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THE WAVE EQUATION WITH OSCILLATING DENSITY: OBSERVABILITY AT LOW FREQUENCY

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Abstract. We prove an observability estimate for a wave equation with rapidly oscillating density, in a bounded domain with Dirichlet boundary condition.

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0. Introduction and results

Let Ω be a smooth bounded domain in $\mathbb{R}^d$, and $\rho(x,y)$ a smooth function on $\mathbb{R}^d \times \mathbb{R}^d$, such that

$$0 < \rho_{\min} \leq \rho(x,y) \leq \rho_{\max} \quad \forall\,(x,y) \tag{0.1}$$

ρ is 2π-periodic with respect to the second variable, *i.e.*

$$\rho(x,y) = \rho(x, y+2\pi\ell) \quad \forall\, \ell \in \mathbb{Z}^d\,. \tag{0.2}$$

For $\varepsilon > 0$, let $(\omega_n^\varepsilon, e_n^\varepsilon(x))$ be the spectrum of the Dirichlet problem for the operator $-\rho^{-1}(x,x/\varepsilon)\Delta_g$ on $L^2\left(\Omega; \rho(x,x/\varepsilon) d_g x\right)$ normalized in the form

$$\begin{cases} \rho(x,x/\varepsilon)(\omega_n^\varepsilon)^2\, e_n^\varepsilon(x) = -\Delta_g e_n^\varepsilon(x) & \text{in } \Omega \\ e_n^\varepsilon(x) = 0 & \text{on } \partial\Omega \\ \int_\Omega e_n^\varepsilon(x)\overline{e_m^\varepsilon(x)}\rho(x,x/\varepsilon) d_g x = \delta_{n,m}; \qquad 0 < \omega_1^\varepsilon \leq \omega_2^\varepsilon \leq \dots & \end{cases} \tag{0.3}$$

Here, Δ_g denotes the Laplace operator for some fixed smooth metric g on $\overline{\Omega}$, and $d_g x$ is the volume form associated to g.

For any given $\gamma_0 > 0$, we shall denote by $J^\varepsilon_{\gamma_0}$ the space of solutions $u^\varepsilon(t,x)$ of the wave equation with oscillating density ρ

$$\begin{cases} \left(\rho(x,x/\varepsilon)\partial_t^2 - \Delta_g\right) u^\varepsilon(t,x) = 0 \quad \text{in } \mathbb{R}_+ \times \Omega \\ u^\varepsilon(t,x)_{|x\in\partial\Omega} = 0 \end{cases} \tag{0.4}$$

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with maximum frequency less than γ_0/ε.

In other words, $J^\varepsilon_{\gamma_0}$ is the set

$$J^\varepsilon_{\gamma_0} = \left\{ u^\varepsilon(t,x) = \sum_{\varepsilon\omega^\varepsilon_n \leq \gamma_0} \left(u_{+,n} e^{it\omega^\varepsilon_n} + u_{-,n} e^{-it\omega^\varepsilon_n} \right) e^\varepsilon_n(x) \right\} . \quad (0.5)$$

Let $\{u^{\varepsilon_k}_k\}$ be a bounded sequence (in $L^2_{\text{loc}}(\mathbb{R}_t, L^2(\Omega))$, of solutions of (0.4), with $\lim \varepsilon_k = 0$. It is well known that any weak limit of this sequence will satisfy the homogenized wave equation in Ω

$$\begin{cases} (\underline{\rho}(x)\partial_t^2 - \Delta_g)u(t,x) = 0 \quad \text{in } \mathbb{R} \times \Omega \\ u(t,x)_{|x\in\partial\Omega} = 0 \end{cases} \quad (0.6)$$

where $\underline{\rho}(x) = \oint \rho(x,y)dy$ is the mean value of ρ.

Let V be an open subset of Ω, and $T_0 > 0$.

One says that waves solution of (0.6) are observable from V in time T_0 if there exists a constant C_0 s.t for any L^2-solution of (0.6) one has

$$\int_0^{T_0} \int_\Omega |u(t,x)|^2 \underline{\rho}(x) dt d_g x \leq C_0 \int_0^{T_0} \int_V |u(t,x)|^2 \underline{\rho}(x) dt d_g x \,. \quad (0.7)$$

If $u = \sum_{\pm,n} u_{\pm,n} e^{\pm it\omega_n} e_n(x)$ is the Fourier series of u in the spectral decomposition of $(-\underline{\rho})^{-1}(x)\Delta_g$, we deduce from the elementary fact

$$\forall T > 0, \forall \omega_0 > 0, \exists C > 0 \text{ such that } \forall \omega \geq \omega_0, |\ c_+\ |^2 + |\ c_-\ |^2 \leq C \int_0^T |\ c_+ e^{it\omega} + c_- e^{-it\omega}\ |^2\ dt$$

that the condition (0.7) is equivalent to the following

$$\begin{cases} \exists C_0 \text{ s.t. } \ \forall (u_{+,n}, u_{-,n})_n \in \ell^2 \times \ell^2 \\ \sum_n |u_{+,n}|^2 + |u_{-,n}|^2 \leq C_0 \int_0^{T_0} \int_V |u(t,x)|^2 \underline{\rho}(x) dt d_g x. \end{cases} \quad (0.8)$$

It is proved in [4] that (0.7) holds true under the geometric-control hypothesis

$$\begin{cases} 1) & \text{there is no infinite order of contact between the boundary} \\ & \partial\Omega \text{ and the bicharacteristics of } \ \underline{\rho}(x)\partial_t^2 - \Delta_g \\ 2) & \text{any generalized bicharacteristic of } \underline{\rho}(x)\partial_t^2 - \Delta_g \\ & \text{parameterized by } \ t \in]0, T_0[\text{ meets } V. \end{cases} \quad (0.9)$$

Here the generalized bicharacteristic flow is the one defined by Melrose and Sjöstrand in [11].

The main result of this paper is the following theorem, which asserts that the estimate (0.7) remains true under the hypothesis (0.9) for $\underline{\rho}(x)$, for solutions of (0.4) in $J^\varepsilon_{\gamma_0}$, if γ_0 is small enough.

Theorem 0.1. *Let the hypothesis (0.9) be satisfied. There exist small positive constants γ_0, ε_0 and a constant C_0, such that for any $\varepsilon \in]0, \varepsilon_0[$ and any $u^\varepsilon \in J^\varepsilon_{\gamma_0}$*

$$\int_0^{T_0} \int_\Omega |u^\varepsilon(t,x)|^2 \rho(x, x/\varepsilon) dt d_g x \leq C_0 \int_0^{T_0} \int_V |u^\varepsilon(t,x)|^2 \rho(x, x/\varepsilon) dt d_g x \,. \quad (0.10)$$

This is clearly a stability result of the observability estimate (0.7) under the singular perturbation $\underline{\rho}(x) \to \rho(x, x/\varepsilon)$. Let us recall that Theorem 0.1 has been proved in the 1-d case by Castro and Zuazua [6], and that in the 1-d case, the counter-example of Avellaneda *et al.* [1] shows that (0.10) fails for γ_0 large. Indeed, in the 1-d case, when $\rho = \rho(x/\varepsilon)$, Castro [5] has shown that the greatest value of γ_0 such that (0.10) holds true for some T_0 (when $V \Subset [a,b] = \Omega$) is related with the first instability interval of the Hill equation on the line $\left(\frac{d}{dy}\right)^2 + \omega^2\rho(y)$. In the multi-d case, the understanding of the best value of γ_0 such that (0.10) holds true will clearly involve the understanding of the localization and propagation of Bloch waves for the boundary value problem (0.4): this highly difficult problem is out of the scope of the present paper.

The conserved energy for solutions of (0.4) is

$$E(u^\varepsilon) = \frac{1}{2}\int_\Omega \left\{|\partial_t u^\varepsilon|^2 \rho(x, x/\varepsilon) + |\nabla_g u^\varepsilon|^2\right\} d_g x\,. \tag{0.11}$$

Applying the estimate (0.10) to $\partial_t u^\varepsilon$, one easily gets the energy observability estimate

Corollary 0.1. *Under the hypothesis and with the notations of Theorem 0.1, there exists a constant C_0 s.t. for any $\varepsilon \in]0, \varepsilon_0[$ and any $u^\varepsilon \in J^\varepsilon_{\gamma_0}$ one has*

$$E(u^\varepsilon) \le C_0 \int_0^{T_0}\int_V |\partial_t u^\varepsilon|^2 \rho(x, x/\varepsilon) dt d_g x\,. \tag{0.12}$$

The paper is organized as follows:

1. reduction to a semi-classical estimate;
2. the Bloch wave;
3. Lopatinski estimate;
4. propagation estimate;
5. Appendix A: semi-classical o.p.d with operators values;
6. Appendix B: proofs of Lemmas 3.4–3.6.

1. In the first part, using a Littlewood-Paley decomposition, we reduce the proof of the inequality (0.10) to the assertion

$$\left\{\begin{array}{l} \text{there exist } \gamma_0, \varepsilon_0, h_0, C_0 \text{ such that for any } \varepsilon \in]0, \varepsilon_0[, \text{ and} \\ h \in [\varepsilon/\gamma_0, h_0] \text{ the inequality (0.10) holds true for any } u^\varepsilon \in I^\varepsilon_h, \\ \text{where} \quad I^\varepsilon_h = \left\{ u^\varepsilon = \displaystyle\sum_{0.9 \le \omega^\varepsilon_n h \le 2.1} (u_{+,n}\, e^{it\omega^\varepsilon_n} + u_{-,n}\, e^{-it\omega^\varepsilon_n}) e^\varepsilon_n(x) \right\}. \end{array}\right. \tag{0.13}$$

2. In the second part, we introduce the Bloch wave at the boundary $\Gamma(u^\varepsilon)$. We refer to [2] and [7] for the study of Bloch waves in equations with oscillating coefficients. We choose a coordinate system

$$\left\{\begin{array}{l} \partial\Omega \times [0, r_0] \xrightarrow{\Theta} \mathbb{R}^d \\ (x', x_d) \mapsto \Theta(x', x_d) \end{array}\right. \tag{0.14}$$

which satisfies

$$\left\{\begin{array}{l} i)\ \Theta(\partial\Omega \times [0, r_0]) \subset \overline{\Omega} \\ ii) \text{ for } x_d \text{ small }, x_d \mapsto \Theta(x', x_d) \text{ is the geodesic normal to the} \\ \text{boundary at } x' \in \partial\Omega\ , \text{ for the metric } g \text{ on } \overline{\Omega}. \end{array}\right. \tag{0.15}$$

In these coordinates, the Laplace operator takes the form

$$\begin{cases} \Delta_g = \frac{\partial}{\partial x_d}\left(A_0(x)\frac{\partial}{\partial x_d} + A_1(x,\partial_{x'})\right) + A_2(x,\partial_{x'}); \\ \qquad x = (x', x_d)\,,\ x' \in \partial\Omega \end{cases} \tag{0.16}$$

where $A_j(x,\partial_{x'})$ are differential operators of order j on $\partial\Omega$, with x_d as parameter. Let $a_j(x,\xi')$ be the principal symbol of A_j. The dual metric $g^{-1}(x,\xi) \stackrel{\text{def}}{=} \|\xi\|_x^2$ on the cotangent bundle $T^*\Omega$ is

$$\|\xi\|_x^2 = a_0(x)\xi_d^2 + a_1(x,\xi')\xi_d + a_2(x,\xi')\,. \tag{0.17}$$

Let $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ be the d-dimensional torus and for $\varepsilon > 0$, $S_\varepsilon \subset \partial\Omega \times [0,r_0] \times \mathbb{T}_y^d$ the submanifold

$$S_\varepsilon = \left\{(x,y)\,;\, y = \Theta(x)/\varepsilon \bmod (2\pi\mathbb{Z})^d\right\}\cdot \tag{0.18}$$

Let $f(x)$ be a function on $\partial\Omega \times [0,r_0]$. We define a distribution $T(f)$ on $\partial\Omega \times [0,r_0] \times \mathbb{T}_y^d$ by the formula

$$T(f) = \sum_{\ell\in\mathbb{Z}^d} e^{i\ell(y-\Theta(x)/\varepsilon)} f(x) = (2\pi)^d \delta_{y=\Theta(x)/\varepsilon} \otimes f(x). \tag{0.19}$$

If X is a vector field on $\partial\Omega \times [0,r_0]$, we shall denote by X_ε^* the lift of X on S_ε. If $x' = (x_1,\dots,x_{d-1})$ is a local coordinate system on $\partial\Omega$, and $(\Theta_1(x),\dots,\Theta_d(x)) = \Theta(x)$ are the Cartesian coordinates of Θ, one has

$$\left(\frac{\partial}{\partial x_k}\right)_\varepsilon^* = \frac{\partial}{\partial x_k} + \frac{1}{\varepsilon}\sum_{j=1}^d \frac{\partial\Theta_j}{\partial x_k}(x)\frac{\partial}{\partial y_j} \quad \text{for } 1 \le k \le d \tag{0.20}$$

and

$$\left(\frac{\partial}{\partial x_k}\right)_\varepsilon^* T(f) = T\left(\frac{\partial}{\partial x_k} f\right) \quad \text{for } 1 \le k \le d\,. \tag{0.21}$$

The Bloch operator on $\partial\Omega \times [0,r_0] \times \mathbb{T}^d$ is defined by

$$\begin{cases} \mathbb{B}_\varepsilon(x,\varepsilon\partial_x,\varepsilon\partial_t;y,\partial_y) = \hat\rho(x,y)(\varepsilon\partial_t)^2 - \varepsilon^2(\Delta_g)_\varepsilon^*\,;\ \hat\rho(x,y) = \rho(\Theta(x),y) \\ (\Delta_g)_\varepsilon^* = \left(\frac{\partial}{\partial x_d}\right)_\varepsilon^*\left(A_0(x)\left(\frac{\partial}{\partial x_d}\right)_\varepsilon^* + A_1(x,(\partial_{x'}))_\varepsilon^*\right) + A_2\left(x,(\partial_{x'})_\varepsilon^*\right). \end{cases} \tag{0.22}$$

It satisfies the identity

$$\mathbb{B}_\varepsilon\left(T(u(x,t))\right) = \varepsilon^2 T(\left(\rho(\Theta(x),\Theta(x)/\varepsilon)\partial_t^2 - \Delta_g\right)(u(x,t))). \tag{0.23}$$

Let $\widetilde{A}_j$ be the operators

$$\widetilde{A}_j = A_j(x,(\partial_{x'})_\varepsilon^*) \tag{0.24}$$

and let $e_k(x)$ $1 \le k \le d$ be the vectors of $\mathbb{R}^d$

$$e_k(x) = \frac{\partial\Theta}{\partial x_k}(x). \tag{0.25}$$

If $v(t,x,y)$ is a distribution on $\overset{\circ}{X} \times \mathbb{T}^d$, with $X = \mathbb{R}_t \times (\partial\Omega \times [0, r_0])$, we shall write the equation $\mathbb{B}_\varepsilon(v) = 0$ as a 2×2 system for the vector $w = \mathcal{A}(v)$.

$$\mathcal{A}(v) = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} v \\ (A_0(x)(\varepsilon \frac{\partial}{\partial x_d})^*_\varepsilon + \varepsilon \widetilde{A}_1)v \end{bmatrix}. \tag{0.26}$$

This system takes the form

$$\begin{cases} \varepsilon \frac{\partial}{\partial x_d} w + \mathbb{M} w = 0 \\ \mathbb{M} = \begin{bmatrix} e_d(x) \cdot \partial_y + \varepsilon A_0^{-1}(x) \widetilde{A}_1 & -A_0^{-1}(x) \\ \varepsilon^2 \widetilde{A}_2 - \hat{\rho}(x,y)(\varepsilon \partial_t)^2 & e_d(x) \cdot \partial_y \end{bmatrix}. \end{cases} \tag{0.27}$$

The operator $\mathbb{M}$ will be seen as a semi-classical operator in $t, x, \frac{\varepsilon}{i}\partial_{x'} = \xi'$, $\frac{\varepsilon}{i}\partial_t = \tau$ with operator values in the fiber $\mathbb{T}^d$

$$\mathbb{M} = \sum_{j=0}^{2} \left(\frac{\varepsilon}{i}\right)^j \mathbb{M}^j(x, \xi', \tau; y, \partial_y). \tag{0.28}$$

The differential degree in y of $\mathbb{M}^j$ is at most $2 - j$ and the principal symbol $\mathbb{M}^0$ is the matrix

$$\mathbb{M}^0(x, \xi', \tau; y, \partial_y) = \begin{bmatrix} e_d(x) \cdot \partial_y + \ a_0^{-1}(x) a_1(x, i\xi' + e'(x) \cdot \partial_y) & -a_0^{-1}(x) \\ a_2(x, i\xi' + e'(x) \cdot \partial_y) + \hat{\rho}(x,y)\tau^2 & e_d(x) \cdot \partial_y \end{bmatrix}. \tag{0.29}$$

Let $E^\bullet = \{E^s,\ s \in \mathbb{R}\}$ be the scale of Hilbert spaces on the torus

$$E^s = H^s(\mathbb{T}^d) \oplus H^{s-1}(\mathbb{T}^d). \tag{0.30}$$

For any $\rho = (x, \xi', \tau)$, $\mathbb{M}^j(\rho, y, \partial_y)$ maps E^s into E^{s-1+j} and $\mathbb{M}^0$ is an elliptic operator. Let $\mathbb{M}^0_0$ be the restriction of $\mathbb{M}^0$ to the zero section $\xi' = \tau = 0$.

$$\mathbb{M}^0_0(x, \partial_y) = \mathbb{M}^0(x, 0, 0, y, \partial_y) = \begin{bmatrix} e_d(x) \cdot \partial_y + \ a_0^{-1} a_1(x, e'(x) \cdot \partial_y) & -a_0^{-1}(x) \\ a_2(x, e'(x) \cdot \partial_y) & e_d(x) \cdot \partial_y \end{bmatrix}. \tag{0.31}$$

The eigenvalues $\lambda^0_{\pm,\ell}(x)$ of $\frac{1}{i}\mathbb{M}^0_0(x, \partial_y)$ on the space $e^{i\ell y}\mathbb{C}^2$, for $\ell \in \mathbb{Z}^d$ are the complex roots of the equation

$$a_0(x)(-\lambda + e_d.\ell)^2 + (-\lambda + e_d.\ell) a_1(x, e'.\ell) + a_2(x, e'.\ell) = 0 \tag{0.32}$$

which is equivalent to

$$\|{}^t d\Theta(x)(\ell) - \lambda(0, \cdots, 0, 1)\|^2_x = 0. \tag{0.33}$$

In particular we have

$$\inf_x \min_{\ell \neq 0} |\lambda^0_{\pm,\ell}(x)| > 0 \tag{0.34}$$

so the double eigenvalue $\lambda^0_{\pm,0}(x) = 0$ is isolated in the spectrum of $\mathbb{M}^0_0(x, \partial_y)$.

In the sequel, we shall restrict the values of the Sobolev index of regularity s on the torus to some fixed large interval, $s \in [-\sigma_0, \sigma_0]$, $\sigma_0 \gg \frac{d}{2}$.

Let $X = \partial\Omega \times \mathbb{R}_t \times [0, r_0]$. We denote by ${}^tT^*X$ the tangential cotangent bundle

$$ {}^tT^*X = T^*(\partial\Omega \times \mathbb{R}_t) \times [0, r_0]. \tag{0.35} $$

Let $W_1 \Subset W_0$ be two small neighborhoods of the set $\{\xi' = \tau = 0\} \times \{t \in [-T_0, 2T_0]\}$ in ${}^tT^*X$.

We choose a non-negative function $\chi_0 \in C_0^\infty(W_0)$, such that $\chi_0 \equiv 1$ on W_1.

If W_0 is small enough, we define the map $p_0(x, t, \xi', \tau) : E^\bullet \to \mathbb{C}^2$ by the formula

$$ p_0[w] = \chi_0 . \oint_{\mathbb{T}^d} \left\{ \frac{1}{2i\pi} \int_{\partial D} \frac{dz}{z - \mathbb{M}^0} \right\} [w] \quad w \in E^s \ , \ s \in [-\sigma_0, \sigma_0] \tag{0.36} $$

(where $D \subset \mathbb{C}$ is a small disk with center $z = 0$).

It satisfies the estimates

$$ \exists\, C\, \forall\, s \in [-\sigma_0, \sigma_0] \ \forall\, w \in E^s \quad \| p_0(w) - \chi_0 \oint_{\mathbb{T}^d} w \|_{\mathbb{C}^2} \le C\tau^2 \|w\|_{E^s} \tag{0.37} $$

and there exists $L^0(x, t, \xi', \tau) \in C^\infty({}^tT^*X; M_2(\mathbb{C}))$, defined near $\xi' = \tau = 0$ such that (see (2.29–2.31))

$$ p_0 \circ \mathbb{M}^0 = L^0 \circ p_0 \, . \tag{0.38} $$

By a Taylor expansion near $\xi' = \tau = 0$, one gets

$$ L^0 = \begin{bmatrix} a_0^{-1}(x) a_1(x, i\xi') & -a_0^{-1}(x) \\ a_2(x, i\xi') + \underline{\hat{\rho}}(x)\tau^2 & 0 \end{bmatrix} + O\left(\tau^4\right). \tag{0.39} $$

We then suitably quantize the above construction and we obtain tangential pseudo differential operators (see Append. A1)

$$ \begin{cases} \Pi_0(\varepsilon, t, x, \varepsilon\partial_t, \varepsilon\partial_{x'}) & : L^2(X; E^s) \to L^2(X, \mathbb{C}^2) \, , \, s \in [-\sigma_0, \sigma_0] \\ L(\varepsilon, t, x, \varepsilon\partial_t, \varepsilon\partial_{x'}) & : L^2(X; \mathbb{C}^2) \to L^2(X, \mathbb{C}^2) \end{cases} \tag{0.40} $$

with principal symbol $\sigma(\Pi_0) = p_0$, $\sigma(L) = L^0$, which satisfy the relation

$$ \Pi_0(\varepsilon\partial_{x_d} + \mathbb{M}) = (\varepsilon\partial_{x_d} + L)\Pi_0 + R(\varepsilon, t, x, \varepsilon\partial_t, \varepsilon\partial_{x'}). \tag{0.41} $$

In (0.41), the error term $R : L^2(X; E^s) \to L^2(X, \mathbb{C}^2)$ will be a tangential pseudo differential operator such that for any tangential o.p.d. Q with essential support in W_1 and any $s \in [-\sigma_0, \sigma_0]$, one has

$$ \| Q \circ R;\, L^2(X; E^s) \to L^2(X, \mathbb{C}^2) \| \in .O(\varepsilon^\infty). \tag{0.42} $$

Definition 0.1. For $u^\varepsilon \in I_h^\varepsilon$, we define the Bloch wave $\Gamma(u^\varepsilon) \in L^2(X; \mathbb{C}^2)$ by the formula

$$ \Gamma(u^\varepsilon) = \begin{bmatrix} \Gamma_0(u^\varepsilon) \\ \Gamma_1(u^\varepsilon) \end{bmatrix} = \Pi_0 \mathcal{T}(u^\varepsilon) \quad (\mathcal{T} = \mathcal{A} \circ T). \tag{0.43} $$

Let $\gamma_0, \varepsilon_0, h_0$ be given small enough, $\varepsilon \in]0, \varepsilon_0]$, $h \in [\varepsilon/\gamma_0, h_0]$. For $u_\varepsilon \in I_h^\varepsilon$, $u^\varepsilon = \sum_{0.9 \le \omega_n^\varepsilon h \le 2.1} (u_{+,n} e^{it\omega_n^\varepsilon} + u_{-,n} e^{-it\omega_n^\varepsilon}) e_n^\varepsilon(x)$, we define $\|u^\varepsilon\|^2 \left(\simeq \int_0^{T_0} \int_\Omega |u^\varepsilon|^2 \right)$ by

$$ \|u^\varepsilon\|^2 = \sum_{0.9 \le \omega_n^\varepsilon h \le 2.1} |u_{+,n}|^2 + |u_{-,n}|^2. \tag{0.44} $$

Let $X_{T_0} = \partial\Omega \times [-T_0, 2T_0] \times [0, r_0]$, and let K be the compact subset of ${}^tT^*X$, $K = \partial\Omega \times [0, T_0] \times [0, r_0/2] \times\{\xi' = 0, \tau = 0\}$. The following proposition will be proven in Section 2.

