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A mathematical proof of a formula of Aspinwall and Morrison

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Abstract. We give a rigorous proof of Aspinwall–Morrison formula, which expresses the cubic derivatives of the Gromov–Witten as a series depending only on the number of rational curves in each homology class, for a Calabi–Yau threefold with only rigid immersed rational curves.

Key words: Calabi–Yau varieties, rational curves, Gromov–Witten potential

1. Introduction

Let X be a Calabi–Yau variety of dimension three, and let $\phi: \mathbb{P}^1 \to X$ be a holomorphic immersion: the normal bundle $N_\phi = \phi^* T_X / \phi_* T_{\mathbb{P}^1}$ splits into the direct sum of two line bundles, $N_\phi = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$, and by the adjunction formula $a + b = -2$. We will assume that $\phi(\mathbb{P}^1)$ is infinitesimally rigid, that is N_ϕ has no holomorphic section, or equivalently $a = b = -1$. In this case, for any holomorphic map $\psi: \mathbb{P}^1 \to \mathbb{P}^1$ of degree k, the deformations of the map $\phi \circ \psi$ consist of maps $\phi \circ \psi'$, where $\psi': \mathbb{P}^1 \to \mathbb{P}^1$ is a deformation of ψ. It follows by compactness of the Chow variety of curves in X of bounded degree, or by [4], that for any $\alpha \in H_2(X, \mathbb{Z})$, there is a neighbourhood V of $\mathbb{P}^1$ in X such that the only rational curves of class α such that $d^0\alpha \leqslant k d^0 A$ are supported on $\phi(\mathbb{P}^1)$, where the degree is counted with respect to any ample line bundle on X, and $A = \phi_*([\mathbb{P}^1])$.

Now consider a small general perturbation J_ϵ of the pseudocomplex structure J of X and let ν be small general $\mathcal{C}^\infty$ section of the bundle $pr_1^* \Omega^{0,1}(\mathbb{P}^1) \otimes pr_2^*(T^{1,0}_{X_\epsilon})$ on $\mathbb{P}^1 \times X$, where $\Omega^{0,1}$ denotes complex $(0,1)$-forms, and $T^{1,0}_{X_\epsilon}$ denotes vector fields of type $(1,0)$ for the pseudocomplex structure J_ϵ. Then it is known (cf. [4], [9], [12]) that the space $W_{kA, J_\epsilon, \nu}$ of solutions to the equation

$$\overline{\partial}_\epsilon \psi = (\mathrm{Id}, \psi)^* \nu \tag{1.1}$$

for $\psi: \mathbb{P}^1 \to X$ such that $\psi_*([\mathbb{P}^1]) = kA$, is smooth, naturally oriented of dimension six and can be compactified with a boundary of dimension $\leqslant 4$. By compactness, for (J_ϵ, ν) close enough to $(J, 0)$, and for V as above the subspace $W^V_{kA, J_\epsilon, \nu}$ of

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$W_{kA,J_\epsilon,\nu}$ consisting of maps ψ whose image is contained in V is a component (non necessarily connected) of $W_{kA,J_\epsilon,\nu}$.

Let x_1, x_2, x_3 be three distinct points of $\mathbb{P}^1$, and consider the evaluation map

$$ev\colon W^V_{kA,J_\epsilon,\nu} \to X^3$$
$$\psi \mapsto (\psi(x_1), (\psi(x_2), (\psi(x_3)). \tag{1.2}$$

Again the image of ev is six dimensional oriented, and can be compactified with a boundary of dimension $\leqslant 4$, so has a homology class in $H_6(X^3)$ (which in fact is in the image of $H_6(V^3) \to H_6(X^3)$, which is generated by $A \times A \times A$). This paper gives a proof of the following

THEOREM 1.1 *This class is equal to* $A \times A \times A \in H_6(X^3)$.

In [10], Manin already proved this statement, admitting the possibility to apply Bott formula to stacks (which may be only a formal point to verify) and using some ideas due to Kontsevich ([5]). It may be nevertheless interesting to have a proof close to Aspinwall and Morrison argument ([1]), and justifying a posteriori their computation.

This theorem is, as in the paper by Aspinwall and Morrison [1], a consequence of a more precise statement, namely that as a space of curves in $\mathbb{P}^1 \times X$, the component $W^V_{kA,J_\epsilon,\nu}$ is homologous to any cycle in $M_k := \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k,1)))$, Poincaré dual to the top Chern class of the bundle with fiber at $\psi\colon \mathbb{P}^1 \to \mathbb{P}^1$, the space $H^1((\phi\circ\psi)^*T_X)$. Here we view M_k as a compactification of the space M_k^0 parametrizing degree k coverings $\psi\colon \mathbb{P}^1 \to \mathbb{P}^1$, and we identify it to a set of curves in $\mathbb{P}^1 \times X$, via ϕ. This statement is quite natural, since this vector bundle, at least on M_k^0, is exactly the excess bundle for the too large family of holomorphic curves M_k. However, the proof shows that one has to be careful with the singular curves in $\mathbb{P}^1 \times X$ parametrized by $M_k - M_k^0$, and especially with non reduced curves: for a special choice of ν (and for $J_\epsilon = J$) we will exhibit a section s of this bundle on $M_k^0 \subset M_k$ such that $W^V_{kA,J,\nu}$ identifies naturally to the zero set of s. However, this section is not even continuous at non reduced curves in M_k. The result is that, nevertheless, the closure of the zero locus of s in M_k has for homology class the Poincaré dual of the top Chern class of this bundle.

