ESAIM: Control, Optimisation and Calculus of Variations September 1995, Vol.1, pp.1-15
URL: http://www.emath.fr/cocv/

APPROXIMATE CONTROLLABILITY OF A HYDRO-ELASTIC COUPLED SYSTEM

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Abstract. We analyze the controllability of the motion of a fluid by means of the action of a vibrating shell coupled at the boundary of the fluid. The model considered is linear. We study its approximate controllability, i.e. whether the fluid may reach a dense set of final configurations at a given time. We show that this problem can be reduced to a unique continuation question for the Stokes system. We prove that this unique continuation property holds generically among analytic domains and therefore, that there is approximate controllability generically. We also prove that this result fails when Ω is a ball showing that the analyticity assumption on the domain is not sufficient.

1. Introduction and Main Result

1.1. Preliminaries

We consider a bounded domain Ω of $\mathbb{R}^3$, not necessarily simply connected, with smooth boundary $\Gamma = \partial\Omega$. For the main result we will actually assume that Γ is real analytic.

We divide Γ in two pieces $\Gamma = \Gamma_0 \cup \Gamma_1$. The subset Γ_0 will play the role of a vibrating shell.

We study a very approximated and simplified linear model of an incompressible viscous fluid flowing in Ω and, in particular, we analyze the possibility of controlling its behavior by means of a control function acting on Γ_0.

We give directly the variational formulation of the problem that we will interpret later on in classical terms. For doing that we need to introduce some functional spaces.

First of all we define the space V:

$$V = \Big\{ v \in (H^1(\Omega))^3 \ : \ \operatorname{div} v = 0 \quad \text{in } \Omega \, ,$$

$$v = 0 \quad \text{on } \Gamma_1 \ \text{ and } \ v|_{\Gamma_0} \text{ is perpendicular to } \Gamma_0 \Big\} \, . \tag{1}$$

The vector space V is endowed with the norm induced by the Hilbert space $(H^1(\Omega))^3$.

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Received by the journal February 1, 1995. Accepted for publication April 24, 1995.

The work of the second author was supported by Project PB93-1203 of the DGICYT (Spain) and Projects SC1*-CT91-0732 and CHRX-CT94-0471 of the European Community.

We also introduce the following bilinear form defined on $V \times V$:

$$a(u,v) = \mu \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}\, dx\ , \tag{2}$$

where μ is a given positive constant (we use the convention of summation of repeated indexes).

We will denote by $\cdot$ the scalar product in $\mathbb{R}^3$.

We denote by n the unit outward normal vector to Ω and introduce a subspace W of V consisting of those elements v of V whose normal component $v \cdot n$ satisfies some further regularity properties. More precisely

$$W = \left\{ v \in V : \quad v \cdot n \in H^2(\Gamma_0) \cap H_0^1(\Gamma_0) \right\} . \tag{3}$$

By $H^s(\Gamma_0)$ and $H_0^s(\Gamma_0)$ we are denoting the Sobolev spaces of order s over $L^2(\Gamma_0)$ equal to $H^0(\Gamma_0)$ considering Γ_0 as a Riemannian manifold with boundary (see for instance J.L. Lions and E. Magenes [8] Chap. 1, n° 7.3 and, in particular, Remark 7.5).

By means of the Laplace-Beltrami operator Δ_{Γ_0} over Γ_0, we can rewrite W as follows:

$$W = \left\{ v \in V \ :\ \Delta_{\Gamma_0}(v \cdot n) \in L^2(\Gamma_0), \quad v \cdot n = 0 \quad \text{on} \quad \partial\Gamma_0 \right\} . \tag{4}$$

The subspace W is not closed in V. In fact, W is dense in V.

We endow W with the Hilbert structure induced by the bilinear form

$$a(u,v) + a_{\Gamma_0}(u,v) \tag{5}$$

where

$$a_{\Gamma_0}(u,v) = \int_{\Gamma_0} \Delta_{\Gamma_0}(u \cdot n)\, \Delta_{\Gamma_0}(v \cdot n)\, d\Gamma_0\ . \tag{6}$$

REMARK 1. Since $\operatorname{div} v = 0$ in Ω and

$$\int_\Omega \operatorname{div} v\, dx = \int_\Gamma v \cdot n \cdot d\Gamma = \int_{\Gamma_0} v \cdot n\, d\Gamma_0\ ,$$

we have

$$\int_{\Gamma_0} v \cdot n\, d\Gamma_0 = 0, \quad \forall v \in V\ . \tag{7}$$

On the other hand, if g is a smooth scalar function defined on Γ_0 that vanishes on $\partial\Gamma_0$ and satisfying

$$\int_{\Gamma_0} g\, d\Gamma_0 = 0\ , \tag{8}$$

then, there exists $v \in W$ such that

$$v \cdot n = g \quad \text{on} \ \Gamma_0\ . \tag{9}$$

(see O.A. Ladyzhenskaya [4], Section 2.1).

1.2. Variational formulation

In the sequel, we will denote by $(\cdot,\cdot)$ both the scalar product in $L^2(\Omega)$ and in $(L^2(\Omega))^3$.

