ESAIM: Control, Optimisation and Calculus of Variations June 1999, Vol. 4, p. 361–376
URL: `http://www.emath.fr/cocv/`

STABILITY AND STABILIZATION OF DISCONTINUOUS SYSTEMS AND NONSMOOTH LYAPUNOV FUNCTIONS

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Abstract. We study stability and stabilizability properties of systems with discontinuous righthand side (with solutions intended in Filippov's sense) by means of locally Lipschitz continuous and regular Lyapunov functions. The stability result is obtained in the more general context of differential inclusions. Concerning stabilizability, we focus on systems affine with respect to the input: we give some sufficient conditions for a system to be stabilized by means of a feedback law of the Jurdjevic-Quinn type.

Résumé. On étudie les propriétés de stabilité et stabilisation des systèmes avec second membre discontinu (les solutions étant prises dans le sens de Filippov) au moyen des fonctions de Lyapunov lipchitziennes et régulières. Le résultat de stabilité est obtenu dans le contexte plus général des inclusions différentielles. En ce qui concerne la stabilisation, on étudie des systèmes affines par rapport au contrôle : on donne des conditions suffisantes pour la stabilisation au moyen d'un retour d'état du type de Jurdjevic et Quinn.

AMS Subject Classification. 34D20, 93D15.

Received September 8, 1998. Revised April 9, 1999.

1. Introduction

In one of the first papers devoted to nonlinear feedback stabilization, Jurdjevic and Quinn introduced the idea that the stability properties of an affine system

$$\dot{x} = f(x) + G(x)u = f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{1}$$

(where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f, g_1, ..., g_m$ are vector fields of $\mathbb{R}^n$, and G is the matrix whose columns are $g_1, ..., g_m$) can be enhanced by setting

Keywords and phrases: Stability, stabilization, Clarke gradient, Lyapunov functions, LaSalle principle.

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$$u = u(x) = -\alpha(\nabla V(x)G(x))^T \tag{2}$$

where V is a Lyapunov function for the unforced system

$$\dot{x} = f(x), \tag{3}$$

the row vector $\nabla V(x)$ denotes its gradient and α is a positive real parameter (see [14]; see also [3] for subsequent developments and improvements). More precisely, assume that

(A) the origin is Lyapunov stable for (3) and a positive definite Lyapunov function $V \in C^1$ such that $\dot{V}$ is negative semi-definite is known;
(B) an additional condition, involving Lie brackets of the vector fields $f, g_1, ..., g_m$, holds.

Then, Jurdjevic and Quinn proved that (1) can be asymptotically stabilized by means of the feedback law (2). A crucial step of the proof is provided by LaSalle's invariance principle.

Although it has been largely and successfully exploited in the literature both from a practical and a theoretical point of view, a weakness of the method related to assumption (A) should be pointed out. Indeed, it is well known that, even for $f \in C^\infty$, Lyapunov stability does not imply in general the existence of a (not even) continuous Lyapunov function (see [2,4]).

When it is known that the unforced system is stable but the existence of a C^1 Lyapunov function cannot be guaranteed, two alternative ways can be pursued:

1) to introduce time dependent Lyapunov functions. In this case the Jurdjevic and Quinn method can be extended (see [16]) but it gives rise, of course, to a time dependent feedback;
2) to replace the (classical) gradient in (2) by some type of generalized gradient. This in general leads to discontinuous feedback, so that Filippov solutions and differential inclusions are involved in the treatment.

The present paper is devoted to the second point of view. The organization is the following. Section 2 contains a stability result for differential inclusions and nonsmooth Lyapunov functions. It involves a new type of "set-valued derivative" and extends an early result in [20]. A new version of an invariance theorem for differential inclusions (a generalization of LaSalle's principle) is presented in Section 3. Moreover it is compared with both Shevitz-Paden's and Ryan's early versions of the invariance principle (see [19,20]). Finally, in Section 4 we discuss the problem of asymptotic stabilization of affine input systems. First of all, we show by an example that for discontinuous systems and nonsmooth Lyapunov functions, the application of the feedback law (2) may actually result in a destabilizing action. This is quite unexpected since it is easy to see that, in the classical smooth case, a feedback of Jurdjevic-Quinn type always preserves stability. Secondly, we identify some alternative conditions which are automatically fulfilled in the smooth case and which allow us to prove that, under the feedback law (2)

- the stability is preserved;
- the stability performances are improved, in the sense that the "bad" set where the set-valued derivative vanishes becomes smaller.

For reader's convenience, we include an Appendix where some fundamental facts about Filippov solutions and Clarke generalized gradient are recalled.

Before closing this introduction, some comments are appropriate. As already remarked, replacing the classical gradient by a generalized gradient leads to introduce discontinuities in the feedback law. On the other hand, it is well known (see [9]) that if an affine system admits an asymptotically stabilizing discontinuous feedback, then it admits an asymptotically stabilizing continuous one, as well. However, in general, there are no constructive methods to define directly a continuous stabilizing feedback.

2. Differential inclusions and stability

In this section we estabilish a stability result for differential inclusions by means of locally Lipschitz continuous and regular Lyapunov functions.

Let $F:\mathbb{R}^n \to 2^{\mathbb{R}^n}\backslash\emptyset$ be an upper semi-continuous multivalued map with compact, convex values.

Recall that a solution of the differential inclusion

$$\dot{x} \in F(x) \tag{4}$$

on a nondegenerate interval $I \subseteq \mathbb{R}$ is a function $\varphi : I \to \mathbb{R}^n$ such that $\varphi(\cdot)$ is absolutely continuous on any interval $[t_1, t_2] \subseteq I$ and $\dot{\varphi}(t) \in F(\varphi(t))$ for almost all $t \in I$. We denote by S_{x_0} the set of solutions of (4) such that $\varphi(0) = x_0$.

As far as the general theory of differential inclusions is concerned, the reader is referred to [1, 10, 12]. We limit ourselves to recall the definition of stability which is of interest for this paper.

Definition 1. The differential inclusion (4) is said to be *stable* at $x = 0$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that for each initial condition x_0 and each solution $\varphi(\cdot) \in S_{x_0}$

$$\|x_0\| < \delta \Rightarrow \|\varphi(t)\| < \epsilon \quad \forall t \geq 0.$$

The kind of stability defined above is often referred to as *strong* stability in the literature. Note that if (4) is stable at $x = 0$, then the origin is an equilibrium position, *i.e.* $0 \in F(0)$.

We now state the definition of Lyapunov function adopted in this paper.

Definition 2. A *Lyapunov function* for (4) is a positive definite, continuous function $V : \mathbb{R}^n \to \mathbb{R}$ such that for each solution $\varphi(\cdot)$ of (4) on $I \subseteq \mathbb{R}$ and for all $t_1, t_2 \in I$

$$t_1 \leq t_2 \;\Rightarrow\; V(\varphi(t_2)) \leq V(\varphi(t_1)). \tag{5}$$

The following theorem is an easy generalization of first Lyapunov theorem.

Theorem 1. *Let $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}\backslash\emptyset$ be an upper semi-continuous multivalued function with compact, convex values.*

If there exists a Lyapunov function for (4), then (4) is stable at $x = 0$.

This theorem does not have an inverse, in general. Other theorems concerning the stability of differential inclusions can be found in [10] and [8].

Here we are interested in the following question: how can condition (5) be checked without involving explicitly the solutions of (4)?

If the Lyapunov function $V \in C^1$, then it is well known that (5) is implied by the condition

$$\nabla V(x) \cdot v \leq 0 \quad \forall v \in F(x) \quad \forall x \in \mathbb{R}^n$$

(see [12]).