We mention at this point an essential difference between Manin's computation [10] and ours: Manin works with the moduli space of stable maps to get a complicated, but more satisfactory from the point of view of moduli spaces, compactification of the space of smooth ramified covers of $\mathbb{P}^1$. As in [1], we work with the naive compactification $M_k \cong \mathbb{P}^{2k+1}$, on which the Chern classes computations are quite easy, but which is not a good moduli space at the boundary.

The Theorem 1.1 is one version of Aspinwall–Morrison formula [1], which we now explain: let $\omega \in H^2(X, \mathbb{Z})$ such that $Re\,\omega \cong \alpha$ is a sufficiently large kähler class on X. The Gromov–Witten potential is the function on $H^{\text{even}}(X)$ defined by the series (expected to be convergent for large α)

$$\Psi_\omega(\eta) = \sum_{\substack{A\in H_2(X,\mathbb{Z})\\ k\geqslant 3}} \frac{1}{k!}\,\mathrm{e}^{-\int_A \omega}\Phi_A(\eta,\eta,\eta|\underbrace{\eta\dots\eta}_{k-3}) \tag{1.3}$$

([7], [13]) where the mixed Gromov–Witten invariants $\Phi_A(\eta,\eta,\eta|\underbrace{\eta\dots\eta}_{k-3})$ ([12]) are defined as follows: for (J,ν) generic, J a pseudocomplex structure, ν a section of $pr_1^*\Omega^{0,1}(\mathbb{P}^1)\otimes pr_2^*(T^{1,0}_{X_J})$ on $\mathbb{P}^1\times X$ and $A\in H_2(X,\mathbb{Z})$, consider the evaluation map

$$\begin{aligned} &ev_{k-3}\colon W_{A,J,\nu}\times {\mathbb{P}^1}^{k-3}\to X^k\\ &(\psi,z_1,\dots,z_{k-3})\mapsto(\psi(x_1),\psi(x_2),\psi(x_3),\psi(z_1),\dots,\psi(z_{k-3})), \end{aligned} \tag{1.4}$$

the points x_i being fixed on $\mathbb{P}^1$. Then $\operatorname{Im} ev_{k-3}$ is as before oriented, smooth of real dimension $6+2(k-3)$, and can be compactified with a boundary of codimension two, so defines a homology class in X^k on which one can integrate $\eta^{\otimes k}$, which gives the invariant. For $A=0, k>3$, one has $\Phi_A(\eta,\eta,\eta|\underbrace{\eta\dots\eta}_{k-3})=0$, essentially because the map ev_{k-3} has positive dimensional fibers, at least when $\nu=0$, and for $A=0$, $k=3$ one has $\Phi_A(\eta,\eta,\eta)=\int_X\eta^3$ because $W_{A,J,0}$ identifies to the constant maps, and $ev(W_{A,J,0})$ is then simply the diagonal in X^3.

Now assume that all generically immersed rational curves in X are immersed and infinitesimally rigid, and let $n(A)$ be the number of immersed rational curves of class $A\neq 0$. Then all rational curves on X are multiple covers of immersed infinitesimally rigid curves, and we can apply the Theorem 1.1, which says that for $l\geq 1$, $A\neq 0$, $W_{lA,J,\nu}$ is made of $n(A)$ components whose contribution to $\Phi_{lA}(\eta,\eta,\eta\mid\underbrace{\eta\dots\eta}_{k-3})$ is equal to

$$l^{k-3}\left(\int_A\eta\right)^{k-3}\int_{A\times A\times A}\eta^{\otimes 3} \tag{1.5}$$

It follows that

$$\begin{aligned} \Psi_\omega(\eta) &= \frac{1}{6}\int_X\eta^3 + \sum_{\substack{A\in H_2(X,\mathbb{Z})-\{0\}\\ k\geqslant 3, l\geqslant 1}} n(A)\frac{1}{k!}\mathrm{e}^{-\int_{lA}\omega}l^{k-3}\left(\int_A\eta\right)^k\\ &= \frac{1}{6}\int_X\eta^3 + \sum_{\substack{A\in H_2(X,\mathbb{Z})-\{0\}\\ l\geqslant 1}} \frac{1}{l^3}n(A)\,\mathrm{e}^{\int_{lA}-\omega+\eta} \end{aligned} \tag{1.6}$$

modulo a quadratic term in η. So if we consider the cubic derivatives $\partial^3\Psi_\omega/\partial t_i \partial t_j\partial t_k(\eta)$ w.r.t. linear coordinates on $H^{\mathrm{even}}(X)$ corresponding to a basis η_i of $H^{\mathrm{even}}(X)$, we find

