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RELAXATION OF OPTIMAL CONTROL PROBLEMS IN L^p-SPACES

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Abstract. We consider control problems governed by semilinear parabolic equations with pointwise state constraints and controls in an L^p-space ($p < \infty$). We construct a correct relaxed problem, prove some relaxation results, and derive necessary optimality conditions.

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1. Introduction

This paper is concerned wih the relaxation of Robin boundary controls for semilinear parabolic equations in the presence of pointwise state constraints. More precisely, we consider the following control problem

$$(P) \qquad \text{Inf} \;\; J(y,v) = \int_Q F(\cdot, y)\, dx\, dt + \int_\Sigma G(\cdot, y, v)\, ds\, dt + \int_\Omega L(\cdot, y(T))\, dx,$$

subject to

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay + \Phi(\cdot, y) = 0 & \text{in } Q, \\ \dfrac{\partial y}{\partial n_A} + \Psi(\cdot, y, v) = 0 \;\text{ on }\; \Sigma, \quad y(0) = y_o & \text{in } \Omega, \end{cases} \tag{1.1}$$

$$g(y) \in Z, \tag{1.2}$$

$$v \in V_{ad} = \{v \in L^p(\Sigma) \mid v(s,t) \in K_V(s,t) \text{ for a.a. } (s,t) \in \Sigma\},$$

where T is a fixed positive constant, Ω is an open bounded subset of $\mathbb{R}^N$ ($N \geq 2$), Γ its boundary, $Q = \Omega \times]0,T[$, $\Sigma = \Gamma \times]0,T[$, A is a second order differential operator, $\frac{\partial y}{\partial n_A}$ is the conormal derivative of y with respect to A, Φ and Ψ are Carathéodory functions (*i.e.* $\Phi(\cdot, y)$ and $\Psi(\cdot, y, v)$ are measurable, and $\Phi(x,t,\cdot)$ and $\Psi(s,t,\cdot,\cdot)$ are continuous), $y_o \in C(\overline{\Omega})$, g is a continuous mapping from $C(\overline{Q})$ into $C(\overline{Q})$, Z is a closed convex subset of

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$C(\overline{Q})$ with nonempty interior in $C(\overline{Q})$, and K_V is a measurable multimapping with closed and nonempty values in $P(\mathbb{R})$.

Since neither convexity of $G(s,t,y,\cdot)$ nor linearity of $\Psi(s,t,y,\cdot)$ is assumed, the original control problem (P) need not have solutions. The idea of the relaxation is to make an extension in order to ensure existence of solutions (in a reasonable sense) in a class large than the original one.

The general compactification theory represents a basic tool for relaxation of problems appearing in variational calculus and optimization of systems governed by differential equations. Following Roubíček [19], we can construct a correct relaxed control problem by considering a convex σ-compactification envelope of the set of classical controls, and by extending the original cost functional and the original state equation. This problem can formally be written as

$$(RP_E)\begin{cases} \text{Inf} \quad \widetilde{J}(y,\mu) \\ \text{subject to} \\ \begin{cases} \dfrac{\partial y}{\partial t} + Ay + \Phi(\cdot,y) = 0 & \text{in } Q, \\ \dfrac{\partial y}{\partial n_A} + \widetilde{\Psi}(\cdot,y,\mu) = 0 \ \text{ on } \ \Sigma, \quad y(0) = y_o & \text{in } \ \Omega, \end{cases} \\ g(y) \in Z, \\ \mu \in \overline{V}_{E,ad} \subset Y_E^p, \end{cases}$$

where

$$Y_E^p = \text{w}^*\text{-bcl}\ (i_E(L^p(\Sigma))) = \left\{ \mu \in E^* \mid \exists \text{ bounded net } (v_\alpha)_\alpha \subset L^p(\Sigma) \ \text{ s.t. } w^*\text{-}\lim_\alpha i_E(v_\alpha) = \mu \right\},$$

E is a Hausdorff locally convex space, E^* its dual space, i_E an imbedding from $L^p(\Sigma)$ into E^*, $\overline{V}_{E,ad}$ is the set of admissible relaxed controls, Y_E^p (the boundedness closure if $i_E(L^p(\Sigma))$ in the weak-star topology of E^*) is a convex, σ-compact subset of E^*, and $\widetilde{J}$ and $\widetilde{\Psi}$ are regarded as extensions of J and Ψ. (See Sect. 5 for a precise setting of the relaxed control problem.)

Different compactifications may be used to define (RP_E), and depend on the choice of E. This choice is related to the properties of G, Ψ, and V_{ad}, and can yield abstract problems which are not easy to interpret. As noticed by Roubíček [19]: "the general dilemma is typically between a finer convex compactification (which contains more information, enables to treat more problems, but has a loss concrete interpretation), and a coarse convex compactification (which works just conversely)".

Historically, the first relaxation method for variational calculus and optimal control problems is based on Young measures [25]. In [23], the relaxation of nonconvex problems in optimal control theory when the controls take value in a compact set $K \subset \mathbb{R}$ is developed (see also [2, 3, 5, 10, 11, 15]). The Young measures are weakly measurable mappings from Σ to the set of all probability Radon measures on K. They are obtained by setting $E_1 = L^1(\Sigma; C(K))$ in the definition of $Y_{E_1}^\infty$, and represent an interesting tool to hold a certain limit information about oscillations of minimizing sequences. These measures have been widely studied and their explicit characterization is well known (see [4, 23], and [22]).

Characterization of the so-called L^p-Young measures (for $1 \le p < \infty$), associated with minimizing sequences bounded in $L^p(\Sigma)$, has been studied by Schonbeck [21]. (See also [12] and [13] for the analysis of Young measures associated with sequences of gradient bounded in $L^p(\Sigma)$.) These measures correspond to $Y_{E_2}^p$, where

E_2 is defined by

$$E_2 = \{\psi \otimes \phi \mid \psi \in C_o(\Sigma),\ \phi \in C(\mathbb{R}) \ \text{ s.t. } |\phi(w)| \le C(1 + |w|^{p_o}),\ 1 \le p < p_o\} \cdot$$

The main drawback of the L^p-Young measures constructed in this way is that concentration effects appearing in some nonlinear problems may be neglected, because the test functions which intervene in the definition of E_2 have growth strictly less than p. (For the definition of concentration, see Sect. 9.) In their pioneering work [7], Diperna and Majda constructed a generalization of the L^p-Young measures to handle both oscillations and concentration effects. Other ways of manipulating concentrations have been proposed. We refer the reader to [14] and [8].

To simplify the writing, the "generalized Young measure" we consider here are constructed by setting $E = Ca^p(\Sigma)$ in the definition of Y_E^p ($Ca^p(\Sigma)$ is the space of all Carathéodory functions with at most p-growth). The relaxed problem $(RP_{Ca^p(\Sigma)})$, denoted for simplicity by (RP), is exactly defined in Section 5. The following questions will be pursued:

- *Well-posedness of the relaxation.* In Section 4, we recall the construction of a convex compactification of $L^p(\Sigma)$. This will enable us to define a correct relaxation of (P) in Section 5.
- *Analysis of the relaxed state equation.* Section 6 is devoted to the study of the relaxed state equation. Existence, regularity and uniqueness results are proved.
- *Existence and stability of solutions of* (RP), *properness of the relaxation.* In Section 7, we state some relaxation results. In particular, we analyze the topological properties of the relaxed trajectories (compactness and denseness properties). We prove existence of a solution for the relaxed control problem, and we analyze the relation between (P) and (RP). (In particular the so-called properness of the relaxation.)
- *First-order optimality conditions for* (RP) are stated in the form of a Pontryagin's principle in Section 8. To prove these results, we use a Lagrangian method based on a geometrical version of the Hahn-Banach theorem.
- In Section 9, we prove that the results stated through the paper are still valid for other choices of E. In particular, nonconcentration of the optimal solution of (RP_E) is proved under some additional assumptions.

2. Notation and assumption

In all the sequel, C denotes a generic constant, q, p, and γ are positive numbers satisfying $q > \frac{N}{2} + 1$, $p < \infty$, and $\gamma > N + 1$. The domain Ω is of class C^2 (the boundary Γ of Ω is an $(N-1)$-dimensional manifold of class C^2 such that Ω lies locally on one side of Γ). The operator A is defined by $Ay(x) = -\sum_{i,j=1}^N D_i(a_{ij}(x)D_j y(x))$. The coefficients a_{ij} belong to $C^1(\overline{\Omega})$ and satisfy the conditions:

$$a_{ij}(x) = a_{ji}(x) \quad \text{for all} \ \ i, j \in \{1, \dots, N\}, \qquad m_o|\xi|^2 \le \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j,$$

for every $\xi \in \mathbb{R}^N$ and every $x \in \overline{\Omega}$, with $m_o > 0$. The conormal derivative of y with respect to A is defined by

$$\frac{\partial y}{\partial n_A}(s,t) = \sum_{i,j} a_{ij}(s) D_j y(s,t) n_i(s),$$

where $n = (n_1, \cdots, n_N)$ is the unit normal to Γ outward Ω. We denote by Q the cylinder $\Omega\times]0,T[$ and by Σ the lateral surface $\Gamma\times]0,T[$. We set $\overline{\Omega}_0 = \overline{\Omega} \times \{0\}$, $\overline{\Omega}_T = \overline{\Omega} \times \{T\}$, $Q_{\varepsilon T} = Q\times]\varepsilon, T[$, for every $\tau \in]0,T[$. For every $1 \le \theta \le \infty$, the usual norms in the spaces $L^\theta(\Omega)$, $L^\theta(Q)$, $L^\theta(\Sigma)$ will be denoted by $\|\cdot\|_{\theta,\Omega}$, $\|\cdot\|_{\theta,Q}$, $\|\cdot\|_{\theta,\Sigma}$. The Hilbert space $W(0,T;H^1(\Omega),(H^1(\Omega))') = \{y \in L^2(0,T;H^1(\Omega)) \mid \frac{dy}{dt} \in L^2(0,T;(H^1(\Omega))')\}$ will be denoted by $W(0,T)$. If O is a locally compact subset of $\overline{Q}$, we denote by $M_b(O)$ the space of bounded Radon measures on O.

A1 - Φ is a Carathéodory function from $Q\times\mathbb{R}$ into $\mathbb{R}$. For almost every $(x,t)\in Q$, $\Phi(x,t,\cdot)$ is of class C^1. The following estimates hold

$$|\Phi(x,t,0)|\le\Phi_1(x,t),\qquad C_o\le\Phi'_y(x,t,y)\le\Phi_1(x,t)\eta(|y|),$$

where $C_o\in\mathbb{R}$, $\Phi_1\in L^q(Q)$, and η is a nondecreasing function from $\mathbb{R}^+$ into $\mathbb{R}^+$.

A2 - Ψ is a Carathéodory function from $\Sigma\times\mathbb{R}^2$ into $\mathbb{R}$. For almost every $(s,t)\in\Sigma$ and all $w\in\mathbb{R}$, $\Psi(s,t,\cdot,w)$ is of class C^1, and

$$|\Psi(s,t,0,w)|\le\Psi_1(s,t)+C|w|^{\frac{p}{\gamma}},$$

$$C_o\le\Psi'_y(s,t,y,w)\le\left(\Psi_1(s,t)+C|w|^{\frac{p}{\gamma}}\right)\ \eta(|y|)$$

$$\left|\Psi'_y(s,t,y_1,w)-\Psi'_y(s,t,y_2,w)\right|\le\left(\Psi_1(s,t)+C|w|^{\frac{p}{\gamma}}\right)\ \ell(|y_1-y_2|),$$

where $\Psi_1\in L^\gamma(\Sigma)$, $\gamma>N+1$, C is a positive constant, and ℓ is an increasing continuous function from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $\ell(0)=0$.