Proposition 0.1. *Let $Q(\varepsilon, t, x, \varepsilon\partial_{x'}, \varepsilon\partial_t)$ be a zero order tangential opd on X, equal to Id near K. If $\gamma_0, \varepsilon_0, h_0$ are small enough, there exists a constant $C > 0$, such that for any $\varepsilon \in]0, \varepsilon_0]$, $h \in [\varepsilon/\gamma_0, h_0]$, one has*

$$\|u^\varepsilon\|^2 \le C\left[\|Q\Gamma_0(u^\varepsilon)\|^2_{L^2(X_{T_0})} + \|u^\varepsilon\|^2_{L^2((0,T_0)\times V)}\right] \ \forall\, u^\varepsilon \in I^\varepsilon_h. \tag{0.45}$$

3. By Proposition 0.1, we shall obtain the inequality (0.10), if we are able to estimate the L^2 norm of the first component $\Gamma_0(u^\varepsilon)$ of the Bloch wave near the set K.

The formula (0.41) shows that $\Gamma(u^\varepsilon)$ satisfies the equation

$$(\varepsilon\partial_{x_d} + L)\Gamma(u^\varepsilon) \in O(\varepsilon^\infty L^2) \text{ (microlocally in } W_1). \tag{0.46}$$

By (0.39) this equation is very closed to the homogenized equation $(\underline{\rho}(x)\partial_t^2 - \Delta_g)[\Gamma_0(u^\varepsilon)] = 0$.

As one can see, all the difficulty in our problem is thus to obtain an estimate on the first Dirichlet data of $\Gamma(u^\varepsilon)$ on the boundary $x_d = 0$, in order to apply propagation arguments to the equation (0.46). We shall prove the following proposition.

Proposition 0.2. *If $\gamma_0, \varepsilon_0, h_0$ are small enough, there exists a constant C such that for any $\varepsilon \in]0, \varepsilon_0[$, $h \in [\varepsilon/\gamma_0, h_0]$ the following estimate holds true*

$$\|\Gamma_0(u^\varepsilon)_{|x_d=0}\|_{L^2(X_{T_0}\cap x_d=0)} \le C\,\varepsilon/h\|u^\varepsilon\| \quad \forall\, u^\varepsilon \in I^\varepsilon_h\,. \tag{0.47}$$

The above estimate will be obtained as a consequence of a uniform Lopatinski estimate on $w^\varepsilon = \mathcal{T}(u^\varepsilon) = \begin{bmatrix} w_0^\varepsilon \\ w_1^\varepsilon \end{bmatrix}$. We shall prove

Theorem 0.2. *Let Q be a scalar tangential o.p.d. with essential support in W_0; if $W_0, \gamma_0, \varepsilon_0, h_0$ are small enough, there exist $s_1 < 0$ and a constant C such that for any $u^\varepsilon \in I^\varepsilon_h$ the following estimate holds true*

$$\|Q(t, x, \varepsilon\partial_{x'}, \varepsilon\partial_t)(w_1^\varepsilon)_{|x_d=0}\|_{L^2(X_{T_0}\cap x_d=0, H^{s_1}(\mathbb{T}^d))} \le C\|u^\varepsilon\|. \tag{0.48}$$

Notice that w^ε satisfies the equation (0.27), with Dirichlet data $w^\varepsilon_{0|x_d=0} = 0$ on the boundary.

The weaker estimate

$$\|Q(w_1^\varepsilon)_{|x_d=0}\| \le C\,\varepsilon^{-1/2}\|u^\varepsilon\| \tag{0.49}$$

is easy to obtain (it is sufficient to commute the Eq. (0.4) with the normal vector field $\frac{\partial}{\partial n}$).

The proof of (0.48) is the most technical part of our work. It involves a detailed study of how the spectral theory of $\mathbb{M}^0(x, \xi', \tau; y, \partial y)$ (see (0.29)) depends on the parameter (x, ξ', τ).

4. This part will be devoted to the proof of the following proposition.

Proposition 0.3. *Let $Q(\varepsilon, t, x, \varepsilon\partial_{x'}, \varepsilon\partial_t)$ be a zero order opd equal to Id near K, with essential support in W_1. There exist $\gamma_0, \varepsilon_0, h_0$, and a constant C_0 such that, for any $\varepsilon \in]0, \varepsilon_0]$, $h \in [\varepsilon/\gamma_0, h_0]$ and $u^\varepsilon \in I^\varepsilon_h$, the following estimate holds true*

$$\|Q\Gamma_0(u^\varepsilon)\|^2_{L^2(X_{T_0})} \leq C_0\left[\|\Gamma_0(u^\varepsilon)_{|x_d=0}\|^2_{L^2(X_{T_0}\cap x_d=0)} + \|u^\varepsilon\|^2_{L^2(0,T_0)\times V}\right]. \tag{0.50}$$

This estimate will be obtained by rather classical arguments in the theory of control of linear waves, for the rescale equation

$$\begin{cases} \left(h\frac{\partial}{\partial x_d} + \mathcal{L}\right)\begin{bmatrix} g_0 \\ g_1 \end{bmatrix} \sim 0 & \begin{bmatrix} g_0 \\ g_1 \end{bmatrix} = \begin{bmatrix} \Gamma_0(u^\varepsilon) \\ \frac{h}{\varepsilon}\Gamma_1(u^\varepsilon) \end{bmatrix} \\ \mathcal{L} = \frac{h}{\varepsilon}\begin{pmatrix} 1 & 0 \\ 0 & h/\varepsilon \end{pmatrix} \circ L \circ \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon/h \end{pmatrix}. \end{cases} \tag{0.51}$$

We shall verify that $\mathcal{L}$ is still a h-pseudo differential operator, with ε/h as parameter. (We use this rescaling in order to be able to use propagation arguments in the range $\varepsilon \ll h$.)

5. In Appendix A.1, we recall the properties of the semi-classical calculus with operators values which is used in **2.** In Appendix A.2, we extend this calculus to a larger class of symbols; this exotic calculus will be used in **3.**

To end this introduction, we finally remark that the validity of (0.13), hence the proof of Theorem 0.1, is a direct consequence of the Propositions 0.1, 0.2 and 0.3.

1. Semi-classical reduction

In this part, we verify that (0.13) implies the Theorem 1.

Let $e^\varepsilon_n(x)$ be a normalized eigenfunction of the Dirichlet problem (0.3), and let μ_1 be the first eigenvalue of the Dirichlet problem for Δ_g in Ω . One has

$$\int_\Omega |\nabla_g e^\varepsilon_n|^2 d_g x = \int_\Omega \rho(x, x/\varepsilon)(\omega^\varepsilon_n)^2 |e^\varepsilon_n(x)|^2 d_g x \leq \rho_{\max}(\omega^\varepsilon_n)^2 \int_\Omega |e^\varepsilon_n(x)|^2 d_g x. \tag{1.1}$$

So we get the uniform lower bound

$$\omega^\varepsilon_n \geq (\rho_{\max})^{-1/2}\mu_1^{1/2}\,. \tag{1.2}$$

The Sobolev spaces $L^2(\Omega), H^1_0(\Omega), H^{-1}(\Omega)$, with norms $(\int_\Omega |f|^2\rho d_g x)^{1/2}$, $(\int_\Omega |\nabla_g f|^2 d_g x)^{1/2}$, $\sup\{\int_\Omega f\bar{h}\,\rho d_g x$, $\|h\|_{H^1_0} \leq 1\}$ are characterized in terms of Fourier series by

$$\begin{cases} f^\varepsilon_n = \int_\Omega f\,\overline{e^\varepsilon_n(x)}\rho d_g x \quad \text{for } f \in H^{-1}(\Omega) \\ \|f\|^2_{L^2} = \sum\limits_n |f^\varepsilon_n|^2\,;\ \|f\|^2_{H^1_0} = \sum\limits_n (\omega^\varepsilon_n)^2 |f^\varepsilon_n|^2\,;\ \|f\|^2_{H^{-1}} = \sum\limits_n (\omega^\varepsilon_n)^{-2}|f^\varepsilon_n|^2\,. \end{cases} \tag{1.3}$$

Any solution u^ε of the wave equation (0.4) with data $(u^\varepsilon(0), \partial_t u^\varepsilon(0)) \in L^2(\Omega) \oplus H^{-1}(\Omega)$ is of the form

$$u^\varepsilon = \sum_n u^\varepsilon_n(t)e^\varepsilon_n(x) = \sum_n (u^\varepsilon_{+,n}e^{it\omega^\varepsilon_n} + u^\varepsilon_{-,n}\,e^{-it\omega^\varepsilon_n})e^\varepsilon_n(x) \tag{1.4}$$

with $(u^\varepsilon_{\pm,n})_n \in \ell^2$, and (1.2) implies that there exists a constant C independent of ε, s.t.

$$\frac{1}{C}\sum_{n,\pm} |u^\varepsilon_{\pm,n}|^2 \leq \int_0^{T_0}\int_\Omega |u^\varepsilon|^2\rho\,dtd_g x \leq C\sum_{n,\pm}|u^\varepsilon_{\pm,n}|^2. \tag{1.5}$$

If the geometric hypothesis (0.9) holds true for T_0, it remains valid for $T_0 - 2\delta$, for $\delta > 0$ small enough; we can therefore assume that (0.13) is valid on $[\delta, T_0 - \delta]$.

Take $\varphi(t) \in C_0^\infty(]0, T_0[)$, $\varphi(t) \equiv 1$ on $[\delta, T_0 - \delta]$ and $\psi(\sigma) \in C_0^\infty(]0.9, 2.1[)$, $\psi(\sigma) \equiv 1$ on $[1, 2]$. Let $\chi(\sigma) = \psi(\sigma) + \psi(-\sigma)$. For $u^\varepsilon \in J^\varepsilon_{\gamma_0}$, one has $\chi(2^{-k}D_t)u^\varepsilon \in I^\varepsilon_{2^{-k}}$, so there exists C_0 s.t.

$$\begin{cases} \forall\, \varepsilon \in]0, \varepsilon_0]\,,\ \forall\, k \in \mathbb{N} \text{ s.t. } 2^{-k} \in [\varepsilon/\gamma_0, h_0] \\ \forall\, u^\varepsilon = \sum\limits_{\varepsilon\omega_n^\varepsilon \leq \gamma_0} \left(u^\varepsilon_{+,n} e^{it\omega_n} + u^\varepsilon_{-,n}\, e^{-it\omega_n}\right) e^\varepsilon_n(x) \in J^\varepsilon_{\gamma_0} \\ \sum\limits_{2^k \leq \omega_n^\varepsilon \leq 2^{k+1}} |u^\varepsilon_{+,n}|^2 + |u^\varepsilon_{-,n}|^2 \leq C_0 \int_{-\infty}^{+\infty} dt \int_V d_g x |\varphi(t)\chi(2^{-k}D_t)u^\varepsilon|^2. \end{cases} \tag{1.6}$$

On the other hand,using classical estimates as in ([9], Sect. 4), one gets $\exists\, C_1, C_2, k_0$ s.t. for any $k_1 \geq k_0$, and any $u^\varepsilon \in J^\varepsilon_{\gamma_0}$

$$\sum_{k \geq k_1} \int_{-\infty}^{+\infty} dt \int_V |\varphi(t)\chi(2^{-k}D_t)u^\varepsilon|^2 d_g x \leq C_1 \int_0^{T_0} \int_V |u^\varepsilon|^2 d_g x + C_2 2^{-2k_1} \left(\sum_n |u^\varepsilon_{\pm,n}|^2\right). \tag{1.7}$$

Let $\gamma_1 = \gamma_0/2$; for $u^\varepsilon \in J^\varepsilon_{\gamma_1}$ and $2^{-k} < \varepsilon/\gamma_0$ one has $\chi(2^{-k}D_t)u^\varepsilon \equiv 0$, so putting together (1.6) and (1.7) we get

$$\begin{cases} \exists\, n_0,\ \exists\, C_3\,,\ \forall\, \varepsilon \in]0, \varepsilon_0]\,,\ \forall\, u^\varepsilon \in J^\varepsilon_{\gamma_1} \\ \sum\limits_{n \geq n_0, \varepsilon\omega_n^\varepsilon \leq \gamma_1} |u^\varepsilon_{+,n}|^2 + |u^\varepsilon_{-,n}|^2 \leq C_3 \left(\int_0^{T_0} dt \int_V d_g x |u^\varepsilon|^2 + \sum\limits_{n \leq n_0} |u^\varepsilon_{\pm,n}|^2\right) \end{cases} \tag{1.8}$$

and (1.8) is equivalent to

$$\begin{cases} \exists\, n_0,\ \exists\, C_4, C_5\,,\ \forall\, \varepsilon \in]0, \varepsilon_0]\,,\ \forall\, u^\varepsilon \in J^\varepsilon_{\gamma_1} \\ \int_0^T \int_\Omega |u^\varepsilon|^2 \rho\, dt d_g x \leq C_3 \int_0^{T_0} \int_V |u^\varepsilon|^2 \rho\, dt d_g x \\ +C_4 \left(\sum\limits_{n \leq n_0} |u^\varepsilon_{+,n}|^2 + |u^\varepsilon_{-,n}|^2\right). \end{cases} \tag{1.9}$$

It is now easy to conclude the proof of Theorem 1 by a uniqueness argument. In fact if (0.10) is untrue, there exist a sequence $\varepsilon_k \to 0$ and $u^{\varepsilon_k} \in J^{\varepsilon_k}_{\gamma_1}$ such that $\int_0^{T_0} \int_\Omega |u^{\varepsilon_k}|^2 \rho dt d_g x = 1$ and $\int_0^{T_0} \int_V |u^{\varepsilon_k}|^2 \rho dt d_g x \to 0$; let u be a weak limit in L^2 of $\{u^{\varepsilon_k}\}$; u satisfies

$$\begin{cases} \underline{\rho}(x)\partial_t^2 u - \Delta_g u = 0 \quad \text{on} \quad \mathbb{R}_t \times \Omega \\ u_{|\partial\Omega} = 0\,; \quad u_{|]0,T_0[\times V} = 0 \end{cases} \tag{1.10}$$

and from the observability inequality (0.7), we get $u \equiv 0$. Then (1.9) implies that $u \equiv 0$ is the strong limit in L^2 of u^{ε_k}, which contradicts $\int_0^{T_0} \int_\Omega |u^{\varepsilon_k}|^2 \rho dt d_g x \equiv 1$.

2. The Bloch wave

We shall now recall how one can quantize the principal symbols maps p_0, L^0 defined in (0.36, 0.38) in order to obtain the pseudo differential relation (0.41).

Let

$$I = [-\sigma_0, \sigma_0]. \tag{2.1}$$

For any $s \in I$, we split $E^s = H^s(\mathbb{T}^d) \oplus H^{s-1}(\mathbb{T}^d)$ into the decomposition

$$\begin{cases} E^s &= E_0 \oplus E^s_\perp \quad E_0 = \mathbb{C}^2 \\ w &= w_{(0)} + w_\perp \quad w_{(0)} = \oint_{\mathbb{T}^d} w. \end{cases} \tag{2.2}$$

In other words we write $w = \sum_\ell w_{(\ell)}\, e^{i\ell y}$ and $w_\perp = \sum_{\ell \neq 0} w_{(\ell)}\, e^{i\ell y}$.

We then construct tangential pseudo differential operators defined near $\varepsilon\partial_t = i\tau = 0$, $\varepsilon\partial_{x'} = i\xi' = 0$, semi-classical in ε

$$A_0(\varepsilon, x, \varepsilon\partial_t, \varepsilon\partial_{x'}) \;:\; L^2(X, E_0) \;\to\; L^2(X, \bigcap_{s\in I} E^s) \tag{2.3}$$

$$A_\perp(\varepsilon, x, \varepsilon\partial_t, \varepsilon\partial_{x'}) \;:\; L^2(X, E^s_\perp) \;\to\; L^2(X, E^s) \quad (\forall\, s \in I) \tag{2.4}$$

$$L(\varepsilon, x, \varepsilon\partial_t, \varepsilon\partial_{x'}) \;:\; L^2(X, E_0) \;\to\; L^2(X, E_0) \tag{2.5}$$

$$L_\perp(\varepsilon, x, \varepsilon\partial_t, \varepsilon\partial_{x'}) \;:\; L^2(X, E^s_\perp) \;\to\; L^2(X, E^{s-1}_\perp) \quad (\forall\, s \in I) \tag{2.6}$$

with symbols admitting asymptotic expansions

$$\sum_k \left(\frac{\varepsilon}{i}\right)^k A^k_0(x, \tau, \xi') \quad A^k_0 \quad \text{bounded from } E_0 \quad \text{to } \bigcap_{s\in I} E^s \tag{2.7}$$

$$\sum_k \left(\frac{\varepsilon}{i}\right)^k A^k_\perp(x, \tau, \xi') \quad A^k_\perp \text{ bounded from } E^s_\perp \quad \text{to } E^s \quad (\forall\, s \in I) \tag{2.8}$$

$$\sum_k \left(\frac{\varepsilon}{i}\right)^k L^k(x, \tau, \xi') \quad L^k \text{ bounded from } E_0 \quad \text{to } E_0 \tag{2.9}$$

$$\sum_k \left(\frac{\varepsilon}{i}\right)^k L^k_\perp(x, \tau, \xi') \quad L^k_\perp \text{ bounded from } E^s_\perp \quad \text{to } E^{s-1}_\perp \; (\forall\, s \in I) \tag{2.10}$$

such that near the zero section $\tau = \xi' = 0$, the two following identities hold true, in the algebra of tangential pseudo differential operators

$$\begin{cases} \left(\varepsilon \frac{\partial}{\partial_{x_d}} + \mathbb{M}\right) A_0 \;=\; A_0\; (\varepsilon\partial_{x_d} + L) \\ \left(\varepsilon \frac{\partial}{\partial_{x_d}} + \mathbb{M}\right) A_\perp \;=\; A_\perp(\varepsilon\partial_{x_d} + L_\perp). \end{cases} \tag{2.11}$$

Using the formula (0.28) $\mathbb{M} = \sum_{j=0}^{2} (\frac{\varepsilon}{i})^j \mathbb{M}^j(x, \xi', \tau; y, \partial y)$, and the rules of composition of pseudo differential operators, one gets that (2.11) is equivalent to the following set of equations (2.12, 2.13)

$$k = 0 \qquad \begin{cases} \mathbb{M}^0\, A^0_0 \;=\; A^0_0\, L^0 \\ \mathbb{M}^0\, A^0_\perp \;=\; A^0_\perp\, L^0_\perp \end{cases} \tag{2.12}$$

$$\begin{cases} \sum_{j+\ell+|\alpha|=k} \frac{1}{\alpha!}\,\partial^\alpha_{\xi'}\mathbb{M}^j\,\partial^\alpha_{x'}A^\ell_0 + i\partial_{x_d}A^{k-1}_0 & = \sum_{j+\ell+|\alpha|=k} \frac{1}{\alpha!}\,\partial^\alpha_{\xi'}A^j_0\,\partial^\alpha_{x'}L^\ell \\ \sum_{j+\ell+|\alpha|=k} \frac{1}{\alpha!}\,\partial^\alpha_{\xi'}\mathbb{M}^j\,\partial^\alpha_{x'}A^\ell_\perp + i\partial_{x_d}A^{k-1}_\perp & = \sum_{j+\ell+|\alpha|=k} \frac{1}{\alpha!}\,\partial^\alpha_{\xi'}A^k_\perp\,\partial^\alpha_{x'}L^\ell_\perp. \end{cases} \tag{2.13}$$

Let j_0 and $j_\perp$ be the inclusion maps

$$E_0 \overset{j_0}{\hookrightarrow} E^s \qquad E^s_\perp \overset{j_\perp}{\hookrightarrow} E^s \tag{2.14}$$

and let $\pi_0 = \pi_0(x, \xi', \tau)$ be the spectral projector of $\mathbb{M}^0$, which is defined near $(\xi', \tau) = (0, 0)$, by

$$\pi_0 = \frac{1}{2i\pi}\int_{\partial D} \frac{dz}{z - \mathbb{M}^0} \tag{2.15}$$

where $D \subset \mathbb{C}$ is a small disk with center at $z = 0$.