$$\begin{aligned}
&\partial^3\Psi_\omega/\partial t_i\partial t_j\partial t_k(\eta)\\
&= \int_X \eta_i\wedge\eta_j\wedge\eta_k + \sum_{\substack{A\in H_2(X,\mathbb{Z})-\{0\}\\ l\geqslant 1}} \frac{n(A)}{l^3}\,\mathrm{e}^{\int_{lA}-\omega+\eta}\int_{lA}\eta_i\int_{lA}\eta_j\int_{lA}\eta_k\\
&= \int_X \eta_i\wedge\eta_j\wedge\eta_k + \sum_{\substack{A\in H_2(X,\mathbb{Z})-\{0\}\\ l\geqslant 1}} n(A)\mathrm{e}^{\int_{lA}-\omega+\eta}\int_{A}\eta_i\int_{A}\eta_j\int_{A}\eta_k\\
&= \int_X \eta_i\wedge\eta_j\wedge\eta_k + \sum_{A\neq 0} n(A)\mathrm{e}^{\int_A-\omega+\eta}/(1-\mathrm{e}^{\int_A-\omega+\eta})\\
&\int_A\eta_i\int_A\eta_j\int_A\eta_k
\end{aligned}\tag{1.7}$$

which is Aspinwall–Morrison formula for the Yukawa couplings of the 'A-model' of X, at the point $\omega-\eta$ (see [1], [15], [3],[12]).

2. Choice of the Parameter ν

We will assume in this section that $\phi\colon \mathbb{P}^1\to X$ is an embedding, and consider the general case in Section 4. We will use the following result ([8])

THEOREM 2.1. *Let $\phi:\mathbb{P}^1\hookrightarrow X$ such that $N_\phi\cong \mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$; then a neighbourhood V of $\mathbb{P}^1$ in X is holomorphically isomorphic to a neighbourhood of the zero section of the total space W of the bundle $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$ on $\mathbb{P}^1$.*

Since the Theorem 1.1 is a local statement, we may assume from now on that $X=W$. Now let $\pi\colon W\to\mathbb{P}^1$ be the natural projection, with fiber $\pi^{-1}(x)=N_{\phi(x)}$; we get an inclusion

$$\pi^*(\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1))\subset T_W \tag{2.8}$$

as the vertical tangent space of π (T_W is the bundle of $(1,0)$-vector fields on W). We choose now two $\mathcal{C}^\infty$ sections σ_1,σ_2 of $pr_1^*\Omega^{0,1}(\mathbb{P}^1)\otimes pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1))$ on $\mathbb{P}^1\times\mathbb{P}^1$, and we define $\nu=(\nu_1,\nu_2)$ where $\nu_i=(\mathrm{Id}\times\pi)^*\sigma_i$. ν is then a $\mathcal{C}^\infty$ section of $pr_1^*\Omega^{0,1}(\mathbb{P}^1)\otimes pr_2^*T_W$ via the inclusion (2.8).

We study now the solutions to the equation

$$\overline{\partial}\psi=(\mathrm{Id}\times\psi)^*\nu \tag{2.9}$$

for $\psi:\mathbb{P}^1\to W$ a $\mathcal{C}^\infty$ map such that $\psi_*([\mathbb{P}^1])=kA$, where A is the homology class of the zero section. Since by construction $\pi_*(\nu)$ vanishes as a section of

$pr_1^*\Omega^{0,1}(\mathbb{P}^1)\otimes pr_2^*\circ\pi^*T_{\mathbb{P}^1}$ on $\mathbb{P}^1\times W$, we get $\overline{\partial}(\pi\circ\psi)=0$, so $\pi\circ\psi$ is holomorphic, of degree k. Let $\psi'=\pi\circ\psi$; then ψ is described by a couple (ψ_1,ψ_2), where ψ_i are C^∞ sections of the bundle $\psi'^*\mathcal{O}_{\mathbb{P}^1}(-1)$. The equation (2.9) rewrites then simply as

$$\overline{\partial}\psi_i=(\mathrm{Id}\times\psi')^*\sigma_i,\ i=1,2 \tag{2.10}$$

Since $H^0(\psi'^*\mathcal{O}_{\mathbb{P}^1}(-1))=\{0\}$, ψ_i are determined by ψ' and exist if and only if $(\mathrm{Id}\times\psi')^*\sigma_i$, which are $(0,1)$-forms with values in $\psi'^*\mathcal{O}_{\mathbb{P}^1}(-1)$, vanish in $H^1(\psi'^*\mathcal{O}_{\mathbb{P}^1}(-1))$.