A3 - F is a Carathéodory function from $Q\times\mathbb{R}$ into $\mathbb{R}$. For almost all $(x,t)\in Q$, $F(x,t,\cdot)$ is of class C^1, and

$$|F(x,t,y)|+\left|F'_y(x,t,y)\right|\le F_1(x,t)\eta(|y|)\qquad\text{where } F_1\in L^1(Q).$$

A4 - G is a Carathéodory function from $\Sigma\times\mathbb{R}^2$ into $\mathbb{R}$. For almost all $(s,t)\in\Sigma$ and all $w\in\mathbb{R}$, $G(s,t,\cdot,w)$ is of class C^1, and

$$|G(s,t,y,w)|+\left|G'_y(s,t,y,w)\right|\le(G_1(s,t)+C|w|^p)\eta(|y|),$$

$$\left|G'_y(s,t,y_1,w)-G'_y(s,t,y_2,w)\right|\le(G_1(s,t)+C|w|^p)\ \ell(|y_1-y_2|),$$

where $G_1\in L^1(\Sigma)$, and ℓ is as in **A2**.

A5 - L is a Carathéodory function from $\Omega\times\mathbb{R}$ into $\mathbb{R}$. For almost all $x\in\Omega$, $L(x,\cdot)$ is of class C^1, and

$$|L(x,y)|+\left|L'_y(x,y)\right|\le L_1(x)\eta(|y|)\qquad\text{where } L_1\in L^1(\Omega).$$

A6 - $g:C(\overline{Q})\longrightarrow C(\overline{Q})$ is of class C^1.

A7 - The infimum of (P) is finite (there exists at least one admissible pair (y,v)).

3. State equation

We begin this section by recalling some results concerning linear equations. Let (a,b) be in $L^q(Q)\times L^\gamma(\Sigma)$ such that $a\ge C_o$ and $b\ge C_o$. Let ϕ be in $L^q(Q)$, f in $L^\gamma(\Sigma)$, w in $C(\overline{\Omega})$, and consider the following equation:

$$\frac{\partial y}{\partial t}+Ay+ay=\phi\ \text{ in } Q,\quad \frac{\partial z}{\partial n_A}+bz=f\ \text{ on }\Sigma,\quad y(0)=w\ \text{ in }\Omega. \tag{3.1}$$

Definition 3.1. A function $y \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$ is a weak solution of (3.1) if, and only if, $ay \in L^1(Q)$, $by \in L^1(\Sigma)$, and

$$\int_Q \left(-y\frac{\partial z}{\partial t} + \sum_{i,j=1}^{N} a_{ij} D_i y D_j z + azy \right) dx\, dt - \int_\Omega y_o z(0)\, dx = \int_\Sigma (f - by) z\, ds\, dt$$

for all $z \in C^1(\overline{Q})$ such that $z(T) = 0$.

Proposition 3.2 ([18], Prop. 3.3). *Equation* (3.1) *admits a unique weak solution* $y \in W(0,T) \cap C(\overline{Q})$ *satisfying*

$$||y||_{C(\overline{Q})} + \|y\|_{W(0,T)} \le C \left(||\phi||_{q,Q} + ||f||_{\gamma,\Sigma} + ||w||_{C(\overline{\Omega})} \right),$$

where $C \equiv C(T,\Omega,N,q,\gamma,C_o)$ *does not depend on* a *and* b.

Proposition 3.3 ([6], Chap. 3, Th. 1.3). *For every* $\tau > 0$*, the weak solution* y *of* (3.1) *is Hölder continuous on* $\overline{Q_{\varepsilon T}}$ *and satisfies*

$$||y||_{C^{\nu,\nu/2}(\overline{Q_{\varepsilon}T})} \le C(\tau) \left(\|\phi\|_{q,Q} + \|f\|_{\gamma,\Sigma} + \|w\|_{C(\overline{\Omega})} \right) \quad \textit{for some } 0 < \nu < 1,$$

where $C(\varepsilon) \equiv C(T,\Omega,N,C_o,q,\gamma,\varepsilon)$*. Moreover, if* w *is Hölder continuous on* $\overline{\Omega}_0$*, then* y *is Hölder continuous on* $\overline{Q}$.

Now, we recall some existence, uniqueness and regularity results concerning the (original) state equation (1.1).

Definition 3.4. A function $y \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$ is a weak solution of (1.1) if, and only if, $\Phi(\cdot, y(\cdot)) \in L^1(Q)$, $\Psi(\cdot, y(\cdot), v(\cdot)) \in L^1(\Sigma)$, and

$$\int_Q \left(-y\frac{\partial z}{\partial t} + \sum_{i,j=1}^{N} a_{ij} D_i y D_j z + \Phi(x,t,y) z \right) dx\, dt - \int_\Omega y_o z(0)\, dx = -\int_\Sigma \Psi(s,t,y,v) z\, ds\, dt$$

for all $z \in C^1(\overline{Q})$ such that $z(T) = 0$.

Theorem 3.5 ([18], Th. 3.1). *If* **A1**, **A2** *are satisfied, if* $v \in L^p(\Sigma)$*, and* $y_o \in C(\overline{\Omega})$*, then equation* (1.1) *admits a unique weak solution* $y_v \in W(0,T) \cap C(\overline{Q})$ *satisfying*

$$\|y_v\|_{C(\overline{Q})} + \|y_v\|_{W(0,T)} \le C \left(\|\Psi(0,v)\|_{\gamma,\Sigma} + \|y_o\|_{C(\overline{\Omega})} + 1 \right),$$

where $C \equiv C(T,\Omega,N,q,p,\gamma,C_o)$.

Theorem 3.6 ([6], Chap. 3, Th. 1.3). *For every* $M > 0$ *and every* $\varepsilon > 0$*, there exist* $0 < \nu < 1$ *and* $C(\varepsilon) \equiv C(T,\Omega,N,q,p,\gamma,\varepsilon,\nu,M)$ *such that, for every* v *satisfying* $||v||_{p,\Sigma} \le M$*, the weak solution* y_v *of* (1.1) *corresponding to* v *is Hölder continuous on* $\overline{Q_{\varepsilon T}}$ *and:*

$$|y_v\|_{C^{\nu,\nu/2}(\overline{Q_{\varepsilon T}})} \le C(\varepsilon).$$

4. Convex compactifications of $\mathbf{L^p}(\Sigma)$

In this section, we recall the construction of a natural convex σ-compact envelope of $L^p(\Sigma)$ due to Roubíček (for more details, see Chap. 3 in [19]). Denote by $Ca^p(\Sigma)$ the linear space of all Carathéodory functions $h : \Sigma \times \mathbb{R} \longrightarrow \mathbb{R}$ (*i.e.* $h(\cdot, w)$ is measurable and $h(s,t,\cdot)$ is continuous) with at most p-growth

$$|h(s,t,w)| \le a_h(s,t) + d_h |w|^p \quad \text{for some } a_h \in L^1(\Sigma) \text{ and } d_h < +\infty.$$

Let $(Ca^p(\Sigma))^*$ be the dual space of $Ca^p(\Sigma)$ and consider the imbedding $i: \ L^p(\Sigma) \longrightarrow (Ca^p(\Sigma))^*$ defined by

$$\left\langle i(v), h \right\rangle_{*,\Sigma} = \int_\Sigma h(s,t,v(s,t))\, ds\, dt \qquad (h,v) \in Ca^p(\Sigma) \times L^p(\Sigma), \tag{4.1}$$

where $\langle \cdot, \cdot \rangle_{*,\Sigma}$ denotes the canonical duality pairing. For $r > 0$, let B_r be the ball of radius r in $L^p(\Sigma)$, and let us set

$$Y_r^p = \text{w}^*\text{-cl}\ (i(B_r)) \quad \text{ and } \quad Y^p = \cup_r Y_r^p = \text{w}^*\text{-cl}\ (i(L^p(\Sigma))).$$

The set Y^p is convex and locally compact. We will address the elements of Y^p as generalized Young measures (or relaxed controls). The space $Ca^p(\Sigma)$ can be normed by

$$||h|| = \inf\ \left\{ ||a||_{1,\Sigma} + d \mid (a,d) \in L^1(\Sigma) \times \mathbb{R}, \quad |h(s,t,w)| \le a(s,t) + d|w|^p \text{ for all } (s,t,w) \in \Sigma \times \mathbb{R} \right\} \cdot \tag{4.2}$$

This norm satisfies

$$||\chi \cdot h|| \le C ||\chi||_{C(\overline{\Sigma})}\ ||h|| \qquad \text{for all } (h,\chi) \in Ca^p(\Sigma) \times C(\overline{\Sigma}), \tag{4.3}$$

where $\chi \cdot h$ stands for $(\chi \otimes 1)h$. This property implies that the mapping $\chi \ \longmapsto \chi \cdot h$ is continuous, and ensures that for all $(h,\sigma) \in Ca^p(\Sigma) \times (Ca^p(\Sigma))^*$, the bilinear mapping $(h,\sigma) \longmapsto h \bullet \sigma$ given by

$$\left\langle h \bullet \sigma, \chi \right\rangle_{M(\overline{\Sigma}) \times C(\overline{\Sigma})} = \left\langle \sigma, \chi \cdot h \right\rangle_{*,\Sigma} \qquad \text{for all } \chi \in C(\overline{\Sigma}),$$

is well defined.

5. Correct relaxation of (P)

From definition of the composition $\bullet$, we can easily see that the control problem (P) can be (formally) written in the form

$$(P)\ \ \inf \quad \widetilde{J}(y,\sigma) = \int_Q F(x,t,y)\, dx\, dt + \left\langle \sigma, G \circ y \right\rangle_{*,\Sigma} + \int_\Omega L(x, y(T))\, dx$$

where $(y,\sigma) \in C(\overline{Q}) \times i(V_{ad})$ satisfies (1.2) and

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay + \Phi(\cdot, y) = 0 & \text{in } Q, \\ \dfrac{\partial y}{\partial n_A} + (\Psi \circ y) \bullet \sigma = 0 \quad \text{on } \ \Sigma, \qquad y(\cdot, 0) = y_o & \text{in } \ \Omega. \end{cases} \tag{5.1}$$

Following [19], we define the relaxed control problem as:

$$(RP)\ \ \inf \quad \left\{ \widetilde{J}(y,\sigma) \mid (y,\sigma) \in C(\overline{Q}) \times \overline{V}_{ad} \text{ satisfies } \ (5.1) \text{ and } (1.2) \right\},$$

where the set of admissible relaxed controls $\overline{V}_{ad} \subset Y^p$ is defined as the weak* closure of $i(V_{ad})$. Due to the special form of the control constraints involved in V_{ad}, the set $\overline{V}_{ad}$ is convex and locally compact. (See [19] for more details.) The functional $\widetilde{J}$ can be rewritten as

$$\begin{aligned}\widetilde{J}(y,\sigma) &= \int_Q F(x,t,y)\,dx\,dt + \langle \sigma, G\circ y\rangle_{*,\Sigma} + \int_\Omega L(x,y(T))\,dx,\\ &= \int_Q F(x,t,y)\,dx\,dt + \langle \sigma \bullet G\circ y, 1\rangle_{M(\overline{\Sigma})\times C(\overline{\Sigma})} + \int_\Omega L(x,y(T))\,dx.\end{aligned}$$

The relaxed state equation is to be understood in the following sense:

Definition 5.1. A function $y \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$ is a weak solution of (5.1) if, and only if, $\Phi(\cdot,y(\cdot)) \in L^1(Q)$, $(\Psi\circ y)\bullet\sigma \in L^1(\Sigma)$, and

$$\int_Q \left(-y\frac{\partial z}{\partial t} + \sum_{i,j=1}^N a_{ij}D_iyD_jz + \Phi(x,t,y)z\right)dx\,dt - \int_\Omega y_o z(0)\,dx = -\int_\Sigma (\Psi\circ y)\bullet\sigma\ z\,ds\,dt$$

for all $z \in C^1(\overline{Q})$ such that $z(T) = 0$.