The range of π_0 is a two-dimensional invariant subspace of $\mathbb{M}^0$, and by the definition formula (0.29) of $\mathbb{M}^0$, one gets for $|\tau|$ small enough

$$\left\| \oint_{\mathbb{T}^d} \pi_0\, j_0 - Id_{E_0} \right\| \le \mathrm{Cte}\,\tau^2\ ;\ \|\pi_0\, j_\perp; E^s_\perp \to E^s\| \le \mathrm{Cte}\,\tau^2. \tag{2.16}$$

In order to obtain the relations (2.12), it is clearly sufficient to select isomorphisms

$$\begin{cases} A^0_0 \ :\ E_0 \xrightarrow{\sim} & \text{range } (\pi_0) \\ A^0_\perp \ :\ E^s_\perp \xrightarrow{\sim} & \text{range } (Id - \pi_0). \end{cases} \tag{2.17}$$

We can choose in view of (2.16), for $|\tau|$ small enough

$$\begin{cases} A^0_\perp & = (Id - \pi_0)j_\perp \\ A^0_0 & = \pi_0 j_0 \alpha \end{cases} \tag{2.18}$$

where $\alpha = \alpha(x, \tau, \xi')$ is the unique endomorphism of E_0, such that

$$\oint_{\mathbb{T}^d} A^0_0 = \oint_{\mathbb{T}^d} \pi_0 j_0 \alpha = Id_{E_0}. \tag{2.19}$$

(This choice of A^0_0 will insure the consistency with the definition (0.36) of p_0.)

The maps $L^0(x, \tau, \xi') : E_0 \to E_0$ and $L^0_\perp(x, \tau, \xi') : E^s_\perp \to E^{s-1}_\perp$ are then uniquely determined by (2.12). $L^0_\perp$ is a smooth function of (x, τ, ξ') defined near $\tau = \xi' = 0$, taking its values in the set of pseudo-differential operators of order 1 for the scale $\{E^s_\perp\}$ on the torus: for any $w_\perp \in \underset{s}{\cup} E^s_\perp$ one has

$$\mathbb{M}^0 j_\perp(w_\perp) - j_\perp L^0_\perp(w_\perp) = \mathbb{M}^0\pi_0 j_\perp(w_\perp) - \pi_0 j_\perp L^0_\perp(w_\perp) \in \underset{s}{\cap}\, E^s. \tag{2.20}$$

The map $A^0 = A^0_0 \oplus A^0_\perp$

$$E^s = E_0 \oplus E^s_\perp \xrightarrow{A^0} E^s \tag{2.21}$$

is an isomorphism; by (2.16) it satisfies

$$\|A^0 - Id\|_{E^s} \le \mathrm{Cte}\,\tau^2 \quad (\forall\, s \in I). \tag{2.22}$$

The equation (2.13) is equivalent to

$$\begin{cases}\mathbb{M}^0\, A_0^k - A_0^k L^0 - A_0^0 L^k = R_0^k\,, & R_0^k \text{ bounded from } E_0 \text{ to } E^s \\ \mathbb{M}^0\, A_\perp^k - A_\perp^k L_\perp^0 - A_\perp^0\, L_\perp^k = R_\perp^k\,, & R_\perp^k \text{ bounded from } E_\perp^s \text{ to } E^{s-1}\end{cases} \tag{2.23}$$

where the right hand side is given by induction by the formula

$$k \geq 1 \qquad R_{0,\perp}^k = \sum_{\substack{j+\ell+|\alpha|=k \\ j\neq k, \ell\neq k}} \frac{1}{\alpha!}\,\partial_{\xi'}^\alpha\, A_{0,\perp}^j \partial_{x'}^\alpha L_{\cdot,\perp}^\ell . - \sum_{\substack{j+\ell+|\alpha|=k \\ \ell\neq k}} \frac{1}{\alpha!}\,\partial_{\xi'}^\alpha\, \mathbb{M}^j \partial_{x'}^\alpha A_{0,\perp}^\ell - i\partial_{x_d} A_{0,\perp}^{k-1}. \tag{2.24}$$

Let $A^k = A_0^k \oplus A_\perp^k$, $\widetilde{A}^k = (A^0)^{-1} A^k$, $\mathcal{L}^k = L^k \oplus L_\perp^k$, $R^k = R_0^k \oplus R_\perp^k$ and $\widetilde{R}^k = (A^0)^{-1} R_k$. The equation (2.23) can be rewritten $\mathbb{M}^0\, A^k - A^k\, \mathcal{L}^0 - A^0 \mathcal{L}^k = R^k$, which is equivalent by (2.12) $[\mathbb{M}^0\, A^0 = A^0\, \mathcal{L}^0]$ to $\mathcal{L}^0\, \widetilde{A}^k - \widetilde{A}^k\, \mathcal{L}_0 - \mathcal{L}^k = \widetilde{R}^k$. The matrix form of this equation on $E_0 \oplus E_\perp^s$ is

$$\begin{cases} L^0\,(\widetilde{A}^k)_{1,1} - (\widetilde{A}^k)_{1,1}\, L^0 &= L^k + (\widetilde{R}^k)_{1,1} \\ L_\perp^0\,(\widetilde{A}^k)_{2,2} - (\widetilde{A}^k)_{2,2}\, L_\perp^0 &= L_\perp^k + (\widetilde{R}^k)_{2,2}\end{cases} \tag{2.25}$$

$$\begin{cases} L^0\,(\widetilde{A}^k)_{1,2} - (\widetilde{A}^k)_{1,2}\, L_\perp^0 &= (\widetilde{R}^k)_{1,2} \\ L_\perp^0\,(\widetilde{A}^k)_{2,1} - (\widetilde{A}^k)_{2,1}\, L^0 &= (\widetilde{R}^k)_{2,1}.\end{cases} \tag{2.26}$$

The choice $(\widetilde{A}^k)_{1,1} = 0$, $(\widetilde{A}^k)_{2,2} = 0$ gives then L^k, $L_\perp^k$ by (2.25). The unique solvability of (2.26) is a consequence of (0.34) which implies for $|\tau| + |\xi'|$ small enough

$$\begin{cases} L_\perp^0 \text{ is invertible and } \|(L_\perp^0)^{-1}; E_\perp^{s-1} \to E_\perp^s\| \leq C & (\forall\, s \in I) \\ \text{Spectrum } (L^0) \subset \{z \in \mathbb{C}; |z| \leq \text{ Cte}(|\tau| + |\xi'|)\}. \end{cases} \tag{2.27}$$

Thus, solving the second equation in (2.26) is equivalent to find a linear map $u : E_0 = \mathbb{C}^2 \to \bigcap_{s\in I} E_\perp^s = E_\perp^{\sigma_0}$ such that

$$u - (L_\perp^0)^{-1} \circ u \circ L^0 = v \tag{2.28}$$

where $v : E_0 \to E_\perp^{\sigma_0}$ is given and (2.27) implies for $|\tau| + |\xi'|$ small the existence of a unique solution u to (2.28). The first equation in (2.26) can be reduced to the second one by taking adjoints.

Remark. We have chosen to work with a fixed interval of regularity on the torus, $s \in [-\sigma_0, \sigma_0] = I$ in order to work in the classical theory of semi-classical (in ε) peudo-differential operators with values in bounded operators between Hilbert spaces. On the other hand, the neighborhood of the zero section $\tau = \xi' = 0$ where the above construction applies may depends on I.

In view of (2.22), the tangential pseudo-differential operator $A = A_0 \oplus A_\perp$ is elliptic near the zero section $\tau = 0, \xi' = 0$. Let A^{-1} be a pseudo-differential inverse and $\mathcal{L} = L \oplus L_\perp$.

By construction we have $(\varepsilon\partial_{x_d} + \mathbb{M})\, A \equiv A\,(\varepsilon\partial_{x_d} + \mathcal{L})$ near the zero section, and $\mathcal{L}$ is diagonal in the decomposition $E_0 \oplus E_\perp$. Therefore, one deduces that the following identity holds true near the zero section

$$\oint_{\mathbb{T}^d} A^{-1} \left(\varepsilon \frac{\partial}{\partial_{x_d}} + \mathbb{M}\right) \equiv \left(\varepsilon \frac{\partial}{\partial_{x_d}} + L\right) \oint_{\mathbb{T}^d} A^{-1}. \tag{2.29}$$

If we choose $W_1 \Subset W_0$ two sufficiently small neighborhoods of the set $\{\xi' = 0\,, \tau = 0\} \times \{t \in [-T_0, 2T_0]\}$ in T^*X, and Q_0, Q_1 two scalars tangentials o.p.d., with $SE(Q_j) \subset W_j$, $j = 0, 1$, such that $Q_0 \equiv Id$ on $\overline{W}_1$, we then get, with

$$\Pi_0 \stackrel{\text{def}}{=} Q_0 \oint_{\mathbb{T}^d} A^{-1}. \tag{2.30}$$

$$\Pi_0 \left(\varepsilon \frac{\partial}{\partial_{x_d}} + \mathbb{M} \right) = \left(\varepsilon \frac{\partial}{\partial_{x_d}} + L \right) \Pi_0 + R \tag{2.31}$$

where R is such that $\|Q_1\, R; L^2(X, E^s) \to L^2(X, \mathbb{C}^2\| \in O(\varepsilon^\infty)$ for any $s \in I$.

The principal symbol p_0 of Π_0 is easy to compute:

If $w = (Id - \pi_0) j_\perp(w_\perp) + \pi_0 j_0 \alpha(w_{(0)})$, one has $\pi_0(w) = \pi_0 j_0 \alpha(w_{(0)})$ and $(A^0)^{-1}(w) = w_{(0)} \oplus w_\perp$, so we get $\oint_{\mathbb{T}^d} (A^0)^{-1}(w) = w_{(0)} =$ (using (2.19)) $\oint_{\mathbb{T}^d} \pi_0 j_0 \alpha(w_{(0)}) = \oint_{\mathbb{T}^d} \pi_0(w)$ and we recover the definition formula (0.36) of p_0, if one takes χ_0 equal to the principal symbol of Q_0.

Lemma 2.1. *The tangential o.p.d.* $L \simeq \sum_k (\frac{\varepsilon}{i})^k L^k(x, \tau, \xi')$ *satisfies*

$$\begin{aligned} \text{i)} \qquad L \quad &\equiv \quad \oint_{\pi^d} (\mathbb{M}_{|\tau=0})\, j_0 \quad \text{modulo } \tau^2 \\ \text{ii)} \qquad L^0 \quad &= \quad \begin{bmatrix} a_0^{-1}(x) a_1(x, i\xi') & -a_0^{-1}(x) \\ a_2(x, i\xi') + \underline{\hat{\rho}}(x)\tau^2 & 0 \end{bmatrix} + O(\tau^4). \end{aligned} \tag{2.32}$$

Proof. For i), we observe that τ^2 is a smooth parameter in the above construction, and that by formulas (0.27, 0.28), the restriction $\mathbb{M}_{|\tau=0}$ is a constant coefficient operator on the torus $\mathbb{T}^d_y$.

We thus get $\pi_{0|\tau=0} = \oint_{\mathbb{T}^d}$, $\alpha_{|\tau=0} = Id$, $A_{0|\tau=0} = j_0$, $A_{\perp|\tau=0} = j_\perp$, $L_{|\tau=0} = \oint_{\mathbb{T}^d} (\mathbb{M}_{|\tau=0}) j_0$, $L_{\perp|\tau=0} = (Id - \oint_{\mathbb{T}^d})_{|\tau=0} (\mathbb{M}_{|\tau=0}) j_\perp$.

One has $\oint_{\mathbb{T}^d} A_0^0 = Id_{E_0}$ and $A_0^0 = j_0 + O(\tau^2)$, so there exists a map $\theta(x, \tau^2, \xi') : E_0 \to E_\perp$ such that $A_0^0 = j + \tau^2 \theta$. Using (2.12), we get

$$L^0 = \oint_{\mathbb{T}^d} \mathbb{M}^0 A_0^0 \tag{2.33}$$

so for any $w \in E_0$

$$L^0(w) = \oint_{\mathbb{T}^d} \mathbb{M}^0 j_0(w) + \tau^2 \oint_{\mathbb{T}^d} \mathbb{M}^0 \theta(w). \tag{2.34}$$

The definition formula (0.29) of $\mathbb{M}^0$ and (2.34) gives the second part of the lemma. □

For $u^\varepsilon \in I_h^\varepsilon$, we define $\underline{u}^\varepsilon$ by

$$\underline{u}^\varepsilon = \begin{bmatrix} u_0^\varepsilon \\ u_1^\varepsilon \end{bmatrix} = \begin{bmatrix} u^\varepsilon \\ A_0(x) \varepsilon \partial_{x_d} u^\varepsilon + \varepsilon A_1(x, \partial_{x'}) u^\varepsilon \end{bmatrix} \tag{2.35}$$

and $w^\varepsilon = \mathcal{T}(u^\varepsilon) = T(\underline{u}^\varepsilon)$ by

$$w^\varepsilon = \begin{bmatrix} w_0^\varepsilon \\ w_1^\varepsilon \end{bmatrix} = \begin{bmatrix} T(u_0^\varepsilon) \\ T(u_1^\varepsilon) \end{bmatrix} \tag{2.36}$$

where T is the transformation (0.19):

$$T(f)(t,x,y) = \sum_{\ell\in\mathbb{Z}^d} e^{i\ell(y-\Theta((x)/\varepsilon)} f(t,x)\,. \tag{2.37}$$

Then w^ε satisfies, for $s_0 < -d/2$.

$$w^\varepsilon(t,x,y) \in L^2\left([t_1,t_2]\times\partial\Omega\times[0,r_0]\ ;\ [H^{s_0}(\mathbb{T}^d)]^2\right) \quad \forall\, t_1,t_2\in\mathbb{R} \tag{2.38}$$

$$\begin{cases} \left(\varepsilon \frac{\partial}{\partial_{x_d}} + \mathbb{M}\right) w^\varepsilon = 0 \quad \text{on} \quad \mathbb{R}_t\times\partial\Omega\times]0,r_0[\times\mathbb{T}^d_y \\ w^\varepsilon_{0|x_{d=0}} = 0. \end{cases} \tag{2.39}$$

We recall that we define the Bloch wave $\Gamma(u^\varepsilon)\in L^2(X;\mathbb{C}^2)$ by

$$\Gamma(u^\varepsilon) = \begin{bmatrix}\Gamma_0(u^\varepsilon)\\ \Gamma_1(u^\varepsilon)\end{bmatrix} = \Pi_0\circ T(\underline{u}^\varepsilon). \tag{2.40}$$

Proof of Proposition 1.

(We denote by C various constants which are independent of ε, h.)

For $u^\varepsilon = \sum_{0.9\le\omega^\varepsilon_n h\le 2.1}(u_{+,n}e^{it\omega^\varepsilon_n} + u_{-,n}e^{it\omega^\varepsilon_n})e^\varepsilon_n(x)$ we put $\|u^\varepsilon\|^2 = \sum|u_{+,n}|^2+|u_{-,n}|^2$. For any $t_1<t_2$, there exists a constant C such that for any ε, h and $u^\varepsilon\in I^\varepsilon_h$ one has

$$\int_\Omega\int_{t_1}^{t_2} |h\nabla u^\varepsilon|^2 + |h\partial_t u^\varepsilon|^2 dt d_g x \le C\|u^\varepsilon\|^2. \tag{2.41}$$

Let $\gamma=\varepsilon/h$; we rewrite (2.41) on the form

$$\int_\Omega\int_{t_1}^{t_2} |\varepsilon\nabla u^\varepsilon|^2 + |\varepsilon\partial_t u^\varepsilon|^2 dt d_g x \le C\gamma^2\|u^\varepsilon\|^2. \tag{2.42}$$

Let $K=\partial\Omega\times[0,T_0]\times[0,r_0/2]\times\{\xi'=0,\tau=0\}$ and $Q(\varepsilon,t,x,\varepsilon\partial_{x'},\varepsilon\partial_t)$ be a scalar tangential o.p.d. on X, equal to Id near K.

Let α small such that the geometric control hypothesis (0.9) holds true for $T_0-4\alpha$, and let $Y=\partial\Omega\times[\alpha,T_0-\alpha]\times[0,r_0/2]$. By (2.42), for γ small,the L^2 norm of u^ε on Y is concentrated near the set $\xi'=0,\tau=0$ where Q is equal to Id; so we get

$$\|u^\varepsilon\|^2_{L^2(Y)} \le C\left[\|Q(u^\varepsilon)\|^2_{L^2(X_{T_0})} + (\gamma+\varepsilon)^2\|u^\varepsilon\|^2\right]\,. \tag{2.43}$$

By construction of Π_0, one has

$$\Pi_0 = Q_0\left[\oint_{\mathbb{T}^d} Id + R_0(\varepsilon\partial_t) + \varepsilon R_1\right] \tag{2.44}$$

where $R_{0,1}$ are tangential o.p.d. from $L^2(X;E^s)$ in $L^2(X;E_0)$ $(s\in I)$. Therefore we get

$$\left\|\Gamma(u^\varepsilon) - Q_0\begin{pmatrix}u^\varepsilon_0\\ u^\varepsilon_1\end{pmatrix}\right\|_{L^2(X;E_0)} \le C[\gamma+\varepsilon]\|u^\varepsilon\| \tag{2.45}$$

(here we have used the fact that $\varepsilon\partial_t$ commutes with T and is bounded by $O(\gamma = \varepsilon/h)$ on I_h^ε). Since QQ_0 is equal to Id near K, we deduce from (2.43, 2.45), for γ_0, ε_0 small enough

$$\|u^\varepsilon\|^2_{L^2(Y)} \leq C\left[\|Q\Gamma_0(u^\varepsilon)\|^2_{L^2(X_{T_0})} + (\gamma+\varepsilon)^2\|u^\varepsilon\|^2\right]. \tag{2.46}$$

We are now ready to prove (0.45) by a contradiction argument. If (0.45) is untrue, there exist sequences $\varepsilon_k \to 0, \gamma_k \to 0, h_k \to 0$, $h_k \geq \varepsilon_k/\gamma_k$, $u^k \in I_{h_k}^{\varepsilon_k}$ such that

$$\begin{cases} \|u^k\| = 1 \\ \lim\limits_{k\to\infty} \|Q\Gamma_0(u^k)\|^2_{L^2(X_{T_0})} + \|u^k\|^2_{L^2((0,T_0)\times V)} = 0. \end{cases} \tag{2.47}$$

Moreover, we can suppose that the weak limit $u = \text{weak} - \lim(u^k)$ exist. Then u satisfies (0.6) and is equal to 0 on $(0, T_0) \times V$. By the geometric control hypothesis (0.9) of [4], the estimate (0.7) holds true for u, so we get $u = 0$. We deduce from (2.46)

$$\lim_{k\to\infty} \|u^k\|_{L^2(Y)} = 0. \tag{2.48}$$

We are thus reduced to an interior problem in Ω.

Let $Z = \{x \in \Omega\ ;\ \text{dist}(x, \partial\Omega) > r_0/4\} \times \mathbb{R}_t$. We denote by $\widetilde{\mathbb{M}} = \rho(x,y)(\varepsilon\partial_t)^2 - \varepsilon^2(\Delta_g)^*_\varepsilon$ the Bloch operator on Z, and $G^s = H^s(\mathbb{T}^d)$. By the same construction as above, there exist a ε-pseudo-differential operator $\widetilde{\Pi}_0(x,\xi,\tau,y,\partial_y) : L^2(Z, G^\bullet) \to L^2(Z,\mathbb{C})$ and a scalar ε-o.p.d. $\widetilde{L}(x,\xi,\tau) : L^2(Z;\mathbb{C}) \to L^2(Z,\mathbb{C})$, defined near the zero section $\xi = \tau = 0$, such that

$$\widetilde{\Pi}_0\widetilde{\mathbb{M}} = \widetilde{L}\widetilde{\Pi}_0 + \widetilde{R}. \tag{2.49}$$

The principal symbol of $\widetilde{\Pi}_0$ is $\tilde{\chi}_0 \oint_{\pi^d} \frac{1}{2i\pi} \int_{\partial D} \frac{dz}{z-\widetilde{\mathbb{M}}^0}$ with $\tilde{\chi}_0 \in C_0^\infty(\widetilde{W}_0), \tilde{\chi}_0 \equiv 1$ on $\widetilde{W}_1$, where $\widetilde{W}_1 \Subset \widetilde{W}_0$ are two small neighborhood of the set $\{\xi = \tau = 0\} \times \{t \in [-T_0, 2T_0]\}$ in T^*Z. The scalar operator $\widetilde{L}$ satisfies

$$\begin{cases} \widetilde{L} \simeq \sum\limits_k (\frac{\varepsilon}{i})^k \widetilde{L}^k(x,\tau,\xi) \\ \widetilde{L}_{|\tau=0} = -\varepsilon^2\Delta_g \quad \text{modulo } \tau^2 \\ \widetilde{L}^0(x,\tau,\xi) = -\underline{\rho}(x)\tau^2 + \|\xi\|^2 + 0(\tau^4). \end{cases} \tag{2.50}$$

The error terms $\widetilde{R}$ in (2.49) is such that for any ε-o.p.d. $\widetilde{Q}$ with essential support in $\widetilde{W}_1$, one has

$$\|\widetilde{Q} \circ \widetilde{R}\ ; L^2(Z; G^s) \to L^2(Z;\mathbb{C})\| \in \mathcal{O}(\varepsilon^\infty) \qquad \forall s \in [-\sigma_0, \sigma_0]. \tag{2.51}$$

Let $v^k(t,x,y)$ be the distribution on $Z \times \mathbb{T}^d$

$$v^k = T(u^k) = \sum_{\ell\in\mathbb{Z}^d} e^{i\ell(y-x/\varepsilon_k)} u^k(t,x). \tag{2.52}$$

We deduce from (2.50) that $(\frac{h}{\varepsilon})^2\widetilde{L} \overset{\text{def}}{=} \widetilde{\mathcal{L}}$ is an h-o.p.d.; writing $\frac{\varepsilon}{i}\partial_x = \frac{\varepsilon}{h}(\frac{h}{i}\partial_x)$, and using $\frac{h_k}{\varepsilon_k} \geq \frac{1}{\gamma_k} \to \infty$ (2.49, 2.51) we get, for any h-o.p.d. Q compactly supported in $\{\xi,\tau\}$ and with support in $Z \times \{t \in (-T_0, 2T_0)\}$

$$\|Q\widetilde{\mathcal{L}}\widetilde{\Pi}_0 v^k\|_{L^2(Z)} \in 0(h_k^\infty). \tag{2.53}$$

By the analogue of (2.45) in the interior case, we also have

$$\|\widetilde{\Pi}_0 v^k - \widetilde{Q}_0 u^k\|_{L^2(Z\cap\{t\in[-T_0,2T_0]\})} \leq C[\gamma_k + \varepsilon_k] \tag{2.54}$$

where $\widetilde{Q}_0$ is an ε-o.p.d. with principal symbol χ_0, with essential support in $\widetilde{W}_0$.