If we have a Lyapunov function which is only Lipschitz continuous, then the previous condition can be weakened by making use of the upper right directional Dini derivative (see [4, 10]), so that

$$\overline{D^+}V(x, v) \leq 0 \quad \forall v \in F(x) \; \forall x \in \mathbb{R}^n$$

implies (5).

The main result of this section is an extension of a theorem due to [20]. It makes use of Clarke generalized gradient (see the Appendix for definitions and notations).

Recall (see [6], p. 39) that a function $V : \mathbb{R}^n \to \mathbb{R}$ is called *regular* at $x \in \mathbb{R}^n$ if

(i) for all $v \in \mathbb{R}^n$ there exists the usual right directional derivative $V'_+(x,v)$;
(ii) for all $v \in \mathbb{R}^n$, $V'_+(x,v) = V^o(x,v)$.

V is called regular if it is regular at each $x \in \mathbb{R}^n$. Let us remark that a convex function is not only Lipschitz continuous, but also regular.

Throughout this paper the *set-valued derivative of* V *with respect to* (4) is defined as

$$\overline{\dot{V}}^{(4)}(x) = \{a \in \mathbb{R} : \exists v \in F(x) \text{ such that } p \cdot v = a \quad \forall p \in \partial V(x)\}\cdot$$

It is not difficult to prove that for each fixed $x \in \mathbb{R}^n$, $\overline{\dot{V}}^{(4)}(x)$ is a closed and bounded interval, possibly empty. In the case V is differentiable at x, one has $\overline{\dot{V}}^{(4)}(x) = \{\nabla V(x) \cdot v,\ v \in F(x)\}$. Moreover $\overline{\dot{V}}^{(4)}(x)$ is in general a proper subset of the set $\dot{\tilde{V}}(x)$ used by Shevitz and Paden [20].

Lemma 1. *Let* $\varphi(\cdot)$ *be a solution of the differential inclusion* (4) *and let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a locally Lipschitz continuous and regular function.*

Then $\frac{d}{dt}V(\varphi(t))$ *exists almost everywhere and* $\frac{d}{dt}V(\varphi(t)) \in \overline{\dot{V}}^{(4)}(\varphi(t))$ *almost everywhere.*

Proof. $V \circ \varphi : \mathbb{R} \to \mathbb{R}$ is absolutely continuous in every interval $I \subset \mathbb{R}$ because it is the composition of a locally Lipschitz continuous function and an absolutely continuous function. Then $\frac{d}{dt}V(\varphi(t))$ exists almost everywhere. Moreover there exists a set N of measure zero such that, for all $t \in I \backslash N$, both $\dot{\varphi}(t)$ and $\frac{d}{dt}V(\varphi(t))$ exist, and $v = \dot{\varphi}(t) \in F(\varphi(t))$.

Let us fix $t \in I \backslash N$. From the fact that V is locally Lipschitz continuous we get that

$$\frac{d}{dt}V(\varphi(t)) = \lim_{h \to 0} \frac{V(\varphi(t) + h\dot{\varphi}(t)) - V(\varphi(t))}{h}\cdot$$

Because of the regularity of V, by letting h tending to zero respectively from the right and the left handside, we get

$$\frac{d}{dt}V(\varphi(t)) = V'_+(\varphi(t), \dot{\varphi}(t)) = V^o(\varphi(t), v) = \max\{p \cdot v,\ p \in \partial V(\varphi(t))\}$$

and

$$\frac{d}{dt}V(\varphi(t)) = V'_-(\varphi(t), \dot{\varphi}(t)) = V_o(\varphi(t), v) = \min\{p \cdot v,\ p \in \partial V(\varphi(t))\}\cdot$$

Then $\frac{d}{dt}V(\varphi(t)) = p \cdot v$ for all $p \in \partial V(\varphi(t))$ and hence $\frac{d}{dt}V(\varphi(t)) \in \overline{\dot{V}}^{(4)}(\varphi(t))$ almost everywhere. □

Proposition 1. *Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a positive definite, locally Lipschitz continuous and regular function. If for all* $x \in \mathbb{R}^n$ *one has either* $\max \overline{\dot{V}}^{(4)}(x) \leq 0$ *or* $\overline{\dot{V}}^{(4)}(x) = \emptyset$, *then* V *is a Lyapunov function for* (4).

The proof of this proposition is an easy consequence of the fact that the function $V \circ \varphi$ is absolutely continuous and Lemma 1.

In the following we agree that $\max \overline{\dot{V}}^{(4)}(x) = -\infty$ where $\overline{\dot{V}}^{(4)}(x) = \emptyset$.

Remark. For locally Lipschitz continuous and regular functions, for each $x \in \mathbb{R}^n$, it holds

$$\max \overline{\dot{V}}^{(4)}(x) \leq \max_{v \in F(x)} \overline{D^+} V(x,v).$$

This means that for regular Lyapunov functions, the stability criterion is more general when it is based on the set-valued derivative $\overline{\dot{V}}^{(4)}$ instead of the Dini derivative.

Let us also remark that the converse inequality does not hold. This is shown by Example 1 below.

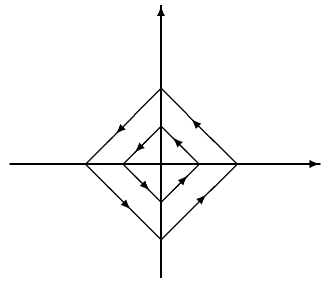

FIGURE 1

From Theorem 1 and Proposition 1 it immediately follows

Theorem 2. *If $V:\mathbb{R}^n\to\mathbb{R}$ is a positive definite, locally Lipschitz continuous and regular function such that, for all $x\in\mathbb{R}^n$, $\max \dot{\overline{V}}^{(4)}(x)\le 0$, then (4) is stable at $x=0$.*

Example 1. *Nonsmooth harmonic oscillator* (Fig. 1). Let us consider a system of the form (3) in $\mathbb{R}^2$ where $f(x_1,x_2)=(-\operatorname{sgn} x_2, \operatorname{sgn} x_1)^T$. According to the Filippov's approach (see Appendix), this leads to the differential inclusion (4), where

$$F(x_1,x_2)=Kf(x_1,x_2)=\begin{cases}\{-\operatorname{sgn} x_2\}\times\{\operatorname{sgn} x_1\} & \text{at } (x_1,x_2),\ x_1\ne 0 \text{ and } x_2\ne 0\\ [-1,1]\times\{\operatorname{sgn} x_1\} & \text{at } (x_1,0),\ x_1\ne 0\\ \{-\operatorname{sgn} x_2\}\times[-1,1] & \text{at } (0,x_2),\ x_2\ne 0\\ \overline{\text{co}}\{(1,1),(-1,1),(-1,-1),(1,-1)\} & \text{at } (0,0).\end{cases}$$

Let us now consider $V(x_1,x_2)=|x_1|+|x_2|$. We have

$$\partial V(x_1,x_2)=\begin{cases}\{\operatorname{sgn} x_1\}\times\{\operatorname{sgn} x_2\} & \text{at } (x_1,x_2),\ x_1\ne 0 \text{ and } x_2\ne 0\\ \{\operatorname{sgn} x_1\}\times[-1,1] & \text{at } (x_1,0),\ x_1\ne 0\\ [-1,1]\times\{\operatorname{sgn} x_2\} & \text{at } (0,x_2),\ x_2\ne 0\\ \overline{\text{co}}\{(1,1),(-1,1),(-1,-1),(1,-1)\} & \text{at } (0,0)\end{cases}$$

so that

$$\dot{\overline{V}}^{(4)}(x_1,x_2)=\begin{cases}\{0\} & \text{at } (x_1,x_2),\ x_1\ne 0 \text{ and } x_2\ne 0\\ \emptyset & \text{at } (x_1,0),\ x_1\ne 0\\ \emptyset & \text{at } (0,x_2),\ x_2\ne 0\\ \{0\} & \text{at } (0,0).\end{cases}$$

Since for all $(x_1,x_2)\in\mathbb{R}^2$ one has $\max\dot{\overline{V}}^{(4)}(x_1,x_2)\le 0$, by Theorem 2, the system is stable at $x=0$. Let us remark that $\max\dot{\overline{V}}^{(4)}(0,x_2)=-\infty<\overline{D^+}V((0,x_2),(-1,1))=2\le\max_{v\in F(0,x_2)}\overline{D^+}V((0,x_2),v)$ so that a test based on Dini derivative is inconclusive.