Given $T>0$ and a scalar function $h=h(x,t)$ we look for u such that

$$\left\{\begin{array}{l} u \in L^2(0,T;\ V), \\ \displaystyle\int_0^t u\cdot n\, ds \in L^2\left(0,T;\ H^2(\Gamma_0)\cap H_0^1(\Gamma_0)\right), \end{array}\right. \tag{10}$$

and

$$\frac{d}{dt}\left[(u,v)+(u\cdot n,\ v\cdot n)_{\Gamma_0}\right]+a(u,v)+a_{\Gamma_0}\left(\int_0^t u\cdot n\, ds,\ v\cdot n\right), \\ = (h,\ v\cdot n)_{\Gamma_0}, \quad \forall v\in W, \tag{11}$$

with

$$\left|\begin{array}{rl} u(0)=0 & \text{in} \quad \Omega, \\ u\cdot n\,(0)=0 & \text{on} \quad \Gamma_0. \end{array}\right. \tag{12}$$

This variational problem is not completely standard since there is an obvious asymmetry between the space of test functions and the space where the solution is required to be. But we will see later on that nevertheless it has a unique solution.

In the next section we provide an interpretation of system (10)-(12) in classical terms.

In (11), h represents *the control function.* We assume that h runs over the space of functions such that

$$h\in L^2(\Gamma_0\times(0,T)). \tag{13}$$

We will see later on that, when u satisfies (10)-(12), then

$$\left\{\begin{array}{l} u(T) \ \text{ is well defined in } \ (L^2(\Omega))^3, \\ u\cdot n\,(T)|_{\Gamma_0} \ \text{ is well defined in } \ L_0^2(\Gamma_0), \end{array}\right. \tag{14}$$

where $L_0^2(\Gamma_0)$ denotes the subspace of $L^2(\Gamma_0)$ of zero mean functions.

1.3. The main result

The goal of this paper is to prove the following result of *approximated controllability*:

Theorem 2. *Assume that the boundary Γ of Ω is real analytic and that the spectrum of the laplacian $-\Delta$ in $H_0^1(\Omega)$ is simple.*

Under these conditions, when the control h runs over the space defined in (13),

$$\left\{u(T),\quad u\cdot n\,(T)|_{\Gamma_0}\right\} \quad \textit{is dense in} \quad (L^2(\Omega))^3\times L_0^2(\Gamma_0). \tag{15}$$

Remark 3. The hypotheses we have made on Ω and, in particular the analyticity one, are probably too restrictive. We conjecture that the approximate controllability result (15) holds *generically with respect to* Ω, i.e. given any domain Ω of class C^2 and fixing the subset Γ_0 of Γ property (15) holds after a possible arbitrarily small perturbation of Γ_1.

Notice that the simplicity of the Dirichlet spectrum of the Laplacian holds generically with respect to Ω (see J. H. Albert [1], A.M. Micheletti [9] and K. Uhlenbeck [10]).

Unfortunately, the analyticity assumption on Γ excludes the case where Γ_0 is plane.

REMARK 4. The analyticity assumption on Ω is not sufficient to guarantee the controllability property (15). Indeed, as we will see in the last section, when Ω is a ball (15) fails even when Γ_0 is the whole boundary.

REMARK 5. Under the hypotheses of Theorem 2, once the approximate controllability is known, given any $\{w, b\}$ in $(L^2(\Omega))^3 \times L_0^2(\Gamma_0)$ and $\varepsilon > 0$ we can look for the optimal control h_{op} among the admissible ones such that

$$||h_{op}||_{L^2(\Gamma_0\times(0,T))} = \min_{h\in U_{ad}} ||h||_{L^2(\Gamma_0\times(0,T))} ,$$

with

$$U_{ad} = \Big\{ h \text{ satisfying (13) s.t. the solution } v \text{ of (10)-(12)}$$

$$\text{satisfies } ||(v(T),\ v\cdot n(T)|_{\Gamma_0}) - (w,b)||_{(L^2(\Omega))^3\times L_0^2(\Gamma_0)} \leq \varepsilon \Big\} .$$

Such an optimal control exists and is unique. It can be characterized by a system of optimality that can be derived via duality theory as in J.L. Lions [7]. This optimality condition allows also to analyze the dependence of the control with respect to a number of parameters of the problem (see, for instance, C. Fabre, J.P. Puel and E. Zuazua [3] where a class of semilinear heat equations is considered).

We do not pursue in this direction in this work.

The rest of the paper is organized as follows. In section 2 we interpret the variational problem in classical terms. In section 3 we prove the basic existence and uniqueness result.

In Section 4 we prove the approximate controllability result. In Section 5 we show that the controllability result does not hold when Ω is a ball even if Γ_0 is the whole boundary.

2. Interpretation of the Variational Problem

First, taking $v \in \mathcal{D}(\Omega)$ in (11) we deduce that

$$\frac{\partial u}{\partial t} - \mu\,\Delta u = -\nabla p \quad \text{in} \quad \Omega\times(0,T) \tag{16}$$

where p is the pressure which is defined up to an additive time-dependent function.

On the other hand, one can deduce that

$$u = 0 \quad \text{on } \Gamma_1 , \tag{17}$$

$$\text{the tangential components of } u \text{ vanish on } \Gamma_0 , \tag{18}$$

$$u = 0 \quad \text{at } t = 0 . \tag{19}$$

But it remains a condition over the restriction of $u\cdot n$ to Γ_0 that we describe now.