Let μ be a h-semi classical measure associated to $\{u^k\}$ (see [8]). (The hypothesis $u^k \in I^{\varepsilon_k}_{h_k}$ implies that μ is supported in $|\tau| \in [0.9, 2.1]$.) From (2.47) and (2.46) we deduce that

$$\mu_{|Y\cap Z} \equiv 0 \text{ and } \mu_{|]0,T_0[\times V} \equiv 0. \tag{2.55}$$

Let ν be a h-semiclassical measure associated to $\widetilde{\Pi}_0 v^k$. Using (2.54) and $\lim\limits_{k\to\infty} \varepsilon_k/h_k = 0$ we get

$$\nu = \widetilde{\chi}_0^2(t,x;\xi' = 0, \tau = 0)\mu. \tag{2.56}$$

The principal symbol of $\widetilde{\mathcal{L}}$ is $-\underline{\rho}(x)\tau^2 + \|\xi\|^2 + \gamma_k^2 0(\tau^4)$. In the equation (2.53) we view $\gamma_k = \varepsilon_k/h_k$ as a small parameter. We can then use the proof of the interior propagation theorem (see [8]) with the additional parameter γ_k going to zero. We get from (2.53) that the support of ν is contained in the set $\underline{\rho}(x)\tau^2 - \|\xi\|^2 = 0$, and that the support of ν propagates along the bicharacteristic flow of $\underline{\rho}(x)\tau^2 - \|\xi\|^2$. Using (2.55, 2.56), and the hypothesis (0.9) we obtain for β small

$$\mu_{]T_0/2-\beta, T_0/2+\beta[} \equiv 0. \tag{2.57}$$

Using (2.41), we get that the sequence u^k is h-oscillatory (see [7]), so from (2.57) we deduce

$$\lim_{k\to\infty} \|u^k\|_{L^2(Z\times]T_0/2-\beta, T_0/2+\beta[)} \equiv 0.$$

Then from (2.48), we obtain $\lim\limits_{k\to\infty} \|u^k\|_{L^2(\Omega\times(T_0/2-\beta, T_0/2+\beta))} = 0$ which contredicts $\|u^k\| \equiv 1$. □

3. Lopatinski estimate

3.1. **Proof of Proposition 2**

We first verify the implication Theorem 2 ⇒ Proposition 2. For $u^\varepsilon \in I^\varepsilon_h$, we have

$$w^\varepsilon = \begin{bmatrix} w_0^\varepsilon \\ w_1^\varepsilon \end{bmatrix} = \begin{bmatrix} T(u^\varepsilon) \\ T(A_0(\varepsilon\partial_{x_d} u^\varepsilon) + \varepsilon A_1(x, \partial_{x'})u^\varepsilon) \end{bmatrix} \tag{3.1}$$

and by (2.44)

$$\Gamma(u^\varepsilon) = Q_0 \left[\oint_{\mathbb{T}^d} w^\varepsilon + R_0(\varepsilon\partial_t) w^\varepsilon + \varepsilon R_1 w^\varepsilon \right]. \tag{3.2}$$

The Dirichlet boundary condition $u^\varepsilon|_{x_d=0}$ implies $w_0^\varepsilon|_{x_d=0} = 0$, so we get

$$\Gamma_0(u^\varepsilon)|_{x_d=0} = Q_0 \left[\oint_{\mathbb{T}^d} (R_0(\varepsilon\partial_t) + \varepsilon R_1) \begin{bmatrix} 0 \\ w_1^\varepsilon|_{x_d=0} \end{bmatrix} \right]_{1^{st}\text{component}}. \tag{3.3}$$

If one multiplies the equation (0.4) by $\varepsilon^3\varphi(x_d)\frac{\partial}{\partial x_d}$ where $\varphi \in C_0^\infty(]-r_0/2, r_0/2[)$ is equal to 1 near the boundary $x_d = 0$, and integrates by part, one gets

$$\begin{cases} \text{For any } \ t_1, t_2, \text{there exist } \quad C \text{ s.t.} \quad \forall\varepsilon \\ \\ \|\varepsilon\partial_n u^\varepsilon\|_{L^2((t_1,t_2)\times\partial\Omega)} \leq C\varepsilon^{-1/2}\|u^\varepsilon\| \quad \forall u^\varepsilon \in I_h^\varepsilon. \end{cases} \tag{3.4}$$

Therefore, by (3.1) we get for $s_0 < -d/2$

$$\|w_1^\varepsilon|_{x_d=0}; L^2((t_1,t_2)\times\partial\Omega\ ; H^{s_0}(\mathbb{T}_y^d)\| \leq C\varepsilon^{-1/2}\|u^\varepsilon\|. \tag{3.5}$$

If R is an o.p.d from $L^2(X_{T_0\cap x_{d=0}}\ ;\ H^{s_0}(\mathbb{T}^d))$ in $L^2(X_{T_0\cap x_{d=0}})$, using the classical calculus of Appendix A.1, we get from the *a priori* bound (3.5) on the trace $w_1^\varepsilon|_{x_{d=0}}$

$$\|[Q_0, R]w_1^\varepsilon|_{x_{d=0}}\ ;\ L^2(X_{T_0\cap x_{d=0}})\| \leq C\varepsilon^{1/2}\|u^\varepsilon\|. \tag{3.6}$$

If Theorem 2 holds true, we have

$$\|Q_0 w_1^\varepsilon|_{x_{d=0}}\ ;\ L^2(X_{T_0\cap x_{d=0}}\ ;\ H^{s_1}(\mathbb{T}^d))\| \leq C\|u^\varepsilon\|. \tag{3.7}$$

Now using the fact that $\varepsilon\partial_t$ commutes with T and is bounded by $\mathcal{O}(\gamma = \varepsilon/h)$ on I_h^ε, (3.3, 3.6, 3.7) and $\varepsilon \leq h_0\varepsilon/h$, we get (0.47), *i.e.*

$$\|\Gamma_0(u^\varepsilon)|_{x_{d=0}}; L^2(X_{T_0} \cap x_d = 0)\| \leq C\tfrac{\varepsilon}{h}\|u^\varepsilon\|.$$

3.2. **Proof of Theorem 2**

In this part, we work with a family $\{u^\varepsilon\}_\varepsilon, u^\varepsilon \in I_h^\varepsilon$ with $\varepsilon \in]0, \varepsilon_0], h \in [\varepsilon/\gamma_0, h_0]$; we always assume $\|u^\varepsilon\| \leq 1$. We first remark that the Theorem 2 is local near any $\rho_0 = (t_0, x_0', \tau_0 = 0, \xi_0' = 0) \in T^*(\mathbb{R}_t \times \partial\Omega)$. Let Q_1 be a tangential scalar *o.p.d* equal to Id near ρ_0, and with essential support close to ρ_0, and contained in W_0. By (2.38, 2.39) we get (see (0.30) for the definition of E^s)

$$\begin{cases} \left(\varepsilon\frac{\partial}{\partial x_d} + \mathbb{M}^0\right)Q_1 w^\varepsilon = \tilde{g}^\varepsilon \\ \\ \tilde{g}^\varepsilon = \left[\varepsilon\frac{\partial}{\partial x_d} + \mathbb{M}, Q_1\right]w^\varepsilon - \frac{\varepsilon}{i}\sum\limits_{j=1}^{2}\mathbb{M}^j Q_1 w^\varepsilon \end{cases} \tag{3.8}$$

and for any $s_0 + 1 < -d/2$ and any t_1, t_2

$$\sup_\varepsilon \|Q_1 w^\varepsilon; L^2([t_1,t_2]\times\partial\Omega\times[0,r_0]; E^{s_0+1}\| < +\infty \tag{3.9}$$

$$\sup_\varepsilon \varepsilon^{-1}\|\tilde{g}^\varepsilon; L^2([t_1,t_2]\times\partial\Omega\times[0,r_0]; E^{s_0}\| < +\infty. \tag{3.10}$$

We define $f^\varepsilon, g^\varepsilon$ by

$$Q_1 w^\varepsilon = \begin{pmatrix} f_0^\varepsilon \\ i f_1^\varepsilon \end{pmatrix}, f^\varepsilon = \begin{pmatrix} f_0^\varepsilon \\ f_1^\varepsilon \end{pmatrix}, g^\varepsilon = \begin{pmatrix} \frac{\tilde{g}_0^\varepsilon}{i} \\ -\tilde{g}_1^\varepsilon \end{pmatrix}. \tag{3.11}$$

We may assume that f^ε is supported in a small neighborhood $U = U_0 \times [0, r_1[$ of (t_0, x_0') in $\mathbb{R}_t \times \partial\Omega \times [0, r_0]$, and we denote by $(x_1, \cdots, x_{d-1})$ a local coordinate system near x_0' in $\partial\Omega$. Near the boundary by the choice of coordinates (0.15), we have $a_0(x) \equiv 1$ and $a_1(x, \xi') \equiv 0$, so equation (3.8) may be rewritten as

$$\begin{cases} \dfrac{\varepsilon}{i}\dfrac{\partial}{\partial x_d} f^\varepsilon + \mathbb{N} f^\varepsilon = g^\varepsilon \\ \mathbb{N} = \begin{pmatrix} e_d(x).D_y & -1 \\ a_2(x, \frac{\varepsilon}{i}\partial_{x'} + e'(x)D_y) - \hat{\rho}(x,y)\left(\dfrac{\varepsilon\partial_t}{i}\right)^2 & e_d(x).D_y \end{pmatrix} \end{cases} \tag{3.12}$$

with $D_y = \frac{1}{i}\frac{\partial}{\partial y}$, $e'(x)D_y = (e_1(x).D_y, \cdots, e_{d-1}(x).D_y)$, we define the trace operators Tr_0, Tr_1 by

$$Tr_0(f^\varepsilon) = f_0^\varepsilon|_{x_d=0} \quad Tr_1(f^\varepsilon) = f_1^\varepsilon|_{x_d=0}. \tag{3.13}$$

We have $Tr_0(f^\varepsilon) \equiv 0$ and we have to prove

$$\begin{cases} \text{If } W_0 \subset \{|\xi'| + |\tau| < \alpha_0\}, \text{ with } \alpha_0 \text{ small enough, there exist } s_1, C, s.t. \\ \sup_\varepsilon \|Tr_1(f^\varepsilon); L^2(U_0; H^{s_1}(\mathbb{T}^d))\| \leq C. \end{cases} \tag{3.14}$$

For any $\ell \in \mathbb{Z}^d$, we define $\ell_x^\perp$ and $\ell_x^{''}$

$$\ell_x^\perp = e_d(x).\ell\ ,\ \ \ell_x'' = (e_1(x).\ell, \cdots, e_{d-1}(x).\ell). \tag{3.15}$$

We have by (32), with $\|\ell_x''\|^2 = a_2(x, \ell_x'')$

$$\|{}^t d\theta(x)(\ell)\|_x^2 = (\ell_x^\perp)^2 + \|\ell_x''\|^2. \tag{3.16}$$

Let $\mathbb{N}_0(x)$ be the restriction of $\mathbb{N}$ to the zero section $\xi' = \tau = 0$. We have

$$\begin{cases} \mathbb{N}_{0,\ell}(x) = \begin{pmatrix} \ell_x^\perp & -1 \\ \|\ell_x''\|^2 & \ell_x^\perp \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}) \\ \mathbb{N}_0(x)\left(\sum_\ell z_\ell e^{i\ell y}\right) = \sum_\ell \mathbb{N}_{0,\ell}(x)(z_\ell) e^{i\ell y} \end{cases} \tag{3.17}$$

and the eigenvalues of $\mathbb{N}_{0,\ell}(x)$ are

$$\lambda_{\pm,\ell}^0(x) = \ell_x^\perp \pm i\|\ell_x''\|. \tag{3.18}$$

Our strategy of proof of the estimate (3.14) is to split f^ε into two pieces. The first one will be concentrate near $\|\ell_x''\|$ small, where the spectrum of $\mathbb{N}$ is close to the real axis; we shall treat this part by a perturbation argument on the spectral theory of $\mathbb{N}$. The second one $\|\ell_x''\| \geq c^{te} > 0$ will be handle by elliptic estimates on $\mathbb{N}$.

To achieve this program, we shall use the "exotic" pseudo-differential calculus of Appendice A.2, with $Z = \mathbb{R}_t \times \mathbb{R}_{x'}^{d-1} \times [0, r_0]_{x_d}$; to simplify notation we denote by $\mathcal{S}^{t,m}$ (resp. $\mathcal{B}^{t,m}$) the class of symbols (resp. operators) defined in (A.15) (resp. (A.17)). The restriction on $x_d = 0$ of these class of symbols and operators will be denoted by $\mathcal{S}^m, \mathcal{B}^m$.

We first conjugate the equation (3.12) so that the natural scale of space on the torus will be

$$\mathcal{H}^s \stackrel{\text{def}}{=} [H^s(\mathbb{T}^d)]^2. \tag{3.19}$$

Let $\langle \ell''_x \rangle = (1 + \|\ell''_x\|^2)^{1/2}$. We define the operators $\Lambda = \Lambda(x)$ and $\mathbb{E}_0 = \mathbb{E}_0(x)$ on the torus by

$$\Lambda(\sum_\ell z_\ell e^{i\ell y}) = \sum_\ell \begin{pmatrix} 1 & 0 \\ 0 & \langle \ell''_x \rangle \end{pmatrix} (z_\ell) e^{i\ell y} \tag{3.20}$$

$$\mathbb{E}_0(\sum_\ell z_\ell e^{i\ell y}) = \sum_\ell \begin{pmatrix} \ell^\perp_x & -\langle \ell''_x \rangle \\ \frac{\|\ell''_x\|^2}{\langle \ell''_x \rangle} & \ell^\perp_x \end{pmatrix} (z_\ell) e^{i\ell y}. \tag{3.21}$$

Let F^ε be

$$F^\varepsilon = \Lambda^{-1}(f^\varepsilon). \tag{3.22}$$

We have $Tr_0(F^\varepsilon) = 0$ and by (3.9), and the fact that Λ^{-1} maps clearly E^{s+1} in $\mathcal{H}^s$,we get

$$\sup_\varepsilon \|F^\varepsilon ; L^2(U; \mathcal{H}^{s_0})\| < \infty. \tag{3.23}$$

Lemma 3.1. *There exist $q \in \mathcal{S}^{t,0}$, with*

$$q_{|\xi'=0,\tau=0} \equiv 0 \tag{3.24}$$

such that, for any scalar tangential symbol $\theta(t, x, \tau, \xi')$ equal to Id near the essential support of Q_1 and with support in $\{|\xi'| + |\tau| \le \alpha_0\}$, F^ε satisfies

$$G^\varepsilon = \frac{\varepsilon}{i} \frac{\partial}{\partial x_d} F^\varepsilon + \left(\mathbb{E}_0 + \begin{pmatrix} 0 & 0 \\ Op(q\theta) & 0 \end{pmatrix} \right) F^\varepsilon \tag{3.25}$$

$$\sup_\varepsilon \varepsilon^{-1} \|G^\varepsilon ; L^2(U; \mathcal{H}^{s_0 - 1})\| < +\infty\ . \tag{3.26}$$

Proof. We conjugate (3.12) by Λ and we obtain

$$\frac{\varepsilon}{i} \frac{\partial}{\partial x_d} F^\varepsilon + \Lambda^{-1} \mathbb{N} \Lambda F^\varepsilon = \Lambda^{-1} g^\varepsilon - \Lambda^{-1} \frac{\varepsilon}{i} \left(\frac{\partial}{\partial x_d} \Lambda \right) F^\varepsilon. \tag{3.27}$$

We have

$$\left(\frac{\partial}{\partial x_d} \Lambda \right) \left(\sum_\ell z_\ell e^{i\ell y} \right) = \sum_\ell \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial}{\partial x_d} \langle \ell''_x \rangle \end{pmatrix} (z_\ell) e^{i\ell y}$$

and $|\frac{\partial}{\partial x_d} \langle \ell''_x \rangle| \le c^{te} (1 + |\ell|^2)^{1/2}$; therefore (by (3.10, 3.19)) we get

$$\sup_\varepsilon \varepsilon^{-1} \|\Lambda^{-1} g^\varepsilon - \Lambda^{-1} \frac{\varepsilon}{i} \left(\frac{\partial}{\partial x_d} \Lambda \right) F^\varepsilon ; L^2(U; \mathcal{H}^{s_0 - 1})\| < +\infty. \tag{3.28}$$

A simple computation gives

$$\left\{\begin{array}{l} \Lambda^{-1}\mathbb{N}\Lambda = \mathbb{E}_0 + \begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix} \\ R = Op(\bigoplus_\ell \langle \ell''_x \rangle)^{-1} \left[a_2\left(x, \frac{\varepsilon}{i}\frac{\partial}{\partial x'}\right) + \sum_{j=1}^{d-1} \frac{\partial a_2}{\partial \xi'_j}\left(x, \frac{\varepsilon}{i}\frac{\partial}{\partial x'}\right)(e_j(x).D_y) + \hat{\rho}(x,y)(\varepsilon\partial_t)^2 \right] \end{array}\right. \tag{3.29}$$

with

$$Op\left(\bigoplus_\ell \langle \ell''_x \rangle\right)^{-1}\left(\sum_\ell z_\ell e^{i\ell y}\right) = \sum_\ell \langle \ell''_x \rangle^{-1} z_\ell e^{i\ell y}.$$

Let $\theta(t,x,\tau,\xi')$ be a classical tangential *o.p.d.* with support in $\{|\xi'|+|\tau| \leq \alpha_0\}$ and equal to Id near the essential support of Q_1. By (3.11) we have

$$\|Op(\theta)F^\varepsilon - F^\varepsilon; L^2(U,\mathcal{H}^{s_0})\| \in \mathcal{O}(\varepsilon^\infty). \tag{3.30}$$

Therefore we can move $R((1-Op(\theta)F^\varepsilon)$ from the left to the right of (3.27). So we just have to verify

$$R \circ Op(\theta) = Op(q\theta) + \varepsilon Op(\bigoplus_\ell \langle \ell''_x \rangle)^{-1} \circ Op(b) \tag{3.31}$$

with $q \in \mathcal{S}^{t,0}$ so that (3.24) holds true, and $b \in \mathcal{S}^{t,1}$. The b term in (3.31) is defined by $[Op(a_2(x,\xi') + \cdots] \circ Op(\theta) = Op(\theta(a_2 + \cdots)) + \varepsilon Op(b)$ and belongs clearly to $\mathcal{S}^{t,1}$ (there is no loose in the x derivatives of b in (A.15)). Let $\chi(\tau,\xi') \in C_0^\infty$ equal to 1 for $(|\tau|+|\xi'|) \leq 2\alpha_0$. We define q by

$$q = (\bigoplus_\ell \langle \ell''_x \rangle^{-1})\left[a_2(x,\xi') + \sum_{j=1}^{d-1} \frac{\partial a_2}{\partial \xi'_j}(x,\xi')(e_j(x).D_y) - \hat{\rho}(x,y)\tau^2 \right].\chi(\tau,\xi'). \tag{3.32}$$

The estimates $|e_j(x).\ell| \leq C^{te}\langle \ell''_x \rangle$, $j \leq d-1$, and

$$\forall\alpha\ \exists C_\alpha\ |\partial_x^\alpha(\langle \ell''_x \rangle^{-1})| \leq C_\alpha(1+|\ell|)^{|\alpha|}(\langle \ell''_x \rangle^{-1}) \tag{3.33}$$

implies $q \in \mathcal{S}^{t,0}$. The function $a_2(x,\xi')$ is quadratic in ξ' so (3.24) follows from (3.32). □

The eigenvalues of $\mathbb{E}_0 = \Lambda^{-1}\mathbb{N}_0\Lambda$ are $\lambda^0_{\pm,\ell}(x) = \ell^\perp_x \pm i\|\ell''_x\|$. For any x, the set $(e_1(x),\cdots,e_d(x))$ is a basis of $\mathbb{R}^d$, so by the definition (3.15) of $\ell^\perp_x$ and ℓ''_x, there exist $c_1 > 0$ such that

$$|\ell^\perp_x - k^\perp_x| + \|\ell''_x - k''_x\| \geq 4c_1|\ell - k| \qquad \forall x, \forall k,\ell \in \mathbb{Z}^d. \tag{3.34}$$

This implies the following separation property for the spectrum of $\mathbb{E}_0$ near the real axis

Lemma 3.2. *For any $x,\ell \in \mathbb{Z}^d$ such that $\|\ell''_x\| \leq c_1$, one has*

$$\text{dist}\ (\{\lambda^0_{\pm,\ell}(x)\}\ ,\{\lambda^0_{\pm,k}(x)\}) \geq c_1 \qquad \forall k \neq \ell. \tag{3.35}$$

Proof. If (3.35) is false, one has $|\ell^\perp_x - k^\perp_x| < c_1$ and

$$|\|\ell''_x\| - \|k''_x\|| < c_1, \text{ so we get } |\ell^\perp_x - k^\perp_x| + \|\ell''_x - k''_x\| < c_1 + 3c_1$$

in contradiction with (3.34). □

Let $Sp_0(x)$ be the spectrum of $\mathbb{E}_0(x)$

$$Sp_0(x) = \bigcup_{\pm,\ell} \lambda^0_{\pm,\ell}(x). \tag{3.36}$$

By (3.21), for $\lambda \notin Sp_0(x)$ the resolvant $(\lambda - \mathbb{E}_0(x))^{-1}$ is diagonal with respect to the decomposition $\bigoplus_\ell e^{i\ell y}\mathbb{C}^2$, $(\lambda - \mathbb{E}_0(x))^{-1} = \bigoplus_\ell (\lambda - \mathbb{E}_{0,\ell}(x))^{-1}$ with

$$(\lambda - \mathbb{E}_{0,\ell}(x))^{-1} = \frac{1}{(\lambda - \lambda^0_{+,\ell})(\lambda - \lambda^0_{-,\ell})} \begin{pmatrix} \lambda - \ell^\perp_x & -\langle \ell''_x \rangle \\ \frac{\|\ell''_x\|^2}{\langle \ell''_x \rangle} & \lambda - \ell^\perp_x \end{pmatrix}. \tag{3.37}$$

Lemma 3.3. *For any $c_0 > 0$, there exist M such that for any x, $\operatorname{dist}(\lambda, Sp_0(x)) \geq c_0$ implies*

$$\|(\lambda - \mathbb{E}_{0,\ell}(x))^{-1}\| \leq M \qquad \forall \ell. \tag{3.38}$$

Proof. We may suppose $Im\lambda \geq 0$. Then we have $|\lambda - \lambda^0_{+,\ell}||\lambda - \lambda^0_{-,\ell}| \geq c_0|\lambda - \ell^\perp_x + i\|\ell''_x\|\,|$, so

$$\frac{|\lambda - \ell^\perp_x|}{|\lambda - \lambda^0_{+,\ell}||\lambda - \lambda^0_{-,\ell}|} \leq \frac{1}{c_0}$$

and

$$\frac{c_0\langle \ell''_x \rangle}{|\lambda - \lambda^0_{+,\ell}||\lambda - \lambda^0_{-,\ell}|} \leq \frac{\sqrt{1 + \|\ell''_x\|^2}}{\max(c_0, \|\ell''_x\|)}.$$

The lemma follows from these two inequalities by (3.37). □

Let us define $\mathbb{E} = \mathbb{E}(t, x, \tau, \xi')$ by (see (3.25))

$$\mathbb{E} = \mathbb{E}_0 + \begin{pmatrix} 0 & 0 \\ q\theta & 0 \end{pmatrix}. \tag{3.39}$$

For $\beta > 0$, let $\Sigma_\beta(x) \subset \mathbb{Z}^d$ be the set

$$\Sigma_\beta(x) = \{\ell \in \mathbb{Z}^d, \|\ell''_x\| < \beta\} \tag{3.40}$$

and for $\ell \in \mathbb{Z}^d$, let $\gamma_\ell(x)$ be the circle

$$\gamma_\ell(x) = \{z \in \mathbb{C}, |z - \ell^\perp_x| = c_1/4\} \tag{3.41}$$

where c_1 is the constant of Lemma 3.2.