As in [1], let us introduce $M_k=\mathbb{P}(H^0(\mathcal{O}_Q(k,1)))$, where $Q=\mathbb{P}^1\times\mathbb{P}^1$, $\mathcal{O}_Q(k,1)=pr_1^*\mathcal{O}_{\mathbb{P}^1}(k)\otimes pr_2^*\mathcal{O}_{\mathbb{P}^1}(1)$. M_k is a compactification of the family of holomorphic maps of degree k from $\mathbb{P}^1$ to $\mathbb{P}^1$: indeed the general member of M_k is a smooth curve in Q, isomorphic to $\mathbb{P}^1$ by the first projection, and of degree k over $\mathbb{P}^1$ by the second projection.

In $M_k\times Q$ we consider as in [1] the universal divisor D defined as the zero set of the natural section of $p_M^*\mathcal{O}_{M_k}(1)\otimes p_Q^*\mathcal{O}_Q(k,1)$ corresponding to the identification $H^0(\mathcal{O}_{M_k}(1))^*\cong H^0(\mathcal{O}_Q(k,1))$, where p_M and p_Q are the projections to M_k and Q respectively. Let $pr_2:Q\to\mathbb{P}^1$ be the second projection, and let $E:=R^1p_{M*}(pr_2\circ p_Q)^*(\mathcal{O}_{\mathbb{P}^1}(-1))_{|D}$; then since $R^1p_{M*}(pr_2\circ p_Q)^*(\mathcal{O}_{\mathbb{P}^1}(-1))=\{0\}$ and $R^2p_{M*}(pr_2\circ p_Q)^*(\mathcal{O}_{\mathbb{P}^1}(-1))=\{0\}$ we conclude by the long exact sequence associated to

$$\begin{aligned}0\to(pr_2\circ p_Q)^*(\mathcal{O}_{\mathbb{P}^1}(-1))\otimes\mathcal{I}_D&\to(pr_2\circ p_Q)^*(\mathcal{O}_{\mathbb{P}^1}(-1))\\&\to(pr_2\circ p_Q)^*(\mathcal{O}_{\mathbb{P}^1}(-1))_{|D}\to 0\end{aligned}\tag{2.11}$$

that $E\cong R^2p_{M*}((pr_2\circ p_Q)^*(\mathcal{O}_{\mathbb{P}^1}(-1))\otimes\mathcal{I}_D)$. Since $\mathcal{I}_D\cong p_M^*\mathcal{O}_{M_k}(-1)\otimes p_Q^*\mathcal{O}_Q(-k,-1)$, we get

LEMMA 2.2. ([1]) $E\cong\mathcal{O}_{M_k}(-1)\otimes H^2(Q,\mathcal{O}_Q(-k,-2))$. *In particular E is a vector bundle on M_k of rank $k-1$.*

Let M_k^0 be the open set parametrizing smooth curves in Q, that is maps ψ': $\mathbb{P}^1\to\mathbb{P}^1$ of degree k; in M_k^0, we have two sections of E, denoted by $s_{\sigma_1},s_{\sigma_2}$, defined by $s_{\sigma_i}(\psi')=$ class of $(\mathrm{Id}\times\psi')^*(\sigma_i)$ in $H^1(\psi'^*(\mathcal{O}_{\mathbb{P}^1}(-1))\cong E_{\psi'}$. We have shown that the solutions of (2.9) in M_k^0 are in bijection with the zeroes of the section $(s_{\sigma_1},s_{\sigma_2})$ of $E\times E$; since $\dim_{\mathbb{C}}M_k=2k+1$, $\mathrm{rank}_{\mathbb{C}}E=k-1$, the zero set of $(s_{\sigma_1},s_{\sigma_2})$ is expected to be of real dimension 6, as we want.

3. Study of the Section s_σ

The behaviour of the section s_σ of E on M_k^0, for σ a C^∞ section of $pr_1^*\Omega^{0,1}(\mathbb{P}^1)\otimes pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)$ on $Q=\mathbb{P}^1\times\mathbb{P}^1$ is easily described by the following

LEMMA 3.1. *s_σ is of class $\mathcal{C}^\infty$ on M_k^0.*

Proof. By definition, for $(C) \in M_k^0$, $s_\sigma((C))$ is represented by a $(0,1)$-form on C, which varies in a $\mathcal{C}^\infty$ way with (C). Now, we have the isomorphism $E_{(C)} \cong H^1(C, pr_2^*(\mathcal{O}_{\mathbb{P}^1}(-1)_{|C}))$, where $C \subset Q$ corresponds to $(C) \in M_k^0$, and we have shown that the rank of this space is independant of (C). This implies that s_σ is of class $\mathcal{C}^\infty$, because we have then the isomorphism $E^* \cong R^0 p_{M*}(K_{D/M_k} \otimes (pr_2 \circ p_Q)^* \mathcal{O}_{\mathbb{P}^1}(-1)) \cong H^0(Q, \mathcal{O}_Q(k-2,0)) \otimes \mathcal{O}_{M_k}(1)$, and it is immediate to see that for a holomorphic section η of the right hand side, the function $\langle s_\sigma, \eta\rangle$ is given by integrals over the curves C of forms varying in a $\mathcal{C}^\infty$ way with (C).