6. Relaxed state equation

In this section, we are interested in existence, uniqueness, and regularity results concerning the relaxed state equation. As in the case of classical Young measures, we prove that the regularity properties for the relaxed trajectories (solutions of the relaxed state equation) are inherited from those of the classical trajectories. In particular, we prove that the relaxed state equation admits a unique continuous solution.

6.1. Preliminary results

Let σ be in Y^p. Suppose for a moment that equation (5.1) admits a solution $y_\sigma \in C(\overline{Q})$. It is obvious that $y_{\sigma|\overline{\Sigma}}$ belongs to $C(\overline{\Sigma})$, and from Definition 5.1, that $(\Psi\circ y_{\sigma|\overline{\Sigma}})\bullet\sigma$ belongs to $L^1(\Sigma)$. In the following lemma, we show that due to the growth condition in **A2**, for every $y \in C(\overline{\Sigma})$, the function $(\Psi\circ y)\bullet\sigma$ (which is naturally in $M(\overline{\Sigma})$) belongs in fact to $L^\gamma(\Sigma)$. This result is proved in [19], Proposition 3.3.6. We rewrite the proof for the convenience of the reader.

Lemma 6.1. *Suppose that* **A2** *is satisfied. Then, for every $\sigma \in Y^p$ and every $y \in C(\overline{\Sigma})$, the function $(\Psi\circ y)\bullet\sigma$ belongs to $L^\gamma(\Sigma)$.*

Proof. Since σ belongs to Y^p, there exists a bounded net $(v_\alpha)_\alpha \subset L^p(\Sigma)$ such that $(i(v_\alpha))_\alpha$ converges to σ in the weak-star topology of $(Ca^p(\Sigma))^*$. Thus, due to **A2**, we have

$$||(\Psi\circ y)\bullet i(v_\alpha)||_{\gamma,\Sigma} = ||\Psi(y,v_\alpha)||_{\gamma,\Sigma} \le C,$$

where $C > 0$ is independent of α. Then there exist a subnet, still indexed by α, and $b \in L^\gamma(\Sigma)$, such that $((\Psi\circ y)\bullet i(v_\alpha))_\alpha$ converges to b in the weak topology of $L^\gamma(\Sigma)$. On the other hand, we have

$$\lim_\alpha \langle(\Psi\circ y)\bullet i(v_\alpha),\chi\rangle_{M(\overline{\Sigma})\times C(\overline{\Sigma})} = \lim_\alpha\ \langle i(v_\alpha), \Psi\circ y\cdot\chi\rangle_{*,\Sigma} = \langle\sigma,\Psi\circ y\cdot\chi\rangle_{*,\Sigma} = \langle(\Psi\circ y)\bullet\sigma,\chi\rangle_{M(\overline{\Sigma})\times C(\overline{\Sigma})}$$

for all $\chi \in C(\overline{\Sigma})$.

Since $L^\gamma(\Sigma)$ is imbedded into $M(\overline{\Sigma})$, we deduce that $b \equiv (\Psi\circ y)\bullet\sigma$. ☐

The convergence result stated below will be very useful for the analysis of the relaxed state equation (5.1).

Lemma 6.2. *Suppose that* **A2** *is satisfied. Let* $(\sigma_\alpha)_\alpha$ *be a net converging to* σ *in the weak-star topology of* Y_r^p $(r > 0)$*, and let* $(y_\alpha)_\alpha$ *be a bounded net in* $C(\overline{\Sigma})$ *converging to* y *uniformly on* $\overline{\Sigma}$*. Then,*

$$\lim_\alpha \int_\Sigma (\Psi \circ y_\alpha) \bullet \sigma_\alpha \; \chi \, ds \, dt = \int_\Sigma (\Psi \circ y) \bullet \sigma \; \chi \, ds \, dt \tag{6.1}$$

$$\lim_\alpha \int_\Sigma (\Psi'_y \circ y_\alpha) \bullet \sigma_\alpha \; \chi \, ds \, dt = \int_\Sigma (\Psi'_y \circ y) \bullet \sigma \; \chi \, ds \, dt \tag{6.2}$$

for all $\chi \in C(\overline{\Sigma})$.

Proof. The proof is split into two steps.

Step 1. To prove (6.1), observe that for all $\chi \in C(\overline{\Sigma})$, we have

$$\begin{aligned}\int_\Sigma (\Psi \circ y_\alpha) \bullet \sigma_\alpha \; \chi \, ds \, dt &= \int_\Sigma (\Psi \circ y) \bullet \sigma_\alpha \; \chi \, ds \, dt + \int_\Sigma (\Psi \circ y - \Psi \circ y_\alpha) \bullet \sigma_\alpha \; \chi \, ds \, dt \\ &= \big\langle \sigma_\alpha, (\Psi \circ y) \cdot \chi \big\rangle_{*,\Sigma} + \int_\Sigma (\Psi \circ y - \Psi \circ y_\alpha) \bullet \sigma_\alpha \; \chi \, ds \, dt = I_\alpha^1 + I_\alpha^2.\end{aligned}$$

First, let us observe that

$$\lim_\alpha I_\alpha^1 = \big\langle \sigma, (\Psi \circ y) \cdot \chi \big\rangle_{*,\Sigma} = \int_\Sigma (\Psi \circ y) \bullet \sigma \; \chi \, ds \, dt.$$

It remains to prove that $\lim_\alpha I_\alpha^2 = 0$. By definition of σ_α, there exists a net $(v_{\beta,\alpha})_\beta \subset B_r$ such that $(i(v_{\beta,\alpha}))_\beta$ converges to σ_α in the weak-star topology of $(Ca^p(\Sigma))^*$. Therefore

$$\begin{aligned} I_\alpha^2 &= \int_\Sigma (\Psi \circ y - \Psi \circ y_\alpha) \bullet \sigma_\alpha \; \chi \, ds \, dt = \lim_\beta \int_\Sigma (\Psi \circ y - \Psi \circ y_\alpha) \bullet i(v_{\beta,\alpha}) \; \chi \, ds \, dt \\ &= \lim_\beta \int_\Sigma (\Psi(y_\alpha, v_{\beta,\alpha}) - \Psi(y, v_{\beta,\alpha})) \; \chi \, ds \, dt = \lim_\beta \int_\Sigma \Psi_{\beta,\alpha}(y_\alpha - y) \; \chi \, ds \, dt, \end{aligned} \tag{6.3}$$

where $\Psi_{\beta,\alpha} = \int_0^1 \Psi'_y(\cdot, \theta y_\alpha + (1-\theta)y, v_{\beta,\alpha}) \, d\theta$. Due to **A2**, we have

$$\begin{aligned} |I_{\beta,\alpha}| &= \left| \int_\Sigma \Psi_{\beta,\alpha}(s,t)(y_\alpha - y) \; \chi \, ds \, dt \right| \le ||\Psi_{\beta,\alpha}||_{1,\Sigma} \; ||(y_\alpha - y)\chi||_{\infty,\Sigma} \\ &\le C(||\Psi_1||_{\gamma,\Sigma} + ||v_{\beta,\alpha}||_{p,\Sigma})\eta\big(\max(||y||_{\infty,\Sigma}, ||y_\alpha||_{\infty,\Sigma})\big) \; ||(y_\alpha - y)\chi||_{\infty,\Sigma} \le C(r) \; ||y_\alpha - y||_{\infty,\Sigma}, \end{aligned} \tag{6.4}$$

where $C(r)$ is a positive constant independent of α and β. From (6.3) and (6.4), we deduce that

$$|I_\alpha^2| \le C(r) \; ||y_\alpha - y||_{\infty,\Sigma}.$$

The conclusion follows from the convergence of $(y_\alpha)_\alpha$ to y in $C(\overline{\Sigma})$.

Step 2. With arguments similar to those used in Step 1, and using the estimate relative to Ψ'_y in **A2**, we may prove that

$$\int_\Sigma (\Psi'_y \circ y_\alpha) \bullet \sigma_\alpha \; \chi \, ds \, dt = \int_\Sigma (\Psi'_y \circ y) \bullet \sigma_\alpha \; \chi \, ds \, dt + \int_\Sigma (\Psi'_y \circ y - \Psi'_y \circ y_\alpha) \bullet \sigma_\alpha \; \chi \, ds \, dt = I_\alpha^3 + I_\alpha^4,$$

with

$$I_\alpha^3 = \left\langle \sigma_\alpha, (\Psi'_y \circ y) \cdot \chi \right\rangle_{*,\Sigma} \longrightarrow 0,$$

$$I_\alpha^4 = \int_\Sigma (\Psi'_y \circ y - \Psi'_y \circ y_\alpha) \bullet \sigma_\alpha \ \chi \, ds \, dt \le C(r) \quad \ell(||y_\alpha - y||_{C(\overline{Q})}) \longrightarrow 0.$$

The proof is complete. □

Lemma 6.3. *Suppose that* **A1**, **A2** *are satisfied. Suppose in addition for every $\sigma \in Y^p$, equation (5.1) admits a weak solution in $C(\overline{Q}) \cap W(0,T)$. Let σ_1, σ_2 be in Y^p, and let y_1 and y_2 be solutions of (5.1) corresponding to σ_1 and σ_2. Then, the function $z = y_1 - y_2$ satisfies*

$$\begin{cases} \dfrac{\partial z}{\partial t} + Az + az = 0 & \text{in } Q, \\[2mm] \dfrac{\partial z}{\partial n_A} + b \bullet \sigma_1 z = (\Psi \circ y_2) \bullet (\sigma_2 - \sigma_1) \ \text{ on } \Sigma, \quad z(0) = 0 & \text{in } \Omega, \end{cases} \tag{6.5}$$

where

$$a = \int_0^1 \Phi'_y(\cdot, \theta y_1 + (1-\theta) y_2) d\theta \ge C_o,$$

$$b(\cdot, w) = \int_0^1 \Psi'_y(\cdot, \theta y_1 + (1-\theta) y_2, w) d\theta \ge C_o.$$

Moreover, $y_1 - y_2$ is Hölder continuous on $\overline{Q}$, and satisfies

$$||y_1 - y_2||_{C^{\nu,\nu/2}(\overline{Q})} \le C||(\Psi \circ y_2) \bullet (\sigma_2 - \sigma_1)||_{\gamma,\Sigma} \ \ \textit{for some} \ \ 0 < \nu < 1.$$

where $C \equiv C(T, \Omega, N, C_o, q, \gamma, p)$ is independent of σ_1 and σ_2.

