: *Géométrie algébrique réelle*. Berlin, Heidelberg, New York, Springer (1987).

The first bounds on the number of inequalities needed to describe a semialgebraic set appeared in

[B1] Bröcker L.: *Minimale Erzeugung von Positivbereich.* Geom. Dedicate **16**, 335-350 (1984).

Finally the sharpest estimations were obtained in

[S] Scheiderer C.: *Stability index of real varieties.* Invent. Math. **97**, 467-483 (1989).

For more information on the semialgebraic case, an excellent reference is the recent survey

[B2] Bröcker L. : *On basic semialgebraic sets.* To apperar in Geom. Dedicata.

Concerning the semianalytic case, Lojasiewicz's Lecture Notes quoted above is again the classical reference for local results. The global ones have appeared in the following papers

[R1] Ruiz J. : *On Hilbert's 17th problem an real Nullstellensatz for global analytic functions.* Math. Z. **190**, 447-459 (1985).

[A B R 1] Andradas C., Bröcker L., Ruiz R. : *Minimal generation of basic open semianalytic sets.* Invent. Math. **92**, 409-430 (1988).

[R 2] Ruiz J. : *On the connected components of a global semianalytic set.* Journal Reine Angew. Math. **392**, 137-144 (1988).

[R 3] Ruiz J. : *On the topology of global semianalytic sets.* In Real analytic and algebraic geometry, Proc. Trento 1988, M. Galbiati, A. Tognoli (Eds.) Springer-Verlag LNM **1420**, 237-246.

Finally, there is an abstract theory of real constructible sets that generalizes both the algebraic and the analytic cases. A symmetric presentation of it is the subject of the forthcoming book.

[A B R 2] Andradas C., Bröcker L., Ruiz J., *Real constructible sets.*