It is unfortunately not true that s_σ extends continuously over M_k. The rest of this section is devoted to the study of the singularities of s_σ and to the proof of the following

THEOREM 3.2. *Let σ_1, σ_2 be general $\mathcal{C}^\infty$ section of $pr_1^*\Omega^{0,1}(\mathbb{P}^1) \otimes pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)$ on $Q = \mathbb{P}^1 \times \mathbb{P}^1$; let $\overline{V}_{\sigma_1,\sigma_2}$ be the closure in M_k of the zero locus $V(s_{\sigma_1}, s_{\sigma_2}) \subset M_k^0$ of the section $(s_{\sigma_1}, s_{\sigma_2})$ of $E \times E$ on M_k^0; then $V(s_{\sigma_1}, s_{\sigma_2})$ is smooth of dimension 6, and $\overline{V}_{\sigma_1,\sigma_2} - V(s_{\sigma_1}, s_{\sigma_2})$ can be stratified by subsets contained in locally closed subvarieties of dimension $\leqslant 4$ of M_k, so $\overline{V}_{\sigma_1,\sigma_2}$ has a homology class in $H_6(M_k)$, which is Poincaré dual to the top Chern class of $E \times E$.*

The proof of this theorem will be based on the following Proposition 3.3, for which we introduce a few notations: for any $(C) \in M_k$, one can write $C = C' \cup V_C$, where $C' \subset Q$ is a smooth member of $|\mathcal{O}_Q(l,1)|$, $l \leqslant k$ and the vertical part $V_C = pr_1^{-1}(D_C)$ for some divisor D_C of degree $k-l$ on $\mathbb{P}^1$. We will denote by D'_C the intersection $C' \cap V_C$, and by $\psi_{C'} : C' \to \mathbb{P}^1$ the second projection, which is a morphism of degree l; writing $D'_C = \Sigma_i n_i p_i$ for distinct points p_i of C' we will denote by B_C the divisor $\Sigma_i (n_i - 1) p_i$ that we will view as a divisor either on C' or on $\mathbb{P}^1 \overset{pr_1^{-1}}{\cong} C'$. There is a natural structure of scheme on $Z := \bigcup_{C \in M_k} D'_C \subset M_k \times \mathbb{P}^1$ defined as follows: choosing homogeneous coordinates Y_0, Y_1 on $\mathbb{P}^1$, a section of $\mathcal{O}_Q(k,1)$ can be written as $pr_1^* P_0 \, pr_2^* Y_0 + pr_1^* P_1 pr_2^* Y_1$, where $P_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(k))$ depend algebraically on $(C) \in M_k$; Z is then defined by $P_0 = P_1 = 0$. This induces immediately a scheme structure on $B := \bigcup_{C \in M_k} B_C \subset M_k \times \mathbb{P}^1$, the ideal being generated by the partial derivatives of P_0, P_1 w. r. t. homogeneous coordinates on $\mathbb{P}^1$.

We have already used the isomorphism

$$\begin{aligned} E^* &\cong R^0 p_{M*}(K_{D/M_k} \otimes (pr_2 \circ p_Q)^* \mathcal{O}_{\mathbb{P}^1}(-1)) \\ &\cong H^0(Q, \mathcal{O}_Q(k-2,0)) \otimes \mathcal{O}_{M_k}(1) \end{aligned} \tag{3.12}$$

which depends essentially on the choice of an isomorphism $K_Q \cong \mathcal{O}_Q(-2,-2)$. Since $H^0(Q, \mathcal{O}_Q(k-2,0)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-2))$ we get

$$E^* \cong R^0 \pi_{M*}(\pi_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(k-2)) \otimes \mathcal{O}_{M_k}(1), \tag{3.13}$$

where π_M, $\pi_{\mathbb{P}^1}$ are the projections of $M_k \times \mathbb{P}^1$ onto its factors. We have then the following

PROPOSITION 3.3. *Let ϕ be a holomorphic section of $R^0\pi_{M*}(\pi_{\mathbb{P}^1}^*\mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_B) \otimes \mathcal{O}_{M_k}(1)$ on an open set U of M_k. Then the function $< s_\sigma, \phi >$ defined on $U \cap M_k^0$ extends continuously on U.*

Proof. Let $(C_0) \in U$ and F_{C_0} an equation for $C_0 \subset Q$. One can write $F_{C_0} = pr_1^* P_{C_0} \cdot F'_{C_0}$ where P_{C_0} is an equation for D_{C_0} and $F'_{C_0} \in |\mathcal{O}_Q(k-l_0, 1)|$ defines a smooth curve in Q, $l_0 = d^0 D_{C_0}$. Using a partition of unity on Q, one may assume that σ is compactly supported in a product of disks $D_1 \times D_2$ with affine coordinates z_1, z_2 such that $D_i = \{z_i, |z_i| < 1\}$ and $(0,0) \in C_0 \cap D_1 \times D_2$, and the inhomogeneous polynomials corresponding to P_{C_0}, F'_{C_0} satisfy