Proof. The function $z = y_1 - y_2$ satisfies $z(0) = 0$, and

$$\begin{cases} \dfrac{\partial z}{\partial t} + Az + az = 0 & \text{in } Q, \\[2mm] \dfrac{\partial z}{\partial n_A} = (\Psi \circ y_2) \bullet (\sigma_2 - \sigma_1) + (\Psi \circ y_2 - \Psi \circ y_1) \bullet \sigma_1 & \text{on } \Sigma, \end{cases}$$

where $a = \displaystyle\int_0^1 \Phi'_y(\cdot, \theta y_1 + (1-\theta) y_2) d\theta \ge C_o$. Let $(v_{1,\alpha})_\alpha$ and $(v_{2,\alpha})_\alpha$ be two bounded nets in $L^p(\Sigma)$, such that $(i(v_{1,\alpha}))_\alpha$ and $(i(v_{2,\alpha}))_\alpha$ converge to σ_1 and σ_2 in the weak-star topology of $(Ca^p(\Sigma))^*$. Let z_α be such that $z_\alpha(0) = 0$, and

$$\begin{cases} \dfrac{\partial z_\alpha}{\partial t} + Az_\alpha + az_\alpha = 0 & \text{in } Q, \\[2mm] \dfrac{\partial z_\alpha}{\partial n_A} = (\Psi \circ y_2) \bullet (i(v_{2,\alpha}) - i(v_{1,\alpha})) + (\Psi \circ y_2 - \Psi \circ y_1) \bullet i(v_{1,\alpha}) & \text{on } \Sigma. \end{cases}$$

Due to Proposition 3.3, z_α belongs to $C^{\nu,\nu/2}(\overline{Q})$ (for some $0 < \nu < 1$) and satisfies

$$||z_\alpha||_{C^{\nu,\nu/2}(\overline{Q})} \le C||(\Psi \circ y_2) \bullet i(v_{2,\alpha}) - (\Psi \circ y_1) \bullet i(v_{1,\alpha})||_{\gamma,\Sigma} \le C, \tag{6.6}$$

where C is independent of α. Since the imbedding from $C^{\nu,\nu/2}(\overline{Q})$ into $C(\overline{Q})$ is compact, there exist a subnet, and $z_o \in C(\overline{Q})$ such that $(z_\alpha)_\alpha$ converges uniformly to z_o in $\overline{Q}$. Moreover, since $(z_\alpha)_\alpha$ is bounded in $W(0,T)$, it converges to z_o in the weak-star topology of $W(0,T)$. With Lemma 6.2, by passing to the limit in the variational formulation satisfied by z_α, we easily see that $z_o \equiv z$. On the other hand, observe that z_α also satisfies

$$\begin{cases} \dfrac{\partial z_\alpha}{\partial t} + Az_\alpha + az_\alpha = 0 & \text{in } Q, \\ \dfrac{\partial z_\alpha}{\partial n_A} + b \bullet i(v_{1,\alpha})\ z = (\Psi \circ y_2) \bullet [i(v_{2,\alpha}) - i(v_{1,\alpha})] & \text{on } \Sigma, \end{cases} \tag{6.7}$$

where $b(\cdot, w) = \displaystyle\int_0^1 \Psi'_y(\cdot, \theta y_1 + (1-\theta) y_2, w) d\theta \geq C_o$. Using Lemma 6.2, and passing to the limit in the variational formulation of (6.7), we show that z satisfies (6.5). The estimate follows from Proposition 3.2.

□

6.2. Existence, uniqueness and regularity of the relaxed state

Theorem 6.4. *If Assumptions* **A1**, **A2** *are fulfilled, if $\sigma \in Y^p$, and if $y_o \in C(\overline{\Omega})$, then equation* (5.1) *admits a unique weak solution y_σ in $W(0,T) \cap C(\overline{Q})$. This solution satisfies*

$$||y_\sigma||_{C(\overline{Q})} + ||y_\sigma||_{W(0,T)} \leq C \left(\left|\left|\overline{\Psi} \bullet \sigma\right|\right|_{\gamma,\Sigma} + ||y_o||_{C(\overline{\Omega})} + 1 \right),$$

where $C \equiv C(T, \Omega, N, q, p, \gamma, C_o)$, and where $\overline{\Psi}(s,t,w) = \Psi(s,t,0,w)$.

Proof. The proof is split into three steps.

Step 1. *Existence of a solution.* By definition of σ, there exists a bounded net $(v_\alpha)_\alpha \subset L^p(\Sigma)$ such that $(i(v_\alpha))_\alpha$ converges to σ in the weak-star topology of $(Ca^p(\Sigma))^*$. Let y_α be the solution of (1.1) corresponding to v_α. Due to Theorem 3.6, there exists $0 < \nu < 1$ such that $(y_\alpha)_\alpha$ is bounded in $C^{\nu,\nu/2}(\overline{Q_{\varepsilon T}})$. Since the imbedding from $C^{\nu,\nu/2}(\overline{Q_{\varepsilon T}})$ into $C(\overline{Q_{\varepsilon T}})$ is compact, there exist a subnet, still indexed by α, and $y \in C(\overline{Q_{\varepsilon T}})$ such that $(y_\alpha)_\alpha$ converges uniformly to y in $\overline{Q_{\varepsilon T}}$, for all $\varepsilon \in]0,T[$. Moreover, since $(y_\alpha)_\alpha$ is bounded in $W(0,T)$, it converges to y in the weak-star topology of $W(0,T)$. By passing to the limit in the variational formulation satisfied by y_α, we easily see that $y \equiv y_\sigma$ if the following equalities hold:

$$\lim_\alpha \int_\Sigma \Psi(s,t,y_\alpha,v_\alpha)\ \phi\, ds\, dt = \int_\Sigma (\Psi \circ y) \bullet \sigma\ \phi\, ds\, dt, \tag{6.8}$$

$$\lim_\alpha \int_Q \Phi(x,t,y_\alpha)\ \phi\, dx\, dt = \int_Q \Phi(x,t,y)\ \phi\, dx\, dt, \tag{6.9}$$

for all $\phi \in C^1(\overline{Q})$ such that $\phi(T) = 0$. It is clear that (6.8) immediately follows from Lemma 6.2 by setting $\sigma_\alpha = i(v_\alpha)$. To prove (6.9), notice that due to **A1**, for all $\varepsilon \in]0,T[$, we have

$$\begin{aligned} \left| \int_Q (\Phi(x,t,y_\alpha) - \Phi(x,t,y))\, \phi\, dx\, dt \right| &= \left| \int_Q \Phi_\alpha(x,t)(y_\alpha - y)\ \phi\, dx\, dt \right| \\ &\leq \int_{Q_{\varepsilon T}} |\Phi_\alpha(x,t)(y_\alpha - y)\phi|\, dx\, dt + \int_{Q \setminus Q_{\varepsilon T}} |\Phi_\alpha(x,t)(y_\alpha - y)\phi|\, dx\, dt \\ &\leq C \left(||y_\alpha - y||_{C(\overline{Q}_{\varepsilon T})} + \int_{Q \setminus Q_{\varepsilon T}} \Phi_1(x,t)|\phi|\, dx\, dt \right) \end{aligned}$$

where C is a positive constant independent of α, and where $\Phi_\alpha = \int_0^1 \Phi'_y(\cdot, \theta y_\alpha + (1-\theta)y)\, d\theta$. By the absolute continuity of the Lebesgue integral, for every $\delta > 0$, there exists $\varepsilon_\delta \in]0, T[$ such that $\int_{Q \setminus Q_{\varepsilon_\delta T}} \Phi_1(x,t) g\, dx\, dt \leq \delta$. Therefore,

$$\left| \int_Q (\Phi(x,t,y_\alpha) - \Phi(x,t,y))\, \phi\, dx\, dt \right| \leq C \left(||y_\alpha - y||_{C(\overline{Q}_{\tau_\delta T})} + \delta \right). \tag{6.10}$$

By passing to the limit in (6.10), we deduce that

$$\lim_\alpha \left| \int_Q (\Phi(x,t,y_\alpha) - \Phi(x,t,y))\, \phi\, dx\, dt \right| \leq \delta,$$

and since δ is a arbitrary, we obtain (6.9).

Step 2. *Uniqueness of the solution.* Let y_1 and y_2 be two weak solutions in $W(0,T) \cap C(\overline{Q})$ of (5.1) corresponding to σ. From Lemma 6.3, the function $z = y_1 - y_2$ is the solution of

$$\frac{\partial z}{\partial t} + Az + az = 0 \text{ in } Q, \qquad \frac{\partial z}{\partial n_A} + b \bullet \sigma\ z = 0 \text{ on } \Sigma, \qquad z(0) = 0 \text{ in } \Omega,$$

where

$$a = \int_0^1 \Phi'_y(\cdot, \theta y_1 + (1-\theta) y_2)\, d\theta \geq C_o,$$
$$b(\cdot, w) = \int_0^1 \Psi'_y(\cdot, \theta y_1 + (1-\theta) y_2, w)\, d\theta \geq C_o.$$

From Proposition 3.2, we deduce that $y_1 \equiv y_2$.

Step 3. *The estimate in* $C(\overline{Q})$. From Lemma 6.3, it is easy to see that the weak solution of (5.1) is also the weak solution of

$$\begin{cases} \dfrac{\partial z}{\partial t} + Az + \widetilde{a} z = -\Phi(\cdot, 0) & \text{in } Q, \\ \dfrac{\partial z}{\partial n_A} + \widetilde{b} z = -\overline{\Psi} \bullet \sigma \quad \text{on } \Sigma, \qquad z(0) = y_o & \text{in } \Omega, \end{cases}$$

where

$$\widetilde{a} = \int_0^1 \Phi'_y(\cdot, \theta y_\sigma)\, d\theta \geq C_o, \qquad \widetilde{b}(\cdot, w) = \int_0^1 \Psi'_y(\cdot, \theta y_\sigma, w)\, d\theta \geq C_o,$$

and $\overline{\Psi}(\cdot, w) = \Psi(\cdot, 0, w)$. The estimate follows from Proposition 3.2. □

Theorem 6.5 ([6], Chap. 3, Th. 1.3)**.** *For every $M > 0$ and every $\varepsilon > 0$, there exist $0 < \nu < 1$ and $C(\varepsilon) \equiv C(T, \Omega, N, q, p, \gamma, \varepsilon, \nu, M)$ such that, for every $\sigma \in Y^p$ such that $||\overline{\Psi} \bullet \sigma||_{\gamma, \Sigma} \leq M$, the weak solution y_σ of* (5.1) *corresponding to σ is Hölder continuous on $\overline{Q_{\varepsilon T}}$ and satisfies:*

$$\|y_\sigma\|_{C^{\nu, \nu/2}(\overline{Q_{\varepsilon T}})} \leq C(\varepsilon).$$

7. Relaxation results

In this section we set an existence result for the relaxed control problem (RP) and we study the relation between this problem and the classical problem (P). We prove that (RP) is closely related to some classical perturbed problems and that under some stability conditions the infimum of (RP) and (P) are identical. Through the sequel, for $\delta \geq 0$, we set

$$(P_\delta) \inf \left\{ J(y,v) \mid (y,v) \in C(\overline{Q}) \times V_{ad} \text{ satisfying } (1.1) \text{ and } d_Z(g(y)) \leq \delta \right\},$$

where $d_Z(g(y)) = \inf_{\phi \in Z} ||\phi - g(y)||_{\infty,Q}$. We will denote by (RP_δ), the relaxed control problem corresponding to (P_δ).

7.1. **Continuity results**

We start with a result describing the dependence of the relaxed trajectories with respect to the corresponding relaxed controls. This continuity result gives us informations about the topological structure of the set of relaxed trajectories, and is the main tool to establish the existence of solutions for the relaxed control problem.