Multiplying in (16) by v and integrating by parts we obtain

$$\left(\frac{\partial u}{\partial t}, \ v\right) - \mu \int_{\Gamma_0} \frac{\partial u}{\partial n} v d\Gamma_0 + a(u,v) = -\int_{\Gamma_0} p v \cdot n \, d\Gamma_0 \, . \tag{20}$$

On the other hand $v = (v \cdot n)\, n$ on Γ_0, therefore

$$\int_{\Gamma_0} \frac{\partial u}{\partial n} v \, d\Gamma_0 = \int_{\Gamma_0} \left(n \cdot \frac{\partial u}{\partial n}\right) (v \cdot n) \, d\Gamma_0 \, . \tag{21}$$

Using (20), (21) and (11) we obtain

$$\frac{d}{dt}(u \cdot n, \ v \cdot n)_{\Gamma_0} + \mu \int_{\Gamma_0} \left(n \cdot \frac{\partial u}{\partial n}\right) (u \cdot n) \, d\Gamma_0 + a_{\Gamma_0}\left(\int_0^t u \cdot n \, ds, \ v \cdot n\right)$$

$$= (h, \ v \cdot n)_{\Gamma_0} + \int_{\Gamma_0} p v \cdot n d\Gamma_0, \quad \forall v \in W \, . \tag{22}$$

But in (22), the test function v appears only through the value of $v \cdot n$ over Γ_0. Thus, in view of 1 we can replace $v \cdot n$ by g, where g is a smooth scalar function defined on Γ_0 such that $g = 0$ on $\partial\Gamma_0$ and $\displaystyle\int_{\Gamma_0} g \, d\Gamma_0 = 0$. Therefore

$$\frac{\partial}{\partial t}(u \cdot n) + \Delta^2_{\Gamma_0}\left(\int_0^t u \cdot n \, ds\right) + \mu \left(n \cdot \frac{\partial u}{\partial n}\right) = h + p + c \quad \text{on } \ \Gamma_0 \times (0,T) \, , \tag{23}$$

where c is a function which depends only on time.

In (23) we have to add the boundary conditions.

$$u \cdot n = 0, \quad \Delta_{\Gamma_0}(u \cdot n) = 0 \quad \text{on } \ \partial\Gamma_0 \, , \tag{24}$$

and the initial condition

$$(u \cdot n)\,(0) = 0 \quad \text{in } \ \Gamma_0 \, . \tag{25}$$

REMARK 6. If we set

$$\frac{\partial \varphi}{\partial t} = u \cdot n \, , \tag{26}$$

we have

$$\frac{\partial^2 \varphi}{\partial t^2} + \Delta^2_{\Gamma_0}\varphi + \mu \left(n \cdot \frac{\partial u}{\partial n}\right) = h + p + c \, . \tag{27}$$

The function φ represents the displacement of Γ_0 in the normal direction n and therefore (26) states that the normal component of the velocity of the fluid $u \cdot n$ coincides with the velocity of the deformation of Γ_0.

REMARK 7. When Γ_0 is flat,

$$n \cdot \frac{\partial u}{\partial n} = 0 \, ,$$

since div $u = 0$ in Ω and u is perpendicular to Γ_0 over Γ_0. In this particular case (27) becomes

$$\frac{\partial^2 \varphi}{\partial t^2} + \Delta^2_{\Gamma_0}\varphi = h + p + c \quad \text{in} \quad \Gamma_0 \times (0,T) \, ,$$

and $\Delta_{\Gamma_0} = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ if Γ_0 is parallel to $x_3 = 0$. However, as we said in Remark 3, the analyticity assumption of Theorem 2 excludes the case when Γ_0 is flat.

Putting all equations above together we obtain the following system:

$$\begin{cases} u_t - \mu\,\Delta u = -\nabla p & \text{in} \quad \Omega \times (0,T)\,, \\ \text{div } u = 0 & \text{in} \quad \Omega \times (0,T)\,, \\ u = 0 & \text{on} \quad \Gamma_1 \times (0,T)\,, \\ u = \varphi_t n & \text{on} \quad \Gamma_0 \times (0,T)\,, \\ \varphi_{tt} + \Delta^2_{\Gamma_0}\varphi + \mu n \cdot \dfrac{\partial u}{\partial n} = h + p + c(t) \ \text{ in } \ \Gamma_0 \times (0,T) & , \\ \displaystyle\int_{\Gamma_0} \varphi_t \, d\Gamma_0 = 0 & \text{for} \quad t \in (0,T)\,, \\ \varphi = \Delta_{\Gamma_0}\varphi = 0 & \text{on} \quad \partial\Gamma_0 \times (0,T)\,, \\ u(0) = 0 & \text{in} \quad \Omega\,, \\ \varphi(0) = \varphi_t(0) = 0 & \text{on} \quad \Gamma_0\,. \end{cases} \tag{28}$$

REMARK 8. In [6], chapter I.9 a somewhat similar coupled parabolic-hyperbolic system is analyzed. That system is motivated by the problem of the flow of blood in arteries as introduced by H. Cohen and S.I. Rubinow [2].

REMARK 9. A completely similar problem can be formulated for a system like (28) in which the first equation is replaced by

$$u_t - \Delta u + b(x,t)\nabla u = -\nabla p\,,$$

where $b(x,t)$ is given such that $\text{div} b(\cdot,t) = 0$.

The variational formulation of this new system is similar to (10)-(12) except that we have to add the term $\int_\Omega (b \cdot \nabla) u v dx$ in the left hand side of (11).

We *conjecture* an analogous of Theorem 2 still to be true but the proof given here does not apply to this situation.

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR THE VARIATIONAL PROBLEM

In this section we apply a classical Galerkin method to prove the existence and uniqueness of solutions of the variational problem (10)-(12). The method being by now rather standard we only give an outline of the proof.