Example 2. *Gradient vector fields.* It is well known that if $V:\mathbb{R}^n\to\mathbb{R}$ is a positive definite smooth function then the equation

$$\dot{x}=-\nabla V(x)$$

has an asymptotically stable equlibrium at the origin. For a locally Lipschitz, regular positive definite function V, a natural substitute of the previous equation is the differential inclusion

$$\dot{x}\in-\partial V(x).$$

Let $a \in \overline{\dot{V}}^{(4)}(x)$, where $F(x) = -\partial V(x)$. Then there exists $v \in -\partial V(x)$ such that $p \cdot v = a$ for each $p \in \partial V(x)$. In particular the equality must be true for $p = v$. But then $a = -|v|^2 \leq 0$. According to Theorem 1, we conclude that any differential inclusion of the form $\dot{x} \in -\partial V(x)$ is stable at the origin.

Remark. Beside [20], nonsmooth Lyapunov functions and generalized derivatives have been previously used in the literature on stability mainly in connection with the problem of asymptotic stability and stabilization: see for instance [7, 13, 19, 21, 22].

3. An invariance theorem

As shown in the previous section, if a Lyapunov function is known, one can get some conclusions about stability, but nothing can be said in general about asymptotic stability. The so-called invariance principle provides some additional information about the behaviour of solutions. Therefore, by virtue of the invariance principle, and provided that further conditions are satisfied, it may be possible to deduce that a system is actually asymptotically stable even if only a Lyapunov function in the sense of Definition 2 is known. As indicated in Section 1, the invariance principle is a basic tool for several stabilization achievements. The main result of this section is a version of the invariance principle based on the notion of the set-valued derivative of a function with respect to a differential inclusion introduced in the previous section. As before, let (4) be such that $F : \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \emptyset$ is a locally bounded, upper semi-continuous multivalued map with compact, convex values.

The following definitions (see [12], p. 129) are useful to formulate and prove such an invariance theorem.

Definition 3. A point $q \in \mathbb{R}^n$ is said to be a limit point for a solution $\varphi(\cdot)$ of (4) if there exists a sequence $\{t_i\}$, $t_i \to +\infty$ as $i \to +\infty$, such that $\varphi(t_i) \to q$ as $i \to +\infty$.

The set of the limit points of $\varphi(\cdot)$ is said to be the limit set of $\varphi(\cdot)$ and is denoted by $\Omega(\varphi)$.

Definition 4. A set Ω is said to be a weakly invariant set for (4) if through each point $x_0 \in \Omega$ there exists a maximal solution of (4) lying in Ω.

We recall that under the assumption that F is an upper semi-continuous multivalued map with compact, convex values, if $\varphi(\cdot)$ is a solution of (4) and $\Omega(\varphi)$ is its limit set, then $\Omega(\varphi)$ is weakly invariant and if $\varphi(t), t \in \mathbb{R}_+$, lies in a bounded domain, then $\Omega(\varphi)$ is nonempty, bounded, connected and $\mathrm{dist}(\varphi(t), \Omega(\varphi)) \to 0$ as $t \to +\infty$ (see [12], p. 129).

Theorem 3. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous and regular Lyapunov function for* (4)*. Let us assume that for some $l > 0$, the connected component L_l of the level set $\{x \in \mathbb{R}^n : V(x) \leq l\}$ such that $0 \in L_l$ is bounded. Let $x_0 \in L_l$, $\varphi(\cdot) \in S_{x_0}$. Let*

$$Z_V^{(4)} = \left\{ x \in \mathbb{R}^n : 0 \in \overline{\dot{V}}^{(4)}(x) \right\}$$

and let M be the largest weakly invariant subset of $\overline{Z_V^{(4)}} \cap L_l$.

Then $\mathrm{dist}(\varphi(t), M) \to 0$ *as* $t \to +\infty$.

Proof. Let $\Omega(\varphi)$ be the limit set of $\varphi(\cdot)$. Let us remark that $\varphi(\cdot)$ is bounded. In fact otherwise there would exist $t_1 > 0$ such that $\varphi(t_1) \notin L_l$ and, since $\varphi(\cdot)$ is continuous, $\varphi(t_1)$ is not in any other connected component of $\{x \in \mathbb{R}^n : V(x) \leq l\}$. Then $V(\varphi(t_1)) > l \geq V(x_0)$, that is impossible since $V \circ \varphi$ is decreasing.

Let us prove that $\Omega(\varphi) \subseteq \overline{Z_V^{(4)}} \cap L_l$. Because of the definition of L_l, $\Omega(\varphi) \subseteq L_l$.

We now prove that $\Omega(\varphi) \subseteq \overline{Z_V^{(4)}}$.

Let us remark that V is constant on $\Omega(\varphi)$. Indeed, since $V \circ \varphi$ is decreasing and bounded from below, there exists $\lim_{t \to +\infty} V(\varphi(t)) = c \geq 0$. Let $y \in \Omega(\varphi)$. There exists a sequence $\{t_n\}, t_n \to +\infty$, such that $\lim_{n \to +\infty} \varphi(t_n) = y$ and, by the continuity of V, $V(y) = c$.

Let $y \in \Omega(\varphi)$ and $\psi(\cdot)$ be a solution of (4) lying in $\Omega(\varphi)$ such that $\psi(0) = y$. Since $V(\psi(t)) = c$ for all t, we have $\frac{d}{dt}V(\psi(t)) = 0$ for all t. Therefore, by Lemma 1, $0 \in \overline{\dot{V}}^{(4)}(\psi(t))$ almost everywhere, namely $\psi(t) \in Z_V^{(4)}$ almost everywhere.

Let $\{t_i\}$, $t_i \to 0$, be a sequence such that $\psi(t_i) \in Z_V^{(4)}$ for all i. Since ψ is continuous $\lim_{i\to+\infty} \psi(t_i) = \psi(0) = y \in \overline{Z_V^{(4)}}$.

From the fact that $\Omega(\varphi)$ is weakly invariant it follows that $\Omega(\varphi) \subseteq M$ and from the fact that $\mathrm{dist}(\varphi(t), \Omega(\varphi)) \to 0$ as $t \to +\infty$ it follows that $\mathrm{dist}(\varphi(t), M) \to 0$ as $t \to +\infty$. □

Remark. Early versions of the invariance principle for differential inclusions can be found in [19,20]. Although the result presented in this paper has been largely inspired from both of them, certain differences should be pointed out. First of all, we emphasize that Theorem 3 of this paper is more general than Theorem 3.2 of [20] since no assumption about uniqueness of solutions is required. As far as Ryan's invariance principle is concerned, essentially two remarks have to be done. On one hand Ryan's result refers to merely locally Lipschitz continuous Lyapunov functions, while we deal with locally Lipschitz continuous and also regular Lyapunov functions. On the other hand our identification of the "bad" set $Z_V^{(4)}$ is sharper than Ryan's one. Finally, Example 4 shows a case in which Theorem 3 can be used in order to compute the limit set, while Ryan's invariance principle doesn't help.

Remark. Example 1 of the previous Section shows that, in the conclusion of Theorem 3, we cannot avoid to take, in general, the closure of $Z_V^{(4)}$. Indeed, in Example 1 each trajectory is a closed path that coincides with its limit set and crosses the coordinates axis.

