ESAIM: Control, Optimisation and Calculus of Variations March 1997, Vol. 2, pp. 57–85
URL: http://www.emath.fr/cocv/

INPUT TO STATE STABILITY PROPERTIES OF NONLINEAR SYSTEMS AND APPLICATIONS TO BOUNDED FEEDBACK STABILIZATION USING SATURATION

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Abstract. The concepts of stability, attractivity and asymptotic stability for systems subject to restrictions of the input values are introduced and analyzed in terms of Lyapunov functions. A comparison with the well known input-to-state stability property introduced by Sontag is provided. We use these concepts in order to derive sufficient conditions for global stabilization for triangular and feedforward systems by means of saturated bounded feedback controllers and also recover some recent results due to Teel.

1. Introduction

Input to state stability analysis of nonlinear control systems has been a subject of research by many authors and the corresponding results consist powerful tools for the feedback stabilizability problem (see for instance and references therein). The well known input–to–state–stability (ISS) property proposed by Sontag in [15] and studied further in [12, 16, 18, 19, 20] play an important role to the global stabilization procedure for a wide class of systems like those having triangular structure, cascade connections, feedforward systems and linear systems subject to actuator saturations.

Our purpose is to analyze a weaker version of the ISS property, that has been originally introduced in [31, 32, 33], and give applications to stabilization of nonlinear interconnected systems by means of nested saturation functions.

In Section 2 we give the concepts of stability, attractivity, asymptotic and uniform asymptotic stability of control systems

$$\dot{x} = f(x, u), \quad (x, u) \in \mathbb{R}^n \times \mathbb{R}^k \tag{1.1}$$

$$f(0,0) = 0$$

subject to the restriction that each admissible control is an essentially bounded map $t \to u(t) \in \mathbb{R}^k$ with the property that for any initial state $x_0 \in \mathbb{R}^n$ and time t the following holds

$$(x(t, x_0, u), u(t)) \in L \tag{1.2}$$

where $x(t, x_0, u)$ denotes the trajectory of (1.1) of initial value x_0 and input u and L is a subregion of $\mathbb{R}^n \times \mathbb{R}^k$ containing zero.

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Reveived by the journal March 20, 1996. Acceptated for publication January 18, 1997.

Thanks to recent contributions of Sontag, Lin and Wang (see [12, 18, 19, 20]) it is possible to analyze the notion of the "uniform global asymptotic stability for (1.1) under the restriction (1.2)" (L–UGAS) in terms of Lyapunov functions (Theorems 2.1 and 2.3) and to provide some links between L–UGAS and ISS. Comparisons with the a–$\mathcal{L}_\infty$ stability as well as with the "robust global uniform asymptotic stability" introduced by Teel [4, 28] and Freeman–Kokotovic [5], respectively, are also given. In Theorem 2.5 of the present work it is shown that L–UGAS is equivalent to ISS for the particular case where L has the form

$$L := \{(x, u) : |u| \leq \gamma(|x|)\}$$

$\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ being an increasing positive definite function with $\gamma(s) \to +\infty$ as $s \to +\infty$.

Using the concepts of L–attractivity, L-GAS and L–UGAS and by extending the feedback design procedure of [30–33] we derive in Section 3 sufficient conditions for global stabilization for interconnected systems by means of combination of simple saturations, namely functions of the form

$$\sigma(s) = \begin{cases} a\varepsilon sgns & , |s| > \varepsilon \\ as & , |s| \leq \varepsilon \end{cases}$$

for certain constants ε and a. Particularly, in Theorem 3.1 of this work sufficient conditions are obtained for global bounded stabilization by using saturations for single–input systems of the form:

$$\begin{aligned} \dot{x} &= f(x, y_1) \\ \dot{y}_1 &= y_2 + g_1(x, y_1) \\ &\vdots \\ \dot{y}_{m-1} &= y_m + g_{m-1}(x, y_1, \ldots, y_{m-1}) \\ \dot{y}_m &= u + g_m(x, y_1, \ldots, y_{m-1}, u) \end{aligned} \tag{1.3}$$

$$(x; y_1, \ldots, y_m) \in \mathbb{R}^n \times \mathbb{R}^m,$$

where $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $g_i : \mathbb{R}^{n+i} \to \mathbb{R}$ are continuously differentiable (C^1) vanishing at zero and each g_i is bounded over $\mathbb{R}^{n+i}$. Boundedness hypothesis for the terms g_i is motivated by the possibility of cascade interconnections subject to actuator saturations.

Theorem 3.1 constitutes a version of the "adding one integrator" result (see [3, 6, 29]), namely the feedback stabilization procedure for systems

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= u, \quad (x, y, u) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \end{aligned} \tag{1.4}$$

and an extension of Teel's main theorem in [24] concerning stabilization of the linear chain of integrators

$$\dot{y}_0 = y_1, \dot{y}_1 = y_2, \ldots, \dot{y}_m = u \tag{1.5}$$

On the other hand, our approach is different from the design procedure developed in [29] and [24] as well as in earlier works (see [1, 7–11, 13, 14, 17, 21–28]), where similar problems are considered. Specifically, the main difference with Teel's approach for the case (1.5) is that our procedure is "backforward" and no change of coordinates is needed, while in [24] is "feedforward".

For the case (1.4) the hypothesis of Theorem 3.1 can be relaxed. In Proposition 3.7 we establish that global stabilization by means of a nested saturation can be succeeded, provided that there exists a C^1 function $y = \phi(x)$ with $\phi(0) = 0$ such that the map $x \to D\phi(x)f(x, \phi(x))$ is bounded and zero is UGAS for the subsystem $\dot{x} = f(x, \phi(x))$.

Using the analysis of Theorem 3.1 we can recover the main results of Teel's works [24] and [25] for the case (1.5) as well as for feedforward single–input systems

$$\dot{y}_i = y_{i+1} + g_i(y_{i+1}, \dots, y_m, u) \tag{1.6}$$
$$0 \le i \le m, \quad y_i \in \mathbb{R}, \quad u := y_{m+1}$$

where each g_i is C^1 and $o(y_{i+1}, \dots, y_m, u)$ at 0. Particularly, Lemma 3.10 presents a general global stabilization approach by using saturation which is applicable to a wide class of feedforward systems including those of the previous form (1.6).

Notations.

- $|x|$ denotes the usual Euclidean norm of a vector $x \in \mathbb{R}^n$ and x' its transpose; $S^n(x, r)$ denotes the open sphere of radius $r > 0$ centered at $x \in \mathbb{R}^n$.
- For any measurable function $u : \mathbb{R}^+ \to \mathbb{R}^m$ we denote $||u||=sup\{|u(t)|, t \ge 0\}$.
- A function $V : \mathbb{R}^n \to \mathbb{R}^+$ is called positive definite, if $V(x) > 0$ for $x \ne 0$ and $V(0) = 0$; V is called positive definite radially unbounded (p.d.r.u.), if it is positive definite with $V(x) \to +\infty$ as $|x| \to +\infty$.
- A function $a : \mathbb{R}^+ \to \mathbb{R}^+$ is of class K, if it is continuous (C^0), positive definite and non–decreasing. By K_∞ we denote the subclass of K consisting of all strictly increasing functions $a \in K$ with $a(s) \to +\infty$ as $s \to +\infty$. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is of class $K\mathcal{L}$, if it is C^0 and such that for each fixed t the map $\beta(\cdot, t)$ is of class K and for each fixed s the function $\beta(s, t)$ is decreasing to zero as $t \to +\infty$.
- $D\phi(x)$ denotes the derivative of a given map ϕ.

Remark 1.1. Usually, K denotes the class of positive definite C^0 functions that are strictly increasing. For the purposes of this paper however, it is preferable to adopt the notation K for the class of positive definite nondecreasing functions in order to include those that are bounded with constant values away from zero.

2. Stability properties of control systems

2.1. Definitions

We consider a system of the form (1.1) whose dynamics $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ are C^0 vanishing at zero. Assume that (1.1) is complete, namely for every initial x_0 and essentially bounded input u each solution $x(t) = x(t, x_0, u)$ of (1.1) is defined for all $t \ge 0$.

Let L be a closed subset of $\mathbb{R}^{n+k}$ with the following properties.

P1. Zero $0 \in \mathbb{R}^{n+k}$ belongs to L.

P2. The projection $\pi(L)$ of L on $\mathbb{R}^n$ along $\mathbb{R}^k$ coincides with the state space, i.e. $\pi(L) = \mathbb{R}^n$.

Next we extend the usual notions of positive invariance, stability, asymptotic stability and uniform asymptotic stability concerning single differential equations, as well as control systems with inputs taking values on a subset I of $\mathbb{R}^k$ (see for instance [12]), for the case (1.1) where the admissible inputs u depend on the initial state x_0 in such a way that

$$(x(t), u(t)) \in L, \ \forall\ t \in [0, t_{x_0,u}] \tag{2.1}$$

for some $0 < t_{x_0,u} \leq +\infty$ also depending on x_0 and u. Of course, the definitions given below have sense provided that for every initial x_0 the set $\mathcal{U}(x_0, L, t_{x_0,u})$ of inputs u satisfying (2.1) for some $t_{x_0,u} > 0$ is nonempty.

L–positive invariance

We say that the set $M \subset \mathbb{R}^n$ is positively invariant with respect to L (L–positively invariant), if $x(t) \in M$, for all initial $x_0 \in M$ and for every $t \geq 0$ and u for which (2.1) holds.

L–stability

We say that zero is stable with respect to L(L–stable), if for any $\varepsilon > 0$ a positive constant $\delta = \delta(\varepsilon)$ can be found such that

$$|x(t)| \leq \varepsilon, \quad \forall\ |x_0| \leq \delta$$

for all $t \geq 0$ and inputs u for which (2.1) holds.

L–attractivity

Zero is called attractor with respect to L(L–attractor), if there exists a neighborhood N of $0 \in \mathbb{R}^n$ such that

$$|x(t)| \to 0 \quad \text{as } t \to +\infty$$

for any initial $x_0 \in N$ and input u for which (2.1) holds for all $t \geq 0$ (i.e., $t_{x_0,u} = +\infty$); if in addition $N = \mathbb{R}^n$ zero is called an L–global attractor.

L–asymptotic stability

Zero is asymptotically stable with respect to L(L–AS) if it is L–stable and L–attractor. It is globally asymptotically stable with respect to L(L–GAS) if it is L–stable and L–global attractor.

L–uniform asymptotic stability

We say that zero is uniformly asymptotically stable with respect to L(L–UAS), if there exists a function $\beta \in K\mathcal{L}$ and a neighborhood N of zero $0 \in \mathbb{R}^n$ such that

$$|x(t)| \leq \beta(|x_0|, t), \quad \forall\ t \geq 0 \tag{2.2}$$

and for every $x_0 \in N$ and input u, provided that (2.1) holds for all $t \geq 0$; if in addition $N = \mathbb{R}^n$ then zero is called L–uniformly globally asymptotically stable (L–UGAS).

The notion of L–UGAS is equivalent to the concept of "robust global uniform asymptotic stability" (RGUAS) introduced in the work [5] of Freeman–Kokotovic. To be precise, given a set–valued map W from $\mathbb{R}^n$ to the subsets of $\mathbb{R}^k$ with $0 \in W(0)$ and $Dom\,(W) = \mathbb{R}^n$, we say that (1.1) satisfies the W–RGUAS property, if (2.2) holds for all x_0 and input $u(t) \in W(x(t))$. It turns out that

$$W\text{–RGUAS} \Longleftrightarrow L\text{–UGAS, provided that } L = Graph\,(W).$$

The following properties are direct consequences of the previous definitions.

- L–UAS, UGAS implies L–AS, GAS, respectively.
- If L_1, L_2 is a pair of subset of $\mathbb{R}^{n+k}$ satisfying properties P1 and P2, then

1. If zero is L_1, L_2–stable, attractor, AS, UAS, GAS, UGAS then it is $L_1 \cup L_2$–stable, attractor, AS, UAS, GAS, UGAS, respectively.
2. If $L_1 \subset L_2$ and zero is stable, attractor, AS, UAS, GAS, UGAS with respect to L_2, then it has the same properties with respect to L_1.

Note that for the particular case

$$L := \mathbb{R}^n \times I, \quad I \subset \mathbb{R}^k \tag{2.3}$$

the concepts of L–AS, L–UAS, L–GAS and L–UGAS coincide with the usual notions of AS, UAS, GAS and UGAS, respectively, as they have been defined in [12, 15, 20]. Moreover, as it has been recently proven by Sontag and Wang in [20], according to our notations and definitions, for the case (2.3) L–UAS is equivalent to L–AS, provided that the map f is Lipschitzian and I is compact.

Note finally that, when L has the form (2.3), completeness assumption for (1.1) implies that for every initial x_0 the set $\mathcal{U}(x_0, L, +\infty)$ consisting of inputs u for which (2.1) holds for all $t \geq 0$ is nonempty; in fact it contains all essentially bounded measurable inputs taking values on I.