THEOREM 10. *For any $h \in L^2(\Gamma_0 \times (0,T))$ the variational problem (10)-(12) admits a unique solution.*

Proof. The uniqueness is standard and for the proof of the existence we proceed in several steps.

STEP 1. CONSTRUCTION OF THE GALERKIN BASIS

The construction of the Galerkin basis is not essential for the proof since we are dealing with a linear problem but the introduction of this basis may be of independent interest.

We consider the following eigenvalue problem: Find the eigenvalues λ_j and the eigenfunctions (w_j, π_j, c_j) such that

$$\left\{ \begin{array}{rll} -\mu\Delta w_j = \lambda_j\, w_j - \nabla \pi_j & \text{in} & \Omega\ , \\ \text{div } w_j = 0 & \text{in} & \Omega\ , \\ w_j = 0 & \text{on} & \Gamma_1\ , \\ w_j \text{ is perpendicular to } \Gamma & \text{on} & \Gamma_0\ , \\ \dfrac{\partial w_j}{\partial n}\cdot n - \pi_j = c_j & \text{on} & \Gamma_0\ . \end{array} \right.$$

where π_j is the pressure that is determined up to an additive constant and c_j is a real number.

This problem admits the following variational formulation: Find λ_j and $w_j \in V$ such that

$$a(w_j, v) = \lambda_j\,(w_j, v)\ , \qquad \forall v \in V\ . \tag{29}$$

It is easy to see that there exists an infinite sequence of positive eigenvalues $\{\lambda_j\}$ (that we repeat according to their multiplity), and that we can construct an orthonormal base $\{w_j\}_{j\geq 1}$ of V with the associated eigenfunctions.

We apply the Faedo-Galerkin method with the following "special basis":

$$\{w_j, \varphi_j\} \quad \text{with} \quad \varphi_j = w_j \cdot n \quad \text{on} \quad \Gamma_0\ .$$

Step 2. Approximated solutions

We define $u_m(t)$ as the solution of the finite-dimensional problem:

$$\left\{ \begin{array}{l} (u_m'(t), w_j) + (u_m'(t)\cdot n,\ w_j\cdot n)_{\Gamma_0} \\ +a(u_m(t), w_j) + a_{\Gamma_0}\left(\displaystyle\int_0^t u_m(s)\cdot n\, ds,\ w_j\cdot n\right) \\ = (h,\ w_j\cdot n)_{\Gamma_0}, \quad 1\leq j\leq m, \quad u_m(t)\in [w_1,\dots,w_m]\ , \\ u_m(0) = 0\ . \end{array} \right. \tag{30}$$

where $[\dots]$ denotes the vector space generated by the functions under the brackets and $'$ the derivative with respect to t.

System (30) admits a unique solution which is globally defined for all $t \in [0, T]$.

On the other hand, the energy identity that the solutions of (28) satisfy formally, i.e.

$$\frac{1}{2}\frac{d}{dt}\left[\int_\Omega |u|^2\, dx + \int_{\Gamma_0} (|\varphi_t|^2 + |\Delta_{\Gamma_0}\varphi|^2)\, d\Gamma_0\right] = \int_{\Gamma_0} h\varphi_t\cdot n\, d\Gamma_0\ , \tag{31}$$

allows us to show that

$$\left\{ \begin{array}{l} u_m \quad \text{remains bounded in} \quad L^2(0,T;V)\cap C\Big([0,T];\ (L^2(\Omega))^3\Big)\ , \\ \displaystyle\int_0^t u_m\cdot n\, ds \quad \text{remains bounded in} \quad C\Big([0,T]\,;\ H^2(\Gamma_0)\cap H_0^1(\Gamma_0)\Big)\ , \\ u_m\cdot n \quad \text{remains bounded in} \quad C\Big([0,T]\,;\ L^2(\Gamma_0)\Big)\ . \end{array} \right. \tag{32}$$

Step 3. Passing to the limit

From (32) and using the equations that $\{u_m,\ u_m\cdot n\}$ satisfy it is easy to see that

$$\{\partial_t u_m\,,\quad \partial_t u_m\cdot n\}\quad\text{remains bounded in}\quad L^2(0,T;\ W')\,, \tag{33}$$

where W' denotes the dual of W.

Classical compactness arguments allow us to show that

$$\left\{\begin{array}{l} u_m \quad \text{is relatively compact in} \quad L^2\Big(0,T;\ (H^{1-\varepsilon}(\Omega))^3\Big)\,,\\ \displaystyle\int_0^t u_m\cdot n\,ds \quad \text{is relatively compact in} \quad L^2\Big(0,T\,;\ H^{2-\varepsilon}(\Gamma_0)\Big)\,,\\ u_m\cdot n \quad \text{is relatively compact in} \quad L^2\Big(0,T\,;\ H^{-\varepsilon}(\Gamma_0)\Big)\,, \end{array}\right. \tag{34}$$

for any $0<\varepsilon<1/2$.

It is then easy to pass to the limit in (30) to get (10)-(12). □

4. Proof of the Approximate Controllability Result

We proceed in several steps. First, applying Hanh-Banach Theorem we reduce the approximate controllability problem to a uniqueness property for solutions of the evolution Stokes system. Then we show that this uniqueness problem can be reduced to the analysis of the eigenfunctions and eigenpressures of the Stokes system. More precisely, we show that it is sufficient to prove that the eigenpressures cannot be identically constant. Finally we show that this property holds generically with respect to the domain Ω.