Example 3. *Smooth oscillator with nonsmooth friction and uncertain coefficients.* Let us consider a differential inclusion of the form (4) in $\mathbb{R}^2$, where

$$F(x_1,x_2) = \begin{cases} [-2x_2-1, -x_2-1] \times \{x_1\} & \text{at } (x_1,x_2), x_1 > 0 \text{ and } x_2 > 0 \\ \{-x_2 - \operatorname{sgn} x_1\} \times \{x_1\} & \text{at } (x_1,x_2) \in \mathbb{R}^2 \backslash (\{(0,x_2), x_2 \in \mathbb{R}\} \\ & \quad \cup \{(x_1,x_2), x_1 > 0 \text{ and } x_2 > 0\}) \\ [-2x_2-1, -x_2+1] \times \{0\} & \text{at } (0,x_2), x_2 > 0 \\ [-x_2-1, -x_2+1] \times \{0\} & \text{at } (0,x_2), x_2 < 0 \\ [-1,1] \times \{0\} & \text{at } (0,0). \end{cases}$$

Let us now consider the smooth function $V(x_1,x_2) = \frac{x_1^2+x_2^2}{2}$. In this case

$$\overline{\dot{V}}^{(4)}(x) = \begin{cases} \{[-1,0]x_1x_2 - x_1\} & \text{at } (x_1,x_2), x_1 > 0 \text{ and } x_2 > 0 \\ \{-|x_1|\} & \text{at } (x_1,x_2) \in \mathbb{R}^2 \backslash (\{(0,x_2), x_2 \in \mathbb{R}\} \\ & \quad \cup \{(x_1,x_2), x_1 > 0 \text{ and } x_2 > 0\}) \\ \{0\} & \text{at } (0,x_2), x_2 \neq 0 \\ \{0\} & \text{at } (0,0), \end{cases}$$

then $Z_V^{(4)} = \{(0,x_2), x_2 \in \mathbb{R}\}$.

Let us now determine the largest weakly invariant subset M of $Z_V^{(4)}$.

Let us remark that, if $|x_2| \leq 1$, then $(0,0) \in F(x_1,x_2)$, hence the segment $\overline{P_1P_2}$, where $P_1 = (0,1)$ and $P_2 = (0,-1)$, is a weakly invariant subset of $Z_V^{(4)}$.

Moreover, if $|x_2| > 1$, all the vectors $v \in F(0,x_2)$ point in the same direction, hence each trajectory, starting in $(0,x_2)$, with $|x_2| > 1$, leaves the x_2-axis.

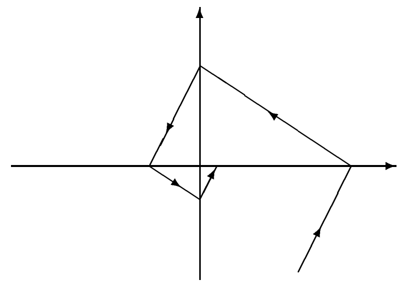

FIGURE 2

We conclude that $M = \overline{P_1P_2}$, *i.e.* all trajectories of the differential inclusion (4) tend to the segment $\overline{P_1P_2}$ as $t \to +\infty$. In fact each solution is attracted by a single point of the segment $\overline{P_1P_2}$. This follows by the proof of Theorem 3. Indeed each solution is attracted by the set $Z_V^{(4)} \cap L_l \cap V^{-1}(c)$ for some c.

Example 4. *Nonsmooth harmonic oscillator with nonsmooth friction* (Fig. 2). Let us consider a system of the form (3) in $\mathbb{R}^2$ where $f(x_1, x_2) = (-\operatorname{sgn} x_2 - \frac{1}{2}\operatorname{sgn} x_1, \operatorname{sgn} x_1)^T$. Filippov solutions of (3) are solutions of the differential inclusion (4), where (see Appendix)

$$F(x_1,x_2) = Kf(x_1,x_2) = \begin{cases} \{-\mathrm{sgn} x_2 - \frac{1}{2}\mathrm{sgn} x_1\} \times \{\mathrm{sgn} x_1\} & \text{at } (x_1,x_2),\ x_1 \neq 0 \text{ and } x_2 \neq 0 \\ \overline{\mathrm{co}}\left\{\left(-\frac{3}{2},1\right),\left(\frac{1}{2},1\right)\right\} & \text{at } (x_1,0),\ x_1 > 0 \\ \overline{\mathrm{co}}\left\{\left(-\frac{1}{2},-1\right),\left(\frac{3}{2},-1\right)\right\} & \text{at } (x_1,0),\ x_1 < 0 \\ \overline{\mathrm{co}}\left\{\left(-\frac{3}{2},1\right),\left(-\frac{1}{2},-1\right)\right\} & \text{at } (0,x_2),\ x_2 > 0 \\ \overline{\mathrm{co}}\left\{\left(\frac{3}{2},-1\right),\left(\frac{1}{2},1\right)\right\} & \text{at } (0,x_2),\ x_2 < 0 \\ \overline{\mathrm{co}}\left\{\left(-\frac{1}{2},-1\right),\left(\frac{1}{2},1\right)\left(-\frac{3}{2},1\right),\left(\frac{3}{2},-1\right)\right\} & \text{at } (0,0). \end{cases}$$

Let us now consider $V(x_1, x_2) = |x_1| + |x_2|$. In this case

$$\dot{\overline{V}}^{(4)}(x_1,x_2) = \begin{cases} \{-\frac{1}{2}\} & \text{at } (x_1,x_2),\ x_1 \neq 0 \text{ and } x_2 \neq 0 \\ \emptyset & \text{at } (x_1,0),\ x_1 \neq 0 \\ \emptyset & \text{at } (0,x_2),\ x_2 \neq 0 \\ \{0\} & \text{at } (0,0) \end{cases}$$

then V is a Lyapunov function for the system, that is stable at $x = 0$. Moreover $Z_V^{(4)} = \{(0,0)\}$, hence the solutions tend to $(0,0)$ as $t \to +\infty$ (see Fig. 2). Let us remark that in this example Ryan's invariance principle doesn't help if we want to compute the limit set of the differential inclusion. In fact, if $x_2 > 0$, we have that $\max\{V^o((0,x_2),v),\ v \in F(0,x_2)\} = \frac{5}{2} > 0$.

4. ASYMPTOTIC STABILIZATION

In this section we apply the stability result and the invariance theorem of the previous sections in order to construct an asymptotically stabilizing feedback of the form (2) for the affine input system (1). We shall assume that

(f0) $f \in L^\infty_{loc}(\mathbb{R}^n; \mathbb{R}^n)$, $0 \in Kf(0)$;

(G0) $G \in C(\mathbb{R}^n; \mathbb{R}^{n\times m})$;
(V0) $V : \mathbb{R}^n \to \mathbb{R}$ is a positive definite, locally Lipschitz continuous and regular function.

Note that, under these assumptions, the associated system (3) has, in general, a discontinuous righthand side.

From now on we adopt the following convention. Solutions of equations with discontinuous righthand side will be intended in the Filippov's sense (see Appendix). Furthermore, when we speak about stability at the origin of a system with discontinuous righthand side, we mean that the differential inclusion associated to the system (according to what explained in the Appendix) is stable at the origin and analogously for asymptotic stability:

Definition 5. System (1) is said to be locally asymptotically stabilizable if there exists $u \in L^\infty_{loc}(\mathbb{R}^n; \mathbb{R}^m)$ such that

(i) system (1) with $u = u(x)$ is stable at $x = 0$;
(ii) there exists $\eta > 0$ such that if $\|x_0\| < \eta$ any solution $\varphi(\cdot)$ of system (1) with $u = u(x)$ is defined on $[0, +\infty)$ and $\varphi(t) \to 0$ as $t \to +\infty$.