2.2. Lyapunov function description

Next, we focus our attention for the case where L is represented as follows:

$$L := \{(x,u) : \ a_i(x) \leq u_i \leq b_i(x), \quad 1 \leq i \leq k\} \tag{2.4}$$

with $f(0,u) = 0$, $\forall\ u : a_i(0) \leq u_i \leq b_i(0)$ for certain continuous functions $a_i, b_i : \mathbb{R}^n \to \mathbb{R}$, which are (locally) *Lipschitzian* for $x \neq 0$ and satisfy $a_i(x) \leq b_i(x)$, $\forall\ x \in \mathbb{R}^n$ and $a_i(0)b_i(0) \leq 0$, $i = 1, 2, \ldots, k$.

This case is of particular interest and arises when one assumes that (1.1) is globally asymptotically stabilizable at the origin by a continuous map $u = (\phi_1(x), \ldots, \phi_k(x))$ vanishing at zero. Then by using standard Lyapunov function based arguments (see for instance [16]) it can be easily established that (1.1) is UGAS with respect to a region L of the form (2.4) for certain continuous functions a_i and b_i with $a_i(x) < \phi_i(x) < b_i(x)$, $x \neq 0$ which are C^1 except possibly at zero. Conversely, L–UGAS for (1.1) with L as above, implies global asymptotic stabilization at the origin by means of a continuous feedback law.

The following theorem, whose proof is based to the converse stability theorem in [12] and Remark 3.1 in [19], gives a Lyapunov characterization of the uniform global asymptotic stability with respect to a given region L of the form (2.4). As it was pointed out by a referee the result is a special case of Theorem 5.1 in [5] under the presence of W–RGUAS. To be more precise, for the case where L has the form (2.4) and a_i, b_i are Lipschitzian everywhere, the proof of the necessity part of the present result is basically the same with that given in [5]. The improvement here is that the function a_i, b_i are assumed to be Lipschitzian except possibly at the origin which is a general hypothesis.

THEOREM 2.1. *Suppose that there exist a positive definite C^1 map $V : \mathbb{R}^n \to \mathbb{R}^+$, a positive definite C^0 function $c : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$DV(x)f(x,u) \leq -c(|x|), \tag{2.5}$$

$$\forall\ (x,u) \in L, \quad x \ near \ zero,$$

where L is a subset of $\mathbb{R}^n \times \mathbb{R}^k$ satisfying properties P1 and P2. Then zero is L–UAS for (1.1); if in addition V is p.d.r.u. and (2.5) holds for all $(x,u) \in L$ then zero is L–UGAS. The converse claims are true, provided that f is Lipschitz continuous and L has the form (2.4), where the functions $a_i, b_i : \mathbb{R}^n \to \mathbb{R}$ are C^0 and Lipschitzian for $x \neq 0$.

Proof. For reasons of simplicity we deal only with the global part of the statement. Suppose first that (2.5) holds for every $(x,u) \in L$ and V is p.d.r.u. Then there exists a pair of functions $c_1, c_2 \in K_\infty$ such that

$$c_1(|x|) \leq V(x) \leq c_2(|x|) \tag{2.6}$$

which by (2.5) implies $DV(x)f(x,u) \leq -c(c_1^{-1}(V(x)))$ for all $(x,u) \in L$. Consequently, by evaluating the time derivative $\dot{V}$ of V along the solutions of (1.1) we find

$$\dot{V}(x(t,x_0,u)) \leq -c(c_1^{-1}(V(x(t,x_0,u))))$$

as long as (2.1) holds. From the comparison principle in [12] it follows that there exists a function $\gamma \in K\mathcal{L}$ depending only on c and c_1 such that $V(x(t,x_0,u)) \leq \gamma(V(x_0),t)$ as long as (2.1) holds. The latter in conjunction with (2.6) implies that zero is L–UGAS for (1.1). Conversely, suppose that L has the form (2.4) and zero is L–UGAS for (1.1). Consider the system

$$\dot{x} = F(x,v) := f(x, a_1(x)v_1 + b_1(x)(1-v_1), \ldots, a_k(x)v_k + b_k(x)(1-v_k)) \tag{2.7}$$

with input $v = (v_1, \ldots, v_k) \in \mathbb{R}^k$ taking values on the compact set $I := [0,1]^k$. Then L–UGAS of zero for (1.1) implies $(\mathbb{R}^n \times I)$–UGAS of zero for (2.7), which in turns guarantees completeness of (2.7). Indeed, each trajectory $x(t) := x(t,x_0,v)$ of (2.7) is also a trajectory of (1.1) with the same initial x_0 and input $u(t) = (u_1(t), \ldots, u_k(t))$ with components

$$u_i(t) := a_i(x(t))v_i(t) + b_i(x(t))(1-v_i(t))$$

This is an immediate consequence of the fact that $a_i(x(t)) \leq u_i(t) \leq b_i(x(t))$, or equivalently $(x(t),u(t)) \in L$. It turns out that the origin is $(\mathbb{R}^n \times I)$–UGAS for (2.7), hence the latter is complete. Since the map F is C^0 and Lipschitzian for $x \neq 0$ it follows by the converse stability theorem in [12] and Remark 3.1 in [19] that there exists a p.d.r.u. C^1 map $V : \mathbb{R}^n \to \mathbb{R}^+$ and a positive definite function $c : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$DV(x)F(x, a_1(x)v_1 + b_1(x)(1 - v_1), \dots, a_k(x)v_k + b_k(x)(1 - v_k)) \leq -c(|x|)$$

$$\forall\ x \in \mathbb{R}^n, \quad v \in I,$$

which is equivalent to (2.5). □

REMARK 2.2. It is worthwhile to note that according to the previous discussion it follows that for every x_0 the set $\mathcal{U}(x_0, L, +\infty)$ in nonempty, provided that L has the form (2.4) and zero is L–UGAS for (1.1). Specifically, $\mathcal{U}(x_0, L, +\infty)$ contains all inputs of the form $u(t) = (u_2(t), \dots, u_k(t))$, with $u_i(t) := a_i(x(t))v_i(t) + b_i(x(t))(1 - v_i(t))$, where $x(\cdot) = x(\cdot, x_0, v)$ denotes the trajectory of (2.7) with input $v \in I$.

A consequence of Theorem 2.1 and the main result in [20] is the following theorem of particular interest.

THEOREM 2.3. *If the functions a_i, b_i in (2.4) are bounded then the following statements are equivalent (provided that f is Lipschitzian).*

(i) *Zero is L–GAS.*
(ii) *Zero is L–UGAS.*
(iii) *There exist a p.d.r.u. C^1 map V and a positive definite C^0 function c such that (2.5) holds for all $(x, u) \in L$.*

Proof. Since the functions a_i and b_i are bounded and (1.1) is complete, it follows that (2.7) with inputs v taking values on the compact set $I = [0, 1]^k$ is also complete. According to [20] the latter in conjunction with the assumption that zero is GAS for (1.1) with respect to L implies that zero is UGAS for (2.7) with respect to $\mathbb{R}^n \times I$. The rest part of the proof is a direct consequence of Theorem 2.1. □

REMARK 2.4. By Remark 2.2 and Theorem 2.3 it follows that, if a_i and b_i are bounded and zero is L–GAS for (1.1), the set $\mathcal{U}(x_0, L, +\infty)$ is nonempty.

2.3. COMPARISON WITH ISS AND a–$\mathcal{L}_\infty$ STABILITY

First, we recall the precise definition of ISS as well as of the concept *asymptotic–$\mathcal{L}_\infty$ stability* as introduced by Teel (see for instance [4, 28]). The system (1.1) satisfies the ISS property, if there exist functions $\alpha \in KL$ and $\beta \in K$ such that

$$|x(t, x_0, u)| \leq \alpha(|x_0|, t) + \beta(||u_t||), \quad \forall\ t \geq 0 \tag{2.8}$$

$$u_t(s) := \begin{cases} u(s) & , 0 \leq s \leq t \\ 0 & , s > t \end{cases}$$

The a–$\mathcal{L}_\infty$ stability holds, if there exist functions $\gamma_1, \gamma_2, \gamma_3 \in K$ such that for every initial x_0 and input u each solution $x(t, x_0, u)$ of (1.1) satisfies

$$\sup_{t \geq 0} |x(t, x_0, u)| \leq \max\{\gamma_1(|x_0|), \gamma_2(||u||)\} \ \text{ “global stability”}$$

$$\lim_{t\to+\infty} \sup |x(t, x_0, u)| \leq \gamma_3(\lim_{t\to+\infty} \sup |u(t)|) \ \text{ “asymptotic gain property”}$$

THEOREM 2.5. *Assume that the dynamics f of (1.1) are Lipschitzian. Then*

$$ISS \Longleftrightarrow \alpha\text{-}\mathcal{L}_\infty \text{ stability } \Longleftrightarrow L\text{-}UGAS$$

provided that L has the form (2.4) with $a_i(x) = -p_i(|x|)$ and $b_i(x) = p_i(|x|)$ for certain $p_i \in K_\infty$.

Proof. The equivalence ISS $\Longleftrightarrow$ a–$\mathcal{L}_\infty$ stability has been established in [20]. It remains to establish the equivalence ISS $\Longleftrightarrow$ L–UGAS. (It should be noted here that the sufficient part of this statement has originally been obtained in [5] in the presence of W–RGUAS and under the assumption that p_i are Lipshitzian everywhere; this is a consequence of this equivalence of L–UGAS and W–RGUAS). From [15, 18] it is known that ISS is equivalent to the existence of a p.d.r.u. C^1 map $V : \mathbb{R}^n \to \mathbb{R}^+$, a positive definite C^0 function $c : \mathbb{R}^+ \to \mathbb{R}^+$ and a function $\gamma \in K_\infty$ such that

$$DV(x)f(x,u) \leq -c(|x|), \quad \forall\ |u| \leq \gamma(|x|). \tag{2.9}$$

Obviously, the latter implies (2.5) and thus by Theorem 2.1 zero is UGAS with respect to L having the form (2.4) with $a_i(x) = -p_i(|x|)$ and $b_i(x) = p_i(|x|)$ for certain $p_i \in K_\infty$ being Lipshitzian for $x \neq 0$ and such that $(\sum_{i=1}^k p_i^2(s))^{1/2} \leq \gamma(s)$. Conversely, suppose that zero is L–UGAS and L has the form (2.4) with $a_i(x) = -p_i(|x|)$ and $b_i(x) = p_i(|x|)$ with $p_i \in K_\infty$, $i = 1, \dots, k$. By Theorem 2.1 it follows that there exists a p.d.r.u. C^1 map V and a positive definite C^0 function such that (2.5) holds for all $(x,u) \in L$. The latter implies (2.9) for some $\gamma \in K_\infty$ with $\gamma(s) \leq \min\{p_i(s),\ \ i = 1, \dots, k\}$ and thus ISS property. □

It turns out by Theorem 2.5 that in general L–UGAS is weaker than both ISS and a–$\mathcal{L}_\infty$ stability. It should be remarked that if (2.8) holds then L–UGAS is satisfied with L as described in the statement of Theorem 2.5 where p_i being of class K_∞ in such a way that $\beta((\sum_{i=1}^k p_i^2(s))^{1/2}) \leq \lambda s$, $\forall\ s \geq 0$ for certain constant $0 < \lambda < 1$, (see [30]). Finally, an important consequence of the recent contribution [20] is that the conjunction of local stability and L–attractivity with L as described in the statement of Theorem 2.5 is equivalent to ISS and thus to L'–UGAS with L' being a subregion of L of the same form (see [20, sec. 1–5] for details).

3. APPLICATIONS TO STABILIZATION USING SATURATIONS

3.1. TRIANGULAR SYSTEMS

The following theorem provides sufficient condition for global stabilization for systems (1.3) by means of a combination of saturation functions.

THEOREM 3.1. *Consider the system (1.3), where f and g_i are C^1 vanishing at zero and each g_i is bounded over $\mathbb{R}^{n+i}$. Moreover, assume that the subsystem*

$$\dot{x} = f(x, y_1) \tag{3.1}$$

with y_1 as input is complete and there exist a bounded function $\phi_0 : \mathbb{R}^n \to \mathbb{R}$ with $\phi_0(0) = 0$ which is C^1 around zero and whose derivative exists almost everywhere being bounded over $\mathbb{R}^n$ and a bounded function $\gamma_0 \in K$ such that

A1. *The origin* $0 \in \mathbb{R}^n$ *is a global attractor for*

$$\dot{x} = f(x, \phi_0(x) + y_1) \tag{3.2}$$

with respect to

$$L_0 := \{(x; y_1) : |y_1| \leq \gamma_0(|x|)\}.$$

A2. *If we denote* $A := \frac{\partial f}{\partial x}(0,0)$, $B := \frac{\partial f}{\partial y_1}(0,0)$ *and* $F := D\phi_0(0)$, *the matrix* $A + BF$ *is Hurwitz.*

A3. *The map* $D\phi_0(x) f(x, y_1 + \phi_0(x))$ *is bounded over*

$$\{(x, y_1) : |y_1| \leq \sup_{x \in \mathbb{R}^n} \gamma_0(|x|)\}.$$

Then there exist positive constants E_i, $1 \leq i \leq m$, *simple saturations* $\sigma_i : \mathbb{R} \to \mathbb{R}$, $1 \leq i \leq m$ *and a bounded* C^1 *map* $\widehat{\phi}_0 : \mathbb{R}^n \to \mathbb{R}$ *which is linear near zero and coincides with* ϕ_0 *away from zero such that the bounded feedback*

$$\begin{aligned} u &= \phi(x, y_1, \ldots, y_m) \\ &:= -E_m \sigma_m (y_m + E_{m-1}\sigma_{m-1}(\ldots + E_1\sigma_1(y_1 - \widehat{\phi}_0(x)) \ldots)) \end{aligned} \tag{3.3}$$

globally asymptotically stabilizes (1.3) at zero. Furthermore, there exist a function $\widehat{\gamma} \in K$ *such that the origin of the closed–loop (1.3) with* $u = \phi(\cdot) + v$ *is UGAS with respect to* $\widehat{L} := \{(x, y_1, \ldots, y_m, v) : |v| \leq \widehat{\gamma}(|(x, y_1, \ldots, y_m)|)\}$.