4.1. Step 1. Application of Hahn-Banach Theorem.

We consider a pair

$$\{f,g\}\in\Big(L^2(\Omega)\Big)^3\times L^2_0(\Gamma_0)\,, \tag{35}$$

and suppose that

$$\left\{\begin{array}{l} (f,u(T))+\Big(g,\ u\cdot n(T)\Big)_{\Gamma_0}=0\,,\\ \forall h \quad \text{as in (13)}\,. \end{array}\right. \tag{36}$$

We have to show that (36) implies

$$f=0,\quad g=0 \tag{37}$$

For that we introduce the function ψ such that

$$\left\{\begin{array}{l} \psi\in L^2\left(0,T\,;\ V\right)\,,\\ \displaystyle\int_t^T \psi\cdot n\,ds\in L^2(0,T;\,H^2(\Gamma_0)\cap H^1_0\left(\Gamma_0\right)\Big)\,, \end{array}\right. \tag{38}$$

$$\left\{\begin{array}{l} -\dfrac{d}{dt}\Big[(\psi,\hat\psi)+(\psi\cdot n,\hat\psi\cdot n)_{\Gamma_0}\Big]+a(\psi,\hat\psi)+a_{\Gamma_0}\Big(\displaystyle\int_t^T\psi\cdot n\,ds,\ \hat\psi\cdot n\Big)=0\\ \qquad\qquad\qquad\qquad\forall\hat\psi\in W\,. \end{array}\right. \tag{39}$$

$$\psi(T)=f,\quad \psi\cdot n(T)=g\,. \tag{40}$$

From Section 3 we know that (38)-(40) has a unique solution. On the other hand, without loss of generality we may restrict ourselves to analyze (36) for smooth functions h such that, for instance,

$$\frac{\partial h}{\partial t} \in L^2(0,T;\ L^2(\Gamma_0)), \quad h(0) = 0\ . \tag{41}$$

In this case the solution u of (10)-(12) has, roughly, one more degree of regularity in time. More precisely,

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial t} \in L^2(0,T\,;\ V)\ , \\ u \cdot n \in L^2(0,T;\ H^2(\Gamma_0) \cap H_0^1(\Gamma_0))\ . \end{array} \right. \tag{42}$$

This allows us to take $\hat{\psi} = u(t)$ as test function in (39) for a.e. $t \in [0,T]$, provided that we rewrite first the term

$$\frac{d}{dt}\left[(\psi, \hat{\psi}) + (\psi \cdot n,\ \hat{\psi} \cdot n)_{\Gamma_0} \right]\ ,$$

as

$$\left(\frac{\partial \psi}{\partial t},\ \hat{\psi} \right) + \left(\frac{\partial \psi}{\partial t} \cdot n,\ \hat{\psi} \cdot n \right)_{\Gamma_0}\ .$$

We obtain in this way:

$$-\left(\frac{\partial \psi}{\partial t},\ u(t) \right) - \left(\frac{\partial \psi}{\partial t} \cdot n,\ u(t) \cdot n \right)_{\Gamma_0} + a(\psi, u(t))$$

$$+ a_{\Gamma_0}\left(\int_t^T \psi \cdot n\, ds,\ u(t) \cdot n \right) = 0\ .$$

Integrating this identity with respect to $t \in [0,T]$ we get:

$$-\Big(\psi(T), u(T) \Big) - \Big(\psi(T) \cdot n,\ u(T) \cdot n \Big)_{\Gamma_0}$$

$$+ \int_0^T \left[\left(\frac{\partial u}{\partial t}, \psi \right) + \left(\frac{\partial u}{\partial t} \cdot n,\ \psi \cdot n \right)_{\Gamma_0} + a(u, \psi) \right] dt$$

$$+ \int_0^T a_{\Gamma_0}\left(\int_t^T \psi \cdot n\, ds,\ u \cdot n \right) dt = 0\ . \tag{43}$$

The last term equals

$$\int_0^T a_{\Gamma_0}\left(\int_0^t u \cdot n\, ds,\ \psi \cdot n \right) dt\ ,$$

by integration by parts. On the other hand, taking $v = \psi(t)$ in (11) and integrating in $(0,T)$ we get

$$\int_0^T \left[\left(\frac{\partial u}{\partial t},\ \psi \right) + \left(\frac{\partial u}{\partial t} \cdot n,\ \psi \cdot n \right)_{\Gamma_0} + a(u, \psi) \right] dt$$

$$+ \int_0^T a_{\Gamma_0}\left(\int_0^t u \cdot n\, ds,\ \psi \cdot n \right) dt = \int_0^T (h,\ \psi \cdot n)_{\Gamma_0}\, dt\ . \tag{44}$$

Combining (36), (40), (43) and (44) we conclude that

$$\int_0^T (h,\ \psi \cdot n)_{\Gamma_0}\, dt = 0, \quad \forall h \ \text{ smooth as in (13) and (41)}\ . \tag{45}$$

Therefore

$$\psi \cdot n = 0 \quad \text{on} \quad \Gamma_0 \times (0,T) . \tag{46}$$

Since the tangential components of ψ on Γ_0 vanish we deduce that

$$\psi = 0 \quad \text{on} \quad \Gamma_0 \times (0,T) . \tag{47}$$

Thus, (39) reduces to

$$-\frac{d}{dt}(\psi, \hat{\psi}) + a(\psi, \hat{\psi}) = 0, \quad \forall \hat{\psi} \in W . \tag{48}$$

Taking in (48) test functions $\hat{\psi}$ with compact support in Ω we deduce that

$$\begin{cases} -\dfrac{\partial \psi}{\partial t} - \mu \Delta \psi = -\nabla \rho & \text{in} \quad \Omega \times (0,T) , \\ \text{div } \psi = 0 & \text{in} \quad \Omega \times (0,T) , \\ \psi = 0 & \text{on} \quad \Gamma \times (0,T) , \\ \psi(T) = f . \end{cases} \tag{49}$$

where ρ is the pressure that is defined up to an additive time-dependent function.