We shall assume that (3) admits a locally Lipschitz continuous and regular Lyapunov function V. We recall that the existence of such a Lyapunov function is sufficient (but in general not necessary) for the stability of system (3).

Since V is locally Lipschitz continuous, the gradient of V may fail to exist on a set of zero measure.

Let us remark that, under these assumptions, the feedback law (2) is locally (essentially) bounded and measurable on $\mathbb{R}^n$. In fact, if L_x is the Lipschitz constant of V in a compact neighbourhood U_x of x,

$$\|u(x)\| \le \alpha\|\nabla V(x)\|\|G(x)\| \le \alpha L_x\|G(x)\| \text{ a.e. in } U_x,$$

that is bounded in U_x because G is continuous. As remarked in the Appendix, for every $v \in \mathbb{R}^n$, $\nabla V(\cdot)\cdot v$ is measurable; hence u is measurable.

From this fact it follows that the right hand-side of the equation

$$\dot{x} = f(x) - \alpha G(x)(\nabla V(x)G(x))^T \tag{6}$$

is also locally (essentially) bounded and measurable on $\mathbb{R}^n$.

In the smooth case, the stability property of system (3) is not affected by the application of the feedback law (2). This is not true in the case V is merely locally Lipschitz continuous and regular, as the following example shows.

Example 5. Let us consider a single-input system of the form (1) in $\mathbb{R}^2$, where

$$f(x_1, x_2) = \begin{cases} (\text{sgn}x_1,\ -2)^T & \text{at } (x_1, x_2),\ x_2 \ge 0 \\ (0, 0)^T & \text{at } (x_1, x_2),\ x_2 < 0. \end{cases}$$

$G(x_1, x_2) = (0, 1)^T$, and the function $V(x_1, x_2) = |x_1| + |x_2|$. By computing $\overline{\dot{V}}^{(3)}(x_1, x_2)$, it is easily proved that system (3) is stable at $x = 0$ (see Fig. 3).

Let us now consider system (6).

$$K(f - \alpha G(\nabla V G))(x_1, x_2) = \begin{cases} \{\text{sgn}x_1\} \times \{-2-\alpha\} & \text{at } (x_1, x_2),\ x_1 \ne 0,\ x_2 > 0 \\ \{0\} \times \{\alpha\} & \text{at } (x_1, x_2),\ x_2 < 0 \\ [-1, 1] \times \{-2-\alpha\} & \text{at } (0, x_2),\ x_2 > 0 \\ \overline{\text{co}}\{(\text{sgn}x_1, -2-\alpha)^T, (0, \alpha)^T\} & \text{at } (x_1, 0),\ x_1 \ne 0 \\ \overline{\text{co}}\{(1, -2-\alpha)^T, (0, \alpha)^T, (-1, -2-\alpha)^T\} & \text{at } (0, 0). \end{cases}$$

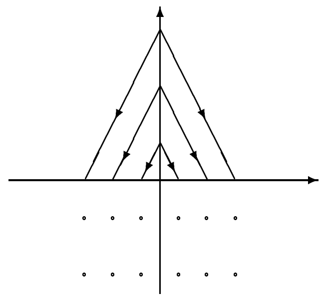

FIGURE 3.

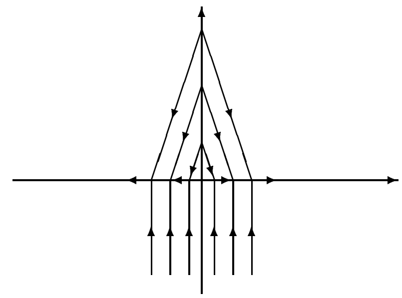

FIGURE 4.

Let us remark that for all $\alpha > 0$ and for all the points $(x_1, 0)$ with $x_1 \neq 0$, there exists a trajectory starting from $(x_1, 0)$ which lies on the x_1-axis and goes to infinity. This is obtained by considering the vector $\left(\frac{\alpha}{2(1+\alpha)}, 0\right) \in K(f - \alpha G(\nabla V G))(x_1, 0)$ if $x_1 > 0$, and the vector $\left(-\frac{\alpha}{2(1+\alpha)}, 0\right) \in K(f - \alpha G(\nabla V G))(x_1, 0)$ if $x_1 < 0$ (see Fig. 4 in the case $\alpha = 1$).

The previous example shows that, in order to guarentee the conservation of stability for the closed loop system in the nonsmooth case, we need to add some extra assumptions. Actually, we do not present a unique condition, but we list some alternative conditions which, combined together in a convenient way, allow us to get not only the stability of system (6), but also the stabilizability of system (1). Note that in these conditions the variable x is not yet quantified. Since the role of x will depend on the circumstances, it is convenient to specify it later. The possible conditions are the following:

(f1) $\max \overline{\dot{V}}^{(3)}(x) \leq 0$;
(f2) for all $v \in Kf(x)$ there exists $p \in \partial V(x)$ such that $p \cdot v \leq 0$;
(f3) for all $v \in Kf(x)$ and for all $p \in \partial V(x)$, $p \cdot v \leq 0$;

(G1) there exists $c \in \mathbb{R}$ such that for all $p, q \in \partial V(x)$, $(pG(x)) \cdot (qG(x)) = c^2$ (c may depend on x);
(G2) either $(pG(x)) \cdot (qG(x)) > 0$ for all $p, q \in \partial V(x)$, or $(pG(x)) \cdot (qG(x)) = 0$ for all $p, q \in \partial V(x)$;
(G3) $(pG(x)) \cdot (qG(x)) \geq 0$ for all $p, q \in \partial V(x)$;
(fG1) there exists $\alpha > 0$ such that for all $v \in Kf(x)$ and for all $q \in \partial V(x)$ there exist $p_1, p_2 \in \partial V(x)$ such that $(p_1 - p_2) \cdot (v - \alpha G(x)(qG(x))^T) \neq 0$.

Note that the symbols $\overline{\dot{V}}^{(3)}$ and $\overline{\dot{V}}^{(6)}$ indicate the set-valued derivatives of V associated to (3) and (6) respectively.

By definition of $\overline{\dot{V}}^{(3)}$, (f1) can be restated by saying that if there exists $v \in Kf(x)$ such that for all $p \in \partial V(x)$ one has $p \cdot v = a$, then $a \leq 0$. Conditions (f1), (f2) and (f3) can then be seen as geometric conditions on mutual positions of the sets $Kf(x)$ and $\partial V(x)$. Moreover we have that (f3) $\Rightarrow$ (f2) $\Rightarrow$ (f1).

In order to interpretate conditions (G1), (G2) and (G3), let us consider the set $H(x) = \{pG(x),\ p \in \partial V(x)\}$. (G1) implies that $H(x)$ reduces to a single vector, while (G2) and (G3) are conditions on the size of $H(x)$. For these conditions it holds that (G1) $\Rightarrow$ (G2) $\Rightarrow$ (G3).

The meaning of condition (fG1) is explained by the following lemma.

Lemma 2. *Assume that conditions* (f0), (G0) *and* (V0) *hold for some* $x \in \mathbb{R}^n$. *There exists* $\alpha = \alpha(x) > 0$ *such that condition* (fG1) *holds if and only if* $\overline{\dot{V}}^{(6)}(x) = \emptyset$.

Proof. We prove the statement by contradiction.