Theorem 7.1. *Suppose that* **A1**, **A2** *are satisfied. Then, for every $r > 0$, the mapping $\Lambda : \sigma \longrightarrow y_\sigma$ is continuous from Y_r^p, endowed with its weak-star topology, into $C(\overline{Q})$.*

Proof. Let r be positive. Let $(\sigma_\alpha)_\alpha$ be a net of boundary relaxed controls converging to σ in the weak-star topology of Y_r^p. Let y_α and y_σ be the solutions of (5.1) corresponding to σ_α and σ. Due to Lemma 6.3, the function $z_\alpha = y_\alpha - y_\sigma$ is the solution of:

$$\begin{cases} \dfrac{\partial z_\alpha}{\partial t} + A z_\alpha + a_\alpha z_\alpha = 0 & \text{in } Q, \\ \dfrac{\partial z_\alpha}{\partial n_A} + b_\alpha \bullet \sigma_\alpha \; z_\alpha = (\Psi \circ y_\sigma) \bullet (\sigma - \sigma_\alpha) \text{ on } \Sigma, \quad z_\alpha(0) = 0 & \text{on } \Omega, \end{cases}$$

where

$$a_\alpha = \int_0^1 \Phi'_y(\cdot, \theta y_\alpha + (1-\theta) y_\sigma) d\theta \geq C_o,$$

$$b_\alpha(\cdot, w) = \int_0^1 \Psi'_y(\cdot, \theta y_\alpha + (1-\theta) y_\sigma, w) d\theta \geq C_o.$$

Due to **A1**, **A2**, the net $(a_\alpha, b_\alpha \bullet \sigma_\alpha)_\alpha$ is bounded in $L^q(Q) \times L^\gamma(\Sigma)$. Moreover, from Lemma 6.3, $(z_\alpha)_\alpha$ is bounded in $W(0,T) \cap C^{\nu,\nu/2}(\overline{Q})$. With Lemma 6.2, and compactness results similar to those used in Theorem 6.4, we may prove that $(z_\alpha)_\alpha$ converges uniformly on $\overline{Q}$ to the solution z of

$$\int_Q \left(-z \frac{\partial \phi}{\partial t} + \sum_{i,j=1}^N a_{ij} D_j z D_i \phi + \Phi'_y(x,t,y_\sigma) \; z\phi \right) dx\, dt + \int_\Sigma \left(\Psi'_y \circ y_\sigma \right) \bullet \sigma \; z\phi \, ds\, dt = 0$$

for all $\phi \in C^1(\overline{Q})$ such that $\phi(T) = 0$.

We conclude by observing that $z \equiv 0$. □

Lemma 7.2. *Suppose that* **A4** *is satisfied. Let σ_α be a net converging to σ in the weak-star topology of Y_r^p $(r > 0)$, and let $(y_\alpha)_\alpha$ be a bounded net in $C(\overline{\Sigma})$ converging to y uniformly on $\overline{\Sigma}$. Then,*

$$\lim_\alpha \int_\Sigma (G \circ y_\alpha) \bullet \sigma_\alpha \; \chi \, ds\, dt = \int_\Sigma (G \circ y) \bullet \sigma \; \chi \, ds\, dt$$

$$\lim_{\alpha} \int_{\overline{\Sigma}} (G'_y \circ y_\alpha) \bullet \sigma_\alpha \ \chi \, ds\, dt = \int_{\overline{\Sigma}} (G'_y \circ y) \bullet \sigma \ \chi \, ds\, dt$$

for all $\chi \in C(\overline{\Sigma})$.

Proof. The proof can be adapted from the one given for Lemma 6.2. □

Proposition 7.3. *Suppose that* **A1**–**A7** *are satisfied. Then, for every* $r > 0$, *the mapping* $\sigma \longrightarrow \widetilde{J}(y_\sigma, \sigma)$ *is continuous from* Y_r^p, *endowed with the weak-star topology, into* $\mathbb{R}$.

Proof. Let $(\sigma_\alpha)_\alpha$ be a net in Y_r^p converging to σ in the weak-star topology of $(Ca^p(\Sigma))^*$. Let y_α and y_σ be the corresponding solutions of (5.1). Then,

$$\begin{aligned}\widetilde{J}(y_\alpha, \sigma_\alpha) - \widetilde{J}(y_\sigma, \sigma) = & \int_Q F_\alpha (y_\alpha - y_\sigma)\, dx\, dt + \int_\Omega L_\alpha (y_\alpha - y_\sigma)(T)\, dx \\ & + \int_{\overline{\Sigma}} \big((G \circ y_\alpha) \bullet \sigma_\alpha - (G \circ y_\sigma) \bullet \sigma \big)\, ds\, dt \end{aligned} \tag{7.1}$$

where

$$\begin{aligned} F_\alpha &= \int_0^1 F'_y(\cdot, \theta y_\alpha + (1-\theta) y_\sigma) d\theta, \\ L_\alpha &= \int_0^1 L'_y(\cdot, \theta y_\alpha(T) + (1-\theta) y_\sigma(T)) d\theta. \end{aligned}$$

Due to Theorem 7.1, the net $(y_\alpha)_\alpha$ converges to y_σ uniformly on $\overline{Q}$. From assumptions on F and L, we deduce that there exists a positive constant C independent of α such that

$$\left| \int_Q F_\alpha (y_\alpha - y_\sigma)\, dx\, dt \right| + \left| \int_\Omega L_\alpha (y_\alpha - y_\sigma)(T)\, dx \right| \leq C ||y_\alpha - y_\sigma||_{C(\overline{Q})} \longrightarrow 0. \tag{7.2}$$

On the other hand, due to Lemma 7.2, we have

$$\lim_{\alpha} \int_{\overline{\Sigma}} (G \circ y_\alpha) \bullet \sigma_\alpha \, ds\, dt = \int_{\overline{\Sigma}} (G \circ y) \bullet \sigma \, ds\, dt. \tag{7.3}$$

The conclusion follows from (7.1), (7.2) and (7.3). □

7.2. **Existence and stability**

Optimization problems involving controls from Lebesgue spaces usually impose control constraints ensuring the set of admissible controls to be bounded in an L^∞-space. For the problems we consider, the control constraints have more general structure. Boundedness of the set of admissible controls in the L^p norm can be handled if a suitable coercivity property is imposed on the problem. More precisely, to prove existence of solutions for (RP_δ), we need the following assumption:

A8-a $\quad -C_1 |y|^j \leq F(x,t,y) \leq F_1(x,t)\ \eta(|y|)$,

A8-b $\quad C_1(|w|^p - |y|^j) \leq G(s,t,y,w) \leq (G_1(s,t) + C|w|^p)\ \eta(|y|)$,

A8-c $\quad -C_1 |y|^j \leq L(x,y) \leq L_1(x)\ \eta(|y|)$,

where $C_1 > 0$, $j \in [1, r[$, η, F_1, G_1, L_1 are as in **A1**, **A3**, **A4**, and **A5**.

Theorem 7.4. *Suppose that* **A1–A8** *are satisfied. Then for every $\delta \geq 0$, the relaxed problem (RP_δ) admits at least one solution. Moreover, we have*

$$\lim_{\delta \searrow 0} \inf(RP_\delta) = \inf(RP).$$

Proof. The proof is split into three steps.

Step 1. Let us prove that for every $M \in \mathbb{R}$, the level set S_M defined by

$$S_M = \{\sigma \in Y^p \mid \widetilde{J}(y_\sigma, \sigma) \leq M\}$$

is contained in Y_r^p, for some r sufficiently large. We argue by contradiction and suppose that for every $r > 0$, there exists $\sigma_r \in Y^p \setminus Y_r^p$ such that $\widetilde{J}(y_\sigma, \sigma) \leq M$. From the definition of σ_r, there exists a bounded net $(v_{\alpha,r})_\alpha \subset L^p(\Sigma)$ such that $(i(v_{\alpha,r}))_\alpha$ converges to σ_r in the weak-star topology of $(Ca^p(\Sigma))^*$. Hence, there exists $\bar{\alpha}_r$ such that for every $\alpha \geq \bar{\alpha}_r$, $v_{\alpha,r} \notin B_r$ (*i.e.* $i(v_{\alpha,r}) \notin Y_r^p$) and, due to Proposition 7.3 $(J(y_{v_{\alpha,r}}, v_{\alpha,r}))_\alpha$ converges to $\widetilde{J}(y_{\sigma_r}, \sigma_r)$. In particular, there exists v_r such that

$$i(v_r) \notin Y_r^p \quad (\textit{i.e. } v_r \notin B_r) \qquad \text{and} \qquad J(y_{v_r}, v_r) \leq M + 1. \tag{7.4}$$

On the other hand, due to **A8**, one can easily see that

$$\begin{aligned} J(y_{v_r}, v_r) &\geq C_1 ||v_r||_{p,\Sigma}^p - C_1 \left(||y_{v_r}||_{j,Q}^j + ||y_{v_r}||_{j,\Sigma}^j + ||y_{v_r}||_{j,\Omega}^j \right) \geq C_1 ||v_r||_{p,\Sigma}^p - C ||y_{v_r}||_{C(\overline{Q})}^j \\ &\geq C_1 ||v_r||_{p,\Sigma}^p - C \left(1 + ||y_o||_{C(\overline{\Omega})}^j + ||v_r||_{p,\Sigma}^j \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{J(y_{v_r}, v_r)}{||v_r||_{p,\Sigma}^j} &\geq C_1 ||v_r||_{p,\Sigma}^{r-j} - C \left(\frac{1 + ||y_o||_{C(\overline{\Omega})}^j}{||v_r||_{p,\Sigma}^j} + 1 \right) \\ &\geq C_1 r^{r-j} - C \left(\frac{1 + ||y_o||_{C(\overline{\Omega})}^j}{r^j} + 1 \right) \longrightarrow +\infty \ \text{ when } r \to +\infty. \end{aligned}$$

This contradicts (7.4).

Step 2. As (RP_δ) is feasible, there exists a minimizing sequence $(\sigma_{n,\delta})_n$. For every $M > \inf(RP_\delta)$, we can easily see that

$$(\sigma_{n,\delta})_n \ \subset \ S_M \cap \left\{ \sigma \in \overline{V}_{ad} \mid d_Z(g(y_\sigma)) \leq \delta \right\} \ \subset \ \overline{V}_{ad} \cap Y_r^p,$$

where r is the constant defined in Step 1. Since $\overline{V}_{ad} \cap Y_r^p$ is compact, there exist a finer net $(\sigma_{n_\alpha,\delta})_\alpha$ and $\sigma_\delta \in \overline{V}_{ad} \cap Y_r^p$ such that $(\sigma_{n_\alpha,\delta})_\alpha$ converges to σ_δ in the weak-star topology of $(Ca^p(\Sigma))^*$. Let $y_{n_\alpha,\delta}$ and y_δ be the solutions of (5.1) corresponding to $\sigma_{n_\alpha,\delta}$ and σ_δ. Due to Theorem 7.1 and Proposition 7.3, we have

$$\begin{aligned} d_Z(g(y_\delta)) &= \lim_\alpha d_Z(g(y_{n_\alpha,\delta})) \leq \delta, \\ \lim_\alpha \widetilde{J}(y_{n_\alpha,\delta}, \sigma_{n_\alpha,\delta}) &= \widetilde{J}(y_\delta, \sigma_\delta) = \inf(RP_\delta). \end{aligned}$$

In other words, $(y_\delta, \sigma_\delta)$ is an optimal solution for (RP_δ).

Step 3. For $\delta > 0$, let σ_δ be a solution of (RP_δ). From Step 1 and Step 2, we know that $(\sigma_\delta)_{\delta > 0} \subset \overline{V}_{ad} \cap Y_r^p$, for r big enough. Since $\overline{V}_{ad} \cap Y_r^p$ is compact, we can suppose that $(\sigma_\delta)_\delta$ converges to some $\widetilde{\sigma} \in \overline{V}_{ad} \cap Y_r^p$. From Theorem 7.1, it follows that $(y_{\sigma_\delta})_\delta$ converges to $y_{\widetilde{\sigma}}$ in $C(\overline{Q})$. Since $d_Z(g(y_{\sigma_\delta})) \leq \delta$, by passing to the limit, we obtain $d_Z(g(y_{\widetilde{\sigma}})) = 0$. Therefore, $\widetilde{\sigma}$ is admissible for (RP), and

$$\min(RP) \leq \widetilde{J}(y_{\widetilde{\sigma}}, \widetilde{\sigma}) = \lim_{\delta \searrow 0} \min(RP_\delta) \leq \min(RP).$$

The proof is complete. □

7.3. **Denseness results. Properness of the relaxation**

A natural question is whether a relaxed optimal trajectory can be closely approximated by a trajectory of the original control problem. We answer this question by showing that the set of original trajectories is dense, for an appropriate topology, in the set of relaxed trajectories. This result is very useful in the analysis of the properness of the relaxation. In this section, we mention interesting results on the connection between $X = \{y_v \mid v \in V_{ad}\}$, the set of standard (or classical) trajectories, and $\overline{X} = \{y_\sigma \mid \sigma \in \overline{V}_{ad}\}$, the set of relaxed trajectories.