$$
\begin{aligned}
&p_{C_0} = z_1^l q_{C_0}(z_1), \qquad q_{C_0}(z_1) \neq 0 \text{ on } D_1 \\
&f'_{C_0}(z_1) = \tilde{h}_{C_0}(z_1) + z_2 \tilde{g}_{C_0}(z_1), \qquad (3.14)
\end{aligned}
$$

where one of the polynomials $\tilde{f}_{C_0}$, $\tilde{g}_{C_0}$ does not vanish on D_1, since $f'_{C_0} = 0$ has no vertical component. We assume $\tilde{g}_{C_0} \neq 0$ on D_1, the other case working similarly. By shrinking D_1 we may even assume $|q_{C_0}\tilde{g}_{C_0}| \geqslant c > 0$ on D_1. Let $h_{C_0} = q_{C_0}\tilde{h}_{C_0}$, $g_{C_0} = q_{C_0}\tilde{g}_{C_0}$; a small generic deformation f_C of f_{C_0} can be written as

$$
f_C = p_C(z_1)(h_C(z_1) + z_2 g_C(z_1)) + r_C(z_1), \qquad (3.15)
$$

where we can normalize f_C by imposing the condition $g_C(0) = 1$, and $d^0 p_C = l$, $p_C(z_1) = z_1^l + \Sigma_{i<l}\alpha_i z_1^i$, $d^0 r_C \leqslant l-1$, $d^0 h_C \leqslant k-l$, $d^0 g_C \leqslant k-l$; the polynomials p_C, h_C, g_C, r_C vary holomorphically with (C) in a neighbourhood (that we still call U) of (C_0), and $p_{C_0} = z_1^l$, $r_{C_0} = 0$. The variety $Z \cap U \times D_1$ is described by the equations $p_C(z_1) = r_C(z_1)$ and the variety $B \cap U \times D_1$ is described by the equations $p_C(z_1) = r_C(z_1) = \partial p_C/\partial z_1(z_1) = \partial r_C/\partial z_1(z_1) = 0$. The restriction to $U \times D_1$ of a section ϕ of $\pi_{\mathbb{P}^1}^*\mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_B$ can then be written as

$$
\begin{aligned}
\phi(z_1, (C)) = {} & \phi_C^p(z_1)p_C + \psi_C^p(z_1)\frac{\partial p_C}{\partial z_1} + \phi_C^r(z_1)r_C \\
& + \psi_C^r(z_1)\frac{\partial r_C}{\partial z_1}, \qquad (3.16)
\end{aligned}
$$

where ϕ_C^p, ψ_C^p, ϕ_C^r, ψ_C^r are holomorphic functions of $((C), z_1)$. We can write $\sigma = \psi(z_1, z_2)d\bar{z}_1$, where ψ is a compactly supported function of class $\mathcal{C}^\infty$ in $D_1 \times D_2$. The couplings $\gamma((C)) := \langle s_\sigma, \phi \rangle$ defined on $U \cap M_k^0$ are obtained by taking the residue along C of the $pr_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$-valued meromorphic form $\phi.\eta/f_C$, and integrating over C the cup-product of this form with $\sigma_{|C}$; hence $\gamma((C))$ has the following form

$$\begin{aligned}\gamma((C)) &= \gamma_C^p + \gamma_C^{p'} + \gamma_C^r + \gamma_C^{r'} \\ &= \int_{D_1} \phi_C^p(z_1)\psi\left(z_1, \frac{-r_C - p_C h_C}{p_C g_C}\right) \cdot \frac{1}{g_C(z_1)} \\ &+\psi_C^p(z_1)\psi\left(z_1, \frac{-r_C - p_C h_C}{p_C g_C}\right) \frac{\partial_{z_1} p_C}{p_C g_C} \\ &+\phi_C^r(z_1)\psi\left(z_1, \frac{-r_C - p_C h_C}{p_C g_C}\right) \frac{r_C}{p_C g_C} \\ &+\psi_C^r(z_1)\psi\left(z_1, \frac{-r_C - p_C h_C}{p_C g_C}\right) \frac{\partial_{z_1} r_C}{p_C g_C} \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \end{aligned} \tag{3.17}$$

and it suffices to show that each function γ_C^p, $\gamma_C^{p'}$, γ_C^r, $\gamma_C^{r'}$ extends continuously at $(C_0) \in U$. This is in fact obvious for γ_C^p and γ_C^r since the functions $\phi_C^p(z_1)\psi(z_1, (-r_C - p_C h_C)/(p_C g_C))/g_C(z_1)$ and $\phi_C^r(z_1)\psi(z_1, (-r_C - p_C h_C)/(p_C g_C))r_C/(p_C g_C)$ are bounded by a constant independant of (C) and are continuous along $(C_0) \times D_1^*$. To show that $\gamma_C^{p'}$ extends continuously at (C_0), consider the degree l covering $\tilde{U} \xrightarrow{r} U$, $\tilde{U} \subset U \times D_1^l$ obtained by taking the roots of p_C (which are all in D_1 for (C) close to (C_0)), that is $\tilde{U} = \{((C), \lambda_1, \dots, \lambda_l) | p_C = \Pi_i(z_1 - \lambda_i)\}$. It suffices to show that $r^*(\gamma_C^{p'})$ extends continuously at $((C_0), 0, \dots, 0) \in \tilde{U}$; but

$$\begin{aligned} r^*(\gamma_C^{p'}) &= \int_{D_1} \frac{\psi_C^p(z_1)}{g_C(z_1)} \psi\left(z_1, \frac{-r_C - p_C h_C}{p_C g_C}\right) \\ &\times \left(\sum_{i=1}^{i=l} (1/z_1 - \lambda_i)\right) \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1. \end{aligned} \tag{3.18}$$