Let us suppose that for all $\alpha > 0$ one has $\overline{\dot{V}}^{(6)}(x) \neq \emptyset$. Then there exist $a \in \mathbb{R}$, $w \in K(f - \alpha G(\nabla V G)^T)(x)$ such that, for all $p \in \partial V(x)$, $p \cdot w = a$. By properties (ii) and (iii) of Proposition 2 and property (iii) of Proposition 3 in the Appendix it follows that there exist $v \in Kf(x)$ and $q \in \partial V(x)$ such that for all $p \in \partial V(x)$, $p \cdot (v - \alpha G(x)G(x)^T q) = a$. Let $p_1, p_2 \in \partial V(x)$. We have $p_1 \cdot (v - \alpha G(x)G(x)^T q) = p_2 \cdot (v - \alpha G(x)G(x)^T q) = a$, hence $(p_1 - p_2) \cdot (v - \alpha G(x)G(x)^T q) = 0$, which is a contradiction to (fG1).

The *vice versa* is easily proved by contradiction. □

4.1. Conservation of stability

From the previous discussion it follows that, in order to prove a stabilization result for system (1) by means of the feedback law (2), the first step is to give some sufficient conditions for system (6) being stable. We do that in the following lemma.

Lemma 3. *Let us assume that* (f0), (G0), (V0) *hold and* (f1) *holds for all* $x \in \mathbb{R}^n \backslash N$, *where*

$$N = \{x \in \mathbb{R}^n \ \ \textit{such that } V \textit{ is not differentiable at } x\}\cdot$$

Let us suppose further that for each $x \in N$ *one of the following combinations of conditions holds: (i)* (f1) *and* (G1)*, (ii)* (fG1) *for some* α *independent of* x*, (iii)* (f2) *and* (G3)*, (iv)* (f3).

Then for each $x \in \mathbb{R}^n$*,* $\max \overline{\dot{V}}^{(6)}(x) \leq 0$.

Moreover, if the use of (fG1) *can be avoided, the choice of* α *can be arbitrary.*

Proof. Let $a \in \overline{\dot{V}}^{(6)}(x)$. Then there exists $w \in K(f - \alpha G(\nabla V G)^T)(x)$ such that, for all $p \in \partial V(x)$, $p \cdot w = a$. From properties (ii) and (iii) of Proposition 2 and property (iii) of Proposition 3 in the Appendix it follows that there exist $v \in Kf(x)$ and $q \in \partial V(x)$ such that $w = v - \alpha G(x)(qG(x))^T$. In the following we will use this representation for w without mentioning it explicitly.

We distinguish five cases: (o) for $x \in \mathbb{R}^n \backslash N$ and (i), (ii), (iii), (iv) for $x \in N$.

(o) In this case $\partial V(x) = \{\nabla V(x)\}$, then $a = \nabla V(x) \cdot w = \nabla V(x) \cdot (v - \alpha G(x)(\nabla V(x)G(x))^T)$ and $\nabla V(x) \cdot v = a + \alpha \|(\nabla V(x)G(x))^T\|^2 = b$, where $b \in \overline{\dot{V}}^{(3)}(x)$. Since by assumption $\max \overline{\dot{V}}^{(3)}(x) \leq 0$, we also have that $b \leq 0$, hence $a = b - \alpha \|(\nabla V(x)G(x))^T\|^2 \leq 0$.

(i) In this case $a = p \cdot w = p \cdot v - \alpha(pG(x))(qG(x))^T = p \cdot v - \alpha c^2$ for each $p \in \partial V(x)$. Hence the proof that $\max \overline{\dot{V}}^{(6)}(x) \leq 0$ is analogous to the one in (o).

(ii) From assumption (fG1) and Lemma 2 it follows that $\overline{\dot{V}}^{(6)}(x) = \emptyset$ for suitable choice of α.

(iii) Since (f2) implies (f1), clearly it is sufficient to prove that for all $w \in K(f + Gu)(x)$ there exists $p \in \partial V(x)$ such that $p \cdot w \leq 0$. Let $p \in \partial V(x)$ such that $p \cdot v \leq 0$ (such a p exists because of (f2)). By (G3) we get $a = p \cdot w = p \cdot v - \alpha(pG(x))^T \cdot (qG(x))^T \leq 0$, as required.

(iv) For all $p \in \partial V(x)$ $a = p \cdot w = p \cdot v - \alpha(pG(x))^T \cdot (qG(x))^T$. In particular, for $p = q$ we get $a = q \cdot w = q \cdot v - \alpha \|(qG(x))^T\|^2$ that is non-positive because of (f3). □

From Lemma 3 and Theorem 2 it follows that system (6) is stable at $x = 0$.

4.2. Improvement of stability

In order to study asymptotic stabilization of system (1) let us introduce the sets

$$Z_V^{(6)} = \{x \in \mathbb{R}^n : 0 \in \overline{\dot{V}}^{(6)}(x)\}$$

and

$$Z_V^{(3)} = \{x \in \mathbb{R}^n : 0 \in \overline{\dot{V}}^{(3)}(x)\}.$$

Let us recall that, by the invariance theorem stated in Section 3, the solutions of systems (3) and (6), with initial condition x_0 such that $\|x_0\| < l$, respectively tend to $\overline{Z_V^{(3)}} \cap L_l$ and $\overline{Z_V^{(6)}} \cap L_l$, where L_l is a connected and bounded level set of V.

Lemma 4. *Let us assume that* (f0), (G0), (V0) *hold and that* (f1) *holds for all* $x \in \mathbb{R}^n\backslash N$. *Let us suppose that for each* $x \in N$ *one of the following pairs of conditions holds: (i)* (f1) *and* (G1)*, (ii)* (f1) *and* (fG1) *for some* α *independent of* x*, (iii)* (f2) *and* (G2)*, (iv)* (f3) *and* (G3).

Then $Z_V^{(6)} \subseteq Z_V^{(3)}$.

Proof. $x \in Z_V^{(6)}$ means that there exists $w \in K(f - \alpha G(\nabla V G)^T)(x)$ such that, for all $p \in \partial V(x)$, $p \cdot w = 0$. Using the decomposition of w already mentioned in the proof of Lemma 3, we get that there exist $v \in Kf(x)$ and $q \in \partial V(x)$ such that $p \cdot w = p \cdot v - \alpha(pG(x))^T \cdot (qG(x))^T = 0$, *i.e.* $p \cdot v = \alpha(pG(x))^T \cdot (qG(x))^T$. Again, we distinguish five cases: (o) $x \in \mathbb{R}^n\backslash N$, (i), (ii), (iii), (iv).

(o) In this case $\partial V(x) = \{\nabla V(x)\}$ then $\nabla V(x) \cdot v = \alpha \|(\nabla V(x) G(x))^T\|^2 = b \geq 0$. On the other hand, since $b \in \overline{\dot{V}}^{(3)}(x)$ and $\max \overline{\dot{V}}^{(3)}(x) \leq 0$, $b \leq 0$, hence $b = 0$, *i.e.* there exists $v \in Kf(x)$ such that $\nabla V(x) \cdot v = 0$ and $x \in Z_V^{(3)}$.

(i) The proof is analogous to the one in (o).

(ii) By Lemma 2, $\overline{\dot{V}}^{(6)}(x) = \emptyset$, so that $0 \notin \overline{\dot{V}}^{(6)}(x)$ and $x \notin Z_V^{(6)}$.

(iii) $p \cdot v = \alpha(pG(x))^T \cdot (qG(x))^T$ implies that x is such that for all $p, q \in \partial V(x)$, $(pG(x))^T \cdot (qG(x))^T = 0$, otherwise for all $p \in \partial V(x)$ one has $p \cdot w > 0$, which contradicts (f2). We conclude that, for all $p \in \partial V(x)$, $p \cdot v = 0$, *i.e.* $x \in Z_V^{(3)}$.

(iv) $p \cdot v = \alpha(pG(x))^T \cdot (qG(x))^T \geq 0$ because of (G3). On the other hand, by condition (f3), for all $p \in \partial V(x)$ we have $p \cdot v \leq 0$, hence, for all $p \in \partial V(x)$, $p \cdot v = 0$, *i.e.* $x \in Z_V^{(3)}$. □

We can finally summarize the results of the present section in the following theorem.