Proposition 7.5. *Suppose that* **A1**, **A2** *are satisfied. Then, X is dense in $\overline{X}$ endowed with the usual topology of $C(\overline{Q})$.*

Proof. Let $\sigma \in \overline{V}_{ad}$ and let y_σ be the corresponding solution (5.1). Since $\overline{V}_{ad}$ is the bounded closure of V_{ad} (in the weak-star topology of $(Ca^p(\Sigma))^*$), there exists a bounded net $(v_\alpha)_\alpha \subset V_{ad} \cap B_{r_o}$ (for some $r_o > 0$) such that $(i(v_\alpha))_\alpha$ converges to σ for the weak-star topology of $\overline{V}_{ad} \cap Y_{r_o}^p$. Theorem 7.1 yields that $(y_\alpha)_\alpha$ converges to y_σ uniformly in $\overline{Q}$. Since $C(\overline{Q})$ is a metric space, there exists a sequence $(y_{i(v_n)} \equiv y_{v_n})_n \subset (y_\alpha)_\alpha \setminus \{y_\sigma\}$ converging to y_σ in $C(\overline{Q})$. □

The next result links together the set of admissible relaxed trajectories and the set of perturbed admissible classical trajectories. As a consequence, we see that the relaxed control problem (RP) gives some informations on the limit behavior of the perturbed control problems (P_δ) associated with the initial one. More precisely, we have the following proposition.

Corollary 7.6. *Suppose that* **A1**–**A8** *are satisfied. Then,*

$$\{y \in \overline{X} \mid g(y) \in Z\} \subset \mathbf{cl}\ \{y \in X \mid d_Z(g(y)) \leq \delta\} \qquad \textit{for all } \delta > 0,$$

where **cl** *denotes the closure for the usual topology of $C(\overline{Q})$. Moreover, we have*

$$\inf(RP) = \lim_{\delta \searrow 0}\ \inf(P_\delta).$$

Proof. The Proof of the denseness result is based on Proposition 7.5 and is the same as for Proposition 6.1 in [3]. The stability result follows by using arguments similar in [5]. □

Generally, on account of the state constraints, the relaxation is not proper, in the sense that $\min(RP)$ is not equal to $\inf(P)$. However, Theorem 7.4 and Proposition 7.6 yield a necessary and sufficient condition for the properness of the relaxation. Indeed, $\inf(P) = \inf(RP)$ if, and only if, (P) is weakly stable on the right (*i.e.* $\inf(P) = \lim_{\delta \searrow 0}\ \inf(P_\delta)$).

8. Optimality conditions

8.1. **Adjoint equation**

In this section, we recall some existence, uniqueness and regularity results for the adjoint equation. Let (a, b) be in $L^q(Q) \times L^\gamma(\Sigma)$ such that $a \geq C_o$ and $b \geq C_o$. We consider the following terminal boundary value problem

$$
\begin{cases}
-\dfrac{\partial \zeta}{\partial t} + A\zeta + a\zeta = \mu_Q & \text{in } Q,\\[2ex]
\dfrac{\partial \zeta}{\partial n_A} + b\zeta = \mu_\Sigma \quad \text{on } \Sigma, \qquad \zeta(T) = \mu_{\overline{\Omega}_T} & \text{on } \overline{\Omega},
\end{cases}
\tag{8.1}
$$

where $\mu = \mu_Q + \mu_\Sigma + \mu_{\overline{\Omega}_T}$ is a bounded Radon measure on $\overline{Q} \setminus \overline{\Omega}_0$, μ_Q is the restriction of σ to Q, μ_Σ is the restriction of σ to Σ and $\mu_{\overline{\Omega}_T}$ the restriction of σ to $\overline{\Omega} \times \{T\}$.

Definition 8.1. A function $\zeta \in L^1(0,T;W^{1,1}(\Omega))$ is a weak solution of (8.1) if, and only if, $a\zeta \in L^1(Q)$, $b\zeta \in L^1(\Sigma)$, and

$$
\int_Q \left(\zeta \frac{\partial z}{\partial t} + \sum_{i,j=1}^{N} a_{ij} D_j \zeta D_i z + a y \zeta \right) dxdt + \int_\Sigma b\zeta z \, dsdt = \langle \mu, z \rangle_{b,\overline{Q}\setminus\overline{\Omega}_0}
$$

for all $z \in C^1(\overline{Q})$ satisfying $z(0) = 0$ on $\overline{\Omega}$.

We recall an existence theorem for parabolic equations with measures as data proved in [17].

Theorem 8.2. *Let (a,b) be in $L^q(Q) \times L^\gamma(\Sigma)$ such that $a \geq C_o$ and $b \geq C_o$, and let μ be in $M_b(\overline{Q} \setminus \overline{\Omega}_0)$. The equation (8.1) admits a unique solution ζ in $L^1(0,T;W^{1,1}(\Omega))$. For every (δ, d) satisfying $\delta \geq 1$, $d \geq 1$, $\frac{N}{2d} + \frac{1}{\delta} > \frac{N+1}{2}$, ζ belongs to $L^\delta(0,T;W^{1,d}(\Omega))$ and*

$$
||\zeta||_{L^\delta(0,T;W^{1,d}(\Omega))} \leq C ||\mu||_{M_b(\overline{Q}\setminus\overline{\Omega}_0)},
$$

where $C \equiv C(\Omega, T, \delta, d, q, \gamma, C_o)$ is a positive constant independent of a and b. Moreover, there exists a function in $L^1(\Omega)$, denoted by $\zeta(0)$, such that:

$$
\int_Q \left(\frac{\partial z}{\partial t} + Az + az \right) \zeta \, dx\, dt + \int_\Sigma \left(\frac{\partial z}{\partial n_A} + bz \right) \zeta \, dsdt = \langle \mu, z \rangle_{b,\overline{Q}\setminus\overline{\Omega}_0} - \int_\Omega z(0)\zeta(0) \, dx \qquad \textit{for all } z \in Y_{q,r},
$$

where $Y_{q,\gamma} = \left\{ y \in W(0,T) \mid \dfrac{\partial y}{\partial t} + Ay \in L^q(Q),\ \dfrac{\partial y}{\partial n_A} \in L^\gamma(\Sigma) \ \text{ and } y(0) \in L^\infty(\Omega) \right\}.$

8.2. Differentiability results

Linearity induced by Young measures simplifies the technical aspects related to Taylor expansions with respect to the controls, for the state variable and the cost functional. Since the set of relaxed controls is convex, Lagrangian perturbations are considered.

Theorem 8.3. *Suppose that* **A1–A7** *are satisfied. Let σ and σ_o be in Y^p. For $\tau \in]0,1[$ set $\sigma_\tau = \sigma + \tau(\sigma_o - \sigma)$. Let y_σ and y_{σ_τ} be the solutions of* (5.1) *corresponding to σ and σ_τ. Then, we have*

$$
y_{\sigma_\tau} = y_\sigma + \tau z + r_\tau \qquad \textit{with} \quad \lim_{\tau \to 0} \frac{1}{\tau} ||r_\tau||_{C(\overline{Q})} = 0, \tag{8.2}
$$

$$
J(y_{\sigma_\tau}, \sigma_\tau) = J(y_\sigma, \sigma) + \tau \Delta J + o(\tau),
$$

where z is the weak solution of

$$\begin{cases} \dfrac{\partial z}{\partial t} + Az + \Phi'_y(x,t,y_\sigma)z = 0 & \text{in } Q, \\ \dfrac{\partial z}{\partial n_A} + (\Psi'_y \circ y_\sigma) \bullet \sigma z = (\Psi \circ y_\sigma) \bullet (\sigma - \sigma_o) \quad \text{on } \Sigma, \quad z(0) = 0 & \text{in } \Omega, \end{cases}$$

and where

$$\Delta J = \int_Q F'_y(x,t,y_\sigma)z\,dx\,dt + \int_\Omega L'_y(x,y_\sigma(T))z(T)\,dx + \int_{\Sigma\cup\Gamma_T} (G'_y \circ y_\sigma) \bullet \sigma z\,ds\,dt + \int_{\overline{\Sigma}} (G \circ y_\sigma) \bullet (\sigma_o - \sigma)\,ds\,dt. \tag{8.3}$$

Proof. The proof is split into two steps.

Step 1. *Preliminary convergence results.* Without loss of generality, we can suppose that there exists $r > 0$ such that σ_o and σ belong to Y^p_r. Since Y^p_r is convex, it follows that $\sigma_\tau = \sigma + \tau(\sigma_o - \sigma)$ belongs to Y^p_r. Moreover, $(\sigma_\tau)_\tau$ converges to σ in the weak-star topology of $(Ca^p(\Sigma))^*$. With Theorem 7.1, we deduce that $(y_\tau)_\tau$ converges to y_σ uniformly on $\overline{Q}$. Let us set

$$a_\tau = \int_0^1 \Phi'_y(\cdot, \theta y_{\sigma_\tau} + (1-\theta)y_\sigma)\,d\theta, \qquad a = \Phi'_y(\cdot, y_\sigma),$$
$$b_\tau(\cdot, w) = \int_0^1 \Psi'_y(\cdot, \theta y_{\sigma_\tau} + (1-\theta)y_\sigma, w)\,d\theta, \qquad b(\cdot, w) = \Psi'_y(\cdot, y_\sigma, w).$$

Due to **A1**, with Lebesgue's theorem of dominated convergence, we can prove that $(a_\tau)_\tau$ converges to a in $L^q(Q)$. Moreover, by using arguments similar to those of Lemma 6.2, we may prove that

$$\lim_{\tau \searrow 0} \int_\Sigma (b_\tau \bullet \sigma_\tau)\ \chi\,ds\,dt = \int_\Sigma (b \bullet \sigma)\ \chi\,ds\,dt \qquad \text{for all } \chi \in C(\overline{\Sigma}).$$

Step 2. *Proof of* (8.2). Due to Lemma 6.3, we see that the function $z_\tau = \dfrac{y_{\sigma_\tau} - y_\sigma}{\tau}$ is the solution of:

$$\begin{cases} \dfrac{\partial z_\tau}{\partial t} + Az_\tau + a_\tau z_\tau = 0 & \text{in } Q, \\ \dfrac{\partial z_\tau}{\partial n_A} + (b_\tau \bullet \sigma_\tau)\ z_\tau = (\Psi \circ y_\sigma) \bullet (\sigma - \sigma_o) \ \text{ on } \Sigma, \quad z_\tau(0) = 0 & \text{in } \Omega, \end{cases}$$

where $a_\tau \geq C_o$ and $b_\tau \geq C_o$. It follows that $\varphi_\tau = z_\tau - z$ is the solution of:

$$\begin{cases} \dfrac{\partial \varphi_\tau}{\partial t} + A\varphi_\tau + a_\tau \varphi_\tau = (a - a_\tau)z & \text{in } Q, \\ \dfrac{\partial \varphi_\tau}{\partial n_A} + (b_\tau \bullet \sigma_\tau)\ \varphi_\tau = (b_\tau \bullet \sigma_\tau - b \bullet \sigma)\ z \ \text{ on } \Sigma, \quad \varphi_\tau(0) = 0 & \text{in } \Omega. \end{cases}$$