For (C) close enough to (C_0), the $\lambda_i's$ are close to zero, so we may assume that $\psi(z_1, z_2) = 0$ outside $|z_1 - \lambda_i| \leqslant 1$. It follows that

$$\begin{aligned} r^*(\gamma_C^{p'}) &= \sum_i \int_{D_1} \frac{\psi_C^p(z_1 + \lambda_i)}{g_C(z_1 + \lambda_i)} \\ &\psi\left(z_1 + \lambda_i, \frac{-r_C - p_C h_C}{(p_C g_C)(z_1 + \lambda_i)}\right) \cdot \frac{1}{z_1} \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1. \end{aligned} \tag{3.19}$$

But the function $(\psi_C^p/g_C)(z_1 + \lambda_i)\psi(z_1 + \lambda_i, (-r_C - p_C h_C)/(p_C g_C)(z_1 + \lambda_i))$ is bounded by a constant on D_1, and the function $1/z_1$ is L^1 on D_1; since for $z_1 \neq 0$, one has

$$\lim_{\substack{(C)\to(C_0)\\ \lambda_i\to 0}} \frac{1}{z_1}\left(\frac{\psi_C^p}{g_C}\right)(z_1+\lambda_i)\psi\left(z_1+\lambda_i, \frac{-r_C-p_Ch_C}{(p_Cg_C)(z_1+\lambda_i)}\right)$$
$$= \frac{\psi_{C_0}^p}{g_{C_0}}(z_1)\psi\left(z_1, -\frac{h_{C_0}}{g_{C_0}}(z_1)\right)\cdot\frac{1}{z_1}, \tag{3.20}$$

one may apply Lebesgue dominated convergence theorem in order to conclude that $\lim_{\substack{(C)\to(C_0)\\ \lambda_i\to 0}} r^*(\gamma_C^{p'})$ exists and is equal to

$$l\int_{D_1}\frac{\psi_{C_0}^p}{g_{C_0}}(z_1)\psi\left(z_1, -\frac{h_{C_0}}{g_{C_0}}(z_1)\right)\cdot\frac{1}{z_1}\,\mathrm{d}z_1\wedge\mathrm{d}\bar{z}_1. \tag{3.21}$$

The proof that $\gamma_C^{r'}$ extends continuously at (C_0) works similarly: in fact, using the result for $\gamma_C^{p'}$ it suffices to prove it for

$$\gamma'^{r'}_C = \int_{D_1}\psi_C^r(z_1)\psi\left(z_1, \frac{-r_C-p_Ch_C}{p_Cg_C}\right)\partial_{z_1}$$
$$\times\frac{r_C+p_C}{p_Cg_C}\,\mathrm{d}z_1\wedge\mathrm{d}\bar{z}_1. \tag{3.22}$$

Now we can write

$$\gamma'^{r'}_C = \int_{D_1}\psi_C^r(z_1)\psi\left(z_1, \frac{-r_C-p_Ch_C}{p_Cg_C}\right)$$
$$\times\,\partial_{z_1}\frac{r_C+p_C}{r_C+p_C(g_C)}\times\frac{r_C+p_C}{p_C}\,\mathrm{d}z_1\wedge\mathrm{d}\bar{z}_1, \tag{3.23}$$

and because ψ is compactly supported in $D_1\times D_2$ the function

$$\psi_C^r(z_1)\psi\left(z_1, \frac{-r_C-p_Ch_C}{p_Cg_C}\right)\frac{r_C+p_C}{p_Cg_C}$$

is bounded in D_1. But $d^0 r_C \leqslant l-1$ and $\lim_{(C)\to(C_0)} r_C = 0$ so the polynomial p_C+r_C is normalized of degree l and has all its roots in D_1 for (C) close to (C_0); as before we can introduce the cover $\tilde{U}\xrightarrow{r}U$ parametrizing an ordering of the roots of r_C+p_C, so $r*(r_C+p_C)=\Pi_{i=1}^{i=l}(z_1-\lambda_i)$, and we get

$$r^*(\gamma'^{r'}_C) = \sum_i\int_{D_1}\psi_C^r(z_1+\lambda_i)\psi\left(z_1+\lambda_i, \frac{-r_C-p_Ch_C}{p_Cg_C}(z_1+\lambda_i)\right)$$
$$\times\frac{r_C+p_C}{p_Cg_C}(z_1+\lambda_i)\times\frac{1}{z_1}\,\mathrm{d}z_1\wedge\mathrm{d}\bar{z}_1, \tag{3.24}$$

and we can apply Lebesgue dominated convergence theorem since the integrand is bounded by $M/|z_1|$ and converges weakly to the L^1 function

$$\psi^r_{C_0}(z_1)\psi\left(z_1, \frac{-h_{C_0}}{g_{C_0}(z_1)}\right)\frac{1}{z_1 g_{C_0}(z_1)} \tag{3.25}$$

outside 0, when (C) tends to (C_0). So the proposition is proved.