Theorem 4. *Let us assume that* (f0), (G0) *and* (V0) *hold and* (f1) *holds for all* $x \in \mathbb{R}^n\backslash N$. *If* N *can be decomposed as a union* $N = N_{11} \cup N_{12} \cup N_2 \cup N_3$ *such that*

- (i) *for all* $x \in N_{11} \cup N_{12}$ (f1) *holds; for all* $x \in N_{11}\backslash\{0\}$, (G1) *holds and for all* $x \in N_{12}\backslash\{0\}$, (fG1) *holds with* α *independent of* x*;*
- (ii) *for all* $x \in N_2$, (f2) *holds and for all* $x \in N_2\backslash\{0\}$, (G2) *holds;*
- (iii) *for all* $x \in N_3$, (f3) *holds and for all* $x \in N_3\backslash\{0\}$, (G3) *holds.*

Then, there exists $\alpha > 0$ *such that*

- (A) (6) *is stable at* $x = 0$*;*
- (B) $Z_V^{(6)} \subseteq Z_V^{(3)}$.

Moreover let us assume that

- (V1) *there exists* $l > 0$ *such that the set* $\{x \in \mathbb{R}^n : V(x) \leq l\}$ *is connected and bounded,*
- (fG2) *the largest weakly invariant subset of* $\overline{Z_V^{(6)}}$ *is* $\{0\}$.

Then

- (C) (1) *is asymptotically stabilizable by means of the feedback law* (2).

Finally, if the use of (fG1) *can be avoided, the choice of* α *is arbitrary.*

Remark. If $V \in C^1$ then $N = \emptyset$, hence we only need to check condition (f1) to get the stability of (6), and conditions (V1) and (fG2) to get the asymptotic stabilization of (1), *i.e.* we have a classical-like stabilization theorem that can be applied in the case the only assumptions on f are measurability and local boundedness.

4.3. **Examples**

In the present subsection we illustrate the various situations described in Theorem 4 by means of some examples.

Example 6. Let us consider a system of the form (1) in $\mathbb{R}^2$, where $f(x_1,x_2) = (-\text{sgn}x_2, \text{sgn}x_1)^T$ and $G(x_1,x_2) = (x_1,x_2)^T$, and the function $V(x_1,x_2) = |x_1| + |x_2|$.

As shown in Example 1, for all $(x_1,x_2) \in \mathbb{R}^2$ we have $\max \overline{\dot{V}}^{(3)}(x_1,x_2) \leq 0$.
$N = \{(x_1,0), x_1 \in \mathbb{R}\} \cup \{(0,x_2), x_2 \in \mathbb{R}\}$. Let us consider $p = (p_1,p_2)$ and $q = (q_1,q_2) \in \partial V(x_1,x_2)$

$$(pG(x_1,x_2)) \cdot (qG(x_1,x_2)) = \begin{cases} x_1^2 & (x_1,0),\ x_1 \neq 0 \\ x_2^2 & (0,x_2),\ x_2 \neq 0 \end{cases}$$

i.e. condition (G1) is verified for all $(x_1,x_2) \in N$. Then, by (A) in Theorem 4, for all $\alpha > 0$ system (1) with the feedback (2) is stable at $x = 0$.

Moreover let us consider the set

$$\overline{\overline{\dot{V}}}^{(6)}(x_1,x_2) = \{a \in \mathbb{R} :\ \exists v \in Kf(x_1,x_2)\ \exists q \in \partial V(x_1,x_2) \text{ such that}$$

$$\forall p \in \partial V(x_1,x_2),\quad p \cdot (v - \alpha G(x_1,x_2)((qG(x_1,x_2))^T) = a\}$$

$$\overline{\overline{\dot{V}}}^{(6)}(x_1,x_2) = \begin{cases} -\alpha(|x_1| + |x_2|)^2 & \text{at } (x_1,x_2),\ x_1 \neq 0 \text{ and } x_2 \neq 0 \\ \emptyset & \text{at } (x_1,0),\ x_1 \neq 0 \\ \emptyset & \text{at } (0,x_2),\ x_2 \neq 0 \\ \{0\} & \text{at } (0,0). \end{cases}$$

From properties (*ii*) and (*iii*) of Proposition 2 and property (*iii*) of Proposition 3 it follows that for all $(x_1,x_2) \in \mathbb{R}^2$, $\overline{\dot{V}}^{(6)}(x_1,x_2) \subseteq \overline{\overline{\dot{V}}}^{(6)}(x_1,x_2)$, hence $Z_V^{(6)} = \{(0,0)\}$ and, by (C) in Theorem 4, the system is asymptotically stabilizable by means of the feedback law (2).

Example 7. Let us consider a system of the form (1) in $\mathbb{R}^2$, where $f(x_1,x_2) = (-\text{sgn}x_2, \text{sgn}x_1)^T$ and $G(x_1,x_2) = (1,0)^T$, and the function $V(x_1,x_2) = |x_1| + |x_2|$.

As shown in Example 1, for all $(x_1,x_2) \in \mathbb{R}^2$ one has $\max \overline{\dot{V}}^{(3)}(x_1,x_2) \leq 0$.
Also in this case $N = \{(x_1,0), x_1 \in \mathbb{R}\} \cup \{(0,x_2), x_2 \in \mathbb{R}\}$. Condition (G1) is verified on $\{(x_1,0), x_1 \in \mathbb{R}\}$ but not on $\{(0,x_2), x_2 \in \mathbb{R}\}$. Nevertheless, for $\alpha \in (0,1)$, condition (fG1) is verified on $\{(0,x_2), x_2 \in \mathbb{R}\}$, in fact

$$\{v - \alpha G(0,x_2)(qG(0,x_2))^T,\ \text{for } v \in Kf(0,x_2),\ q \in \partial V(0,x_2), x_2 \neq 0\}$$

$$= \begin{cases} [-1-\alpha, -1+\alpha] \times [-1,1] & \text{at } (0,x_2), x_2 > 0 \\ [1-\alpha, 1+\alpha] \times [-1,1] & \text{at } (0,x_2), x_2 < 0 \end{cases}$$

and

$$\{(p_1 - p_2);\ p_1, p_2 \in \partial V(0,x_2), x_2 \neq 0\} = ([-2,2],0)^T.$$

By (A) in Theorem 4, it follows that system (6) is stable at $x = 0$ with $\alpha \in (0,1)$. Moreover computations analogous to those of Example 6 show that $Z_V^{(6)} = \{(0,0)\}$. Hence, by (C) in Theorem 4, the system is asymptotically stabilizable by means of the feedback law (2) with a fixed $\alpha \in (0,1)$. Example 4 is actually a

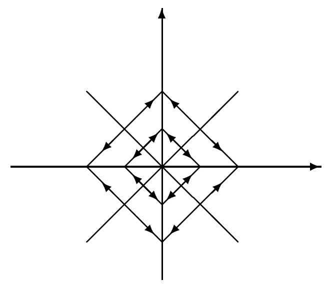

FIGURE 5.

particular case of the present example, with $\alpha = \frac{1}{2}$. Figures 1 and 2 show the behaviour of the system before and after the application of the feedback.

Remark. By direct computation, it is possible to see that the closed loop system considered in the previous example is actually stable for all $\alpha > 0$. However, for $\alpha > 1$, no one of the alternative conditions of Theorem 4 can be applied. This shows that Theorem 4 does not cover all the possible cases.

Example 8. Let us consider a system of the form (1) in $\mathbb{R}^2$, where $f(x_1, x_2) = (-\text{sgn}x_2, \text{sgn}x_1)^T$ and $G(x_1, x_2) = \left(x_1 + \frac{1}{2}x_2, x_2 + \frac{1}{2}x_1\right)^T$, and the function $V(x_1, x_2) = |x_1| + |x_2|$.