Proof. Taking into account Conditions A1, A2 and Lemma 4.1 in the appendix we may assume that γ_0 is linear near zero, constant away from zero and $\phi_0(x) = Fx$ for x near zero. The same conditions in conjunction with Theorem 2.3 and boundedness of ϕ_0 imply that $0 \in \mathbb{R}^n$ is L_0–UGAS for (3.1). We proceed by induction showing that for every $1 \leq i \leq m$ the system

$$\begin{aligned} \dot{x} &= f(x, y_1) \\ \dot{y}_j &= y_{j+1} + g_j(x, y_1, \ldots, y_j) \\ & \quad 1 \leq j \leq i \end{aligned}$$

is globally asymptotically stabilizable from the input y_{j+1}. Consider the case $i = 1$:

$$\begin{aligned} \dot{x} &= f(x, y_1) \\ \dot{y}_1 &= y_2 + g_1(x, y_1) \end{aligned} \tag{3.4}$$

with y_2 as input. Let $G_{11} := \frac{\partial g_1}{\partial x}(0,0)$ and $G_{12} := \frac{\partial g_2}{\partial y_1}(0,0)$. It is not difficult to verify that there exists a constant $E_{10} > 0$ such that for every $E_1 \geq E_{10}$ the matrix

$$\begin{pmatrix} A & B \\ E_1 F + G_{11} & -E_1 F + G_{12} \end{pmatrix} \text{ is Hurwitz.} \tag{3.5}$$

Particularly, Condition A2 implies the existence of a positive definite matrix P such that $P(A+BF) + (A+BF)'P$ is negative definite. Then by evaluating the time derivative $\dot{V}_{\Sigma_1}(x, y_1, y_2, E_1)$ of the positive definite function

$$V(x, y_1) := x'Px + (y - Fx)^2$$

along the trajectories of

$$\Sigma_1 : \quad \begin{aligned} \dot{x} &= Ax + By_1 \\ \dot{y}_1 &= -E_1(y_1 - Fx) + y_2 + G_{11}x + G_{12}y_1 \end{aligned}$$

we can determine positive constants E_{10}, ξ_{10} and ℓ such that

$$\begin{array}{l} \dot{V}_{\Sigma_1}(x,y_1,y_2,E_1) \leq -\ell|(x,y_1)|^2 \\ \forall\ |y_2| \leq \xi_1|(x,y_1)|, \quad \xi_1 \leq \xi_{10}, \quad E_1 \geq E_{10} \end{array} \tag{3.6}$$

which implies (3.5). We now take into account (3.6) and the fact that $\phi_0(x) = Fx$ for x near zero and evaluate the time derivative $\dot{V}_{\Sigma_2}(x,y_1,y_2,E_1)$ of V along the trajectories of

$$\Sigma_2: \begin{array}{l} \dot{x} = f(x,y_1) \\ \dot{y}_1 = -E_1(y_1 - \phi_0(x)) + y_2 + g_1(x,y_1) \end{array} \tag{3.7}$$

For x, y_1 appropriate small we find by (3.6)

$$\begin{array}{l} \dot{V}_{\Sigma_2}(x,y_1,y_2,E_1) = \dot{V}_{\Sigma_1}(x,y_1,y_2,E_1) \\ +DV(x,y_1)\begin{pmatrix} f(x,y_1) - Ax - By_1 \\ g_1(x,y_1) - G_{11}x - G_{12}y_1 \end{pmatrix} \leq -\ell|(x,y_1)|^2 + o(x,y_1), \\ \qquad \forall |y_2| \leq \xi_1|(x,y_1)|,\ \xi_1 \leq \xi_{10}, E_1 \geq E_{10} \end{array}$$

where $o(\cdot)$ is independent of E_1 and satisfies $o(x,y_1)/|(x,y_1)| \to 0$ as $(x,y_1) \to 0$. This implies that

$$\dot{V}_{\Sigma_2}(x,y_1,y_2,E_1) \leq -\frac{\ell}{2}|(x,y_1)|^2 \tag{3.8}$$

$$\forall\ |(x,y_1)| \leq \delta_0, \quad |y_2| \leq \xi_1|(x,y_1)|, \quad \xi_1 \leq \xi_{10}, \quad E_1 \geq E_{10}$$

for certain positive constant δ_0. From Condition A1 there exists a function $\gamma_0 \in K$ of the form

$$\gamma_0(s) = \begin{cases} c_0 & , s > \varepsilon_0 \\ \dfrac{c_0}{\varepsilon_0}s & , s \leq \varepsilon_0 \end{cases} \tag{3.9}$$

for certain positive constants $c_0 < 1$ and $0 < \varepsilon_0 \leq \delta_0$ in such a way that the region

$$\Pi_{\varepsilon_0} := \{(x,y_1) : |x| \leq \varepsilon_0, \quad |y_1 - \phi_0(x)| \leq \gamma_0(|x|)\}$$

is a subset of $S^{n+1}(0,\delta_0)$ and zero is UGAS for (3.2) with respect to L_0 with γ_0 as defined by (3.9). Using (3.8) we can determine positive constants C_0, ξ_1 and E_1 such that

$$\frac{C_0}{\delta_0} \leq \xi_1 \leq \xi_{10}, \quad E_1 \geq E_{10}, \tag{3.10a}$$

$$C_0 + \lambda_1 + \sup\{|D\phi_0(x)f(x,\phi_0(x)+y_1)|, \quad |y_1| \leq c_0\} < \frac{1}{2}E_1c_0, \tag{3.10b}$$

$$\lambda_1 := \sup\{|g_1(x,y_1)|, \quad (x,y_1) \in \mathbb{R}^{n+1}\} \tag{3.10c}$$

From (3.8), (3.9) and (3.10a) it follows that zero is (locally) AS for (3.7) with respect to $\{(x,y_1) : |y_1 - \phi_0(x)| \leq \gamma_0(|x|)\}$ for the specific choice of $\varepsilon_0, c_0, \xi_1$ and E_1. Without loss of generality we may assume that $S^{n+1}(0,\delta_0)$ is contained to the region of attraction $\mathcal{A}_1$ and the restriction of the graph of the function $y_1 = \phi_0(x)$ in $\mathcal{A}_1$ coincides with the graph of the linear map $y = Fx$. Consider now the simple saturation

$$\sigma_1(s) := \begin{cases} c_0 sgns & , |s| > c_0 \\ s & , |s| \leq c_0 \end{cases} \tag{3.11a}$$

and define

$$\phi_1(x,y_1) := -E_1\sigma_1(y_1 - \phi_0(x)). \tag{3.11b}$$

We claim that zero $0 \in \mathbb{R}^{n+1}$ is UGAS for

$$\begin{aligned} \dot{x} &= f(x, y_1) \\ \dot{y}_1 &= \phi_1(x, y_1) + y_2 + g_1(x, y_1) \end{aligned} \tag{3.12}$$

with y_2 as input with respect to

$$L_1 := \{(x, y_1; y_2) : |y_2| \le \gamma_1(|(x, y_1)|)\} \tag{3.13}$$

where

$$\gamma_1(s) := \begin{cases} C_0 & , s > \delta_0 \\ \dfrac{C_0}{\delta_0} s & , s \le \delta_0 \end{cases} \tag{3.14}$$

First, notice that (3.12) with y_2 as input is complete. This is an immediate consequence of boundedness of ϕ_0, ϕ_1 and g_1 and completeness of the subsystem (3.1). In order to prove that zero is L_1–UGAS for (3.12) we need to establish that this system satisfies the following properties. □

Property 1. If we define

$$\Pi := \{(x, y_1) : |y_1 - \phi_0(x)| \le c_0\} \tag{3.15}$$

each trajectory of (3.11) enters Π after some finite time provided that

$$(x(t), y_1(t); y_2(t)) \in L_1. \tag{3.16}$$

Indeed, by (3.10) and (3.11) there exists a constant $\mu > 0$ such that

$$\begin{aligned} \dot{y}_1(t) &= -E_1\sigma_1(y_1(t) - \phi_0(x(t))) + y_2(t) + g_1(x(t), y_1(t)) \\ &\le -\frac{1}{2}E_1 c_0 + C_0 + \lambda_1 < -\mu \end{aligned} \tag{3.17}$$

as long as (3.16) holds and

$$y_1(t) > \phi_0(x(t)) + \frac{1}{2}c_0. \tag{3.18}$$

From (3.17) it follows that $y_1(t) \le y_{10} - t\mu$, as long as (3.16) and (3.18) hold, thus there exists a time $T > 0$ such that $(x(T), y_1(T)) \in \Pi$. Similarly, a constant $\mu' > 0$ can be found such that $y_1(t) > y_{10} + \mu' t$ as long as $y_1(t) < \phi_0(x(t)) - \frac{1}{2}c_0$ and (3.16) hold from which we get the desired conclusion.

Property 2. The set Π as defined by (3.15) is L_1–positively invariant for (3.12).

It suffices to show that

$$\begin{aligned} \dot{y}_1(t) = \; & -E_1\sigma_1(y_1(t) - \phi_0(x(t))) + y_2(t) \\ & + g_1(x(t), y_1(t)) \\ \le \; & \frac{d}{dt}\phi_0(x(t)) = D\phi_0(x(t)) f(x(t), y_1(t)) \\ & \text{for } \frac{1}{2}c_0 \le y_1(t) - \phi_0(x(t)) \le c_0 \end{aligned} \tag{3.19a}$$

and similarly

$$\dot{y}_1(t) \ge \frac{d}{dt}\phi_0(x(t)), \quad \text{for } -\frac{1}{2}c_0 \le y_1(t) - \phi_0(x(t)) \le -c_0 \tag{3.19b}$$

as long as (3.16) holds. Indeed, by taking into account (3.10) we get

$$\sup\{\, |D\phi_0(x) f(x, y_1)|, \;\; |y_1 - \phi_0(x)| \le c_0 \,\} =$$

$$\sup\{\,|D\phi_0(x)f(x,y_1+\phi_0(x))|,\quad |y_1|\le c_0\}<\frac{1}{2}E_1c_0-\lambda_1-C_0$$

therefore

$$|D\phi_0(x)f(x,y_1)|+|y_2|+|g_1(x,y_1)|<\frac{1}{2}E_1c_0\le E_1\sigma_1(|y_1-\phi_0(x)|),$$

$$\forall\ \frac{1}{2}c_0\le|y_1-\phi_0(x)|\le c_0,\quad (x,y_1;y_2)\in L_1.$$

The previous inequality implies both (3.19a) and (3.19b).

Property 3. Each trajectory $x(t)$ of the subsystem (3.1) enters $S^n(0,\varepsilon_0)$ after some finite time, provided that $|y_1(t)-\phi_0(x(t))|\le\gamma_0(|x(t)|)$

This is an immediate consequence of the fact that zero $0\in\mathbb{R}^n$ is L_0–UGAS for the system (3.2).

Taking into account Properties 1 and 2 it follows that each trajectory $(x(t),y_1(t))$ of (3.12) is defined for all $t\ge 0$ and enters Π after some finite time T and remains thereafter, provided that (3.16) holds for all $t\ge 0$. We now distinguish two cases. The first is $(x(T),y_1(T))\in N:=S^{n+1}(0,\delta_0)\cap\Pi$. Since in that region $\phi_1(x,y_1)=-E_1(y_1-Fx)$ and N is contained to the region of attraction of (3.7) it follows by the positive invariance of Π that $(x(t),y_1(t))\to 0$ as $t\to+\infty$. The second case is $(x(T),y_1(T))\in\Pi\setminus N$. In that region $|y_1(t)-\phi_0(x(t))|\le\gamma_0(|x(t)|)=c_0$, hence by Property 3 $(x(T'),y_1(T'))\in N$ for some $T'>T$, and so $(x(t),y_1(t))\to 0$ as $t\to+\infty$. It turns out by taking into account the previous discussion and (3.5) that zero $0\in\mathbb{R}^{n+1}$ is L_1–UGAS for (3.12).