Multiplying in (49) by $\hat{\psi} \in W$ we deduce that

$$\int_{\Gamma_0} \left(\mu \frac{\partial \psi}{\partial n} - \rho n \right) \cdot \hat{\psi} \, d\Gamma_0 = 0 ,$$

or, since $\hat{\psi} = (\hat{\psi} \cdot n)\, n$ on Γ_0,

$$\int_{\Gamma_0} \left(\mu \left(n \cdot \frac{\partial \psi}{\partial n} \right) - \rho \right) \hat{\psi} \cdot n \, d\Gamma_0 = 0 . \tag{50}$$

In view of Remark 1, in (50) we can replace $\hat{\psi} \cdot n$ by any function in $L^2_0(\Gamma_0)$.

Thus,

$$\mu n \cdot \frac{\partial \psi}{\partial n} - \rho = k(t) \quad \text{on} \quad \Gamma_0 \times (0,T) , \tag{51}$$

where $k = k(t)$ is some time-dependent function.

On the other hand, since $\psi(t)|_\Gamma = 0$ we have

$$\nabla \psi_i(t) = n \frac{\partial \psi_i}{\partial n}(t) \quad \text{on} \quad \Gamma_0 \times (0,T)$$

for any component ψ_i of ψ. But, since div $\psi = 0$ in $\Omega \times (0,T)$, we deduce that

$$n_i \frac{\partial \psi_i}{\partial n} = n \cdot \frac{\partial \psi}{\partial n} = 0 \quad \text{on} \quad \Gamma_0 \times (0,T) . \tag{52}$$

Therefore, from (51) and (52) we obtain that

$$\rho = -k(t) \quad \text{on} \quad \Gamma_0 \times (0,T), \quad \text{a function independent of } x . \tag{53}$$

4.2. Step 2.

In order to simplify the notation, we reverse the sense of time in system (49). In this way $\varphi(x,t) = \psi(x, T-t)$ and $\sigma(x,t) = \rho(x, T-t)$ satisfy the Stokes system

$$\left\{ \begin{aligned} \frac{\partial \varphi}{\partial t} - \mu\, \Delta \varphi = -\nabla \sigma \quad & \text{in} \quad \Omega \times (0,T)\,, \\ \operatorname{div} \varphi = 0 \quad & \text{in} \quad \Omega \times (0,T)\,, \\ \varphi = 0 \quad & \text{on} \quad \Gamma \times (0,T)\,, \\ \varphi(0) = f \quad & \text{in} \quad \Omega\,. \end{aligned} \right. \tag{54}$$

and the additional boundary condition

$$\sigma = k_1(t) \quad \text{on} \quad \Gamma_0 \times (0,T), \quad \text{a function independent of } x \tag{55}$$

with $k_1(t) = -k\,(T-t)$.

Since the domain Ω is smooth by the regularizing effect of the Stokes system we know that for any $x \in \Gamma = \partial\Omega$, $\sigma(x,t)$ is a real analytic function of $t \in (0,\infty)$ (at this level we do not need the boundary Γ to be real analytic). In view of (55) this implies that

$$\sigma = k_1(t) \quad \text{on} \quad \Gamma_0 \times (0,\infty) \tag{56}$$

where σ is the pressure obtained by extending the solution of the Stokes initial-boundary value problem to the whole time interval $t \in (0,\infty)$ and $k_1(t)$ is the real analytic continuation of the function $k_1 : (0,T) \to \mathbb{R}$ to $\mathbb{R}^+$ determined by the value of the pressure σ at any point of Ω for all $t \geq 0$.

4.3. Step 3.

We introduce now the spectrum of the Stokes system in Ω:

$$\left\{ \begin{aligned} -\mu\, \Delta w = \lambda w - \nabla \sigma \quad & \text{in} \quad \Omega\,, \\ \operatorname{div}\, w = 0 \quad & \text{in} \quad \Omega\,, \\ w = 0 \quad & \text{on} \quad \Gamma\,. \end{aligned} \right. \tag{57}$$

We denote by $\{\lambda_j\}$ the sequence of distinct eigenvalues of multiplicity $l(j)$.

Let $\{w_j^m\}_{j\in\mathbb{N},\, m=1,\dots,l(j)}$ be an orthonormal basis of V constituted by eigenfunctions and σ_j^m the corresponding eigenpressure.