Now, we fixe β, $0 < \beta \ll c_1/4$. Then for any x and $\ell \in \Sigma_\beta(x)$, one has $|\lambda^0_{\pm,\ell}(x) - \ell^\perp_x| \leq \beta \ll c_1/4$, so the eigenvalues $\lambda^0_{\pm,\ell}(x)$ are the only ones inside the circle $\gamma_\ell(x)$. By Lemma 3.3, one gets

$$\left\{ \begin{array}{l} (\lambda - \mathbb{E}_0(x))^{-1} \in \mathcal{A}^0 \\ \|(\lambda - \mathbb{E}_0(x))^{-1}; \mathcal{H}^0 \to \mathcal{H}^0\| \leq M \end{array} \right\} \quad \begin{array}{l} \forall \lambda \in \bigcup_{\ell \in \Sigma_\beta(x)} \gamma_\ell(x) \\ \forall x. \end{array} \tag{3.42}$$

We then apply Lemma A.1: q vanishes on $\xi' = 0, \tau = 0$ and θ is supported in $|\xi'| + |\tau| \leq \alpha_0$. Therefore, if α_0 is small enough, the resolvant $(\lambda - \mathbb{E}(t, x, \tau, \xi'))^{-1}$ exist for any $(t, x, \tau, \xi', \lambda)$ for $\lambda \in \bigcup_{\ell \in \Sigma_\beta(x)} \gamma_\ell(x)$. Obviously, one has

$$\frac{1}{2i\pi}\int_{\gamma_\ell(x)}(\lambda-\mathbb{E}_0(x))^{-1}d\lambda\left(\sum_k z_k e^{iky}\right)=z_\ell e^{i\ell y}. \tag{3.43}$$

We choose $\psi\in C_0^\infty(]-1,1[)$ equal to 1 on $[-1/2,1/2]$ and we define $pr_0(x), pr(t,x,\tau,\xi')$ by the formulas

$$pr_0(x)\left[\sum_\ell z_\ell e^{i\ell y}\right]=\sum_\ell \psi\left(\frac{\|\ell''_x\|^2}{\beta^2}\right)z_\ell e^{i\ell y} \tag{3.44}$$

$$pr(t,x,\tau,\xi')=\sum_\ell\psi\left(\frac{\|\ell''_x\|^2}{\beta^2}\right)\frac{1}{2i\pi}\int_{\gamma_\ell(x)}(\lambda-\mathbb{E}(t,x,\tau,\xi'))^{-1}d\lambda. \tag{3.45}$$

The next lemma shows that pr is well defined.

Lemma 3.4. *There exists a 2×2 matrix $\delta pr(t,x,\tau,\xi')$ with entries in $\mathcal{S}^{t,0}$, such that*

$$\begin{cases} pr=pr_0+\delta pr\\ \delta pr_{|\xi'=0,\tau=0}=0.\end{cases} \tag{3.46}$$

Proof. See Appendix B.

Let $\varphi(t,x)\in C_0^\infty(U)$ equal to 1 near (t_0,x'_0). We next define $Q_0(t,x)$ and $Q(t,x,\tau,\xi')$ by the formulas, where $\langle\ell_x^\perp\rangle=\sqrt{1+|\ell_x^\perp|^2}$, and $\sigma=2|s_0|+2$

$$Q_0(t,x)\left[\Sigma z_\ell e^{i\ell y}\right]=\varphi\sum_\ell\psi\left(\frac{\|\ell''_x\|^2}{4\beta^2}\right)\frac{1}{\langle\ell_x^\perp\rangle^\sigma}\begin{pmatrix}0 & -\langle\ell''_x\rangle\\ \frac{\|\ell''_x\|^2}{\langle\ell''_x\rangle} & 0\end{pmatrix}(z_\ell)e^{i\ell y} \tag{3.47}$$

$$Q(t,x,\tau,\xi')=\varphi\sum_\ell\psi\left(\frac{\|\ell''_x\|^2}{4\beta^2}\right)\frac{1}{\langle\ell_x^\perp\rangle^\sigma}\frac{1}{2i\pi}\int_{\gamma_\ell(x)}(\lambda-\mathbb{E}(t,x,\tau,\xi'))^{-1}(\mathbb{E}(t,x,\tau,\xi')-\ell_x^\perp)d\lambda. \tag{3.48}$$

Lemma 3.5. *There exist a 2×2 matrix $\delta Q(t,x,\tau,\xi')$ with entries in $\mathcal{S}^{t,-\sigma}$, such that*

$$\begin{cases} Q=Q_0+\delta Q\\ \delta Q_{|\xi'=0,\tau=0}=0.\end{cases} \tag{3.49}$$

Proof. See Appendix B.

We then define $F^{\varepsilon,\mathbb{R}}$ and $F^{\varepsilon,I}$ by

$$\begin{cases} F^{\varepsilon,\mathbb{R}}=Op(pr)F^\varepsilon\\ F^{\varepsilon,I}=F^\varepsilon-F^{\varepsilon,\mathbb{R}}.\end{cases} \tag{3.50}$$

The Lemmas 3.4, A.2, the estimates (3.23) and (3.5), and the assumption $\|u^\varepsilon\|\le 1$ imply

$$\sup_\varepsilon\|F^{\varepsilon,\mathbb{R},I};L^2(U;\mathcal{H}^{s_0})\|<+\infty \tag{3.51}$$

$$\sup_{\varepsilon} \varepsilon^{1/2} \| Tr_{0,1}(F^{\varepsilon,\mathbb{R},I}); L^2(U_0; H^{s_0}(\mathbb{T}^d) \| < +\infty. \tag{3.52}$$

Moreover, $F^{\varepsilon,I}$ satisfies the following elliptic estimate

Lemma 3.6. *There exist $D(t, x', \tau, \xi') \in \mathcal{S}^0$ such that*

$$\sup_{\varepsilon} \varepsilon^{-1/2} \| Tr_1(F^{\varepsilon,I}) - Op(D) Tr_0(F^{\varepsilon,I}); L^2(U_0; H^{s_0-1}(\mathbb{T}^d) \| < +\infty.$$

Proof. See Appendix B.

To simplify notations, for $A \in \mathcal{S}^{t,*}$ we define $\widetilde{A}$ by $\widetilde{A} = Op(A)$, and for g^ε, a family depending on ε in a norm space B, $g^\varepsilon \in \varepsilon^\alpha B$ means $\sup_\varepsilon \varepsilon^{-\alpha} \|g^\varepsilon; B\| < +\infty$. We denote also by δ various symbols in $\mathcal{S}^0$ such that $\delta_{|\xi'=0,\tau=0} = 0$. We first notice that $Tr_0(F^\varepsilon) = 0$ and (3.44) imply $Tr_0(\widetilde{pr}_0(F^\varepsilon)) = 0$, so by Lemma 3.4 we get

$$Tr_0(F^{\varepsilon,\mathbb{R}}) = \tilde{\delta} Tr_1(F^{\varepsilon,\mathbb{R}} + F^{\varepsilon,I}). \tag{3.53}$$

By Lemma 3.6, and the Lemmas A.2 and A.3 on the symbolic calculus, we deduce from (3.53)

$$Tr_0(F^{\varepsilon,\mathbb{R}}) + \tilde{\delta} Tr_1(F^{\varepsilon,\mathbb{R}}) + \tilde{\delta} Tr_0(F^{\varepsilon,I}) \in \varepsilon^{1/2} L^2(U_0, H^{s_0-1}). \tag{3.54}$$

We have $Tr_0(F^{\varepsilon,I}) = -Tr_0(F^{\varepsilon,\mathbb{R}})$, so (3.54) may be rewrite as a boundary condition for $F^{\varepsilon,\mathbb{R}}$

$$\left(1 - \tilde{\delta}\right) Tr_0 \left(F^{\varepsilon,\mathbb{R}}\right) + \tilde{\delta} Tr_1 \left(F^{\varepsilon,\mathbb{R}}\right) \in \varepsilon^{1/2} L^2(U_0, H^{s_0-1}). \tag{3.55}$$

By Lemma 3.1, $F^{\varepsilon,\mathbb{R}}$ satisfy the equation

$$\left\{ \begin{array}{l} \frac{\varepsilon}{i} \partial_{x_d} F^{\varepsilon,\mathbb{R}} + \widetilde{\mathbb{E}} F^{\varepsilon,\mathbb{R}} = G^{\varepsilon,\mathbb{R}} \\ G^{\varepsilon,\mathbb{R}} = \widetilde{pr}(G^\varepsilon) + [\widetilde{E}, \widetilde{pr}] F^\varepsilon + \frac{\varepsilon}{i} (\partial_{x_d} \widetilde{pr}) F^\varepsilon. \end{array} \right. \tag{3.56}$$

By construction, we have $[\mathbb{E}, pr] \equiv 0$, so by Lemma A.3 $[\widetilde{\mathbb{E}}, \widetilde{pr}] \in \varepsilon \mathcal{S}^{t,2}$ and from (3.23, 3.26) and Lemma A.2 we deduce

$$G^{\varepsilon,\mathbb{R}} \in \varepsilon L^2(U, \mathcal{H}^{s_0-2}). \tag{3.57}$$

For $u(x,y) \in L^2(U, \mathcal{H}^s), v(x,y) \in L^2(U, \mathcal{H}^{-s})$ let $\langle u|v \rangle$ be the duality

$$\langle u|v \rangle = \int_U \left(\int_{\mathbb{T}^d} u(x,y) \bar{v}(x,y) dy \right) dx \tag{3.58}$$

and let us define $J : L^2(U, \mathcal{H}^s) \to L^2(U, \mathcal{H}^s)$ by

$$J \begin{pmatrix} u_0(x,y) \\ u_1(x,y) \end{pmatrix} = \begin{pmatrix} u_1(x,y) \\ u_0(x,y) \end{pmatrix}. \tag{3.59}$$

By the choice $\sigma = 2|s_0| + 2$ and Lemma 3.6 (A.2) we have

$$\left\{ \begin{array}{l} J\widetilde{Q} F^{\varepsilon,\mathbb{R}} \in L^2(U; \mathcal{H}^{|s_0|+2}) \\ J\widetilde{Q} F^{\varepsilon,\mathbb{R}} \text{ is compactly supported in } U. \end{array} \right. \tag{3.60}$$

Multiplying (3.56) by $J\widetilde{Q}F^{\varepsilon,\mathbb{R}}$, we obtain (where $(\cdot|\cdot)$ is the duality on $x_d = 0$)

$$\begin{cases} \left\langle G^{\varepsilon,\mathbb{R}}|J\widetilde{Q}F^{\varepsilon,\mathbb{R}}\right\rangle = \left\langle \frac{\varepsilon}{i}\partial_{x_d}F^{\varepsilon,\mathbb{R}}|J\widetilde{Q}F^{\varepsilon,\mathbb{R}}\right\rangle + \left\langle J\widetilde{\mathbb{E}}F^{\varepsilon,\mathbb{R}}|\widetilde{Q}F^{\varepsilon,\mathbb{R}}\right\rangle \\ = -\frac{\varepsilon}{i}\left(F^{\varepsilon,\mathbb{R}}|_{x_d=0}|J\widetilde{Q}F^{\varepsilon,\mathbb{R}}|_{x_d=0}\right) + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 \end{cases} \tag{3.61}$$

$$\mathcal{J}_1 = \left\langle F^{\varepsilon,\mathbb{R}}|\left\{\left(J\widetilde{\mathbb{E}}\right)^* - J\widetilde{\mathbb{E}}\right\}\widetilde{Q}F^{\varepsilon,\mathbb{R}}\right\rangle$$

$$\mathcal{J}_2 = \left\langle JF^{\varepsilon,\mathbb{R}}|\left[\frac{\varepsilon}{i}\partial_{x_d} + \widetilde{\mathbb{E}}, \widetilde{Q}\right]F^{\varepsilon,\mathbb{R}}\right\rangle$$

$$\mathcal{J}_3 = \left\langle JF^{\varepsilon,\mathbb{R}}|\widetilde{Q}G^{\varepsilon,\mathbb{R}}\right\rangle.$$

By (3.57) and (3.60), both $|\langle G^{\varepsilon,\mathbb{R}}|J\widetilde{Q}F^{\varepsilon,\mathbb{R}}\rangle|$ and $\mathcal{J}_3$ are $\mathcal{O}(\varepsilon)$. By construction of Q (see (3.50)) we have $[\mathbb{E}, Q] \equiv 0$ so by Lemma A.3, we get $\mathcal{J}_2 \in \mathcal{O}(\varepsilon)$. Finally, we have

$$J\mathbb{E} = J\mathbb{E}_0 + \begin{pmatrix} q\theta & 0 \\ 0 & 0 \end{pmatrix} \; ;$$

$$J\mathbb{E}_0\left(\sum_\ell z_\ell e^{i\ell y}\right) = \sum_\ell \begin{pmatrix} \frac{\|\ell_x''\|^2}{\langle \ell_x''\rangle} & \ell_x^\perp \\ \ell_x^\perp & -\langle \ell_x''\rangle \end{pmatrix} (z_\ell)e^{i\ell y}$$

so we obtain

$$(J\widetilde{\mathbb{E}})^* - (J\widetilde{E}) = \begin{pmatrix} (\widetilde{q\theta})^* - q\theta & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.62}$$

By formula (3.32) and if we choose $\theta(t, x, \tau, \xi')$ real, $q\theta$ is self adjoint, so from Lemma A.4 we deduce $\mathcal{J}_1 \in \mathcal{O}(\varepsilon)$. Summing up, we have thus

$$\sup_\varepsilon \left| \left(F^{\varepsilon,\mathbb{R}}|_{x_d=0}|J\widetilde{Q}F^{\varepsilon,\mathbb{R}}|_{x_d=0}\right) \right| < +\infty. \tag{3.63}$$

We now remark that if $\delta_1 \in \mathcal{S}^0$ vanishes on $\xi' = 0, \tau = 0$, there exist $\delta_2 \in \mathcal{S}^0$, vanishing on $\xi' = \tau = 0$ such that $(1 + \delta_2)(1 - \delta_1) = 1 - p$, where $p(t, x, \tau, \xi') \in \mathcal{S}^0$, is supported in $c_0 \le |\tau| + |\xi'| \le 1/c_0$ for some $c_0 > 0$. Decreasing α_0, hence W_0, if necessary, we then will have $\tilde{p}Tr_{0,1}(F^{\varepsilon,\mathbb{R}}) \in \varepsilon^{1/2}L^2(U_0, H^{s_0-1})$. Using once more Lemma A.3, we can thus rewrite the boundary condition (3.55) on the form

$$Tr_0\left(F^{\varepsilon,\mathbb{R}}\right) - \tilde{\delta}Tr_1\left(F^{\varepsilon,\mathbb{R}}\right) \in \varepsilon^{1/2}L^2\left(U_0, H^{s_0-1}\right). \tag{3.64}$$

Let $Q = \begin{pmatrix} Q^1 & Q^2 \\ Q^3 & Q^4 \end{pmatrix}$; inserting (3.64) in (3.63) and taking in account the *a priori* estimate (3.52) we get

$$\begin{cases} \sup_\varepsilon |(Tr_1(F^{\varepsilon,\mathbb{R}})|\widetilde{A}Tr_1(F^{\varepsilon,\mathbb{R}}))| < +\infty \\ A = Q^2 + Q^1\delta + \delta^*Q^3\delta + \delta^*Q^4. \end{cases} \tag{3.65}$$

Let (we use Lem. 3.4 for the second equality)

$$F_0^{\varepsilon,\mathbb{R}} = \widetilde{pr}_0(F^\varepsilon) = F^{\varepsilon,\mathbb{R}} + \delta(F^{\varepsilon,\mathbb{R}} + F^{\varepsilon,I}). \tag{3.66}$$

We know already that $Tr_0(F^{\varepsilon,\mathbb{R}}), Tr_0(F^{\varepsilon,I})$ and $Tr_1(F^{\varepsilon,I})$ are of the form $\tilde{\delta} Tr_1(F^{\varepsilon,\mathbb{R}}) + \varepsilon^{1/2} L^2(U_0, H^{s_0-1})$ so we get from (3.66)

$$Tr_1\left(F_0^{\varepsilon,\mathbb{R}}\right) = (1+\delta) Tr_1\left(F^{\varepsilon,\mathbb{R}}\right) + \varepsilon^{1/2} L^2\left(U_0, H^{s_0-1}\right). \tag{3.67}$$

Decreasing α_0 if necessary we get as above

$$Tr_1\left(F^{\varepsilon,\mathbb{R}}\right) = (1+\delta) Tr_1\left(F_0^{\varepsilon,\mathbb{R}}\right) + \varepsilon^{1/2} L^2(U_0, H^{s_0-1}). \tag{3.68}$$

Therefore (3.65) and (3.68) imply

$$\begin{cases} \sup_{\varepsilon} |(Tr_1(F_0^{\varepsilon,\mathbb{R}}) | \widetilde{A}_0 Tr_1(F_0^{\varepsilon,\mathbb{R}}))| < +\infty \\ A_0 = Q_0^2 + \delta^* A_1 + A_2 \delta \end{cases} \tag{3.69}$$

with $A_1, A_2 \in \mathcal{S}^{+\sigma}$.

By (3.66, 3.44), $Tr_1(F_0^{\varepsilon,\mathbb{R}})$ is of the form

$$Tr_1(F_0^{\varepsilon,\mathbb{R}}) = \sum_{\|\ell''_{(x',0)}\| \le \beta} z_\ell(t,x') e^{i\ell y} \tag{3.70}$$

and we may assume that the functions $z_\ell(t,x')$ are supported in $\{\varphi \equiv 1\}$.

For $\|\ell''_x\| \le \beta$ we have $\psi\left(\frac{\|\ell''_x\|^2}{4\beta^2}\right) = 1$, and $\frac{|\langle \ell''_x \rangle|}{\langle \ell_x^\perp \rangle^\sigma} \sim (1+|\ell|)^{-(2|s_0|+2)}$; from (3.47) we therefore get for some $C_0 > 0$

$$|(Tr_1(F_0^{\varepsilon,\mathbb{R}}) | Q_0^2 Tr_1(F_0^{\varepsilon,\mathbb{R}}))| \ge C_0 \|Tr_1(F_0^{\varepsilon,\mathbb{R}}); L^2(U_0, H^{s_0-1})\|^2. \tag{3.71}$$

We now remark that in (3.69), we may replace any $\delta(t,x',\tau,\xi')$ term by $\chi((\tau,\xi')/\alpha_0)\delta(t,x',\tau,\xi')$, with $\chi \in C_0^\infty$ equal to 1 in the unit ball, and

$$\chi((\tau,\xi')/\alpha_0)\delta = \sum_{j=1}^{d-1} \chi((\tau,\xi')/\alpha_0)\xi'_j b_j + \chi((\tau,\xi')/\alpha_0)\tau b_0$$

where $b_* \in \mathcal{S}^0$ so we have, for some $C_1 > 0$

$$|(Tr_1(F_0^{\varepsilon,\mathbb{R}}) | (\widetilde{\delta A_0}) Tr_1(F_0^{\varepsilon,\mathbb{R}}))| \le C_1 \alpha_0 \|Tr_1(F_0^{\varepsilon,\mathbb{R}}); L^2(U_0, H^{s_0-1})\|^2. \tag{3.72}$$

From (3.69, 3.71, 3.72) we get, for α_0 small,

$$\sup_{\varepsilon} \|Tr_1(F_0^{\varepsilon,\mathbb{R}}) \ ; L^2(U_0 \ ; \ H^{s_0-1})\| < +\infty \tag{3.73}$$

so by (3.68), the same estimate holds true for $Tr_1(F^{\varepsilon,\mathbb{R}})$, hence also for

$$Tr_1(F^{\varepsilon,I}) = \tilde{\delta} Tr_1(F^{\varepsilon,\mathbb{R}}) + \varepsilon^{1/2} L^2(U_0, H^{s_0-1}).$$

Thus we have

$$\sup_{\varepsilon} \|Tr_1(F^{\varepsilon}); L^2(U_0, H^{s_0-1})\| < \infty. \tag{3.74}$$

This concludes the proof of Theorem 2. □

4. Propagation estimate

This section is devoted to the proof of Proposition 0.3. We fix a zero order o.p.d. $Q(\varepsilon, t, x, \varepsilon\partial_{x'}, \varepsilon\partial_t)$ equal to Id near K, with essential support in W_1 and we argue by contradiction. If (0.50) is untrue, there exist sequences $\varepsilon_k \to 0$, $\gamma_k \to 0$ $h_k \to 0$, $h_k \geq \varepsilon_k/\gamma_k$, and $u^k \in I^{\varepsilon_k}_{h_k}$ such that

$$\begin{cases} \|u^k\| = 1 \\ \frac{1}{k}\|Q\Gamma_0(u^k)\|^2_{L^2(X_{T_0})} \geq \left[\|\Gamma_0(u^k)_{|x_d=0}\|^2_{L^2(X_{T_0}\cap x_d=0)} + \|u^k\|^2_{L^2((0,T_0)\times V)}\right]. \end{cases} \tag{4.1}$$

In particular the right hand side of the second line in (4.1) goes to zero.