Notice that ϕ_1 and its derivative $D\phi_1$ are bounded over $\mathbb{R}^{n+2}$. Next we show that the map

$$(x,y_1;y_2)\to D\phi_1(x,y_1)\begin{pmatrix} f(x,y_1)\\ y_2+\phi_1(x,y_1)+g_1(x,y_1)\end{pmatrix}$$

is bounded over $\{(x,y_1;y_2):|y_2|\le C_0\}$. Indeed, $D\phi_1$ vanish for $|y_1-\phi_0(x)|>c_0$ whereas for $|y_1-\phi_0(x)|<c_0$ we get by (3.10)

$$\begin{aligned}\left|D\phi_1(x,y_1)\begin{pmatrix} f(x,y_1)\\ y_2+\phi_1(x,y_1)+g_1(x,y_1)\end{pmatrix}\right|&\le\\ &E_1|D\phi_0(x)f(x,y_1)|+E_1|y_2|+E_1^2|y_1-\phi_0(x)|+E_1\lambda_1\\ \le\ &2E_1^2c_0+E_1C_0+E_1\lambda_1\end{aligned}$$

for almost all x for which $D\phi_1(x)$ exists.

We conclude that for the system (3.4) with y_2 as input Conditions A1, A2 and A3 are satisfied with $(f,y_2+g_1)'$, ϕ_1, γ_1 and L_1 instead of f, ϕ_0, γ_0 and L_0, respectively, hence by repeating the previous analysis we can find a saturation σ_2 and a constant $E_2>0$ such that the map $y_3=\phi_2(x,y_1,y_2):=-E_2\sigma_2(y_2-\phi_1(x,y_1))=-E_2\sigma_2(y_2+E_1\sigma_1(y_1-\phi_0(x)))$ globally asymptotically stabilizes the system $\dot{x}=f(x,y_1)$, $\dot{y}_1=y_2+g_1(x,y_1)$, $\dot{y}_2=y_3+g_2(x,y_1,y_2)$.

We proceed similarly by induction. For reasons of completeness we note that for each $1\le i\le m$ we can select appropriare constants $E_j>0$ and sufficiently small positive constants $c_j<C_{j-1}$, ε_j and δ_j, $1\le j\le i-1$ such that

$$C_j+\lambda_{j+1}+2E_j^2c_{j-1}+E_jC_{j-1}+E_j\lambda_j<\frac{1}{2}E_{j+1}c_j \tag{3.20a}$$

$$\lambda_j := \sup\{\, |g_j(x, y_1, \dots, y_j)| \,\}, \tag{3.20b}$$

for each j the matrices

$$\begin{bmatrix} A & B & 0 & . & . & 0 \\ G_{11} & 1{+}G_{12} & 0 & . & . & 0 \\ . & . & 1{+}G_{23} & . & . & 0 \\ . & . & 0 & . & . & . \\ . & . & 0 & . & . & . \\ & & & & & 1{+}G_{(j-1)j} \\ E_jE_{j-1}\dots E_1F{+}G_{j1}, & -E_jE_{j-1}\dots E_1{+}G_{j2}, & . & . & . & -E_j{+}G_{jj} \end{bmatrix} \tag{3.21}$$

where $G_{ji} := \frac{\partial g_j}{\partial y_i}(0, 0, \dots, 0)$ are Hurwitz, the origin $0 \in \mathbb{R}^{n+j}$ is (locally) AS for

$$\begin{aligned} \dot x &= f(x, y_1) \\ \dot y_1 &= y_2 + g_1(x, y_1) \\ &\vdots \\ \dot y_j &= y_{j+1} + \phi_j(x, y_1, \dots, y_j) + g_j(x, y_1, \dots, y_j) \end{aligned}$$

with y_{j+1} as input with respect to

$$L_j := \{\, (x, y_1, \dots, y_j; y_{j+1}) : |y_{j+1}| \le \gamma_j(|(x, y_1, \dots, y_j)|) \,\}$$

where

$$\phi_j(x, y_1, \dots, y_j) := -E_j\sigma_j(y_j - \phi_{j-1}(x, y_1, \dots, y_{j-1})), \quad j \ge 1$$

$$\sigma_j(s) := \begin{cases} c_{j-1} & , |s| > c_{j-1} \\ s & , |s| \le c_{J-1} \end{cases}$$

$$\gamma_j(s) := \begin{cases} C_{j-1} & , s > \delta_{j-1} \\ \dfrac{C_{j-1}}{\delta_{j-1}} s & , 0 \le s \le \delta_{j-1}. \end{cases}$$

Furthermore, the constants δ_j and ε_j have been selected in such a way that $S^{n+j}(0, \delta_{j-1})$ is contained to the region of attraction $\mathcal{A}_j$ of

$$\begin{aligned} \dot x &= f(x, y_1) \\ \dot y_1 &= y_2 + g_1(x, y_1) \\ &\vdots \\ \dot y_{j-1} &= y_j + g_{j-1}(x, y_1, \dots, y_{j-1}) \\ \dot y_j &= -E_j(y_j - \phi_{j-1}(x, y_1, \dots, y_{j-1})) + g_i(x, y_1, \dots, y_j) \end{aligned}$$

and in addition the region

$$\begin{aligned} \Pi_{\varepsilon_{j-1}} := \{ \quad & (x, y_1, \dots, y_j) : |(x, y_1, \dots, y_{j-1})| \le \varepsilon_{j-1}, \\ & |y_j - \phi_{j-1}(x, y_1, \dots, y_{j-1})| \le \gamma_{j-1}(|(x, y_1, \dots, y_{j-1})|) \,\} \end{aligned}$$

is a subset of $S^{n+j}(0, \delta_{j-1})$ and the restriction of ϕ_j on $\mathcal{A}_j$ is linear.

EXAMPLE 3.2. Consider the planar system $\dot x = x + y_1(1 + x^2)^{1/2}$, $\dot y_1 = u + \sin x$. Notice that the first subsystem is complete. Indeed, for any input y_1 its corresponding trajectory x satisfies $|x(t)| \le |x(0)| \exp\Big(t{+}2\int_0^t |y_1(s)|ds\Big)$, provided that $|x(t)| \geq 1$ which implies completeness. Moreover, all assumptions of Theorem 3.1 are satisfied, hence the system is globally asymptotically stabilizable by means of a feedback law of the form (3.3). Indeed, let $y_1 = \phi_0(x) := \frac{-2x}{(1+x^2)^{1/2}}$. Then, obviously zero is UGAS for $\dot{x} = x + (\phi_0(x) + v)(1+x^2)^{1/2} = -x + v(1+x^2)^{1/2}$ with respect to $L_0 := \{ (x; v) : |v| \leq \gamma_0(|x|) \}$ for some $\gamma_0 \in K$ with $\gamma_0(s) = 1/2$ for s away from zero, and the map $(x, v) \to D\phi_0(x)(-x + v(1+x^2)^{1/2})$ is bounded over $\mathbb{R} \times [-1/2, 1/2]$.

Consider next the particular case of systems (1.3) with $n = 1$, $f(y_0, y_1) = y_1$, $y_0 := x$ and $g_i \equiv 0$, $1 \leq i \leq m$, or equivalently the linear chain of integrators (1.5). Teel's theorem in [24] asserts that for arbitrarily small $\varepsilon > 0$ there exist simple saturations $\sigma_i : \mathbb{R} \to \mathbb{R}$ with $|\sigma_i(s)| \leq \varepsilon$, $\forall\ s \in \mathbb{R}$ and matrices $T_i((m+1) \times (m+1))$ such that the map

$$u = -\sigma_m(T_m y + \sigma_{m-1}(T_{m-1}y + \sigma_{m-2}(\ldots \sigma_1(T_1 y + \sigma_0(T_0 y))\ldots))), \quad (3.22)$$

globally asymptotically stabilizes (1.5). This result can be modified by the following corollary which states that stabilization can be succeeded by means of a saturated feedback of the form (3.22), where each T_i is the identity matrix.

COROLLARY 3.3. *For any constant $\varepsilon > 0$ there exist positive constants E_i with $E_i < \varepsilon$, simple saturations $\sigma_i : \mathbb{R} \to \mathbb{R}$ of the form (3.11a) with $|\sigma_i(s)| \leq 1$, $\forall\ s$ such that the map*

$$\begin{aligned} u &= \phi(y_0, y_1, \ldots, y_m) \\ &:= -E_m\sigma_m(y_m + E_{m-1}\sigma_{m-1}(\ldots + E_1\sigma_1(y_1 + E_0\sigma_0(y_0))\ldots)) \quad (3.23) \end{aligned}$$

globally asymptotically stabilizes (1.5) at zero. Furthermore, there exist functions $\gamma_i \in K$ such that if we define

$$\phi_i(y_0, \ldots, y_i) := -E_i\sigma_i(y_i + E_{i-1}\sigma_{i-1}(\ldots + E_1\sigma_1(y_1 + E_0\sigma_0(y_0))\ldots)) \quad (3.24)$$

the origin $0 \in \mathbb{R}^{i+1}$ is UGAS for the system

$$\dot{y}_0 = y_1, \dot{y}_1 = y_2, \ldots, \dot{y}_i = y_{i+1} + \phi_i(y_0, \ldots, y_i) \quad (3.25)$$

with y_{i+1} as input with respect to $L_i := \{ (y_0, \ldots, y_i; y_{i+1}) : |y_{i+1}| \leq \gamma_i(|(y_0, \ldots, y_i)| \}$ and for almost every $y_0, \ldots, y_i$ for which the derivative $D\phi_i$ exist it holds that

$$|D\phi_i \cdot (y_1, y_2, \ldots, y_{i+1} + \phi_i)'| \leq \frac{1}{2}E_i, \quad \forall\ (y_0, \ldots, y_{i+1}) \in L_i. \quad (3.26)$$

Proof. Consider the family of mappings:

$$\begin{aligned} \phi_0(y_0) &:= -E_0\sigma_0(y_0), \quad \sigma_0(y_0) := \begin{cases} \varepsilon_0\, signs & , |s| > \varepsilon_0 \\ s & , |s| \leq \varepsilon_0 \end{cases} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.27) \\ \gamma_0(s) &:= \begin{cases} c_0 & , s > \varepsilon_0 \\ \dfrac{c_0}{\varepsilon_0}s & , s \leq \varepsilon_0 \end{cases} \qquad c_0, \varepsilon_0 > 0, E_0 > c_0\varepsilon_0^{-1}. \end{aligned}$$

Obviously, all conditions of Theorem 3.1 are satisfied with the previous choice of ϕ_0 and γ_0 for arbitrary c_0, ε_0 and E_0 as above. Therefore, we can determine positive constants E_i, C_i, c_i, $i = 1, \dots, m$ and saturations $\sigma_1, \dots, \sigma_m$ such that the map (3.23) globally asymptotically stabilizes (1.5) at zero. Specifically, as in the proof of Theorem 3.1 we can choose appropriate small positive constants $E_i, c_i < C_{i-1}$, such that in addition to the required properties described in the procedure design of Theorem 3.1 the following hold

$$\max\{E_0, E_1, \dots, E_m\} \leq \min\{1, \varepsilon\}, \tag{3.28a}$$

$$c_0 + E_0 < \frac{1}{2}E_1, \quad C_i + 2E_i^2 c_{i-1} + E_i C_{i-1} < \frac{1}{2}E_{i+1}c_i, \quad i \geq 1. \tag{3.28b}$$

The latter imply (3.10) and (3.20a), respectively with $\lambda_i = 0$. In order to complete the proof it suffices to show that the constants E_i can be chosen in such a way that both (3.28a) and (3.28b) are satisfied and for each $1 \leq j \leq m$ the matrix (3.21) with $F = -E_0$, $A = 0$, $B = 1$ and $G_{ji} = 0$ is Hurwitz. But this is a direct consequence of Lemma 3.4 below. □

LEMMA 3.4. *Suppose that $G_{ji} = 0$ for all i, j and assume either that A is Hurwitz and $F = 0$ or $A = 0$, $B = 1$ and $F = -E_0$. Then for every $\varepsilon > 0$ the constants E_j can be selected in such a way that $0 < E_j \leq \varepsilon$ and the matrices (3.21), $1 \leq j \leq m$ are Hurwitz.*

Proof. For reasons of simplicity we deal only with the second case, namely we assume that $A = 0$, $B = 1$, $F = -E_0$. The first case can be dealt quite similarly. To establish the statement it suffices to show that for arbitrarily small constants E_i the polynomials

$$p_i(s) = s^i + a_i s^{i-1} + a_{i-1} s_{i-2} + \dots + a_2 s + a_1, \quad 1 \leq i \leq m$$

with $a_1 := E_i E_{i-1} \dots E_1 E_0$, $a_2 := E_i E_{i-1} \dots E_1, \dots, a_i := E_i$ are Hurwitz. To establish the previous claim we apply the Hurwitz algebraic criterion, namely, for every i we consider the matrix

$$H_i := \begin{bmatrix} a_i & a_{i-2} & a_{i-4} & . & . \\ 1 & a_{i-1} & a_{i-3} & . & . \\ 0 & a_i & a_{i-2} & . & . \\ 0 & 1 & a_{i-1} & . & . \\ 0 & 0 & a_i & . & . \\ . & . & . & . & . \end{bmatrix}$$

and evaluate the determinants of its principal minors

$$H_{i1} = H_{i1}(E_i) := E_i$$

$$H_{i2} = H_{i2}(E_i, E_{i-1}, E_{i-2}) := det \begin{pmatrix} a_i & a_{i-2} \\ 1 & a_{i-1} \end{pmatrix} = E_i E_{i-1}(E_i - E_{i-2}),$$

$$H_{i3} = H_{i3}(E_i, E_{i-1}, E_{i-2}, E_{i-3}, E_{i-4}) := det \begin{pmatrix} a_i & a_{i-2} & a_{i-4} \\ 1 & a_{i-1} & a_{i-3} \\ 0 & a_i & a_{i-2} \end{pmatrix}, \dots$$