Then, the solution φ of (54) can be written as

$$\varphi = \sum_{j=1}^{\infty} \sum_{m=1}^{l(j)} (f, w_j^m)\, e^{-\lambda_j t}\, w_j^m\,. \tag{58}$$

From (58) and system (54) we obtain the following representation for the gradient of the pressure σ:

$$\nabla \sigma = \sum_{j=1}^{\infty} \sum_{m=1}^{l(j)} (f, w_j^m)\, e^{-\lambda_j t}\, \nabla \sigma_j^m\,. \tag{59}$$

If we denote by ∇_τ the tangential component of the gradient on Γ, from (59) we deduce that

$$\nabla_\tau \sigma = \sum_{j=1}^{\infty} \sum_{m=1}^{l(j)} (f, w_j^m)\, e^{-\lambda_j t} \nabla_\tau \sigma_j^m \quad \text{on} \quad \Gamma \times (0, \infty) , \tag{60}$$

and, in view of (55),

$$\sum_{j=1}^{\infty} \sum_{m=1}^{l(j)} (f, w_j^m)\, e^{-\lambda_j t} \nabla_\tau \sigma_j^m = 0 \quad \text{on} \quad \Gamma_0 \times (0, \infty) . \tag{61}$$

From (61) it is easy to deduce that

$$\sum_{m=1}^{l(j)} (f, w_j^m)\, \nabla_\tau \sigma_j^m = 0 \quad \text{on} \quad \Gamma_0, \quad \forall j \geq 1 . \tag{62}$$

Indeed, multiplying in (61) by $e^{\lambda_1 t}$ we get

$$\sum_{m=1}^{l(1)} (f, w_1^m)\, \nabla_\tau \sigma_1^m = - \sum_{j=2}^{\infty} \sum_{m=1}^{l(j)} (f, w_j^m)\, e^{(\lambda_1 - \lambda_j) t} \nabla_\tau \sigma_j^m \quad \text{on} \quad \Gamma_0 \times (0, \infty) .$$

The right hand side converges to zero, for instance, in $L^2(\Gamma_0)$ as $t \to \infty$, while the left hand side is time-independent. This implies that

$$\sum_{m=1}^{l(1)} (f, w_1^m)\, \nabla_\tau \sigma_1^m = 0 \quad \text{on} \quad \Gamma_0 .$$

Repeating this argument, by induction we get (62). Of course, (62) is equivalent to the existence of a sequence of constants $c_j \in \mathbb{R}$ such that

$$\sum_{m=1}^{l(j)} (f, w_j^m)\, \sigma_j^m = c_j \quad \text{on} \quad \Gamma_0 . \tag{63}$$

4.4. Step 4.

We introduce now

$$\omega_j = \sum_{m=1}^{l(j)} (f, w_j^m)\, w_j^m , \tag{64}$$

and

$$\rho_j = \sum_{m=1}^{l(j)} (f, w_j^m)\, \sigma_j^m . \tag{65}$$

We verify that

$$\left\{ \begin{aligned} -\mu \Delta \omega_j &= \lambda_j \omega_j - \nabla \rho_j \quad \text{in} \quad \Omega , \\ \operatorname{div} \omega_j &= 0 \quad \text{in} \quad \Omega , \\ \omega_j &= 0 \quad \text{on} \quad \Gamma . \end{aligned} \right. \tag{66}$$

and

$$\rho_j = c_j \quad \text{on } \Gamma_0 . \tag{67}$$

The problem is then reduced to showing that (66) and (67) imply (generically) that

$$\omega_j = 0$$

i.e. $(f, w_j^k) = 0$ for all j, k, i.e. $f \equiv 0$.

4.5. Step 5.

We can drop the index "j" in (66)-(67). We have to show that (generically) if

$$\begin{cases} -\mu\Delta\omega = \lambda\,\omega - \nabla\rho & \text{in} \quad \Omega\,, \\ \text{div}\ \omega = 0 & \text{in} \quad \Omega\,, \\ \omega = 0 & \text{on} \quad \Gamma\,. \end{cases} \tag{68}$$

and

$$\rho = \ \text{constant on } \Gamma_0\,. \tag{69}$$

then $\omega = 0$, $\rho =$ constant in Ω.

Since ρ is defined up to an additive constant, we do not restrict the generality by assuming that $\rho = 0$ on Γ_0.

We use here (very likely in a non essential way !) the analyticity of $\Gamma = \partial\Omega$. (Γ is assumed to be real analytic). Then ω, ρ are real analytic up to the boundary, so that $\rho = 0$ on Γ and since $\Delta\rho = 0$ in Ω, it follows that $\rho = 0$ in Ω.

Then (68) reduces to

$$\begin{cases} -\mu\Delta\omega = \lambda\,\omega & \text{in} \quad \Omega\,, \\ \omega = 0 & \text{on} \quad \Gamma\,, \end{cases} \tag{70}$$

and

$$\text{div}\omega = 0 \quad \text{in } \Omega\,, \tag{71}$$

and we want to show that it (generically) implies that $\omega = 0$.

4.6. Step 6.

We have assumed that the spectrum of $-\Delta$ for Dirichlet in Ω is simple. Let θ be the normalized eigenfunction of

$$-\mu\Delta\theta = \lambda\theta \quad \text{in } \Omega, \quad \theta = 0 \quad \text{on } \Gamma.$$

Then, since the spectrum is simple, there are real numbers α_i, $i = 1, 2, 3$ such that $\omega_i = \alpha_i\theta$, $i = 1, 2, 3$ so that

$$\text{div}\ \omega = \sum \alpha_i \frac{\partial\theta}{\partial x_i}\,.$$

Therefore one has necessarily

$$\sum \alpha_i \frac{\partial\theta}{\partial x_i} = 0 \quad \text{in } \Omega\,,$$

and $\theta = 0$ on Γ, which is impossible except if $\alpha_i = 0$, $\forall i$, i.e. $\omega = 0$, and the proof is completed.