Let $\mathcal{L}$ and $\begin{bmatrix} g_0 \\ g_1 \end{bmatrix}$ be defined by the formula (0.51) with $u^\varepsilon = u^k$. We have

$$L \sim \sum_n \left(\frac{\varepsilon}{i}\right)^n L^n \ , \ L^n = \begin{pmatrix} L^n_1 & L^n_2 \\ L^n_3 & L^n_4 \end{pmatrix} \tag{4.2}$$

so we get

$$\mathcal{L} \sim \begin{pmatrix} h/\varepsilon L^0_1 & L^0_2 \\ h^2/\varepsilon^2 L^0_3 & h/\varepsilon L^0_4 \end{pmatrix} + \frac{h}{i}\begin{pmatrix} L^1_1 & \varepsilon/h L^1_2 \\ h/\varepsilon L^1_3 & L^1_4 \end{pmatrix} + \sum_{n\geq 2}\left(\frac{h}{i}\right)^n \left(\frac{\varepsilon}{h}\right)^{n-1}\begin{pmatrix} L^n_1 & \varepsilon/h L^n_2 \\ h/\varepsilon L^n_3 & L^n_4 \end{pmatrix}. \tag{4.3}$$

By Lemma 2.1, i) $\frac{h}{\varepsilon}L^1_3$ is a smooth function of $x, \xi' = \frac{h}{i}\partial_{x'}, \tau = \frac{h}{i}\partial_t$, defined for $\frac{h}{\varepsilon}(|\xi'| + |\tau|)$ small. Therefore, $\mathcal{L}(h, x, \frac{h}{i}\partial_{x'}, \frac{h}{i}\partial_t)$ is a h-o.p.d. defined for $\frac{h}{\varepsilon}(|\xi'| + |\tau|)$ small, with asymptotic development

$$\mathcal{L} \sim \sum_{n\geq 0}\left(\frac{h}{i}\right)^n \mathcal{L}^n \tag{4.4}$$

and by Lemma 2.1, ii) we get

$$\mathcal{L}_0 = \begin{pmatrix} a_0^{-1}(x)a_1(x, i\xi') & -a_0^{-1}(x) \\ a_2(x, i\xi') + \underline{\rho}(x)\tau^2 & 0 \end{pmatrix} + 0\left(\left(\frac{\varepsilon}{h}\right)^2\tau^4\right). \tag{4.5}$$

Let $\underline{u}^k$ be the extension of u^k by zero outside Ω. Let μ be a h-semiclassical measure associated to $\{\underline{u}^k\}$ (see [8]). Let $\underline{g}^k_0$ the extension of $g^k_0 = \Gamma_0(u^k)$ by zero on $x_d < 0$ and let ν be a h-semiclassical measure associated to $\underline{g}^k_0$. Using (2.45) and $\lim_{k\to\infty} \varepsilon_k/h_k = 0$ we get

$$\nu = \chi^2_0(t, x; \xi' = 0, \tau = 0)\mu \qquad (\text{for } x \in \partial\Omega\times]-r_0, r_0[). \tag{4.6}$$

We have $u^k \in I^{\varepsilon_k}_{h_k}$ so we know that μ is supported in $|\tau| \in [0.9, 2.1]$; moreover, by the proof of Proposition 1 Section 2, the support of $\mu_{|\Omega}$ is contained in the set $\underline{\rho}(x)\tau^2 - \|\xi\|^2 = 0$, and $\mu|\Omega$ propagates on the bicharacteristic flow of $\underline{\rho}(x)\tau^2 - \|\xi\|^2$. Let $g^k = \Gamma(u^k)$, and let $A(h, t, x, h\partial_{x'}, h\partial_t)$ be any h-o.p.d. compactly supported in $T^*(X_{T_0})$. Using (0.41, 0.42) and $\lim_{k\to\infty} \varepsilon_k/h_k = 0$ we get with $h = h_k$

$$A\left[\left(h\frac{\partial}{\partial x_d} + \mathcal{L}\right)g^k\right] \in \mathcal{O}(h^\infty L^2). \tag{4.7}$$

Writing $\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ \mathcal{L}_3 & \mathcal{L}_4 \end{pmatrix}$, we observe that the principal symbol of $\mathcal{L}_1$ vanishes near $x_d = 0$. Using (4.1, 4.7) we get that g_0^k satisfies near the boundary the second order tangential h-pseudo differential equation,with $h = h_k$

$$\begin{cases} A\left[(h\partial_{x_d})^2 g_0^k + \left(R_2 + hR_1 h\frac{\partial}{\partial x_d}\right) g_0^k\right] \in \mathcal{O}(h^\infty L^2) \\ \lim\limits_{k\to\infty} \|g_0^k|_{x_d=0}\|_{L^2} = 0 \end{cases} \tag{4.8}$$

where $R_{1,2}$ are h-tangential o.p.d. defined for $\frac{h}{\varepsilon}(|\xi'| + |\tau|)$ small, and the principal symbol of R_2 , R_2^0 is given by

$$R_2^0 = a_2(x, i\xi') + \underline{\rho}(x)\tau^2 + 0\left(\left(\frac{\varepsilon}{h}\right)^2 \tau^4\right). \tag{4.9}$$

We can now use the propagation theorem at the boundary for second order Dirichlet problem (see [8] for the localization and propagation at hyperbolic point and [10], Append. or [3], Th. 1 for the propagation result near the glancing set; here we view $\gamma_k = \varepsilon_k/h_k$ as a small parameter in equation (4.8), and we notice that the proof of the propagation theorem allows this additional parameter going to zero). We get that the support of ν is contained in the set $\underline{\rho}(x)\tau^2 - \|\xi\|^2 = 0$, and that the support of ν propagates along the generalized bicharacteristic flow of $\underline{\rho}(x)\tau^2 - \|\xi\|^2$; but (4.1) implies $\mu_{|]0,T_0[\times V} \equiv 0$, so from (0.9) and (4.6) we get $\mu_{|t\in]T_0/2-\alpha,T_0/2+\alpha[} \equiv 0$ for α small. This is in contradiction with $\|u^k\| = 1$ by (2.41). □

A. Semi-classical o.p.d. with operator values

A.1. **Classical calculus**

We recall here some classical properties of semi-classical tangential pseudo differential operators. Let $Z = \mathbb{R}^p_z \times [0, r_0]_{x_d}$ and H_1, H_2 two separable Hilbert spaces.

We denote by $S^t_Z(H_1 \to H_2)$ the vector space of functions $q(\varepsilon, z, \zeta, x_d)$ defined for $\varepsilon \in]0, \varepsilon_0]$ (ε_0 small) smooth in $(z, \zeta) \in T^*\mathbb{R}^p_z$, $x_d \in [0, r_0]$, compactly supported in z, with values in bounded operators from H_1 to H_2 which satisfies the estimates

$$\begin{aligned} &\forall \alpha, k\ \exists C_{\alpha,k}\ \forall \varepsilon, z, \zeta, x_d \\ &\|(1 + |\zeta|)^k \partial^\alpha_{z,\zeta,x_d} q(\varepsilon, z, \zeta, x_d); H_1 \to H_2\| \le C_{\alpha,k} \end{aligned} \tag{A.1}$$

and admitting classical asymptotic expansions in ε

$$q \sim \sum_{n=0}^{\infty} \left(\frac{\varepsilon}{i}\right)^n q_n(z, \zeta, x_d) \Leftrightarrow \forall N \quad q - \sum_{n<N} \left(\frac{\varepsilon}{i}\right)^n q_n \in \varepsilon^N S^t_Z. \tag{A.2}$$

For $f(z, x_d) \in L^2(Z, H_1)$ with compact support in z, the Fourier transform $\hat{f}_\varepsilon(\zeta, x_d)$ is defined by

$$\hat{f}_\varepsilon(\zeta, x_d) = \int e^{-iz\zeta/\varepsilon} f(z, x_d) dz \in L^2(\mathbb{R}^p_\zeta \times [0, x_d], H_1) \tag{A.3}$$

and for $q \in S^t_Z(H_1, H_2)$, $Op(q)(f)$ is defined by

$$Op(q)(f)(\varepsilon, z, x_d) = (2\pi\varepsilon)^{-p} \int e^{iz\zeta/\varepsilon} q(\varepsilon, z, \zeta, x_d)[\hat{f}_\varepsilon(\zeta, x_d)] d\zeta. \tag{A.4}$$

We define the set $\mathcal{E}_Z^t(H_1 \to H_2)$ of tangential pseudo differential operators from $L^2(Z,H_1)$ to $L^2(Z,H_2)$ by

$$\left\{ \begin{array}{l} Q = Q_\varepsilon \in \mathcal{E}_Z^t(H_1 \to H_2) \text{ iff there exist } \varphi(z) \in C_0^\infty(\mathbb{R}_z^p) \\ \text{and } \tilde{q} \in S_Z^t(H_1 \to H_2) \text{ such that} \\ Q_\varepsilon(f)(z) = Op(\tilde{q})[\varphi(z)f] \quad \forall f \in L^2(Z,H_1). \end{array} \right. \tag{A.5}$$

For $Q \in \mathcal{E}_Z^t(H_1 \to H_2)$, one has $Q = Op(q)$ with

$$q(\varepsilon,z,\zeta,x_d) = (2\pi)^{-p} \int e^{-it\theta} \tilde{q}(\varepsilon,z,\zeta+\varepsilon\theta,x_d)\varphi(z+t)dtd\theta$$

and Q is bounded on L^2, *i.e.*

$$\exists C \quad \forall\varepsilon \quad \|Q_\varepsilon(f); L^2(Z,H_2)\| \le C\|f; L^2(Z,H_1)\|. \tag{A.6}$$

For $Q_1 = Op(q_1) \in \mathcal{E}_Z^t(H_1 \to H_2)$ and $Q_2 = Op(q_2) \in \mathcal{E}_Z^t(H_2 \to H_3)$, one has $Q_1 \circ Q_2 = Op(q) = Q \in \mathcal{E}_Z^t(H_1 \to H_3)$ with

$$q(\varepsilon,z,\zeta,x_d) = (2\pi)^{-p} \int e^{-it\theta} q_1(\varepsilon,z,\zeta+\varepsilon\theta,x_d) \circ q_2(\varepsilon,z+t,\zeta,x_d)dtd\theta$$

and the asymptotic expansion of q is given by the rule

$$q \sim \sum_\alpha \left(\frac{\varepsilon}{i}\right)^{|\alpha|} \frac{1}{\alpha!} \partial_\zeta^\alpha q_1 \circ \partial_z^\alpha q_2. \tag{A.7}$$

The set of operators $\mathcal{E}_Z^t(H_1 \to H_2)$ is free of coordinates, *i.e.*, if $z \mapsto \phi(z)$ is a smooth diffeomorphism of $\mathbb{R}_z^p$, and $Q \in \mathcal{E}_Z^t$, then $\phi \circ Q \circ \phi^{-1} \in \mathcal{E}_Z^t$. Thus, in the definition of $\mathcal{E}_Z^t$, we can replace $\mathbb{R}_z^p$ by a smooth manifold M. For $Q = Op(q) \in \mathcal{E}_Z^t(H_1 \to H_2)$ its principal symbol, $q_0(z,\zeta,x_d)$ is then defined as a smooth function of $(z,\zeta,x_d) \in T^*M \times [0,r_0]$, with values in bounded operators from H_1 to H_2. For $Q = Op(q) \in \mathcal{E}_Z^t$, the essential support of Q $SE(Q)$ is the closed subset of $T^*M \times [0,r_0]$ defined by

$$\left\{ \begin{array}{l} \rho_0 = (z_0,\zeta_0,x_{d,0}) \notin SE(Q) \text{ iff there exists a neighborhood} \\ W \text{ of } \rho_0 \text{ such that } q_{|W} \sim 0. \end{array} \right. \tag{A.8}$$

Let K be a compact subset of $T^*M \times [0,r_0]$. One says that $Q_1 \equiv Q_2$ near K if $SE(Q_1 - Q_2) \cap K = \phi$ and if $u : H_1 \to H_2$ is bounded, $Q \equiv u$ near K means $Q - \varphi(y,x_d)u \equiv 0$ for some $\varphi \in C_0^\infty(M \times [0,r_0])$ equal to 1 near the projection of K on $M \times [0,r_0]$. If $Q \equiv 0$ near K, for any scalar tangential o.p.d. $P \in \mathcal{E}_Z^t(\mathbb{C} \to \mathbb{C})$, such that $SE(P) \subset K$ one has

$$\forall N, \exists C_N \|QP \text{ or } PQ; L^2(Z,H_1) \to L^2(Z,H_2)\| \le C_N \varepsilon^N. \tag{A.9}$$

One says that $Q = Op(q) \in \mathcal{E}_Z^t$ is elliptic on K if for any $\rho = (z,\zeta,x_d) \in K$, the principal symbol $q_0(\rho)$ is an isomorphism from H_1 onto H_2. In that case, there exist $E \in \mathcal{E}_Z^t(H_2 \to H_1)$ with principal symbol e_0 equal to q_0^{-1} near K such that $E \circ Q \equiv Id_{H_1}$ and $Q \circ E \equiv Id_{H_2}$ near K.

A.2. **An exotic calculus**

Let $\mathbb{T}_Y^d$ be the d-dimensional torus, and for $s \in \mathbb{R}$, H^s the usual Sobolev space

$$H^s = \left\{ \sum_{\ell \in \mathbb{Z}^d} a_\ell e^{i\ell y} \ , \ \sum_\ell (1+|\ell|^2)^s |a_\ell|^2 < \infty \right\} \cdot \tag{A.10}$$

For any operator $A : \bigcap_s H^s \to \cup H^s$, we denote by $A_{\ell,k}$ the matrix coefficient

$$A_{\ell,k} = \oint_{\mathbb{T}^d} (Ae^{iky}).e^{-i\ell y}. \tag{A.11}$$

For $m \in \mathbb{R}$, let $\mathcal{A}^m$ be the following class of operators on the torus

$$\mathcal{A}^m = \left\{ A \ ; \forall N, \exists C_N \quad |A_{\ell,k}| \leq C_N \frac{(1+|\ell|)^m}{(1+|\ell-k|)^N} \quad \forall \ell, k \in \mathbb{Z}^d \right\}. \tag{A.12}$$

One has $\mathcal{A}^m \circ \mathcal{A}^{m'} \subset \mathcal{A}^{m+m'}$, and for $A \in \mathcal{A}^m$, A is bounded from H^s to H^{s-m} for any $s \in \mathbb{R}$. The identity

$$[D_j, A]_{\ell,k} = (\ell_j - k_j) A_{\ell,k} \quad D_j = \frac{1}{i}\frac{\partial}{\partial y_j} \tag{A.13}$$

shows that $\mathcal{A}^0$ is the class of bounded operators on $L^2 = H^0$ such that all the commutators

$$[D_{j_1}, [D_{j_2}, \cdots [D_{j_p}, A] \cdots]] \tag{A.14}$$

are bounded on L^2.As a consequence, we get

Lemma A.1. *Let $A \in \mathcal{A}^0$, and $\delta < (\|A; L^2 \to L^2\|)^{-1}$. Then $(Id + \delta A)^{-1} \in \mathcal{A}^0$.*

Proof. $B = (Id + \delta A)^{-1}$ is bounded on L^2, and all the commutators (A.14) for B can be expressed in terms of commutators for A by iteration of the formula

$$[D_j, B] = -B\delta[D_j, A]B.$$

$\square$

Let $Z = \mathbb{R}^p_z \times [0, r_0]$.

We denote by $\mathcal{S}^{t,m}_Z$ the vector space of functions $A(\varepsilon, z, \zeta, x_d)$ defined for $\varepsilon \in]0, \varepsilon_0]$ smooth in $(z, \zeta) \in T^*\mathbb{R}^p_z$, $x_d \in [0, r_0]$, with values operators on the torus, which satisfy the estimates

$$\begin{cases} \forall \alpha, \beta, \gamma, N, \exists C \ , \quad \forall \varepsilon, \ell, k, z, \zeta, x_d \\ \\ |(1+|\zeta|)^\gamma \partial^\alpha_{z,x_d} \partial^\beta_\zeta A_{\ell,k}(\varepsilon, z, \zeta, x_d)| \leq C \dfrac{(1+|\ell|)^{m+|\alpha|}}{(1+|\ell-k|)^N}. \end{cases} \tag{A.15}$$

In other words, $A \in \mathcal{S}^{t,m}_Z$ means

$$\forall \alpha, \beta, \gamma \quad (1+|\zeta|)^\gamma \partial^\alpha_{z,x_d} \partial^\beta_\zeta A \in \mathcal{A}^{m+|\alpha|}$$

uniformly in $\varepsilon, z, \zeta, x_d$.

Leibniz formula implies

$$\mathcal{S}^{t,m}_Z \circ \mathcal{S}^{t,m'}_Z \subset \mathcal{S}^{t,m+m'}_Z. \tag{A.16}$$

We denote by $\mathcal{B}^{t,m}_Z$ the class of operators

$$\begin{cases} Op(A) \ ; \ A \in \mathcal{S}^{t,m}_Z \\ Op(A)[f](z, x_d) = (2\pi\varepsilon)^{-p} \int e^{iz\zeta/\varepsilon} A(\varepsilon, z, \zeta, x_d)[\hat{f}_\varepsilon(\zeta, x_d)] d\zeta \end{cases} \tag{A.17}$$

where $f \in L^2(Z; H^s)$ for some s, $Z = \mathbb{R}^p_z \times (0, r_0)_{x_d}$, and $\hat{f}_\varepsilon$ is the partial Fourier transform

$$\hat{f}_\varepsilon(\zeta, x_d) = \int e^{-iz\zeta/\varepsilon} f(z, x_d) dz \in L^2(\mathbb{R}^p_\zeta \times (0, r_0); H^s). \tag{A.18}$$

Lemma A.2. *For any $A \in \mathcal{S}^{t,m}_Z$, $Op(A)$ is bounded from $L^2(Z; H^s)$ in $L^2(Z; H^{s-m})$ for any s, uniformly in $\varepsilon \in]0, \varepsilon_0]$.*

Proof. To avoid the loose of derivative in z, we use the fact that A is a Schwartz function in ζ, so we can write, by Fourier inversion formula

$$A(\varepsilon, z, \zeta, x_d) = (2\pi)^{-p} \int e^{i\zeta\theta} B(\varepsilon, z, \theta, x_d) d\theta \tag{A.19}$$

with $B \in \mathcal{S}^{t,m}_Z$; we obtain

$$Op(A)(f) = (2\pi)^{-p} \int B(\varepsilon, z, \theta, x_d)[f(z + \varepsilon\theta, x_d)] d\theta. \tag{A.20}$$

The bounds (A.15) for B (with $\alpha = \beta = 0, |\gamma| = p + 1$) imply

$$\forall s, \exists C_s \sup_\varepsilon \|B(\varepsilon, \cdot, \theta, \cdot); L^2(Z; H^s) \to L^2(Z; H^{s-m})\| \leq \frac{C_s}{(1 + |\theta|)^{p+1}} \tag{A.21}$$

and the lemma follows from (A.21) and

$$\|f(z + \varepsilon\theta, x_d)\| = \|f(z, x_d)\| \text{ in } L^2(Z; H^s).$$

□

The next lemma gives the principal part of the symbolic calculus

Lemma A.3. *For $A_1 \in \mathcal{S}^{t,m_1}_Z$, $A_2 \in \mathcal{S}^{t,m_2}_Z$, one has*

$$\begin{cases} Op(A_1) \circ Op(A_2) = Op(B) \\ B = A_1 \circ A_2 + \varepsilon R \\ B \in \mathcal{S}^{t,m_1+m_2}_Z, R \in \mathcal{S}^{t,m_1+m_2+1}_Z. \end{cases} \tag{A.22}$$

Proof. We have $Op(A_1) \circ Op(A_2) = Op(B)$ with

$$B(\varepsilon, z, \zeta, x_d) = (2\pi)^{-p} \int\int e^{-it\eta} A_1(\varepsilon, z, \zeta + \varepsilon\eta, x_d) \circ A_2(\varepsilon, z + t, \zeta, x_d) d\eta dt. \tag{A.23}$$

Using the Taylor formula $f(\zeta + \varepsilon\eta) = f(\zeta) + \sum_j \varepsilon\eta_j \int_0^1 \frac{\partial f}{\partial\zeta_j}(\zeta + \varepsilon s\eta) ds$ and integrating by part with respect to t_j, we get $B = A_1 \circ A_2 + \varepsilon R$ with

$$R = \frac{1}{i} \sum_j \int_0^1 ds \int\int (2\pi)^{-p} e^{-it\eta} \frac{\partial A_1}{\partial\zeta_j}(\varepsilon, z, \zeta + \varepsilon s\eta, x_d) \circ \frac{\partial A_2}{\partial z_j}(\varepsilon, z + t, \zeta, x_d) d\eta dt. \tag{A.24}$$

We shall verify $R \in \mathcal{S}^{t,m_1+m_2+1}_Z$ (the proof of $B \in \mathcal{S}^{t,m_1+m_2}_Z$ is similar).

If we define $B_2 = \frac{\partial A_2}{\partial z_j} \in \mathcal{S}^{t,m_2+1}_Z$ and $B_1 \in \mathcal{S}^{t,m_1}_Z$ by

$$\frac{\partial A_1}{\partial\zeta_j}(\varepsilon, z, \zeta, x_d) = (2\pi)^{-p} \int e^{i\theta\zeta} B_1(\varepsilon, z, \theta, x_d) d\theta$$

we are reduce to prove

$$\int_0^1 ds \int e^{i\theta\zeta} B_1(\varepsilon, z, \theta, x_d) \circ B_2(\varepsilon, z + \varepsilon s\theta, \zeta, x_d) d\theta \in \mathcal{S}_Z^{t, m_1+m_2+1}. \tag{A.25}$$

The verification of (A.25) is now easy using (A.16), the Leibniz rule for derivatives and the fact that B_1 (resp. B_2) is in the Schwartz space with respect to θ (resp. ζ). □

Lemma A.4. *Let $\psi \in \mathcal{S}(\mathbb{R}^p)$ and $A \in \mathcal{S}_Z^{t,m}$. One has*

$$Op(A)^* \circ \psi(\frac{\varepsilon}{i}\partial z) = Op(A^*\psi(\zeta)) + \varepsilon\ Op(R) \tag{A.26}$$

with $R \in \mathcal{S}_Z^{t,m+1}$.