Notice that for every $j \geq 2$ we get

$$H_{ij}(E_i, E_{i-1}, \dots, E_{i-j}, 0, E_{i-j-2}, \dots) = E_i E_{i-1} \dots E_{i-j} H_{i(j-1)}$$

The latter implies that for any positive constants $\varepsilon_1, \dots, \varepsilon_m$ we can find constants $E_0, E_1, \dots, E_i$, in such a way that $0 < E_i < \varepsilon_i$ and $H_{ij} > 0$ which imply that for every $1 \leq i \leq n$ the polynomial $p_i(s)$ is Hurwitz. □

REMARK 3.5. Alternatively, in order to prove the statement of the previous lemma we can directly proceed by evaluating for each $0 \leq i \leq m$ the time derivative of the positive definite functions

$$V_0(y_0) := \ell_0 y_0^2$$

$$V_i(y_0, \ldots, y_i) = \ell_i V_{i-1}(y_0, \ldots, y_{i-1}) + (y_i - \phi_{i-1}(y_0, \ldots, y_{i-1}))^2, 1 \leq i \leq m$$

$$\phi_0(y_0) := -E_0 y_0, \quad \phi_i(y_0, \ldots, y_i) := -E_i(y_i + E_{i-1}\phi_{i-1}(y_0, \ldots, y_{i-1}))$$

for appropriate positive constants $\ell_i > 0$, along the trajectories of (3.25). The corresponding procedure is more technical but quite useful for further considerations (see for instance Example 3.11 in Section 3.3).

Corollary 3.3 can be extended for a chain of integrators subject to input saturations, namely for systems of the form

$$\dot{y}_0 = f_0(y_1), \dot{y}_1 = f_1(y_2), \ldots, \dot{y}_2 = f_m(u); \tag{3.29}$$

where each $f_i : \mathbb{R} \to \mathbb{R}$ is C^1 with bounded derivative and the mappings $s \to f_i(s)$ and $-s \to -f_i(-s)$, $s \in \mathbb{R}^+$ are of class K and linear near zero.

COROLLARY 3.6. *There exist arbitrarily small positive constants $E_0, \ldots,$ E_m and simple saturations $\sigma_0, \ldots, \sigma_m$ such that the map (3.23) globally asymptotically stabilizes (3.29) at zero.*

Proof. The proof is a direct consequence of Corollary 3.3. Specifically, taking into account the linearity of f_i near zero, boundedness of Df_i, inequality (3.26) and the fact that E_i can be selected arbitrarily small, we can easily verify that for appropriate small E_i the map (3.23) globally asymptotically stabilizes (3.29). □

3.2. ADDING ONE INTEGRATOR

The approach of Theorem 3.1 is applicable to the design of a global saturated feedback stabilizer for the special case of $(n+1)$–dimensional triangular systems, namely systems of the form

$$\begin{aligned} \dot{x} &= f(x, y_1) \\ \dot{y}_1 &= u + g_1(x, y_1) \end{aligned} \tag{3.30}$$

$$(x, y_1, u) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R},$$

under weaker assumptions than those imposed in the general case.

PROPOSITION 3.7. *Consider the system (3.30), where f and g_1 are C^1 with $f(0,0) = 0$ and g_1 is bounded over $\mathbb{R}^{n+1}$. Assume that the subsystem (3.1) with y_1 as input is complete and there exists a C^1 bounded function ϕ_0 : $\mathbb{R}^n \to \mathbb{R}$ with $\phi_0(0) = 0$ such that Condition A2 of Theorem 3.1 holds and further the rest Conditions A1 and A3 are satisfied with $\gamma_0 = 0$. Then there exists a simple saturation $\sigma_1 : \mathbb{R} \to \mathbb{R}$, a C^1 map $d : \mathbb{R}^+ \to \mathbb{R}^+$, which is strictly positive everywhere, a constant $E_1 > 0$ and a C^1 bounded map $\widehat{\phi}_0 : \mathbb{R}^n \to \mathbb{R}$ which coincides with ϕ_0 away from zero such that the saturated feedback*

$$u = \phi_1(x, y_1) := -E_1 \sigma_1(d(|x|)(y_1 - \widehat{\phi}_0(x))) \tag{3.31}$$

globally asymptotically stabilizes (3.30) at zero and the linearization of the resulting closed–loop system at $0 \in \mathbb{R}^{n+1}$ is asymptotically stable.

Proof. As in the proof of Theorem 3.1 by Condition A2 we may assume that $\phi_0(x) = Fx$ near zero and further (3.8) is satisfied for appropriate δ_0, ξ_{10} and E_{10}. Using our hypothesis and Lemma 4.1 we can establish that for any sufficiently small constant $\varepsilon_0 > 0$ there exist a positive definite C^1 function $\beta : \mathbb{R}^+ \to [0,1)$ and constants $E_1 \geq E_{10}$ and $K > 0$ such that $\beta(s)$ is linear near zero, particularly, $\beta(s) = \sigma s$ for $0 \leq s \leq \varepsilon_0$ and for certain constant $\sigma > 0$, zero is UGAS for (3.1) with respect to

$$L := \{\, (x, y_1) : |y_1 - \phi_0(x)| \leq \beta(|x|) \,\}$$

and the following holds

$$\lambda_1 + |D\phi_0(x) f(x, y_1)| < \frac{1}{4} E_1, \quad \forall\ (x, y_1) \in L \tag{3.32}$$

$$\text{where} \quad \lambda_1 := \sup\{\, |g_1(x, y_1), (x, y_1)| \in \mathbb{R}^{n+1} \,\}.$$

Let k be a function of class K which satisfies

$$\sup\{\, |f(x, \phi_0(x) + y_1)|, |y_1| \leq \beta(|x|) \,\} \leq k(|x|), \quad \forall\ x \in \mathbb{R}^n. \tag{3.33}$$

By Lemma 4.1 there exists a positive definite bounded C^1 function γ which is linear for $s \leq \frac{\varepsilon_0}{2}$, increasing for $s \leq \varepsilon_0$, decreasing for $s > \varepsilon_0$ and such that

$$\gamma(s) \leq \min\{\beta(s), 1\} \tag{3.34a}$$

$$|D\gamma(s)| k(s) \leq \frac{1}{4} E_1 - \lambda_1, \quad \forall\ s \geq 0. \tag{3.34b}$$

In addition to the previous requirements and by selecting ε_0 appropriate small we can construct the function γ in such a way that the region

$$\Pi_{\varepsilon_0} := \{\, (x, y_1) : |x| \leq \varepsilon_0, |y_1 - \phi_0(x)| \leq \gamma(\varepsilon_0) \,\}$$

is contained in $S^{n+1}(0, \delta_0)$. Let

$$c := \max_{s \geq 0} \gamma(s) = \gamma(\varepsilon_0) \tag{3.35}$$

and define

$$\widehat{\gamma}(s) := \begin{cases} \gamma(s) & , s > \varepsilon_0 \\ c & , s \leq \varepsilon_0 \end{cases} \quad , d(s) := \frac{1}{\widehat{\gamma}(s)} \tag{3.36}$$

By (3.32)–(3.36) and the specific definition of β, L and Π_{ε_0} it follows

$$\widehat{\gamma}(|x|) \leq c < 1,$$

zero is UGAS for (3.1) with respect to

$$L_0 := \{\, (x, y_1) : |y_1 - \phi_0(x)| \leq \gamma(|x|) \,\}$$

and the following inequality is satisfied:

$$(|D\widehat{\gamma}(s)| + |D\phi_0(x)|)\, |f(x, \phi_0(x) + y_1)| + \lambda_1 < \frac{1}{2} E_1 \tag{3.37}$$

$$\forall\ x \in \mathbb{R}^n, \quad |y_1| \leq \widehat{\gamma}(|x|).$$

As in the proof of Theorem 3.1 we may assume that $\phi_0(x) = Fx$ for $(y_1, x) \in S^{n+1}(0, \delta_0)$ and because of (3.8) $S^{n+1}(0, \delta_0)$ and thus Π_{ε_0} are both contained to the common region of (local) attraction of zero for the family of systems

$$\dot{x} = f(x, y_1),\ \dot{y}_1 = -E_1(y_1 - Fx) + g_1(x, y_1),\ E_1 \geq E_{10}. \tag{3.38}$$

Consider now the simple saturation

$$\sigma_1(s) := \begin{cases} 1 & , |s| > 1 \\ s & , |s| \le 1 \end{cases}$$

and the map ϕ_1 as defined in (3.31) with E_1 and d as given by (3.32) and (3.36), respectively. We claim that $0 \in \mathbb{R}^{n+1}$ is UGAS for the closed–loop system

$$\dot{x} = f(x, y_1), \quad \dot{y}_1 = \phi_1(x, y_1) + g_1(x, y_1). \tag{3.39}$$

We proceed as in the proof of Theorem 3.1. Notice again that completeness of (3.1) and boundedness of ϕ_0, ϕ_1 and g_1 imply completeness of (3.39). Moreover, similar to the proof of Theorem 3.1 we can establish the following properties.

Property 1. Each trajectory $(x(t), y_1(t))$ of (3.39) enters

$$\Pi := \{\, (x, y) : |y_1 - \phi_0(x)| \le \widehat{\gamma}(|x|) \,\}.$$

This is a consequence of (3.32) and the definition of ϕ_1 which imply

$$\begin{aligned} |\phi_1(x,y)| \;&\ge \min\{\, \tfrac{1}{2} E_1 d(|x|)\widehat{\gamma}(|x|), E_1 \,\} = \tfrac{1}{2}E_1 \\ &> \sup_{(x,y_1)\in\mathbb{R}^{n+1}} |g_1(x,y_1)|, \quad \forall\ (x,y_1) : \frac{1}{2}\widehat{\gamma}(|x|) \le |y_1 - \phi_0(x)|. \end{aligned}$$

Using the previous inequality we can immediately establish as in the proof of Theorem 3.1 the statement.

Property 2. The set Π is positively invariant.
The statement is a consequence of (3.37) from which we get

$$\begin{aligned} \lambda_1 + \sup\{\, &|(D\phi_0(x) \pm D\widehat{\gamma}(|x|)|\ |f(x,y_1)|, \\ &x \in \mathbb{R}^n, \tfrac{1}{4}\widehat{\gamma}(|x|) \le |y_1 - \phi_0(x)| \le \widehat{\gamma}(|x|)\} \\ &\le \frac{1}{2}E_1 = \frac{1}{2}E_1 d(|x|)\widehat{\gamma}(|x|) \le |\phi_1(x,y_1)|, \\ &\forall\ \frac{1}{2}\widehat{\gamma}(|x|) \le |y_1 - \phi_0(x)| \le \widehat{\gamma}(|x|). \end{aligned}$$

The previous inequality leads as in the proof of Theorem 3.1 to the desired property.

Finally, by (3.36) and the fact that zero $0 \in \mathbb{R}^n$ is UGAS for (3.1) with respect to L_0, we can easily verify that the following holds.

Property 3. Each trajectory $x(t)$ of $\dot{x} = f(x, y_1)$ enters $S^n(0, \varepsilon_0)$ after some finite time, provided that $|y_1(t) - \phi_0(x(t))| \le \widehat{\gamma}(|x(t)|)$.

Finally, we take into account that $\phi_1(x, y_1) = -E_1 c^{-1}(y_1 - Fx)$ for $(x, y_1) \in \Pi_{\varepsilon_0}$ and the facts that $E_1 c^{-1} > E_1 \ge E_{10}$ and the region $\Pi_{\varepsilon_0} \subset> S^{n+1}(0, \delta_0)$ is contained to the common region of attraction of zero for (3.38) for all $E_1 \ge E_{10}$, which in turns implies that zero is locally AS for $\dot{x} = f(x, y_1)$, $\dot{y}_1 = -E_1 c^{-1}(y_1 - Fx) + g_1(x, y_1)$. We then proceed as in the proof of Theorem 3.1 to complete the proof. □

EXAMPLE 3.8. The planar system $\dot{x} = f(x) + x + y(1 + x^2)^{1/2}$, $\dot{y} = u$, where $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function with $f(0) = 0$, $Df(0) < 0$, $xf(x) < 0$ for $x \ne 0$ and $|f(x)| \le |x|$ for all x, satisfies the hypothesis of Proposition 3.7, hence it is globally asymptotically stabilizable by means of a feedback law of the form (3.31). Indeed, notice that the first subsystem with y as input is complete, and if we define $\phi_0(x) := \frac{-x}{(1+x^2)^{1/2}}$ then obviously zero is UGAS for $\dot{x} = f(x)+x+(1+x^2)^{1/2}\phi_0(x) = f(x)$ and $D\phi_0(x)(f(x)+x+\phi_0(x)(1+x^2)^{1/2})$ is bounded.