With arguments similar to those used in the proof of Lemma 6.3, we may prove that φ_τ is Hölder continuous on $\overline{Q}$ and satisfies

$$||\varphi_\tau||_{C^{\nu,\nu/2}(\overline{Q})} \leq C \qquad \text{with } \ 0 < \nu < 1,$$

where C is a positive constant independent of τ. Moreover, $(\varphi_\tau)_\tau$ is bounded in $W(0,T)$. Then, there exists a subsequence, still indexed by τ, and φ such that $(\varphi_\tau)_\tau$ converges to φ for the usual topology of $C(\overline{Q})$ and in the weak-star topology of $W(0,T)$. By taking into account the convergence results stated above, and by passing to the limit when τ tends to zero in variational formula satisfied by φ_τ, we obtain

$$\int_Q \left(-\varphi \frac{\partial \phi}{\partial t} + \sum_{i,j=1}^{N} a_{ij}\ D_i\varphi\ D_j\phi + a\varphi\phi \right) dx\, dt + \int_\Sigma (b \bullet \sigma)\ \varphi\ \phi\, ds\, dt = 0$$

for all $\phi \in C^1(\overline{Q})$ such that $\phi(T) = 0$. Therefore $\varphi \equiv 0$. We have proved (8.2). Similar arguments give Taylor's expansion relative to the cost functional. $\square$

8.3. Statement of necessary optimality conditions

For $\lambda \in \mathbb{R}$, $y \in C(\overline{\Sigma})$ and $p \in L^{\gamma'}(\Sigma)$, let us define the Hamiltonian function by:

$$H(\lambda, y, p) = \lambda\ G \circ y + p \cdot \Psi \circ y.$$

We shall say that $(\bar{y}, \bar{\sigma})$ is regular, if there exists $\widetilde{\sigma} \in \overline{V}_{ad}$ such that

$$g(\bar{y}) + g'(\bar{y})(z_{\widetilde{\sigma}} - z_{\bar{\sigma}}) \in \text{int } Z,$$

where z_σ (with $\sigma = \bar{\sigma}$ or $\sigma = \widetilde{\sigma}$) is the solution of

$$\begin{cases} \dfrac{\partial z}{\partial t} + Az + \Phi'_y(\cdot, \bar{y})z = 0 & & \text{in } Q, \\ \dfrac{\partial z}{\partial n_A} + (\Psi'_y \circ \bar{y}) \bullet \bar{\sigma}\ z = (\Psi \circ \bar{y}) \bullet \sigma \quad \text{on } \Sigma, & z(0) = 0 & \text{in } \Omega. \end{cases}$$

Theorem 8.4. *Suppose that* **A1–A8** *are satisfied. If $(\bar{y}, \bar{\sigma})$ is an optimal solution of (RP), then there exist $\bar{\lambda} \geq 0$, $\bar{\mu} \in M(\overline{Q})$, and $\bar{\zeta} \in L^1(0,T; W^{1,1}(\Omega))$, such that*

$$(\bar{\mu}, \bar{\lambda}) \neq 0$$

$$\langle \bar{\mu}, z - g(\bar{y}) \rangle_{M(\overline{Q}) \times C(\overline{Q})} \leq 0 \qquad \textit{for all } z \in Z, \tag{8.4}$$

$$\begin{cases} -\dfrac{\partial \bar{\zeta}}{\partial t} + A\bar{\zeta} + \Phi'_y(\cdot, \bar{y})\bar{\zeta} = -\bar{\lambda} F'_y(\cdot, \bar{y}) - g'(\bar{y})^* \bar{\mu}|_Q & \textit{in } Q, \\ \dfrac{\partial \bar{\zeta}}{\partial n_A} + (\Psi'_y \circ \bar{y}) \bullet \bar{\sigma}\ \bar{\zeta} = -\bar{\lambda}\ (G'_y \circ \bar{y}) \bullet \bar{\sigma}|_\Sigma - g'(\bar{y})^* \bar{\mu}|_\Sigma & \textit{on } \Sigma, \\ \bar{\zeta}(T) = -\bar{\lambda} L'_y(\cdot, \bar{y}(T)) - \bar{\lambda}\ (G'_y \circ \bar{y}) \bullet \bar{\sigma}|_{\Gamma_T} - g'(\bar{y})^* \bar{\mu}|_{\overline{\Omega}_T} & \textit{on } \overline{\Omega}, \end{cases} \tag{8.5}$$

$$\int_{\overline{\Sigma}} H(\bar{\lambda}, \bar{y}, \bar{\zeta}) \bullet \bar{\sigma}\, ds\, dt = \min_{\sigma \in \overline{V}_{ad}} \int_{\overline{\Sigma}} H(\bar{\lambda}, \bar{y}, \bar{\zeta}) \bullet \sigma\, ds\, dt. \tag{8.6}$$

Moreover, if $(\bar{y},\bar{\sigma})$ *is regular, then we can take* $\bar{\alpha}=1$, *and*

$$\int_{\overline{\Sigma}} H(1,\bar{y},\bar{\zeta})\bullet\bar{\sigma}\,ds\,dt=\int_{\Sigma}\min_{w\in K_V(s,t)} H(1,\bar{y},\bar{\zeta})(s,t,w)\,ds\,dt. \tag{8.7}$$

Proof. The proof is split into four steps.

Step 1. Let us set

$$A=\big\{(z,\beta)\in C(\overline{Q})\times\mathbb{R}\mid \text{there exists } \sigma\in\overline{V}_{ad} \text{ such that}$$

$$z=g(\bar{y})+g'_y(\bar{y})(z_\sigma-z_{\bar{\sigma}}),\quad \beta=\widetilde{J}'_y(\bar{y},\bar{\sigma})\ (z_\sigma-z_{\bar{\sigma}})+\widetilde{J}(\bar{y},\sigma-\bar{\sigma})\big\},$$

$$B=\text{ int } Z\times]-\infty,0[.$$

The sets A and B are convex, and B is open. Let us prove that $A\cap B=\emptyset$. We argue by contradiction and suppose that there exists $\sigma_o\in\overline{V}_{ad}$ such that

$$g(\bar{y})+g'_y(\bar{y})(z_{\sigma_o}-z_{\bar{\sigma}})\in\text{int } Z, \tag{8.8}$$

$$\widetilde{J}'_y(\bar{y},\bar{\sigma})\ (z_{\sigma_o}-z_{\bar{\sigma}})+\widetilde{J}(\bar{y},\sigma-\bar{\sigma})<0. \tag{8.9}$$

Let $\sigma_\tau=\bar{\sigma}+\tau(\sigma_o-\bar{\sigma})$, let y_τ be the solution of (5.1) corresponding to σ_τ, and $g_\tau=g(\bar{y})+\frac{1}{\tau}(g(y_\tau)-g(\bar{y}))$. Because of Theorem 8.3, (8.8) and (8.9), it follows that

$$\lim_{\tau\searrow 0} g_\tau\in\text{int } Z\quad\text{and}\quad\lim_{\tau\searrow 0}\frac{\widetilde{J}(y_\tau,\sigma_\tau)-\widetilde{J}(\bar{y},\bar{\sigma})}{\tau}<0.$$

Therefore, there exists $\tau_o>0$ such that, for every $0<\tau\le\tau_o<1$, we have

$$g(y_\tau)=\tau\ g_\tau+(1-\tau)\ g(\bar{y})\in\text{ int } Z,$$

$$\widetilde{J}(y_\tau,\sigma_\tau)<\widetilde{J}(\bar{y},\bar{\sigma})=\min(RP).$$

Since (y_τ,σ_τ) is admissible for (RP), we obtain a contradiction. Thus, $A\cap B=\emptyset$. From a geometric version of the Hahn-Banach theorem (the Eidelheit theorem [19]), there exists $(\bar{\lambda},\bar{\mu})\in\mathbb{R}\times M(\overline{Q})$, such that:

$$\bar{\lambda}\ \beta_1+\big\langle\bar{\mu},z_1\big\rangle_{M(\overline{Q})\times C(\overline{Q})}>\bar{\lambda}\ \beta_2+\big\langle\bar{\mu},z_2\big\rangle_{M(\overline{Q})\times C(\overline{Q})} \tag{8.10}$$

for all $(z_1, \beta_1) \in A$ and all $(z_2, \beta_2) \in B$, and

$$\bar\lambda\ \beta_1 + \langle \bar\mu, z_1 \rangle_{M(\overline{Q})\times C(\overline{Q})} \geq \bar\lambda\ \beta_2 + \langle \bar\mu, z_2 \rangle_{M(\overline{Q})\times C(\overline{Q})} \tag{8.11}$$

for all $(z_1, \beta_1) \in A$ and all $(z_2, \beta_2) \in \overline{B} = Z\times]-\infty, 0]$.

Step 2. With (8.10), we can easily prove that $\bar\lambda$ is nonnegative and that $(\bar\lambda, \bar\mu) \neq 0$. For $z \in Z$, by setting $z_1 = g(\bar y)$, $z_2 = z$, $\beta_1 = \beta_2 = 0$, in (8.11), we establish (8.4).

If $(\bar y, \bar\sigma)$ is regular, by setting $z_1 = z_2 = g(\bar y) + g'(\bar y)(z_{\tilde\sigma} - z_{\bar\sigma})$ in (8.10), we easily see that $\bar\lambda \neq 0$. (Using if necessary a normalization process, we can suppose that $\bar\lambda = 1$.)

Step 3. Let $\sigma \in \overline{V}_{ad}$. By setting $z_1 = g(\bar y) + g_y'(\bar y)(z_\sigma - z_{\bar\sigma})$, $\beta_1 = \widetilde{J}_y'(\bar y, \bar\sigma)\ (z_\sigma - z_{\bar\sigma}) + \widetilde{J}(\bar y, \sigma - \bar\sigma)$, $z_2 = g(\bar y)$, and $\beta_2 = 0$ in (8.11), we obtain

$$\bar\lambda\big(\widetilde{J}_y'(\bar y, \bar\sigma)\ (z_\sigma - z_{\bar\sigma}) + \widetilde{J}(\bar y, \sigma - \bar\sigma)\big) + \langle g_y'(\bar y)^* \bar\mu, z_\sigma - z_{\bar\sigma} \rangle_{b, \overline{Q}\setminus\overline{\Omega}_o} \geq 0. \tag{8.12}$$

Let $\bar\zeta$ be the solution of (8.5). With the Green formula of Theorem 8.2, we have

$$\int_Q \bar\lambda\ F_y'(x,t,\bar y)\ (z_\sigma - z_{\bar\sigma})\, dx\, dt + \int_\Omega \bar\lambda\ L_y'(x, \bar y(T))\ (z_\sigma - z_{\bar\sigma})(T)\, dx + \int_{\Sigma\cup\Gamma_T} \bar\lambda\ (G_y' \circ \bar y) \bullet \bar\sigma\ (z_\sigma - z_{\bar\sigma})\, ds\, dt$$
$$+ \langle g_y'(\bar y)^* \bar\mu, z_\sigma - z_{\bar\sigma} \rangle_{b, \overline{Q}\setminus\overline{\Omega}_o} = \int_\Sigma \bar\zeta\ (\Psi \circ \bar y) \bullet (\sigma - \bar\sigma)\, dx\, dt.$$

This equality together with (8.12) gives (8.6).