Proof. We have $Op(A)^* = Op(B)$ with

$$B(\varepsilon, z, \zeta, x_d) = (2\pi)^{-p} \int\int e^{-it\eta} A^*(\varepsilon, z+t, \zeta + \varepsilon\eta, x_d) d\eta dt. \tag{A.27}$$

Using the Taylor formula as before, we get the identity (A.26) with

$$R = \frac{1}{i} \sum_j \int_0^1 ds \int (2\pi)^{-p} e^{-it\eta} \frac{\partial^2 A^*}{\partial z_j \partial \zeta_j}(\varepsilon, z+t, \zeta + \varepsilon s\eta, x_d)\psi(\zeta) dt d\eta. \tag{A.28}$$

As in the proof of Lemma A.3, we just observe that, for $B \in \mathcal{S}_Z^{t,m}$,we have

$$\int_0^1 ds \int e^{i\theta\zeta} B(\varepsilon, z + \varepsilon s\theta, \theta, x_d)\psi(\zeta) d\theta \in \mathcal{S}_Z^{t,m}. \tag{A.29}$$

□

B. Appendix

B.1. Proof of Lemmas 3.4 and 3.5

Let $pr_0^\ell(x)$ and $pr^\ell(t, x, \tau, \xi')$ be the operators

$$pr_0^\ell(x)\left(\Sigma_k z_k e^{iky}\right) = \psi\left(\frac{\|\ell_x''\|^2}{\beta^2}\right) z_\ell\ e^{i\ell y} \tag{B.1}$$

$$pr^\ell(t, x, \tau, \xi') = \psi\left(\frac{\|\ell_x''\|^2}{\beta^2}\right) \int_{\gamma_\ell(x)} (\lambda - \mathbb{E}(t, x, \tau, \xi'))^{-1} \frac{d\lambda}{2i\pi} \tag{B.2}$$

and

$$\delta pr^\ell(t, x, \tau, \xi') = pr^\ell(t, x, \tau, \xi') - pr_0^\ell(x). \tag{B.3}$$

Let $z = (t, x), \zeta = (\tau, \xi')$, and $(\delta pr^\ell)_{j,k}$ be the matrix of δpr^ℓ as in Appendix A.2. We shall prove:

For any α, β, γ, N, there exist $C_{\alpha,\beta,\gamma,N}$ such that

$$\begin{cases} (1+|\zeta|)^{\gamma}|\partial_z^{\alpha}\partial_{\zeta}^{\beta}(\delta pr^{\ell}(z,\zeta))_{j,k}| \leq C_{\alpha,\beta,\gamma,N}\frac{(1+|\ell|)^{|\alpha}}{(1+|\ell-j|+|\ell-k|)^N} \\ \text{for every } z,\zeta,j,k,\ell \end{cases} \tag{B.4}$$

$$\delta pr^{\ell}(z,0) = 0 \text{ for any } \ell, z. \tag{B.5}$$

The two Lemmas 3.4 and 3.5 are consequences of the two properties (B.4, B.5), by definition of the class $\mathcal{S}^{t,m}$ (see (A.15)). In fact (B.4) implies that the series ((3.45) and (3.48)) are convergent in the class $\mathcal{S}^{t,0}$ and $\mathcal{S}^{t,-\sigma}$.

(B.5) is obvious since $\mathbb{E}(z,0) = \mathbb{E}_0(z)$ by (3.21) so $pr^{\ell}(z,0) = pr_0^{\ell}(z)$. We have also $\mathbb{E}(z,\zeta) = \mathbb{E}_0(z)$ for $|\zeta| \geq \alpha_0$ since $\mathbb{E}(z,\zeta) = \mathbb{E}_0(z) + \begin{pmatrix} 0 & 0 \\ q\theta & 0 \end{pmatrix}$ and $\theta(z,\zeta) = 0$ for $|\zeta| \geq \alpha_0$ so $\delta pr^{\ell}(z,\zeta)$ is compactly supported in ζ and we can forget γ and $(1+|\zeta|)^{\gamma}$ in the proof of (B.4). We firt conjugate $\mathbb{E}(t,x,\tau,\xi')$ by the multiplication by $e^{i\ell y}$. We get by (3.21, 3.32, 3.39)

$$e^{-i\ell y} \circ \mathbb{E}(z,\zeta) \circ e^{i\ell y} = \ell_x^{\perp} Id + \mathbb{E}^{\ell}(z,\zeta) \tag{B.6}$$

$$\mathbb{E}^{\ell}(z,\zeta) = \mathbb{E}_0^{\ell}(z) + \begin{pmatrix} 0 & 0 \\ q^{\ell}\theta & 0 \end{pmatrix} \tag{B.7}$$

$$\mathbb{E}_0^{\ell}(z)[\Sigma w_k e^{iky}] = \sum_k \begin{pmatrix} k_x^{\perp} & -\langle (k+\ell)_x'' \rangle \\ \frac{\|(k+\ell)_x''\|^2}{\langle (k+\ell)_x'' \rangle} & k_x^{\perp} \end{pmatrix} (w_k) e^{iky} \tag{B.8}$$

$$q^{\ell}(z,\zeta) = \left(\bigoplus_k \langle (k+\ell)_x'' \rangle^{-1} \right) \circ \left[a_2(x,\xi') + \sum_{j=1}^{d-1} \frac{\partial a_2}{\partial \xi_j'}(x,\xi')(e_j(x).D_y + e_j(x).\ell) - \hat{\rho}(x,y)\tau^2 \right] \chi. \tag{B.9}$$

We define $\pi_0^{\ell}(z) = e^{-i\ell y} \circ pr_0^{\ell}(z) \circ e^{i\ell y}$,

$$\pi^{\ell}(z,\zeta) = e^{-i\ell y} \circ pr^{\ell}(z,\zeta) \circ e^{i\ell y} \ , \ \delta\pi^{\ell} = \pi^{\ell} - \pi_0^{\ell} \ .$$

We have

$$\pi_0^{\ell}(z) \left[\sum_k z_k e^{iky} \right] = \psi \left(\frac{\|\ell_x''\|^2}{\beta^2} \right) z_0 \tag{B.10}$$

$$\pi^{\ell}(z,\zeta) = \psi \left(\frac{\|\ell_x''\|^2}{\beta^2} \right) \int_{|\lambda| = c_{1/4}} (\lambda - \mathbb{E}^{\ell}(z,\zeta))^{-1} \frac{d\lambda}{2i\pi} \, . \tag{B.11}$$

We are now reduce to prove

$$\left\{\begin{array}{l} \text{For any } \alpha,\beta,N, \text{ there exist } C_{\alpha,\beta,N} \text{ such that} \\ |\partial_z^\alpha\partial_\zeta^\beta(\delta\pi^\ell)_{j,k}(z,\zeta)| \leq C_{\alpha,\beta,N}\dfrac{(1+|\ell|)^{|\alpha|}}{(1+|j|+|k|)^N} \\ \text{for every } z,\zeta,j,k,\ell. \end{array}\right. \tag{B.12}$$

Notice that the spectrum of $\mathbb{E}_0^\ell(z)$, with (ℓ,x) such that $\|\ell''_x\| \leq 2\beta \ll c_1$ can be separate in two pieces: two small eigenvalues $\pm i\|\ell''_x\|$ with associated eigenspace $\mathbb{C}^2 e^{i0y}$, and the other part of the spectrum leaving outside the complex disk $|\lambda| \geq c_1/2$. The same is true for the spectrum $\mathbb{E}^\ell(z,\zeta)$ (the cutt-off function $\chi(z,\zeta)$ localize q^ℓ in $|\zeta| \leq 2\alpha_0 \ll \beta$). In order to prove (B.12), we use a Grushin method.

Let $\widehat{\mathcal{A}}^m$ be the set of operators on $(\mathcal{D}'(\mathbb{T}^d))^2 \oplus \mathbb{C}^2$ of the form

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{B.13}$$

with A a 2×2 matrix with entries in $\mathcal{A}^m$ (see Append. A.2) B a linear map from $\mathbb{C}^2$ in $(C^\infty(\mathbb{T}^d))^2$, C a continuous linear map from $(\mathcal{D}'(\mathbb{T}^d))^2$ in $\mathbb{C}^2$ and $D \in \mathcal{M}^2(\mathbb{C})$. As in Appendix A.2, we remark that $\hat{\mathcal{A}}^0$ is the class of bounded operators L on $\mathcal{H} = (L^2(\mathbb{T}^d))^2 \oplus \mathbb{C}^2$ such that all the commutators $[\widetilde{\mathcal{D}}_{j_1}[\widetilde{D}_{j_1},\cdots,[\widetilde{D}_{j,p},L]]$ are bounded on $\mathcal{H}$, with

$$\widetilde{D}_j = \begin{pmatrix} \frac{1}{i}\partial_{y_j} & 0 \\ 0 & 0 \end{pmatrix}.$$

In particular, Lemma A.1 remain valid. We denote by $\widehat{\mathcal{S}}_V^m$ the vector space of functions of $(z,\zeta) \in V$, V open

$$L(z,\zeta) = \begin{pmatrix} A(z,\zeta) & B(z,\zeta) \\ C(z,\zeta) & D(z,\zeta) \end{pmatrix} \tag{B.14}$$

where B,C,D are as above and depends smoothly on (z,ζ), and $A \in \mathcal{S}_V^m$. In other words, $L(z,\zeta) \in \widehat{S}^m$ means

$$\forall \alpha,\beta \quad \partial_z^\alpha\partial_\zeta^\beta L \in \widehat{\mathcal{A}}^{m+|\alpha|} \text{ uniformly in } (z,\zeta) \in K \Subset V. \tag{B.15}$$

Let j,p be the injection and projection

$$\left\{\begin{array}{lll} j(w) = we^{i0y} & : & \mathbb{C}^2 \to C^\infty(\mathbb{T}^d)^2 \\ p(f) = \oint f dy & & \mathcal{D}'(\mathbb{T}^2) \to \mathbb{C}^2 \end{array}\right. \tag{B.16}$$

and

$$L^\ell(\lambda,z,\zeta) = \begin{pmatrix} \lambda - \mathbb{E}^\ell(z,\zeta) & j \\ p & 0 \end{pmatrix}. \tag{B.17}$$

Then $L^\ell(\lambda,.,.)$ is a holomorphic family in λ with values in $\widehat{\mathcal{A}}^1_{V_\ell}$, with inverse in $\widehat{\mathcal{A}}^{-1}_{V_\ell}$ for $|\lambda| < \frac{c_1}{2}$, with $V_\ell = \{(z,\zeta); \|\ell''_x\| < 2\beta\}$.

Notice that in view of (B.8, B.9), ℓ''_x can be replace by a small parameter in $\mathbb{R}^{d-1}$ in both $\mathbb{E}_0^\ell, q^\ell$, so all the semi-norms of $(L^\ell(\lambda,z,\zeta))^{-1}$ are uniform in (ℓ,x) such that $\|\ell''_x\| < 2\beta$.

Let $\mathcal{L}^\ell(\lambda,z,\zeta) = (L^\ell(\lambda,z,\zeta))^{-1}$

$$\mathcal{L}^\ell = \begin{pmatrix} A^\ell & B^\ell \\ C^\ell & D^\ell \end{pmatrix}. \tag{B.18}$$

Then $\lambda - \mathbb{E}^\ell(z,\zeta)$ is invertible iff $\det(D^\ell(\lambda,z,\zeta)) \neq 0$, and we have the algebraic identity

$$(\lambda - \mathbb{E}^\ell(z,\zeta))^{-1} = [A^\ell - B^\ell(D^\ell)^{-1}C^\ell](\lambda,z,\zeta). \tag{B.19}$$

The function A^ℓ is holomorphic in $\lambda \in \{|z| < \frac{c_1}{2}\}$ so we get by (B.11)

$$\pi^\ell(z,\zeta) = -\psi\left(\frac{\|\ell''_x\|^2}{\beta^2}\right)\int_{|\lambda|=\frac{c_1}{4}} (B^\ell(D^\ell)^{-1}C^\ell)(\lambda,z,\zeta)\frac{d\lambda}{2i\pi}\,\cdot \tag{B.20}$$

This implies that the estimate (B.12) holds true for π^ℓ, hence for $\delta\pi^\ell$ ((B.12) is obvious for π_0^ℓ).

B.2. **Proof of Lemma 3.6**

One has $[pr(t,x,\tau,\xi'),\mathbb{E}(t,x,\tau,\xi')] \equiv 0$, $pr \in \mathcal{S}^{t,0}, \mathbb{E} \in \mathcal{S}^{t,1}$, so Lemma A.3 implies $[Op(pr),Op(\mathbb{E})] \in \varepsilon\mathcal{S}^{t,2}$. In fact, the more precise estimate $[Op(pr),Op(\mathbb{E})] \in \varepsilon\mathcal{S}^{t,1}$ holds true. To see this, we just observe that we have $\mathbb{E}-\mathbb{E}_0 \in \mathcal{S}^{t,0}$; from the definitions ((3.21) and (3.33)) we get $\partial_\zeta\mathbb{E}_0 = 0, \partial_z\mathbb{E}_0 \in \mathcal{S}^{t,1}$ and the result follows from the symbolic calculus formulas (A.23) and (A.24). We then deduce from Lemma 3.1 that $F^{\varepsilon,I} = Op(Id-pr)(F^\varepsilon)$ satisfies the following equation

$$\frac{\varepsilon}{i}\frac{\partial}{\partial x_d}F^{\varepsilon,I} + Op(\mathbb{E})F^{\varepsilon,I} = G^{\varepsilon,I} \tag{B.21}$$

where, $F^{\varepsilon,I}$ and $G^{\varepsilon,I}$ are such that

$$\begin{cases} \sup_\varepsilon \|F^{\varepsilon,I};L^2(U;\mathcal{H}^{s_0})\| < +\infty \\ \sup_\varepsilon \varepsilon^{-1}\|G^{\varepsilon,I};L^2(U,\mathcal{H}^{s_0-1})\| < +\infty. \end{cases} \tag{B.22}$$

We shall first modified $\mathbb{E}$ in (B.21) in order to work with an elliptic equation. Let us define $\widetilde{pr}_0(x)$ and $\widetilde{pr}(t,x,\tau,\xi')$ by formulas (3.44) and (3.45) with $\psi(\frac{4\|\ell''x\|^2}{\beta^2})$ instead of $\psi(\frac{\|\ell''x\|^2}{\beta^2})$. One has

$$\widetilde{pr}\circ(Id-pr)(t,x,\tau,\xi') \equiv 0 \tag{B.23}$$

and by the proof of Lemma 3.4 one gets

$$\widetilde{pr} = \widetilde{pr}_0 + \delta\widetilde{pr}\ ,\ \delta\widetilde{pr} \in \mathcal{M}_{2,2}(\mathcal{S}^{t,0}), \delta\widetilde{pr}|_{\xi'=\tau=0} = 0. \tag{B.24}$$

Let $\chi(\tau,\xi') \in C_0^\infty(|\tau|+|\xi'|<2)$ equal to 1 near $(|\tau|+|\xi'|\le 1), \chi_{\alpha_0}(\tau,\xi') = \chi((\tau,\xi')/\alpha_0)$ and let $K(x)$ be the operator on the torus

$$K(x)(\Sigma_\ell z_\ell e^{i\ell y}) = \sum_\ell \psi\left(\frac{16\|\ell''_x\|^2}{\beta^2}\right)\frac{z_\ell}{\langle\ell''_x\rangle}e^{i\ell y}. \tag{B.25}$$

Let us define $\widetilde{\mathbb{E}}$ by the formula

$$\begin{cases} \widetilde{\mathbb{E}} = \widetilde{\mathbb{E}}_0 + \delta\widetilde{\mathbb{E}} \\ \widetilde{\mathbb{E}}_0 = \mathbb{E}_0 + \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix} \\ \delta\widetilde{\mathbb{E}} = \left[\begin{pmatrix} 0 & 0 \\ q\theta & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}\circ\delta\widetilde{pr}\right]\chi_{\alpha_0}. \end{cases} \tag{B.26}$$

One has $\theta\chi_{\alpha_0} \equiv \theta$ and $\begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix} \circ \widetilde{pr}_0 = \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}$, which implies

$$\widetilde{\mathbb{E}} = \mathbb{E} + \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix} \circ \widetilde{pr}\chi_{\alpha_0} + \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix} (1 - \chi_{\alpha_0}). \tag{B.27}$$

One has $(1-\chi_{\alpha_0})F^{\varepsilon,I} \in \varepsilon L^2(U; \mathcal{H}^{s_0-1})$ so using (B.21, B.23) and Lemma A.3, one gets

$$\begin{cases} \left[\frac{\varepsilon}{i}\frac{\partial}{\partial x_d} + Op(\widetilde{\mathbb{E}})\right] F^{\varepsilon,I} = \widetilde{G}^{\varepsilon,I} \\ \sup\limits_{\varepsilon} \varepsilon^{-1} \|\widetilde{G}^{\varepsilon,I}; L^2(U, \mathcal{H}^{s_0-1})\| < +\infty. \end{cases} \tag{B.28}$$

Notice that $\widetilde{\mathbb{E}}_0$ is a diagonal operator

$$\begin{cases} \widetilde{\mathbb{E}}_0(\Sigma z_\ell e^{i\ell y}) = \sum\limits_{\ell} \widetilde{\mathbb{E}}_{0,\ell}(z_\ell) e^{i\ell y} \\ \widetilde{\mathbb{E}}_{0,\ell} = \begin{pmatrix} \ell_x^\perp & -\langle \ell_x'' \rangle \\ \dfrac{\|\ell_x''\|^2 + \psi\left(\frac{16\|\ell_x''\|^2}{\beta^2}\right)}{\langle \ell_x'' \rangle} & \ell_x^\perp \end{pmatrix}. \end{cases} \tag{B.29}$$

The eigenvalues of $\widetilde{\mathbb{E}}_{0,\ell}$ are

$$\widetilde{\lambda}_\ell^\pm = \ell_x^\perp \pm i \left(\|\ell_x''\|^2 + \psi\left(\frac{16\|\ell'' x\|^2}{\beta^2}\right)\right)^{1/2}. \tag{B.30}$$

In particular, one has with $0 < c_1 < c_2$

$$c_1 \langle \ell_x'' \rangle \le |\mathrm{Im} \widetilde{\lambda}_\ell^\pm| \le c_2 \langle \ell_x'' \rangle \cdot \tag{B.31}$$

We choose the associated eigenvectors

$$e_\ell^\pm(x) = \begin{bmatrix} \frac{-\langle \ell_x'' \rangle}{\widetilde{\lambda}_\ell^\pm(x) - \ell_x^\perp} \\ 1 \end{bmatrix}. \tag{B.32}$$

The map $J_0(x)$ defined by

$$J_0(x) \left(\sum_\ell \begin{pmatrix} z_\ell^+ \\ z_\ell^- \end{pmatrix} e^{i\ell y} \right) = \sum_\ell (z_\ell^+ e_\ell^+(x) + z_\ell^- e_\ell^-(x)) e^{i\ell y} \tag{B.33}$$

is then an isomorphism of $\mathcal{H}^s$ for any s.

Let $D^\pm$ be the operators

$$D^\pm(\Sigma z_\ell e^{i\ell y}) = \Sigma \widetilde{\lambda}_\ell^\pm z_\ell e^{i\ell y}. \tag{B.34}$$

By construction, one has

$$J_0^{-1} \widetilde{\mathbb{E}}_0 J_0 = \begin{pmatrix} D^+ & 0 \\ 0 & D^- \end{pmatrix}. \tag{B.35}$$

Lemma B.1. *If α_0 is small enough, there exist $\delta B, \delta C, \delta D^+, \delta D^-$ in $\mathcal{S}^{t,0}$, with support in $\{|\tau| + |\xi'| \leq 2\alpha_0\}$ vanishing on $\xi' = 0, \tau = 0$ such that the following identity holds true*

$$\begin{pmatrix} Id & \delta B \\ \delta C & Id \end{pmatrix}^{-1} J_0^{-1} \widetilde{\mathbb{E}} J_0 \begin{pmatrix} Id & \delta B \\ \delta C & Id \end{pmatrix} = \begin{pmatrix} D^+ + \delta D^+ & 0 \\ 0 & D^- + \delta D^- \end{pmatrix}. \tag{B.36}$$

Proof. By formulas (B.26, B.35), one has

$$J_0^{-1} \widetilde{\mathbb{E}} J_0 = \begin{pmatrix} D^+ & 0 \\ 0 & D^- \end{pmatrix} + \delta M \ , \ \delta M = \begin{pmatrix} \delta M_1 & \delta M_2 \\ \delta M_3 & \delta M_4 \end{pmatrix}$$

where $\delta M_j \in \mathcal{S}^{t,0}$, vanishes on $\xi' = 0, \tau = 0$, and has support in $\{|\tau| + |\xi'| \leq 2\alpha_0\}$. Equation (B.36) is then equivalent to the following system of equations

$$\left\{ \begin{array}{l} \delta M_1 + \delta M_2 \delta C = \delta D_+ \\ \delta M_4 + \delta M_3 \delta B = \delta D_- \\ \delta M_2 + \delta M_1 \delta B + D^+ \delta B = \delta B D^- + \delta B \delta D^- \\ \delta M_3 + \delta M_4 \delta C + D^- \delta C = \delta C D^+ + \delta C \delta D^+. \end{array} \right. \tag{B.37}$$

We are thus reduce to solve the equation, with unknown $\delta B \in \mathcal{S}^{t,0}$

$$\left\{ \begin{array}{l} D^+ \delta B - \delta B D^- + \delta M_1 \delta B + \delta M_2 - \delta B \delta M_4 - \delta B \delta M_3 \delta B = \phi(\delta B, \delta M) \\ \phi(\delta B, \delta M) = 0. \end{array} \right. \tag{B.38}$$

Let $\mathcal{E}_x, \mathcal{E}$ be the Banach space of operators on the torus: $(A_{\ell,k} = \oint_{\mathbb{T}^d} (A e^{iky}) e^{-i\ell y})$

$$\mathcal{E} = \left\{ A; \|A; \mathcal{E}\| = \sup_{\ell,k} |A_{\ell,k}| (1 + |\ell - k|)^{N_0} < +\infty \right\} \tag{B.39}$$

$$\mathcal{E}_x = \left\{ A; \|A; \mathcal{E}_x\| = \sup_{\ell,k} |A_{\ell,k}| (1 + |\ell - k|)^{N_0} |\widetilde{\lambda}_\ell^+(x) - \widetilde{\lambda}_k^-(x)| < +\infty \right\} \tag{B.40}$$

where N_0 is given, $N_0 \geq d + 1$. By (B.30, B.31), the injection $\mathcal{E}_x \hookrightarrow \mathcal{E}$ is continuous, and the map $(A_1, A_2) \to A_1 A_2$ is continuous on $\mathcal{E}$ by the choice of N_0.