3.3. Remarks for the Feedforward case

In this section we briefly present extensions of the previous stabilization procedure for feedforward single–input systems

$$\begin{aligned} \dot{y}_0 &= d_0(y)f_0(y_1) + g_0(y_1, \ldots, y_m, u) \\ \dot{y}_1 &= d_1(y)f_1(y_2) + g_1(y_2, \ldots, y_{m,u}) \\ &\vdots \\ \dot{y}_{m-1} &= d_{m-1}(y)f_{m-1}(y_m) + g_{m-1}(y_m, u) \\ \dot{y}_m &= d_m(y)f_m(u) + g_m(u) \end{aligned} \tag{3.40}$$

$$y := (y_0, \ldots, y_m) \in \mathbb{R}^{m+1},$$

where the mappings f_i, g_i are C^1 vanishing at zero, each g_i is $o(y_{i+1}, \ldots, y_m, u)$ at 0 (i.e. satisfies $g_i(y_{i+1}, \ldots, y_m, u)/|(y_{i+1}, \ldots, y_m, u)| \to 0$ as $(y_{i+1}, \ldots, y_m, u) \to 0$), f_i satisfy the hypothesis of Corollary 3.6 and the functions d_i are C^0 such that

$$k_1 \le d_i(y) \le k_2, \quad \forall\ i = 0, 1, \ldots, m, \quad y \in \mathbb{R}^{m+1}. \tag{3.41}$$

for certain constants $k_2 \ge k_1 > 0$. Obviously, the feedforward case (1.6) belongs to the previous class (3.40). Under the previous hypothesis, it is possible to show that there exist saturations $\sigma_i : \mathbb{R} \to \mathbb{R}$ with $|\sigma_i(s)| \le 1$ for all s and arbitrarily small constants $E_i > 0$ such that the saturated map ϕ as defined by (3.23) globally asymptotically stabilizes (3.42) and in addition the origin of the system (3.40) with $u = \phi + v$ is UGAS with respect to $\widehat{L} := \{\,(y, v) : |v| \le \widehat{\gamma}(|y|)\,\}$ for certain $\widehat{\gamma} \in K$. The proof of the previous statement follows by extending the induction procedure of Corollary 3.3 and is based on a general technical result (Lemma 3.10 below) which is interest in itself concerning systems of the form

$$\begin{aligned} \dot{x} &= f(x, y, u, \overline{u}) \\ \dot{y} &= ud + g(x, y, u, \overline{u}) \end{aligned} \tag{3.42}$$

$$(x, y) \in \mathbb{R}^n \times \mathbb{R}, \quad (u, \overline{u}) \in \mathbb{R} \times \mathbb{R}^m, \quad d \in \mathbb{R}$$

with $(u, \overline{u}, d)$ as input.

We assume that the mappings $f, g : \mathbb{R}^{n+m+2} \to \mathbb{R}$ are C^1 vanishing at zero and $d = d(t)$ takes values on a compact set, namely

$$k_1 \le d(t) \le k_2, \quad \forall\ t \ge 0 \tag{3.43}$$

for certain constants $k_2 \ge k_1 > 0$. We also make the following assumption.

Assumption 3.9. *The subsystem*

$$\dot{x} = f(x, y, u, \overline{u}) \tag{3.44}$$

with $(y, u, \overline{u})$ as input is complete and there exist a C^0 function $\phi_0 : \mathbb{R}^n \to \mathbb{R}$ with $\phi_0(0) = 0$ which is differentiable almost everywhere, positive constants $E_1, c_0, \overline{c}_0, c_1, \overline{c}_1, \varepsilon_1, \varepsilon_0$ with $\varepsilon_1 \le \varepsilon_0$, $0 < p < 1$ and functions $\gamma_0, \overline{\gamma}_0, \gamma_1, \overline{\gamma}_1 \in K$ with

$$\gamma_0(s) = c_0, \quad \overline{\gamma}_0(s) = \overline{c}_0, \quad \textit{for } s \ge \varepsilon_0$$

$$\gamma_1(s) = c_1, \quad \overline{\gamma}_1(s) = \overline{c}_1, \quad \textit{for } s \geq \varepsilon_1$$

such that

B1. *The origin $0 \in \mathbb{R}^n$ is UGAS for*

$$\dot{x} = f(x, \phi_0(x) + y, u, \overline{u}) \tag{3.45}$$

with $(y, u, \overline{u})$ as input with respect to

$$L_0 := \{ (x; y, u, \overline{u}) : |y| \leq \gamma_0(|x|), |(u, \overline{u})| \leq \overline{\gamma}_0(|x|) \}$$

B2. *The origin $0 \in \mathbb{R}^{n+1}$ is (locally) AS for*

$$\begin{aligned} \dot{x} &= f(x, y, -E_1 k_1^{-1}(y - \phi_0(x)) + u, \overline{u}) \\ \dot{y} &= -E_1 k_1^{-1}(y - \phi_0(x))d + g(x, y, -E_1 k_1^{-1}(y - \phi_0(x)) + u, \overline{u}) \end{aligned} \tag{3.46}$$

with $(u, \overline{u}, d)$ as input with respect to

$$L_1 := \{ (x, y; u, \overline{u}, d) : |u| \leq \gamma_1(|(x, y)|), |\overline{u}| \leq \overline{\gamma}_1(|(x, y)|), \ d \in [k_1, k_2] \}$$

and if we define

$$\Pi_{\varepsilon_0} := \{ (x, y) : |x| \leq \varepsilon_0, \ |y - \phi_0(x)| \leq c_0 p \}$$

then $S^{n+1}(0, \varepsilon_1) \subset \Pi_{\varepsilon_0}$ and Π_{ε_0} is contained to the region of attraction of (3.46).

B3. *By denoting*

$$\begin{aligned} \lambda := \sup\{ \|g(x, y, u - E_1 k_1^{-1}(y - \phi_0(x)), \overline{u})|, \ |u| \leq \gamma_1(|(x, y)|), \\ |\overline{u}| \leq \overline{\gamma}_1(|(x, y)|), \ |y - \phi_0(x)| \leq c_0\} \end{aligned}$$

we assume that

$$\lambda + c_1 + \overline{c}_1 + E_1 k_1^{-1} c_0 p < \overline{c}_0; \tag{3.47a}$$

$$\begin{gathered} |D\phi_0(x) f(x, \phi_0(x) + y, u, \overline{u})| \leq E_1 k_1^{-1} c_0 p, \\ \forall \ |y| \leq c_0, \ |(u, \overline{u})| \leq \overline{c}_0 \end{gathered} \tag{3.47b}$$

and for (almost) all x for which $D\phi_0(x)$ exists.

LEMMA 3.10. *Consider the system (3.42) and suppose that Assumption 3.9 and (3.43) are fulfilled. Then if we define*

$$\phi_1(x, y) := -E_1 k_1^{-1} \sigma(y - \phi_0(x)); \ \sigma(s) := \begin{cases} pc_0 sings & , |s| > c_0 p \\ s & , |s| \leq c_0 p \end{cases} \tag{3.48}$$

- *the origin of the closed–loop system*

$$\begin{aligned} \dot{x} &= f(x, y, \phi_1(x, y) + u, \overline{u}) \\ \dot{y} &= (\phi_1(x, y) + u)d + g(x, y, \phi_1(x, y) + u, \overline{u}) \end{aligned} \tag{3.49}$$

  with $(u, \overline{u}, d)$ as input is L_1–UGAS;
- *if in addition*

$$p k_1^{-1} \left(\left(\frac{E_1}{k_1} \right)^2 c_0 (1 + k_2) + \frac{E_1}{k_1}(1 + \lambda) \right) < E_2 c_1 p' < \overline{c}_1 \tag{3.50}$$

  for certain constants $E_2 > 0$ and $0 < p' < 1$, the following holds

$$\left| D\phi_1(x, y) \begin{pmatrix} f(x, y, \phi_1(x, y) + u, \overline{u}) \\ (\phi_1(x, y) + u)d + g(x, y, \phi_1(x, y) + u, \overline{u}) \end{pmatrix} \right| \leq E_2 c_1 p' < \overline{c}_1 \tag{3.51}$$

  for all $|u| \leq c_1$, $|\overline{u}| \leq \overline{c}_1$, $d \in [k_1, k_2]$ and for (almost) all (x, y) for which $D\phi_1(x, y)$ exists.

Proof. We proceed as in the proof of Theorem 3.1. We define

$$\Pi := \{\, (x,y) : |y - \phi_0(x)| \le c_0 \,\}$$

and establish that the following properties hold.

Property 1. Each trajectory $(x(t), y(t))$ of (3.49) enters Π after some finite time, provided that

$$(x(t), y(t), u(t), \overline{u}(t), d(t)) \in L_1 \tag{3.52}$$

Property 2. Π is L–positively invariant for (3.49).

Property 3. Each trajectory $x(t)$ of the subsystem

$$\dot{x} = f(x, y, \phi_1(x,y) + u, \overline{u})$$

enters after some finite time in $S^n(0, \varepsilon_0)$, provided that $|x(t)| > \varepsilon_0$, $|y(t) - \phi_0(x(t))| \le \gamma_0(|x(t)|)$ and (3.52) hold.

Properties 1 and 2 are consequences of Condition B3, (3.43) and the definition (3.48), which yield

$$\begin{aligned}
&\sup\bigl\{|D\phi_0(x) f(x, y, -E_1 k_1^{-1}(y - \phi_0(x)) + u, \overline{u}),\\
&\qquad\qquad (x, y, u, \overline{u}, d) \in L_1, |y - \phi_0(x)| \le c_0 p\bigr\}\\
&+\sup\bigl\{|g(x, y, -E_1 k_1^{-1}(y - \phi_0(x)) + u, \overline{u})|,\\
&\qquad\qquad (x, y, u, \overline{u}, d) \in L_1, |y - \phi_0(x)| \le c_0 p\bigr\}\\
&\le \sup\bigl\{|D\phi_0(x) f(x, \phi_0(x) + y, -E_1 k_1^{-1} y + u, \overline{u})|,\\
&\qquad\qquad (x, y, u, \overline{u}, d) \in L_1, |y| \le c_0 p\bigr\} + \lambda\\
&\le \sup\bigl\{|D\phi_0(x) f(x, \phi_0(x) + y, u, \overline{u})|,\\
&\qquad\qquad |(u, \overline{u})| \le E_1 k_1^{-1} c_0 p + c_1 + \overline{c}_1 \le \overline{c}_0,\ |y| \le c_0 \xi\,\bigr\} + \lambda\\
&\le E_1 k_1^{-1} c_0 p + \lambda \le E_1 k_1^{-1} \sigma(|y - \phi_0(x)|) d - c_1 - \overline{c}_1 \le\\
&\qquad\qquad \phi_1(x,y) d - c_1 - \overline{c}_1,
\end{aligned}$$

$$\forall\ (x, y, u, \overline{u}, d) \in L_1, \quad |y - \phi_0(x)| \ge c_0 p.$$

Property 3 follows from Condition B1 and the facts that $\gamma(s) = c_0, \gamma(s) = \overline{c}_0$, and (because of (3.47a)) $E_1 k_1^{-1} c_0 p + \gamma_1(s) + \overline{\gamma}_1(s) = E_1 k_1^{-1} c_0 p + c_1 + c_1 < \overline{c}_0$, for $s \ge \varepsilon_0$ from which we obtain

$$(x, y, \phi_1(x,y) + u, \overline{u}) \in L_0 \tag{3.53}$$

$$|(\phi_1(x,y) + u, \overline{u})| < \overline{c}_0 = \gamma_1(|x|) \tag{3.54}$$

$$\forall\ |x| \ge \varepsilon_0, \quad (x, y, u, \overline{u}, d) \in L_1.$$

From (3.53) and (3.54) and our hypothesis that zero is L_0–UGAS for (3.45) we get the desired Property 3. In order to show that zero is L_1–UGAS for (3.49) we use Properties 1,2,3, Condition B2, the fact that $\phi_1(x,y) = -E_1 k_1^{-1}(y - \phi_0(x))$ for $(x,y) \in \Pi_{\varepsilon_0}$ and proceed as in Theorem 3.1. The rest part of the proof is left to the reader. ☐