Step 4. Let us prove (8.7). Consider the function $\widetilde{H}$ defined by:

$$\widetilde{H}(s,t) = \inf_{w \in K_V(s,t)} H(1, \bar y, \bar\zeta)(s,t,w).$$

Due to **A2** and **A4**, we have

$$\begin{aligned}
\big|H(1,\bar y,\bar\zeta)(s,t,w)\big| &= \big|G(s,t,\bar y(s,t),w) + \bar\zeta(s,t)\Psi(s,t,\bar y(s,t),w)\big| \\
&\leq \big(G_1(s,t) + C|w|^p + |\bar\zeta(s,t)|\ (\Psi_1(s,t) + C|w|^{\frac{p}{\gamma}})\big)\ \eta(||\bar y||_{\infty,\Sigma}) \\
&\leq C\big(G_1(s,t) + C|w|^p + |\bar\zeta(s,t)|\ \Psi_1(s,t) + |w|^p + |\bar\zeta(s,t)|^{\gamma'}\big) \\
&\leq C(S(s,t) + |w|^p),
\end{aligned}$$

where S belongs to $L^1(\Sigma)$. It follows that $\widetilde{H}$ belongs to $L^1(\Sigma)$. For $\varepsilon > 0$, let us consider the multivalued mapping K_ε defined by

$$\begin{aligned}
K_\varepsilon(s,t) &= \{w \in K_V(s,t) \mid H(1,\bar y,\bar\zeta)(s,t,w) \leq \widetilde{H}(s,t) + \varepsilon\} \\
&= \{w \in \mathbb{R} \mid H(\bar\lambda,\bar y,\bar\zeta)(s,t,w) \leq \widetilde{H}(s,t) + \varepsilon\} \cap K_V(s,t) = R_\varepsilon(s,t) \cap K_V(s,t).
\end{aligned}$$

Since $K_V(s,t)$ is closed and since $H(1,\bar y,\bar\zeta)(s,t,\cdot)$ is continuous, we deduce that $K_\varepsilon(s,t)$ is closed. Moreover, since K_V and R_ε are measurable, K_ε is also measurable. Then, there exists a measurable selection v_ε. Let us prove that v_ε belongs to $L^p(\Sigma)$. From **A8b**, we have

$$\begin{aligned}
C_1\ |v_\varepsilon(s,t)|^p &\leq G(s,t,\bar y(s,t),v_\varepsilon(s,t)) + C_1|\bar y(s,t)|^j \\
&\leq H(1,\bar y,\bar\zeta)(s,t,v_\varepsilon(s,t)) - \bar\zeta(s,t)\ \Psi(s,t,\bar y(s,t),v_\varepsilon(s,t)) + C_1||\bar y||_{\infty,\Sigma}^j \\
&\leq \widetilde{H}(s,t) + \varepsilon + C - \bar\zeta(s,t)\ \Psi(s,t,\bar y(s,t),v_\varepsilon(s,t)).
\end{aligned} \tag{8.13}$$

Due to Young's inequality and **A2**, one can prove that

$$\begin{aligned}\left|\bar\zeta(s,t)\ \Psi(s,t,\bar y(s,t),v_\varepsilon(s,t))\right| &\le \left(|\bar\zeta(s,t)|\ \Psi_1(s,t) + C|\bar\zeta(s,t)|\ |w|^{\frac{p}{\gamma}}\right)\ \eta(||\bar y||_{\infty,\Sigma})\\ &\le C\ |\bar\zeta(s,t)|\ \Psi_1(s,t) + \frac{C_1}{2}|w|^p + \tilde C\ |\bar\zeta(s,t)|^{\gamma'},\end{aligned} \tag{8.14}$$

where $\tilde C$ depends on γ, and C_1. By taking into account (8.13) and (8.14), we obtain

$$|v_\varepsilon(s,t)|^p \le \frac{2\,C}{C_1}\ \left(\widetilde H(s,t) + 1 + |\bar\zeta(s,t)|\ \Psi_1(s,t) + |\bar\zeta(s,t)|^{\gamma'}\right).$$

Since $\widetilde H$ belongs to $L^1(\Sigma)$, $\bar\zeta_{|\Sigma}$ belongs to $L^{\gamma'}(\Sigma)$, and Ψ_1 belongs to $L^\gamma(\Sigma)$, we deduce that v_ε belongs to $L^p(\Sigma)$, and thus to V_{ad}. From (8.6), it follows that

$$\int_{\overline\Sigma} H(1,\bar y,\bar\zeta)\bullet\bar\sigma\,ds\,dt \le \int_{\overline\Sigma} H(1,\bar y,\bar\zeta)\bullet i(v_\varepsilon)\,ds\,dt \le \int_\Sigma \widetilde H(s,t)\,ds\,dt + \varepsilon\ |\Sigma|,$$

and since ε is arbitrary, one has

$$\int_{\overline\Sigma} H(1,\bar y,\bar\zeta)\bullet\bar\sigma\,ds\,dt \le \int_\Sigma \widetilde H(s,t)\,ds\,dt. \tag{8.15}$$

On the other hand, let $(\bar v_\alpha)_\alpha$ be a bounded net in V_{ad} such that $(i(\bar v_\alpha))_\alpha$ converges to $\bar\sigma$ in the weak-star topology of $(Ca^p(\Sigma))*$. Observe that

$$\widetilde H(s,t) \le H(1,\bar y,\bar\zeta)(s,t,\bar v_\alpha(s,t)) = H(1,\bar y,\bar\zeta)\bullet i(\bar v_\alpha)(s,t).$$

Therefore,

$$\int_\Sigma \widetilde H(s,t)\,ds\,dt \le \int_\Sigma H(1,\bar y,\bar\zeta)\bullet i(\bar v_\alpha)\,ds\,dt,$$

and

$$\int_\Sigma \widetilde H(s,t)\,ds\,dt \le \int_{\overline\Sigma} H(1,\bar y,\bar\zeta)\bullet\bar\sigma\,ds\,dt. \tag{8.16}$$

The result follows from (8.15) and (8.16). □

9. Final remarks

Since $Ca^p(\Sigma)$ is not separable, we select a separable linear subspace E of it, and we equip it with the norm (4.2). Let E^* be the topological dual of E and let $i_E:\ L^p(\Sigma)\longrightarrow E^*$ be the imbedding defined again by (4.1) for $h\in E$ (*i.e.* $i_E(v)=i(v)_{|E}$)). Let Y_E^p be the weak-star closure of $i_E(L^p(\Sigma))$ in E^*. It is well known ([19]) that Y_E^p is a convex, locally compact, and locally sequentially compact subset of E^*. Moreover, if E is $C(\overline\Sigma)$-invariant (*i.e.* $C(\overline\Sigma)\cdot E = E$), then we can define a bilinear mapping $(h,\sigma)\longmapsto h\bullet\sigma$ from $E^*\times E$ into $M(\overline\Sigma)$, by $\langle h\bullet\sigma,\chi\rangle_{M(\overline\Sigma)\times C(\overline\Sigma)} = \langle\sigma,\chi\cdot h\rangle_{*,\Sigma}$ for all $\chi\in C(\overline\Sigma)$. As in Section 5, we can extend the original control problem (P) by setting

$$(RP_E)\ \inf\ \left\{\widetilde J(y,\sigma)\mid (y,\sigma)\in C(\overline Q)\times\overline V_{E,ad}\text{ satisfying }\ (5.1)\text{ and }(1.2)\right\},$$

where $\overline V_{E,ad}$ is the weak* closure of $i_E(V_{ad})$. The set $\overline V_{E,ad}$ is convex and locally compact.

Proposition 9.1. *Let E be a separable normed subspace of $Ca^p(\Sigma)$. Suppose that E is $C(\overline{\Sigma})$-invariant, and that Assumptions* **A1**–**A8** *are fulfilled. Suppose in addition that:*
A9- *The functions $(\Psi \circ y)\cdot\chi$, $(\Psi'_y \circ y)\cdot\chi$, $(G\circ y)\cdot\chi$, and $(G'_y\circ y)\cdot\chi$ belong to E, for all $(y,\chi)\in C(\overline{\Sigma})\times C(\overline{\Sigma})$. Then, the statements of Theorems* 6.4, 7.4, *and* 8.4 *are still valid.*

An interesting property of elements of Y_E^p is their possible nonconcentration. More precisely, we say that $\sigma \in Y_E^p$ is *p-nonconcentrating* if it is attainable by a sequence $(v_k)_k$ (*i.e.* $\sigma = w^*\text{-}\lim_{k\to\infty} i_E(v_k)$) such that the set $\{|v_k|^p;\ k\in\mathbb{N}\}$ is relatively weakly compact in $L^1(\Sigma)$. From Ball's theorem [4], it follows that every p-nonconcentrating measure σ admits an L^p-Young measure representation, in the sense that there exists a weakly measurable mapping $(s,t)\longrightarrow \pi^{(s,t)}$ from Σ to the set of all probability Radon measures on $\mathbb{R}$ such that $(s,t)\longrightarrow \int_{\mathbb{R}}|w|^p\, d\pi^{(s,t)}(w)\in L^1(\Sigma)$, and

$$\langle \sigma, h\rangle_{*,\Sigma} = \int_\Sigma\int_{\mathbb{R}} h(s,t,w)d\,\pi^{(s,t)}(w)\,ds\,dt \qquad \text{for all } h\in E.$$

A measure $\sigma_o\in Y_E^p$ is the *p-nonconcentrating modification* of $\sigma\in Y_E^p$, if σ_o is p-nonconcentrating and

$$\langle \sigma_o, h\rangle_{*,\Sigma} = \langle \sigma, h\rangle_{*,\Sigma} \ \text{ for all } h\in E \ \text{ s.t. } |h(s,t,w)|\le a(s,t)+o(|w|^p),$$

with $a\in L^1(\Sigma)$ and $o:\mathbb{R}^+\longrightarrow\mathbb{R}$ satisfies $\lim\limits_{w\to+\infty}\dfrac{o(w)}{w}=0$. In [19], Roubíček proved that if E is separable, then every $\sigma\in Y_E^p$ admits one p-nonconcentrating modification ([19], Prop. 3.4.18). Moreover,

$$\langle \sigma-\sigma_o, h\rangle_{*,\Sigma} > 0 \quad \forall\, h\in E \ \text{ s.t. } h(s,t,w)\ge a_o(s,t)+b|w|^p, \tag{9.1}$$

where $a_o\in L^1(\Sigma)$ and $b>0$ ([19], Lem. 4.2.3 (ii)). In the following result, we prove that solutions to (RP_E) are p-nonconcentrating.

Theorem 9.2. *Let E be a separable normed subspace of $Ca^p(\Sigma)$. Suppose that E is $C(\overline{\Sigma})$-invariant, and that Assumptions* **A1**–**A9** *are fulfilled. Then every solution of (RP_E) is p-nonconcentrating.*

Proof. Let $(\bar y,\bar\sigma)$ be an optimal solution of (RP_E). (Existence of such a pair follows from Prop. 9.1.) Since E is separable, $\bar\sigma$ admits a unique p-nonconcentrating modification σ_o which belongs to $\overline{V}_{E,ad}$. Let us argue by contradiction and suppose that $\bar\sigma$ is not p-nonconcentrating. Then $\bar\sigma\neq\sigma_o$. By the definition of σ_o, and due to **A2**, it follows that

$$\int_\Sigma (\Psi\circ\bar y)\bullet\bar\sigma\ \chi\,ds\,dt = \int_\Sigma(\Psi\circ\bar y)\bullet\sigma_o\ \chi\,ds\,dt \qquad \text{for all } \chi\in C(\overline{\Sigma}).$$

Thus $\bar y\equiv y_{\sigma_o}$, and $g(\bar y)=g(y_{\sigma_o})$. (In other words, (y_{σ_o},σ_o) is admissible for (RP_E).) On the other hand, due to (9.1), with the coercivity condition **A8**, we can easily prove that

$$\widetilde{J}(y_{\sigma_o},\sigma_o) < \widetilde{J}(\bar y,\bar\sigma). \tag{9.2}$$

Since (y_{σ_o},σ_o) is admissible for (RP_E), (9.2) contradicts the optimality of $(\bar y,\bar\sigma)$. The proof is complete. $\square$

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