We shall first verify that (B.38) has a unique small solution $\delta B \in \mathcal{E}_x$, for (t, τ, x, ξ') fixed, if α_0 is small enough. By construction, one has

$$(D^+ \delta B - \delta B D^-)_{\ell,k} = (\widetilde{\lambda}_\ell^+ - \widetilde{\lambda}_k^-)(\delta B)_{\ell,k} \tag{B.41}$$

so $\delta B \mapsto D^+ \delta B - \delta B D^-$ is an isomorphism of $\mathcal{E}_x$ onto $\mathcal{E}$. The map $(\delta B, \delta M) \to \phi(\delta B, \delta M)$ is differentiable from $\mathcal{E}_x \times (\mathcal{E})^4$ to $\mathcal{E}$ and satisfies

$$\frac{\partial}{\partial \delta B} \phi(0,0) = D^+(\cdot) - (\cdot) D^- \qquad \phi(0,0) = 0. \tag{B.42}$$

By the implicit function theorem, the equation $\phi(\delta B, \delta M) = 0$ has thus a unique small solution $\delta B \in \mathcal{E}_x$, provide $\|\delta M; \mathcal{E}\|$ is small. Using (B.26) (q and $\delta \widetilde{pr}$ vanish on $\xi' = \tau = 0$) and $\chi_{\alpha_0}(\tau, \xi') = \chi(\frac{(\tau,\xi')}{\alpha_0})$ one gets the estimate $\|\delta M_j; \mathcal{E}\| \leq C^{te} \alpha_0$. This shows the existence of δB solution of (B.38).

It remains to prove that for any fixed $z = (t, x), \zeta = (\tau, \xi)$, we have

$$\forall N, \exists C_N \quad |(\delta B)_{\ell,k}(z, \zeta)| \leq \frac{C_N}{(1 + |\ell - k|)^N} \tag{B.43}$$

and that the functions $(z, \zeta) \mapsto (\delta B)_{\ell,k}(z, \zeta)$ are smooth and satisfy

$$\begin{aligned} &\forall \alpha, \beta, \gamma, N \ , \ \exists C \quad \forall z, \zeta, \ell, k \\ &|(1 + |\zeta|)^\gamma \partial_z^\alpha \partial_\zeta^\beta (\delta B)_{\ell,k}(z, \zeta)| \leq C \frac{(1 + |\ell|)^{|\alpha|}}{(1 + |\ell - k|)^N}. \end{aligned} \tag{B.44}$$

Let $\nabla_i = \frac{1}{i} \frac{\partial}{\partial y_i}$ a derivation on the torus. The commutator $[\nabla_i, \delta B]$ satisfies the linear equation

$$\left\{ \begin{array}{l} \mathcal{L}([\nabla_i, \delta B]) \in \mathcal{E} \\ \mathcal{L}(u) = D^+ u - u D^- + \delta M_1 u - u \delta M_4 - u \delta M_3 \delta B - \delta B \delta M u. \end{array} \right. \tag{B.45}$$

The linear map $\mathcal{L}$ is an isomorphism of $\mathcal{E}_x$ onto $\mathcal{E}$ provide $\|\delta M_j; \mathcal{E}\|$ (hence $\|\delta B; \mathcal{E}\|$) is small enough; decreasing α_0 if necessary, we find that (B.45) admits a unique solution $[\nabla_i, \delta B] \in \mathcal{E}_x \hookrightarrow \mathcal{E}$. By iteration, all the commutators $[\nabla_{i1}, [\nabla_{i2}, \cdots [\nabla_{1k}, \delta B] \cdots]$ belongs to $\mathcal{E}_x$ so (B.43) holds true. By construction, the functions $(\delta B)_{\ell,k}(z, \zeta)$ are smooth and compactly supported in $\{|\zeta| \leq 2\alpha_0\}$. For any $m \geq 0$, let $\mathcal{A}^m$, $\mathcal{A}_x^m$ be the vector spaces

$$\mathcal{A}^m = \left\{ A; \forall N, \exists C_N |A_{\ell,k}| \leq C_N \frac{(1 + |\ell|)^m}{(1 + |\ell - k|)^N} \right\} \tag{B.46}$$

$$\mathcal{A}_x^m = \left\{ A; \forall N, \exists C_N |A_{\ell,k}| \leq \frac{C_N}{|\widetilde{\lambda}_\ell^+(x) - \widetilde{\lambda}_k^-(x)|} \frac{(1 + |\ell|)^m}{(1 + |\ell - k|)^N} \right\}. \tag{B.47}$$

In order to prove (B.44), we differentiate (B.38) with respect to (z, ζ) and we are reduce to verify that the following assertion holds true

$$\left\{ \begin{array}{l} \text{There exist } \beta > 0 \text{ such that for } \delta M_j, \delta B \in \mathcal{A}^0, \\ \text{with } \sum\limits_j \|\delta M_j; \mathcal{E}\| + \|\delta B; \mathcal{E}\| \leq \beta, \text{ the map } u \mapsto \mathcal{L}(u) \\ \text{is an isomorphism of } \mathcal{A}_x^m \text{ onto } \mathcal{A}^m \text{ for any } m \geq -1. \end{array} \right. \tag{B.48}$$

(Here we use the fact that $A \mapsto (\partial_x^\alpha D^+) A - A(\partial_x^\alpha D^-)$ maps $\mathcal{A}_x^m$ into $\mathcal{A}^{m+|\alpha|}$ for any α: it is a consequence of the estimates $|\widetilde{\lambda}_\ell^+(x) - \widetilde{\lambda}_k^-(x)| \geq C^{te}(\langle \ell_x'' \rangle + \langle k_x'' \rangle)$ and for $|\alpha| \geq 1 \quad |\partial_x^\alpha \widetilde{\lambda}_\ell^\pm(x)| \leq C_\alpha (1 + |\ell|)^{|\alpha|} \langle \ell_x'' \rangle$.)

Let us first verify that (B.48) holds true for $m = 0$. We remark that $\mathcal{A}^0$ (resp. $\mathcal{A}_x^0$) is the set of operators $A \in \mathcal{E}$ (resp.$\mathcal{E}_x$) such that all the commutators $[\nabla_{i_1} [\nabla_{i_1} [\nabla_{i_2}, \cdots, [\nabla_{i_p}, A]]$ belongs to $\mathcal{E}$ (resp. $\mathcal{E}_x$). For β small $\mathcal{L}$ is an isomorphism between $\mathcal{E}_x$ and $\mathcal{E}$ and for $u \in \mathcal{E}_x$, $v \in \mathcal{E}$ such that $\mathcal{L}(u) - v = 0$, one has $\mathcal{L}([\nabla_i, u]) - [\nabla_i, v] \in \mathcal{E}$. Therefore (B.48) holds true for $m = 0$, and by the same argument for $m = -1$. We now fixe β and we proceed by induction on $m \geq 1$; let us assume that (B.48) holds true for $-1 \leq m' \leq m - 1$. Let Λ be the operator on the torus $\Lambda(\Sigma z_\ell e^{i\ell y}) = \Sigma(1 + |\ell|) z_\ell e^{i\ell y}$. We have $\mathcal{L}(u) = D^+ u - u D^- + pu + uq$ with $p, q \in \mathcal{A}^0$, so $[\Lambda, p] \in \mathcal{A}^0$; from $[\Lambda, D^\pm] = 0$, we get $\mathcal{L}(\Lambda w) - \Lambda \mathcal{L}(w) = [p, \Lambda] w$. Let $J : \mathcal{A}^m \to \mathcal{A}_x^m$ the map

$$J(v) = \Lambda \mathcal{L}^{-1}(\Lambda^{-1} v) + \mathcal{L}^{-1}([\Lambda, p] \mathcal{L}^{-1}(\Lambda^{-1} v)) \tag{B.49}$$

where $\mathcal{L}^{-1} : \mathcal{A}^{m-1} \to \mathcal{A}_x^{m-1}$ is the inverse map of $\mathcal{L}$. We have $\mathcal{L} \circ J(v) \equiv v$, and it remains to show that $u \in \mathcal{A}_x^m$, and $\mathcal{L}(u) = 0$ imply $u = 0$: we have $[\Lambda^{-1}, p] \in \mathcal{A}^{-2}$ so $\mathcal{L}(u) = 0 \Rightarrow \mathcal{L}(\Lambda^{-1} u) = [p, \Lambda^{-1}] u \in \mathcal{A}^{m-2}$ $\Rightarrow \Lambda^{-1} u \in \mathcal{A}_x^{m-2} \Rightarrow u \in \mathcal{A}_x^{m-1}$ and we get $u = 0$. □

Lemma B.2. *Let* $U_0 = \{z \in \mathbb{R}^p, |z| \leq r_0\}$*, and* $U = U_0 \times [0, r_1]$ *with* $r_0, r_1 > 0$*. For any* $\ell \in \mathbb{Z}^d$*, let* $\lambda_\ell(z, x_d) \in C^0(U; \mathbb{C})$ *be given continuous functions such that*

$$\begin{aligned} &\exists c_0 > 0, \exists c_1 > 1, \forall \ell, \forall z, x_d \\ &Im\lambda_\ell(z, x_d) \geq c_0 \text{ and } \frac{|\lambda_\ell(z, x_d)|}{1 + |\ell|} \in \left[\frac{1}{c_1}, c_1\right]. \end{aligned} \tag{B.50}$$

Let $D(z, x_d)$ be the operator on the torus

$$D(z, x_d)[\Sigma u_\ell e^{i\ell y}] = \Sigma \lambda_\ell(z, x_d) u_\ell e^{i\ell y}. \tag{B.51}$$

Let $\sigma \in \mathbb{R}$ be given, and for $\varepsilon \in]0, 1]$, $B_\varepsilon(x_d)$ a family of bounded operator on $E^\sigma = L^2(U_0, H^\sigma(\mathbb{T}^d))$ such that

$$\left\{ \begin{array}{l} i) \quad \forall f \in E^\sigma, \forall \varepsilon \quad x_d \mapsto B_\varepsilon(x_d)[f] \text{ is a continuous function} \\ \text{of } x_d \in [0, r_1] \text{ with values in } E^\sigma \\ ii) \quad \exists \delta, \forall \varepsilon, \forall x_d \quad \|B_\varepsilon(x_d); E^\sigma \to E^\sigma\| \leq \delta. \end{array} \right. \tag{B.52}$$

Then, for $\delta < c_0$ the Cauchy problem

$$\left\{ \begin{array}{l} \left[\frac{\varepsilon}{i}\frac{d}{dx_d} - (D + B_\varepsilon)\right](u^\varepsilon(x_d)) = 0 \ x_d \in]0, r_1[\\ u^\varepsilon(0) = u_0 \in E^\sigma \end{array} \right. \tag{B.53}$$

admits a solution $u^\varepsilon \in C^0([0, r_1], E^\sigma) \cap C^1([0, r_1], E^{\sigma-1})$ such that

$$\|u^\varepsilon(x^d), E^\sigma\| \leq \|u_0, E^\sigma\| e^{-(c_0 - \delta)x_d/\varepsilon}. \tag{B.54}$$

Proof. We first observe that the assumption (B.50) implies that D maps $C^0([0, r_1], E^\sigma)$ onto $C^0[0, r_1], E^{\sigma-1})$ for any σ. We have $\|v; E^\sigma\| = \|(1 + |D_y|^2)^{\sigma/2} v; E^0\|$ and $[D, (1 + |D_y|^2)^{\sigma/2}] = 0$, so if one replace B_ε by $(1 + |D_y|^2)^{\sigma/2} B_\varepsilon (1 + |D_y|^2)^{-\sigma/2}$, we are reduce to the case $\sigma = 0$. For any L, let π_L be the orthogonal projector $\pi_L(\Sigma u_\ell e^{i\ell y}) = \sum_{|\ell| \leq L} u_\ell e^{i\ell y}$. The equation

$$\left\{ \begin{array}{l} \left(\frac{\varepsilon}{i}\frac{d}{dx_d} - \pi_L(D + B_\varepsilon)\pi_L\right) u_L^\varepsilon(x_d) = 0 \\ u_L^\varepsilon(0) = \pi_L(u_0) \end{array} \right. \tag{B.55}$$

is an ordinary differential equation in the Hilbert space $= L^2(U_0, \bigoplus_{|\ell| \leq L} \mathbb{C} e^{i\ell y}) = E_L \hookrightarrow E^s$, so admit a unique solution $u_L^\varepsilon \in C^1([0, r_1], E_L)$. It satisfies the identity,

$$\frac{d}{dx_d}\|u_L^\varepsilon\|^2 = 2\mathrm{Re}\left(\frac{i}{\varepsilon} D u_L^\varepsilon | u_L^\varepsilon\right) + 2\mathrm{Re}\left(i/\varepsilon \pi_L B_\varepsilon \pi_L u_L^\varepsilon | u_L^\varepsilon\right) \tag{B.56}$$

so we get using (B.50) and (B.52) $\frac{d}{dx_d}\|u_L^\varepsilon\|^2 \leq \frac{-2}{\varepsilon}(c_0 - \delta)\|u_L^\varepsilon\|^2$, which implies

$$\|u_L^\varepsilon(x_d), E^0\| \leq \|u_0, E^0\| e^{-(c_0 - \delta)\frac{x_d}{\varepsilon}}. \tag{B.57}$$

Therefore u_L^ε is bounded in $L^2([0, r_1], E^\sigma) \cap H^1([0, r_1], E^{\sigma-1}) = F$ for fixed ε so we can extract a subsequence $u_{L_k}^\varepsilon$ so that $u_{L_k}^\varepsilon \overset{\text{weak}}{\rightharpoonup} u^\varepsilon$ in F and u^ε satisfies (B.53). In particular we have $\frac{\varepsilon}{i}\frac{d}{dx_d} u_\varepsilon - D u^\varepsilon = B_\varepsilon u^\varepsilon \in L^2([0, r_1], E^\sigma)$ so $u^\varepsilon \in C^0([0, r_1], E^\sigma) \cap C^1([0, r_1], E^{\sigma-1})$. The estimate (B.54) is then a consequence of (B.57). □

We can now achieve the verification of Lemma 3.6. We choose a tangential scalar *o.p.d.* Q_2 equal to Id near the support of Q_1 and with essential support closed to ρ_0, and we define $T = Q_2 \delta\widetilde{\mathbb{E}} Q_2$. Then $Op(T)(x_d)$ acts on $L^2(U_0, H^\sigma), \forall\sigma$. Using (B.28), we still have

$$\left[\frac{\varepsilon}{i}\frac{\partial}{\partial x_d} + Op(\widetilde{\mathbb{E}}_0 + T)\right] F^{\varepsilon,I} \in \varepsilon L^2(U, \mathcal{H}^{s_0-1}). \tag{B.58}$$

We then apply Lemma B.1 to $\widetilde{\mathbb{E}}_0^* + T^*$ instead of $\widetilde{\mathbb{E}}$; let $I_0(x)$ be the map

$$I_0(x)\left(\sum_\ell \begin{pmatrix} z_\ell^+ \\ z_\ell^- \end{pmatrix} e^{i\ell y}\right) = \sum_\ell \left[z_\ell^+ \begin{pmatrix} \frac{-\tilde{\lambda}_\ell^+(x) + \ell_x^\perp}{\langle \ell_x'' \rangle} \\ 1 \end{pmatrix} + z_\ell^- \begin{pmatrix} \frac{-\tilde{\lambda}_\ell^-(x) + \ell_x^\perp}{\langle \ell_x'' \rangle} \\ 1 \end{pmatrix} \right] e^{i\ell y}. \tag{B.59}$$

We get the existence of $\delta B, \delta C, \delta D^+, \delta D^-$ in $\mathcal{S}^{t,0}$ such that, with

$$I = I_0 \begin{pmatrix} Id & \delta B \\ \delta C & Id \end{pmatrix}.$$

One has

$$I^{-1}(\widetilde{\mathbb{E}}_0^* + T^*)I = \begin{pmatrix} D^+ + \delta D^+ & 0 \\ 0 & (D^- + \delta D^-) \end{pmatrix}. \tag{B.60}$$

Moreover, by the proof of Lemma B.1, and the fact that $\lim\limits_{\alpha_0 \to 0} \|(|\xi'| + |\tau|)\chi_{\alpha_0}(\xi', \tau)\|_{L^\infty} = 0$, we may suppose that the norm of the tangential operators $\delta D^\pm(x_d)$ acting on $L^2(U_0, H^{|s_0|+1})$ is as small as we want. Taking in account the lower bound (B.31) $-Im\lambda_\ell^-(x) \geq c_1\langle \ell_x'' \rangle \geq c_1$, we can apply Lemma B.2. For every $h \in L^2(U_0, H^{|s_0|+1})$ we get $v^\varepsilon \in L^2(U_0 \times [0, r_1], H^{|s_0|+1})$ such that

$$\left\{ \begin{array}{l} \left[\frac{\varepsilon}{i}\frac{\partial}{\partial x_d} + Op(D^- + \delta D^-)\right] v^\varepsilon = 0 \\ v^\varepsilon|_{x_d=0} = \varepsilon^{-1/2} h \\ \sup\limits_\varepsilon \|v^\varepsilon, L^2(U_0 \times [0, r_1]; H^{|s_0|+1})\| \leq C^{te}\|h; L^2(U_0; H^{|s_0|+1})\|. \end{array} \right. \tag{B.61}$$

We put $\underline{v}^\varepsilon = \begin{bmatrix} 0 \\ v^\varepsilon \end{bmatrix}$.

We choose $\theta(x_d) \in C_0^\infty(]-r_1, r_1[)$ equal to 1 near zero. We denote by $\langle\ |\ \rangle$ the duality between $L^2(V_0, H^\sigma)$ and $L^2(V_0, H^{-\sigma})$. We have by (B.58)

$$\int_0^\infty \left\langle \left(\frac{\varepsilon}{i}\partial_{x_d} + Op(\widetilde{\mathbb{E}}_0 + T)\right) F^{\varepsilon,I} | \theta(x_d) Op(I) \underline{v}^\varepsilon \right\rangle \in 0(\varepsilon)\|h\|. \tag{B.62}$$

We integrate by part, taking into account Lemma A.4 and $\|\theta'(x_d)Op(I)\underline{v}^\varepsilon\ ;\ L^2(U, \mathcal{H}^{|s_0|+1})\| \leq C^{te}\|h\|$, we get

$$\frac{\varepsilon}{i}\langle F^{\varepsilon,I}|_{x_d=0} | Op(I)\underline{v}_\varepsilon|_{x_d=0}\rangle = \int_0^\infty \left\langle \theta(x_d) F^{\varepsilon,I} | \left(\frac{\varepsilon}{i}\partial_{x_d} + Op\left(\widetilde{\mathbb{E}}_0^* + T^*\right)\right) Op(I)\underline{v}^\varepsilon \right\rangle dx_d + 0(\varepsilon\|h\|). \tag{B.63}$$

We have $\|\varepsilon[\frac{\partial}{\partial x_d} Op(I)]\underline{v}^\varepsilon; L^2(U, \mathcal{H}^{|s_0|})\| \leq C^{te}\varepsilon\|h\|$, and the estimates

$$\left\{ \begin{array}{l} \left|\partial_x^\alpha \left[\frac{\tilde{\lambda}_\ell^\pm(x) - \ell_x^\perp}{\langle \ell_x'' \rangle}\right]\right| \leq C_\alpha(1 + |\ell|)^{|\alpha|} \quad \forall\alpha \\ |\partial_x^\alpha \widetilde{\mathbb{E}}_{0,\ell}| \leq C_\alpha(1 + |\ell|)^{|\alpha|} \quad \forall\alpha, |\alpha| \geq 1 \end{array} \right. \tag{B.64}$$

implies

$$\begin{cases} \| \Big[Op(\widetilde{\mathbb{E}}_0^* + T^*) \circ Op(I) - Op((\widetilde{\mathbb{E}}_0^* + T^*)I) \Big] \underline{v}^\varepsilon ; L^2(U, \mathcal{H}^{|s_0|}) \| \le C^{te} \varepsilon \|h\| \\ \| \Big[Op((I \left(\begin{smallmatrix} D^+ + \delta D^+ & 0 \\ 0 & D^- + \delta D^- \end{smallmatrix} \right)) - Op(I) \circ Op \left(\begin{smallmatrix} D^+ + \delta D^+ & 0 \\ 0 & D^- + \delta D^- \end{smallmatrix} \right) \Big] \underline{v}^\varepsilon ; L^2(U, \mathcal{H}^{|s_0|}) \| \le C^{te} \varepsilon \|h\|. \end{cases} \tag{B.65}$$

From (B.61, B.63, B.65), we get

$$\left| \left\langle F^{\varepsilon,I}|_{x_d=0} | Op(I) \begin{bmatrix} 0 \\ h \end{bmatrix} \right\rangle \right| \le C^{te} \varepsilon^{1/2} \|h\|. \tag{B.66}$$

If one use the definition of I, the fact that h is arbitrary in $L^2(U_0, H^{|s_0|+1})$, one get for some $D \in \mathcal{S}^0$

$$\|Op(Id + \delta B)^* Tr_1(F^{\varepsilon,I}) - Op(D) Tr_0(F^{\varepsilon,I}); L^2(U_0, H^{s_0 - 1})\| \le C \varepsilon^{1/2}. \tag{B.67}$$

Lemma 3.6 is then a consequence of the estimates (3.52), Lemmas A.2–A.4, and the fact for α_0 small, $Id + (\delta B)^*$ is invertible in $\mathcal{S}^0$. □

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