EXAMPLE 3.11. Consider the system

$$\begin{aligned} \dot{x} &= d_0(x,y)y + g_0(y,u,\overline{u}) \\ \dot{y} &= d_1(x,y)u + g_1(u,\overline{u}) \end{aligned} \tag{3.55}$$

$$(x,y) \in \mathbb{R}^2, \quad (u,\overline{u}) \in \mathbb{R} \times \mathbb{R}^m,$$

where g_0 and g_1 are C^1 and satisfy

$$\frac{|g_0(y,u,\overline{u})|}{|(y,u,\overline{u})|} \to 0 \text{ as } (y,u,\overline{u}) \to 0 \tag{3.56a}$$

$$\frac{|g_1(u,\overline{u})|}{|(u,\overline{u})|} \to 0 \text{ as } (u,\overline{u}) \to 0 \tag{3.56b}$$

and the functions d_0 and d_1 are C^0 and satisfy

$$k_1 \leq d_i(x,y) \leq k_2, \quad i = 0,1, \quad \forall\ x,y \tag{3.57}$$

It is not difficult to verify that all hypothesis of Assumption 3.9 as well as (3.50) are satisfied for certain appropriate small constants $E_1, E_2, c_0, \overline{c}_0, c_1, \overline{c}_1, \varepsilon_0, \varepsilon_1$ and functions $\gamma_0, \overline{\gamma}_0, \gamma_1, \overline{\gamma}_1 \in K$. Indeed, consider first the positive definite functions

$$V_1(x) = \frac{1}{2}x^2, \quad V_2(x,y) = \frac{1}{2}E_0^2x^2 + \frac{1}{2}(y + E_0x)^2$$

where E_0 is a positive constant and evaluate their derivatives along the trajectories of

$$\dot{x} = d_0(x, -E_0x + y)(-E_0x + y) + g_0(-E_0x + y, u, \overline{u}) \tag{3.58}$$

$$\text{with } (y,u,\overline{u}) \text{ as input;}$$

$$\begin{aligned} \dot{x} &= d_0(x,y)y + g_0(y, -E_1(y + E_0x) + u, \overline{u}) \\ \dot{y} &= d_1(x,y)(-E_1(y + E_0x) + u) + g_1(-E_1(y + E_0x) + u, \overline{u}) \end{aligned} \tag{3.59}$$

$$\text{with } (u,\overline{u}) \text{ as input,}$$

respectively. Taking into account (3.57) we find

$$\dot{V}_1(x) \leq -k_1E_0x^2 + k_2|x||y| + |x|\,|g_0(-E_0x + y, u, \overline{u})| \tag{3.60}$$

$$\begin{aligned} \dot{V}_2(x,y) \leq \quad & -k_1E_0^3x^2 + k_2(E_0 - E_1)|y + E_0x|^2 + k_2|y + E_0x|\,|u| + \\ & (E_0^2|x| + E_0|y + E_0x|)\,|g_0(y, -E_1(y + E_0x) + u, \overline{u})| + \\ & |y + E_0x|\,|g_1(-E_1(y + E_0x) + u, \overline{u})| \end{aligned} \tag{3.61}$$

Consider now positive constants $\alpha, \overline{\alpha}, \beta, \overline{\beta}, \varepsilon$ and $\varepsilon_1 < \varepsilon_0$ and define

$$\phi_0(x) := \begin{cases} -E_0\varepsilon_0 sgn x & , |x| > \varepsilon_0 \\ -E_0x & , |x| \leq \varepsilon_0 \end{cases} \tag{3.62a}$$

$$c_0 := \alpha\varepsilon_0, \quad \overline{c}_0 := \overline{\alpha}\varepsilon_0, \quad c_1 := \beta\varepsilon_1, \quad \overline{c}_1 := \overline{\beta}\varepsilon_1 \tag{3.62b}$$

$$\gamma_i(s) := \begin{cases} c_i & , s > \varepsilon_0 \\ \frac{c_i}{\varepsilon_0}s & , s \leq \varepsilon_0 \end{cases}, \quad \overline{\gamma}_i(s) := \begin{cases} \overline{c}_i & , s > \varepsilon_1 \\ \frac{\overline{c}_i}{\varepsilon_1}s & , s \leq \varepsilon_1 \end{cases}, \quad i = 0,1 \tag{3.62c}$$

We also consider the region

$$N_\xi := \{\, (x,y) : V_2(x,y) \leq \xi \,\}$$

for certain constant $\xi > 0$. From (3.56), (3.57), (3.60), (3.61) and (3.62) it follows that there exist arbitrarily small positive constants $\xi, E_0, E_1, \alpha, \overline{\alpha}, \beta, \overline{\beta}, \varepsilon_0$ and ε_1 such that the following properties hold.

- The system (3.55) satisfies Conditions B1 and B2 of Assumption 3.9. Particularly, (3.56a), (3.59) and (3.60) imply that $\dot{V}_1(x) < 0$ for all $x \neq 0$ and for appropriate small selection of the previous constants, hence zero $0 \in \mathbb{R}$ is L_0–UGAS with respect to

$$\dot{x} = d_0(x, \phi_0(x) + y)(\phi_0(x) + y) + g_0(\phi_0(x) + y, u, \overline{u}).$$

From (3.56b) and (3.61) it also follows that for each sufficiently small ξ it holds

$$\dot{V}_2(x, y) < 0 \ , \quad \forall \ (x, y) \in N_\xi, \tag{3.63}$$

provided that

$$k_1(E_1 - E_0)E_0^3 > \frac{1}{4}\overline{\alpha}^2 \tag{3.64}$$

and for sufficiently small constants $\beta, \overline{\beta}$ and ε_1. Condition (3.63) implies that for fixed E_0 and for each appropriate small choice of the rest constants the region N_ξ consists a positively invariant neighborhood of L_1–(local) attraction for (3.59) as well as for

$$\begin{aligned} \dot{x} &= d_0(x, y)y + g_0(y, -E_1(y + \phi_0(x)), \overline{u}) \\ \dot{y} &= d_1(x, y)(-E_1(y + \phi_0(x)) + u) + g_1(-E_1(y + \phi_0(x)) + u, \overline{u}). \end{aligned}$$

Roughly speaking the latter is a consequence of (3.63) and the fact that for appropriate small ε_0 the restriction of the graph of the map $y = \phi_0(x)$ on the region N_ξ –except of its slight modification near the boundary of N_ξ– coincides with the graph of the map $y = -E_0 x$. Furthermore, we can select the constants α and ε_0 in such a way that

$$\{ \ (x, y) : |x| \leq \varepsilon_0, \ |y - \phi_0(x)| \leq \alpha\varepsilon_0 \ \} \subset N_\xi. \tag{3.65}$$

- In addition to the previous requirements the desired constants can be selected arbitrarily small in such a way that the following inequalities hold for certain constants $0 < p, \, p' < 1$:

$$\widehat{\lambda} + \beta + \overline{\beta} + E_1 k_1^{-1}\alpha p < \overline{\alpha} \tag{3.66}$$

where

$$\widehat{\lambda} := \sup_{\varepsilon > 0}\{\frac{1}{\varepsilon}|g_1(u - E_1 k_1^{-1}(y + E_0 x), \overline{u})|, \ |u| \leq \beta\varepsilon, \ \ |\overline{u}| \leq \overline{\beta}\varepsilon, \ \ |y + E_0 x| \leq \alpha\varepsilon \ \}$$

$$E_0(E_0 + \alpha + \sup_{\varepsilon > 0}\{\frac{1}{\varepsilon}|g_0(y, u, \overline{u})|, \ \ |y| \leq \alpha\varepsilon, \ \ |(\overline{u}, u)| \leq \beta\varepsilon \,\}) < E_1 k_1^{-1}\alpha p, \tag{3.67}$$

$$p k_1^{-1}\left(\left(\frac{E_1}{k_1}\right)^2 \alpha(1 + k_2) + \frac{E_1}{k_1}\beta + \frac{E_1}{k_1}\widehat{\lambda}\right) < E_2\beta p' < \overline{\beta}. \tag{3.68}$$

It should be noted that, because of (3.56), the previous choice of ξ, $E_0, E_1, \alpha, \overline{\alpha}, \beta, \overline{\beta}$ and ε_0 satisfying (3.64)–(3.68) is always feasible. Furthermore, the constants E_0 and E_1 can be selected arbitrarily small. Obviously, (3.66), (3.67), (3.68) in conjunction with (3.62) imply B3 for appropriate small selection of the constants ε_0 and ε_1.

We now briefly describe the applicability of Lemma 3.10 for the feedforward case (1.6). By extending the approach in the Example 3.11 a family of arbitrarily small constants $c_i, \overline{c}_i, 0 < p_i < 1$, and E_i, $0 \leq i \leq m$ can be determined such that

$$\lambda_i + c_{i+1} + \overline{c}_{i+1} + E_{i+1} c_i p_i < \overline{c}_i \tag{3.69}$$

where

$$\begin{aligned} \lambda_i := \ & \sup\{g_i(y_{i+1}, y_{i+2} - E_{i+1}(y_{i+1} - \phi_i(y_0, \dots, y_i)), y_{i+2}, \dots, y_m, u), \\ & |y_{i+2}| \leq c_i,\ |(y_{i+3}, \dots, y_m, u)| \leq \overline{c}_i,\ |y_{i+1} - \phi_i(y_0, \dots, y_i)| \leq c_{i-1}\}; \end{aligned}$$

$$\phi_0(y_0) := -E_0 y_0,$$

$$\phi_i(y_0, \dots, y_i) := -E_i(y_i - \phi_{i-1}(y_0, \dots, y_{i-1})) = -E_i E_{i-1} \dots E_0 y_0 - \dots - E_i y_i,$$

$$p_i(2E_{i+1}^2 c_i + E_{i+1} c_{i+1} + E_{i+1}\lambda_i) \leq E_{i+2} c_{i+1} p_{i+1} < \overline{c}_{i+1} \tag{3.70}$$

for $0 \leq i \leq m-2$ and in addition the polynomials $s^i + E_i s^{i-1} + E_i E_{i-1} s^{i-2} + \dots + E_i E_{i-1} \dots E_0$ are Hurwitz. Furthermore, we can determine functions $\phi_0, \gamma_i, \overline{\gamma}_i$ like those in (3.62); particularly, there exist positive constants ε_i with $\varepsilon_{i+1} > \varepsilon_i$ such that

$$\gamma_i(s) := \begin{cases} c_i & , s > \varepsilon_i \\ \dfrac{c_i}{\varepsilon_i} s & , s \leq \varepsilon_i \end{cases}, \quad \overline{\gamma}_i(s) := \begin{cases} \overline{c}_i & , s > \varepsilon_i \\ \dfrac{\overline{c}_i}{\varepsilon_i} s & , s \leq \varepsilon_i \end{cases},$$

each $S^i(0, \varepsilon_i)$ is contained to the region of (local) attraction of zero for

$$\begin{aligned} \dot{y}_0 &= y_1 + g_0(y_1, \dots, u) \\ &\vdots \\ \dot{y}_i &= y_{i+1} + g_i(y_{i+1}, \dots, u) \\ \dot{y}_{i+1} &= -E_i E_{i-1} \dots E_0 y_0 - \dots - E_i y_i + v + g_{i+1}(y_{i+2}, \dots, u) \end{aligned}$$

with respect to $L_i := \{ (y_0, \dots, y_i; v; y_{i+2}, \dots, u) : |v| \leq \gamma_i(|(y_0, \dots, y_i)|), |(y_{i+2}, \dots, u)| \leq \overline{\gamma}_i(|(y_0, \dots, y_i)|) \}$ and all Conditions B1–B3 are satisfied for the subsystem $\dot{y}_0 = y_1 + g_1(y_1, \dots, u); \dot{y}_2 = y_3 + g_2(y_3, \dots, u)$, where $(y_0, y_1), y_3$ and $(y_4, \dots, u)$ play the role of x, y, u and $\overline{u}$, respectively. We then use Lemma 3.10, conditions (3.69) and (3.70) and proceed by induction.

The same procedure is applicable for the general case (3.40). Our methodology also works for several other cases under different assumptions as the following example shows.

EXAMPLE 3.12. Consider the planar case

$$\begin{aligned} \dot{x} &= a(x)y + b(u) \\ \dot{y} &= u \end{aligned} \tag{3.71}$$

where b is C^1 and satisfies

$$b(u)/|u|^2 \to 0 \text{ as } u \to 0 \tag{3.72}$$

and a is C^0, bounded, positive definite with

$$a_1|x|^2 \le a(x) \le a_2|x|^2, \quad \forall\ |x| \le \delta; \quad a(x) \ge a_0, \quad \forall\ |x| > \delta \tag{3.73}$$

for certain positive constants a_0, a_1, a_2 and δ.