As shown in Example 1, for all $(x_1, x_2) \in \mathbb{R}^2$ we have $\max \dot{\overline{V}}^{(3)}(x_1, x_2) \leq 0$, and also in this case $N = \{(x_1, 0), x_1 \in \mathbb{R}\} \cup \{(0, x_2), x_2 \in \mathbb{R}\}$. On N condition (f2) is satisfied. Moreover for all $(x_1, x_2) \in N$ and for all $p, q \in \partial V(x_1, x_2)$ we have $(pG(x)) \cdot (qG(x)) > 0$, *i.e.* condition (G2) is satisfied in N. By (A) in Theorem 4, it follows that system (6) is stable at $x = 0$ for all $\alpha > 0$. Moreover computations analogous to those of Example 6 show that $Z_V^{(6)} = \{(0, 0)\}$, hence, by (C) in Theorem 4, the system is asymptotically stabilizable by means of the feedback law (2) for all $\alpha > 0$.

Example 9. Let us consider a system of the form (1) in $\mathbb{R}^2$, where

$$f(x_1, x_2) = \begin{cases} (x_2, -x_2)^T & \text{at } (x_1, x_2),\ 0 \leq x_2 \leq x_1 \\ (-x_1, x_1)^T & \text{at } (x_1, x_2),\ 0 \leq x_1 \leq x_2 \\ (-x_1, -x_1)^T & \text{at } (x_1, x_2),\ 0 \leq -x_1 \leq x_2 \\ (-x_2, -x_2)^T & \text{at } (x_1, x_2),\ 0 \leq x_2 \leq -x_1 \\ (x_2, -x_2)^T & \text{at } (x_1, x_2),\ x_1 \leq x_2 \leq 0 \\ (-x_1, x_1)^T & \text{at } (x_1, x_2),\ x_2 \leq x_1 \leq 0 \\ (-x_1, -x_1)^T & \text{at } (x_1, x_2),\ x_2 \leq -x_1 \leq 0 \\ (-x_2, -x_2)^T & \text{at } (x_1, x_2),\ -x_1 \leq x_2 \leq 0 \end{cases}$$

and $G(x_1, x_2) = (x_1 + x_2, x_1 + x_2)^T$, and the function $V(x_1, x_2) = |x_1| + |x_2|$.

By computing $Kf(x_1, x_2)$, it is easy to see that (f3) is verified, then (3) is stable at $x = 0$ (see Fig. 5). Since condition (G3) is satisfied on N (note that (G2) is not satisfied on N), then not only system (6) is stable at $x = 0$, but also $Z_V^{(6)} \subseteq Z_V^{(3)}$. Actually in this case it can be shown that the feedback law (2) does not stabilize system (1) asymptotically.

5. Appendix

5.1. Filippov solutions of equations with discontinuous right handside

In this paper we were concerned with differential equations with discontinuous right handside of the form (3). For these equations the classical notion of solution is not appropriate and needs to be generalized. There are many possibilities of interest for control theory (see [8, 15]). We follow the approach due to Filippov (see [12]).

For each $x \in \mathbb{R}^n$, let us consider the set

$$Kf(x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{\text{co}}\{f(B(x,\delta)\backslash N)\}$$

where $B(x,\delta)$ is the ball of center x and radius δ, $\overline{\text{co}}$ denotes the convex closure and μ the usual Lebesgue measure in $\mathbb{R}^n$.

Definition 6. A Filippov solution of (3) on a nondegenerate interval $I \subseteq \mathbb{R}$ is a function $\varphi : I \to \mathbb{R}^n$ such that $\varphi(\cdot)$ is absolutely continuous on any interval $[t_1, t_2] \subseteq I$ and

$$\dot{\varphi}(t) \in Kf(\varphi(t)) \text{ for almost all } t \in I.$$

In other words Filippov's approach constists in replacing (3) by a differential inclusion defined by means of the "operator" K.

Note that if we change f on a set of $\mathbb{R}^n$ of zero measure then $Kf(x)$ remains unchanged and so does the Filippov solution of (3).

Assume that $f \in L^\infty_{loc}(\mathbb{R}^n;\mathbb{R}^n)$. It is proved in [12] that the multivalued function $Kf : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is upper semi-continuous with nonempty, compact, convex values and locally bounded. Hence (see [1, 10, 12]) for each $x_0 \in \mathbb{R}^n$ there exists at least one solution of the differential inclusion $\dot{x} \in Kf(x)$ with the initial condition $x(0) = x_0$.

In [18] it is provided a calculus which simplify the calculation of the multivalued map associated to a locally essentially bounded function. In particular we report some facts which were used in this paper.

Proposition 2. (i) *If $f \in C(\mathbb{R}^n;\mathbb{R}^n)$ then $Kf(x) = \{f(x)\}$ $\forall x \in \mathbb{R}^n$.*
(ii) *If $f, g \in L^\infty_{loc}(\mathbb{R}^n;\mathbb{R}^n)$ then $K(f+g)(x) \subseteq Kf(x) + Kg(x)$ $\forall x \in \mathbb{R}^n$.*
Moreover if $f \in C(\mathbb{R}^n;\mathbb{R}^n)$ then $K(f+g)(x) = f(x) + Kg(x)$ $\forall x \in \mathbb{R}^n$.
(iii) *If $G \in C(\mathbb{R}^n;\mathbb{R}^{n\times m})$, $u \in L^\infty_{loc}(\mathbb{R}^n;\mathbb{R}^m)$ then $K(Gu)(x) = G(x)Ku(x)$ $\forall x \in \mathbb{R}^n$.*

5.2. Clarke generalized gradient

To deal with nonsmooth Lyapunov functions we have to introduce generalized derivatives and gradients; we mainly focus on Clarke generalized directional derivatives and gradient (see [6]).

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function.

Definition 7. Clarke upper generalized derivative of V at x in the direction of v is

$$V^o(x,v) = \limsup_{y\to x\ h\downarrow 0} \frac{V(y+hv) - V(y)}{h}\cdot$$

Analogously Clarke lower generalized derivative of V at x in the direction of v is

$$V_o(x,v) = \liminf_{y\to x\ h\downarrow 0} \frac{V(y+hv) - V(y)}{h}\cdot$$

If L_x is the Lipschitz constant of V in a compact neighbourhood U_x of x then

$$-L_x\|v\| \leq V_o(y,v) \leq V^o(y,v) \leq L_x\|v\| \quad \forall y \in U_x.$$

Since V is locally Lipschitz, by Rademacher's theorem it is differentiable almost everywhere; moreover, for every $v \in \mathbb{R}^n$, $\nabla V(\cdot)\cdot v$ is measurable (see [11], p. 83). Let N be the set of measure zero where the gradient of V does not exist.

Definition 8. Clarke generalized gradient of V at x is the set

$$\partial V(x) = \text{co}\left\{\lim_{i\to+\infty} \nabla V(x_i) \,:\, x_i \to x, \quad x_i \notin S \; x_i \notin N\right\}$$

where S is any set of zero measure in $\mathbb{R}^n$.

In the following proposition we summarize some useful properties of Clarke generalized directional derivatives and gradient.

Proposition 3. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz.*
(i) $\partial V(x) = \{p \in \mathbb{R}^n : \; V^o(x,v) \geq p\cdot v \quad \forall v \in \mathbb{R}^n\} = \{p \in \mathbb{R}^n : \; V_o(x,v) \leq p\cdot v \quad \forall v \in \mathbb{R}^n\}$
(ii) $V^o(x,v) = \max\{p\cdot v, \; p \in \partial V(x)\}$
$V_o(x,v) = \min\{p\cdot v, \; p \in \partial V(x)\} = -V^o(x,-v)$
(iii) $K(\nabla V)(x) = \partial V(x)$.

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