Remark 11. The proof in Step 6 does *not* assume the boundary Γ of Ω to be real analytic.

5. A Counter-Example

Analyzing the development of section 4 we see that the key point of the proof of Theorem 2, in addition to the real analyticity of Ω, is the following uniqueness property (see (68)-(69) above):

$$\left.\begin{array}{lll} -\Delta\omega = \lambda\omega - \nabla\pi & \text{in} & \Omega\,, \\ \text{div}\ \omega = 0 & \text{in} & \Omega\,, \\ w = 0 & \text{on} & \partial\Omega\,, \\ \pi = \text{constant} & \text{on} & \partial\Omega\,, \end{array}\right\} \Rightarrow w \equiv 0\,. \tag{72}$$

We have seen that (72) holds when the spectrum of $-\Delta$ in $H_0^1(\Omega)$ is simple.

As we have seen in section 4, since π is harmonic in Ω, (72) is equivalent to the following:

$$\left.\begin{array}{lll} -\Delta w = \lambda w & \text{in} & \Omega\,, \\ \text{div}\ w = 0 & \text{in} & \Omega\,, \\ w = 0 & \text{on} & \partial\Omega\,, \end{array}\right\} \Rightarrow w \equiv 0\,. \tag{73}$$

We are going to show that these uniqueness results do not hold in the case of the ball Ω. This is an old example (see, for instance, H.Lamb [5]). We present here a simple self contained proof.

Let us consider first the two-dimensional analog of (73) (that can be formulated exactly in the same terms) and let us see that (73) is false when Ω is a ball in $\mathbb{R}^2$. Of course, this shows that the two-dimensional analog of Theorem (2) is false if we drop the assumption of the simplicity of the spectrum of the Dirichlet Laplacian.

Let $\varphi = \varphi(r)$ be a radially symmetric eigenfunction of the problem

$$\left\{\begin{array}{lll} \Delta^2\varphi = -\lambda\Delta\varphi & \text{in} & \Omega\,, \\ \varphi = \dfrac{\partial\varphi}{\partial n} = 0 & \text{on} & \partial\Omega\,. \end{array}\right. \tag{74}$$

Then, the vector field

$$w = \left(\frac{\partial\varphi}{\partial x_2},\ -\frac{\partial\varphi}{\partial x_1}\right) \tag{75}$$

satisfies

$$\left\{\begin{array}{lll} \text{div}\ w = 0 & \text{in} & \Omega\,, \\ w = 0 & \text{on} & \partial\Omega\,. \end{array}\right. \tag{76}$$

Let us see that

$$-\Delta w = \lambda w \quad \text{in}\ \ \Omega\,. \tag{77}$$

One verifies that

$$\frac{\partial}{\partial x_2}\left(\Delta w_1 + \lambda w_1\right) - \frac{\partial}{\partial x_1}\left(\Delta w_2 + \lambda w_2\right) = \Delta^2\varphi + \lambda\Delta\varphi = 0 \quad \text{in}\ \ \Omega\,.$$

Therefore, there exists a scalar function π such that

$$-\Delta w = \lambda w + \nabla\pi \quad \text{in}\ \ \Omega\,. \tag{78}$$

It is sufficient to show that π is constant in Ω. This is true since, in view of (76), (78) and taking into account that φ is radially symmetric, we have

$$\left\{ \begin{array}{r} \Delta \pi = 0 \quad \text{in} \quad \Omega\ , \\ \dfrac{\partial \pi}{\partial n} = -\Delta w \cdot n = \dfrac{\partial \Delta \varphi}{\partial \tau} = 0 \quad \text{on} \quad \partial\Omega\ . \end{array} \right.$$

where $\dfrac{\partial}{\partial \tau}$ denotes the tangential derivative on $\partial\Omega$. Therefore $\pi =$ constant on Ω and the vector field (75) provides a counter-example to (72) and/or (73) when Ω is a two-dimensional ball.

Let us now consider the three-dimensional problem when Ω is a ball. Let ψ be a (not identically constant) radially symmetric eigenfuncion of $-\Delta$ in $H^1(\Omega)$ with Neumann boundary condition:

$$\left\{ \begin{array}{r} -\Delta \psi = \lambda \psi \quad \text{in} \quad \Omega\ , \\ \dfrac{\partial \psi}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\ . \end{array} \right.$$

Then, clearly $\nabla \psi = 0$ on $\partial\Omega$. Therefore $\partial\psi/\partial x_j$ is an eigenfunction of $-\Delta$ in $H_0^1(\Omega)$ with eigenvalue λ for $j = 1, 2, 3$. Let us now define $w = (w_1, w_2, w_3)$ by

$$\begin{aligned} w_1 &= \partial\psi/\partial x_2 + \partial\psi/\partial x_3\ , \\ w_2 &= -\partial\psi/\partial x_1 + \partial\psi/\partial x_3\ , \\ w_3 &= -\partial\psi/\partial x_2 - \partial\psi/\partial x_1 . \end{aligned}$$

It is clear that

$$\left\{ \begin{array}{r} -\Delta w = \lambda w \quad \text{in} \quad \Omega\ , \\ \operatorname{div} w = 0 \quad \text{in} \quad \Omega\ , \\ w = 0 \quad \text{on} \quad \partial\Omega\ . \end{array} \right.$$

This shows that the uniqueness property (73) fails when Ω is a three-dimensional ball.

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