We claim that (3.71) satisfies Assumption 3.9, with $\overline{u} = 0, g \equiv 0$ and $d \equiv 1$, hence is globally asymptotically stabilizable by saturated feedback. Indeed, notice first that, since a is bounded, the subsystem $\dot{x} = a(x)y + b(u)$, with (y, u) as input is complete. Taking account the properties of the functions a and b we can easily verify that there exist positive constants ε_0, α_0 and β_0 such that if we consider the family of functions:

$$\phi_\varepsilon(x) := \begin{cases} -\varepsilon\, sgn x & , |x| > \varepsilon \\ -x & , |x| \le \varepsilon \end{cases}$$

$$\gamma_{\varepsilon,a}(s) := \begin{cases} \alpha\varepsilon & , s > \varepsilon \\ \alpha s & , s \le \varepsilon \end{cases}$$

$$\overline{\gamma}_{\varepsilon,\beta}(s) := \begin{cases} \beta\varepsilon & , s > \varepsilon \\ \beta s & , s \le \varepsilon \end{cases}$$

the origin is UGAS for the system $\dot{x} = a(x)(y + \phi_\varepsilon(x)) + b(u)$ with respect to

$$L_{\varepsilon,\alpha,\beta} := \{\, (x; y, u) : |y| \le \gamma_{\varepsilon,\alpha}(|x|), |u| \le \overline{\gamma}_{\varepsilon,\beta}(|x|) \,\}$$

for every pair of positive constant $\varepsilon, \alpha, \beta$ with $\varepsilon \le \varepsilon_0$, $\alpha \le \alpha_0$ and $\beta \le \beta_0$. Furthermore, by evaluating the time derivative of the positive definite function $V(x, y) := x^2 + (x + y)^2$ along the trajectories of the closed–loop system

$$\dot{x} = a(x)y + b(-E(y - \phi_\varepsilon(x))), \quad \dot{y} = -E(y - \phi_\varepsilon(x)) \tag{3.74}$$

and taking into account (3.72) and (3.73) we find a positive constant E such that $\dot{V}(x, y) \le 0$ for every nonzero for (x, y) belonging to a neighborhood N of $0 \in \mathbb{R}^2$. Without loss of generality assume that N is positively invariant and is contained in the region of attraction of zero for (3.74). Finally, determine positive constants $\widehat{\varepsilon} \le \min\{\,\varepsilon_0, \delta\,\}$, $\widehat{\alpha} \le \alpha_0$, $\widehat{\beta} \le \beta_0$ and $0 < p < 1$ with

$$\left| a_2 \left(\frac{\widehat{\beta}}{pE} + 1 \right) \widehat{\varepsilon}^3 + \max\{|b(u)|,\ |u| \le \widehat{\beta}\widehat{\varepsilon}\} \right| \le pE\widehat{\alpha}\widehat{\varepsilon} < \widehat{\beta}\widehat{\varepsilon} \tag{3.75}$$

and in such a way that if we define $c_0 := \widehat{\alpha}\widehat{\varepsilon}$, and

$$\phi_0 := \phi_{\widehat{\varepsilon}}, \quad \gamma_0 := \gamma_{\widehat{\varepsilon},a}, \quad \overline{\gamma}_0 := \overline{\gamma}_{\widehat{\varepsilon},\beta}, \tag{3.76}$$

the region $\Pi_{\varepsilon_1} = \{\,(x, y) : |x| \le \widehat{\varepsilon}, |y - \phi_0(x)| \le c_0\,\}$ contained in N. It follows that all conditions of Assumption 3.9 are satisfied with $\gamma_0, \overline{\gamma}_0$ and ϕ_0 as defined by (3.76). For reasons of completeness we note that (3.47) is a direct consequence of (3.72), (3.73), (3.75) and the fact that $D\phi_0(x) = 0$ for $|x| \ge \widehat{\varepsilon}$. We conclude by Lemma 3.10 that the map $\phi_1(x, y) = -E\sigma(y - \phi_0(x))$ with σ and ϕ_0 as defined by (3.48) and (3.76), respectively, globally asymptotically stabilizes (3.71).

3.4. Robustness

The results of Sections 3.1 and 3.3 can directly be extended for the case where the dynamics contain unknown parameters. For instance, let us consider systems of the form

$$\begin{aligned} &\dot{x} = f(x, y_1, \theta) \\ &\dot{y}_i = d_i y_{i+1} + G_i(x, y, \theta)\ ,\ 1 \le i \le m \end{aligned} \tag{3.77}$$

$$u := y_{m+1}$$

$$y := (y_1, \dots, y_m)'$$

where $\theta = \theta(t) \in \mathbb{R}^\mu$ and $d_i = d_i(t)$ are time varying unknown parameters with

$$k_1 \le d_i(t) \le k_2\ ,\ \forall\ t \ge 0 \tag{3.78}$$

for certain constants $k_2 \ge k_1 > 0$, the mappings $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^\mu \to \mathbb{R}^n$ and: $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\mu$ are C^1 vanishing at zero and satisfy $f(0,0,\theta) = 0$, $G_i(0,0,\theta) = 0$, $\forall\ \theta \in \mathbb{R}^\mu$. Furthermore, assume that there exists a pair of functions ϕ_0 and γ_0 as in the statement of Theorem 3.1 such that

- Zero $0 \in \mathbb{R}^n$ is a global attractor for $\dot{x} = f(x, \phi_0(x) + y_1, \theta)$ with respect to $L_0 := \{\ (x; y_1, \theta) : |y_1| \le \gamma_0(|x|),\ |\theta| \le k\ \}$ for certain constant $k > 0$.
- If we denote $A = \frac{\partial f}{\partial x}(0,0,0)$, $B = \frac{\partial f}{\partial y_1}(0,0,0)$ and $F = D\phi_0(0)$ then the matrix $A + BF$ is Hurwitz with $|f(x, y_1, \theta) - Ax - By_1| / |(x, y_1)|^2 \to 0$, uniformly on θ.
- The mapping $D\phi_0(x) f(x, y_1 + \phi_0(x), \theta)$ is bounded over $\{\ (x, y_1, \theta) : |y_1| \le \sup_{x \in \mathbb{R}^n} \gamma_0(|x|),\ |\theta| \le k\ \}$
- There exist positive definite C^1 mappings $g_i(x, y_1, \dots, y_i)$ such that

$$|G_i(x, y, \theta)| \le g_i(x, y_1, \dots, y_i) \text{ for all } x, y \text{ and } \theta.$$

Under the previous assumptions it can be shown as in Theorem 3.1 that (3.77) is globally asymptotically stabilizable (uniformly on θ and d_i) by means of a feedback of the form (3.3) being independent of θ and d_i.

Similarly, it can be shown that global stabilization by means of a saturated feedback law (3.23) is feasible for parameterized systems of the form

$$\dot{y}_i = d_i y_{i+1} + G_i(y_{i+1}, \dots, y_m, u, \theta),\quad 0 \le i \le m$$

$$u := y_{m+1}$$

$$y_i \in \mathbb{R},\quad \theta \in \mathbb{R}^\mu,\quad d_i \in \mathbb{R},$$

where θ and d_i are unknown parameters such that (3.78) holds and each G_i satisfies

$$|G_i(y_{i+1}, \dots, y_m, u, \theta)| \le g_i(y_{i+1}, \dots, y_m, u),\quad \forall\ y_{i+1}, \dots, y_m, \theta, u$$

for certain positive definite C^1 mappings g_i which are $o(y_{i+1}, \dots, y_m, u)$ at 0. The result follows by applying Lemma 3.10 and induction procedure.

4. Appendix

The following result has been extensively used in the previous sections. Its proof is a direct consequence of Theorem 2.1 and its nature is closed related with the results in [16].

Lemma 4.1. *Assume that the system* $\dot{x} = f(x, y, u)$, $(x, y, u) \in \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^m$ *with* (y, u) *as input is complete, the map* $f : \mathbb{R}^{n+k+m} \to \mathbb{R}$ *is* C^1 *vanishing at zero and there exist a* C^1 *map* $\phi : \mathbb{R}^n \to \mathbb{R}^k$ *with* $\phi(0) = 0$ *and* C^0 *functions* $\gamma_0 : \mathbb{R}^+ \to \mathbb{R}^+$ *being positive definite and* $a_i, b_i : \mathbb{R}^n \to \mathbb{R}$, $1 \leq i \leq m$ *being* C^1 *for* $x \neq 0$ *with* $a_i(x) \leq b_i(x)$ *for all* x, *such that*

1. *Zero is UGAS for*

$$\dot{x} = f(x, \phi(x) + y, u)$$

   with respect to $L := \{\, (x; y, u) : |y| \leq \gamma_0(|x|),\ a_i(x) \leq u_i \leq b_i(x) \,\}$.

2. *The matrix* $Df(x, \phi(x), 0)|_{x=0}$ *is Hurwitz and there exists a constant* $K > 0$ *such that*

$$\left| f(x, y, u) - \frac{\partial f}{\partial x}(0, 0, 0)x - \frac{\partial f}{\partial y}(0, 0, 0)y \right| \leq K|(x, y)|^2 \qquad (4.1)$$

   for all $u \in \mathbb{R}^m$ *and* x, y *near zero.*

Then for any bounded function $h \in K$ *there exist a pair of constants* $\delta_1 > \delta_2 > 0$, *a* C^1 *map* $\widehat{\phi} : \mathbb{R}^n \to \mathbb{R}$ *with* $\widehat{\phi}(0) = 0$ *and a positive definite* C^1 *function* $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

- *Both* $\widehat{\phi}(x)$ *and* $\gamma(s)$ *are linear for* $|x| \leq \delta_2$ *and* $s < \delta_2$, *respectively and* $\widehat{\phi}(x) = \phi(x)$ *for* $|x| > \delta_1$.
- *Zero is UGAS for*

$$\dot{x} = f(x, \widehat{\phi}(x) + y, u) \qquad (4.2)$$

  with (y, u) *as input with respect to*

$$L_\gamma := \{(x, y, u) :\ |y| \leq \gamma(|x|),\ \ a_i(x) \leq u_i \leq b_i(x)\}. \qquad (4.3)$$

- γ *is nonincreasing for* $s > \delta_1$ *and satisfies*

$$\gamma(s) \leq \gamma_0(s),\ \ \forall\ s > \delta_2;\ \ |D\gamma(s)|h(s) \leq 1,\ \ \forall\ s \geq 0. \qquad (4.4)$$

Proof. Condition 2 guarantees the existence of a positive definite matrix P and a positive constant ℓ such that $x'P\left(\frac{\partial f}{\partial x}(0, 0, 0) + \frac{\partial f}{\partial y}(0, 0, 0)D\phi(0)\right)x \leq -\ell|x|^2$ for all x which in conjunction with (4.1) implies

$$x'Pf(x, D\phi(0)x + y, u) \leq -\frac{\ell}{2}|x|^2, \qquad (4.5a)$$

$$\forall\ x'Px < \varepsilon_1,\ |y| < 2\xi|x|,\ \ a_i(x) \leq u_i \leq b_i(x),$$

$$|\phi(x) - D\phi(0)x| \leq \xi|x|,\quad \forall x'Px \leq \varepsilon_1 \qquad (4.5b)$$

for certain positive constants ε_1 and ξ. From Condition 1 and Theorem 2.1 we can determine a p.d.r.u. C^1 map $V : \mathbb{R}^n \to \mathbb{R}^+$ and a positive definite function $c : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$DV(x)f(x, \phi(x) + y, u) \leq -c(|x|),\quad \forall\ (x, y, u) \in L. \qquad (4.6)$$

Let $\zeta : \mathbb{R}^+ \to [0,1]$ be a smooth function such that $\zeta(s) = 1$ for $s \le \varepsilon_2$ for certain positive $\varepsilon_2 < \varepsilon_1$, $\zeta(s) = 0$ for $s \ge \varepsilon_1$ and define

$$\widehat{\phi}(x) := \zeta(p(x)) D\phi(0)x + (1 - \zeta(p(x)))\phi(x) \tag{4.7a}$$

where

$$p(x) := x'Px + V(x) \tag{4.7b}$$

From (4.5), (4.6) and (4.7) we can determine a pair of constants $\delta_1 > \delta_2 > 0$, a positive definite C^1 function $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ being linear on $[0, \delta_2]$, increasing on $[0, \delta_1]$, nonincreasing on $[\delta_1, +\infty)$ and in such a way that

$$\Gamma(s) < \gamma_0(s), \quad \forall\ s > \delta_2, \tag{4.8}$$

$$x'Pf(x, \widehat{\phi}(x) + y, u) \le -\frac{\ell}{2}|x|^2, \quad \forall\ (x, y, u) \in L_\Gamma,\ p(x) < \varepsilon_1, \tag{4.9a}$$

$$DV(x)f(x, \widehat{\phi}(x) + y, u) \le -c(|x|), \quad \forall\ (x, y, u) \in \mathrm{L}_\Gamma,\ p(x) > \varepsilon_2', \tag{4.9b}$$

$$\text{where} \quad L_\Gamma := \{(x; y, u) : |y| \le \Gamma(|x|),\ a_i(x) \le u_i \le b_i(x)\}$$

for certain $\varepsilon_2' < \varepsilon_1$. Finally, let λ be a positive constant with

$$\lambda \sup\{|D\Gamma(s)|h(s),\ s > 0\} < 1 \tag{4.10}$$

and define

$$\gamma := \frac{\lambda}{1+\lambda}\Gamma \tag{4.11}$$

Obviously, (4.8) and (4.10) imply (4.4) with γ as defined by (4.11). Moreover, the Lyapunov inequalities (4.9) imply that zero is L_γ–UGAS for (4.2) with L_γ as defined by (4.3). To be more precise the statement follows by checking the time derivatives of $x'Px$ for $p(x) < \varepsilon_1$ and of $V(x) + x'Px$ for $p(x) > \varepsilon_2'$ along the trajectories of (4.2) provided that $(x, y, u) \in L_\gamma$. Details are left to the reader. □

REMARK 4.2. Condition 1 of Lemma 4.1 can be substituted by weaker hypothesis that zero is L–GAS for (4.2) provided that the functions γ_0, a_i and b_i are bounded. This is a direct consequence of Theorem 2.3.

The author would like to thank the anonymous referee who pointed out the relationship between RGUAS property introduced by Freeman–Kokotovic in [5] *and L–UGAS.*

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