In fact, the proof of the proposition gives as well the interpretation of the limit of the functions $\langle s_\sigma, \phi\rangle$: we have the decomposition $C_0 = C_0' \cup pr_1^{-1}(D_{C_0})$, with C_0' smooth and $D'_{C_0} = \Sigma_i n_i p_i$, $n_i \neq 0$, where D'_{C_0} is the inverse image of D_{C_0} under the isomorphism $pr_1 : C_0' \to \mathbb{P}^1$. Let $D''_{C_0} := \Sigma_i p_i$; denote by $\mathcal{C}^\infty_{D'_{C_0}}(pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1))$ the space of $\mathcal{C}^\infty$ sections τ of $pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|C_0'}$ which satisfy the condition: $\tau(p_i) = \tau(p_i) = \cdots = (\partial_z)^{(n_i-1)}\tau(p_i) = 0$ for all p_i and for any coordinate z on C_0' at p_i; similarly, let $\mathcal{C}^\infty_{D''_{C_0}}(pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1))$ the space of $\mathcal{C}^\infty$ sections τ of $pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|C_0'}$ which satisfy the condition: $\tau(p_i) = 0$, $\forall p_i$. We have

LEMMA 3.4. *There are natural isomorphisms*

$$\begin{aligned} &H^1(C_0, pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|C_0}) \cong A^{0,1}_{C_0'}(pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|C_0'})/\bar\partial\mathcal{C}^\infty_{D'_{C_0}}(pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)),\\ &H^1(C_0', pr_2^*\mathcal{O}(-1)_{|C_0'} \otimes \mathcal{I}_{D''_{C_0}})\\ &\quad\cong A^{0,1}_{C_0'}(pr_2^*\mathcal{O}(-1)_{|C_0'})/\bar\partial\mathcal{C}^\infty_{D''_{C_0}}(pr_2^*\mathcal{O}(-1)). \end{aligned} \tag{3.26}$$

Proof. Consider the exact sequence of coherent sheaves on C_0

$$\begin{aligned} 0 \to pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|C_0'} \otimes \mathcal{I}_{D'_{C_0}} &\to pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|C_0}\\ &\to pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|V_{C_0}} \to 0. \end{aligned} \tag{3.27}$$

It is easy to see that the last sheaf has trivial cohomology, and it follows that

$$H^1(C_0, pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|C_0}) \cong H^1(C_0', pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|C_0'} \otimes \mathcal{I}_{D'_{C_0}}) \tag{3.28}$$

so we are reduced to prove the existence of natural isomorphisms

$$\begin{aligned} &H^1(C_0', pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|C_0'} \otimes \mathcal{I}_{D'_{C_0}})\\ &\quad\cong A^{0,1}_{C_0'}(pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|C_0'})/\bar\partial\mathcal{C}^\infty_{D'_{C_0}}(pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)),\\ &H^1(C_0', pr_2^*\mathcal{O}(-1)_{|C_0'} \otimes \mathcal{I}_{D''_{C_0}})\\ &\quad\cong A^{0,1}_{C_0'}(pr_2^*\mathcal{O}(-1)_{|C_0'})/\bar\partial\mathcal{C}^\infty_{D''_{C_0}}(pr_2^*\mathcal{O}(-1)) \end{aligned} \tag{3.29}$$

which is immediate because we have the fine resolution

$$0 \to pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|C'_0} \otimes \mathcal{I}_{D'_{C_0}} \to \mathcal{A}^0_{D'_{C_0}}(pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)) \stackrel{\overline{\partial}}{\to} \mathcal{A}^{0,1}_{C'_0}(pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|C'_0}) \to 0, \tag{3.30}$$

where $\mathcal{A}^0_{D'_{C_0}}$, $\mathcal{A}^{0,1}$ are now the sheaves of $\mathcal{C}^\infty_{D'_{C_0}}$ sections and of $(0,1)$-forms respectively. One gets similarly the second isomorphism.

Now, by the Lemma 3.4, $\sigma_{|C'_0}$ gives a class $s_\sigma(C_0) \in H^1(C'_0, pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|C'_0} \otimes \mathcal{I}_{D''_{C_0}})$, and this group is naturally a quotient of $E_{(C_0)} = H^1(C_0, pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|C_0})$. It is immediate to verify that $H^1(C'_0, pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)_{|C'_0} \otimes \mathcal{I}_{D"_{C_0}})$ identifies to the dual of $H^0(\mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_{B_{C_0}}) \subset H^0(\mathcal{O}_{\mathbb{P}^1}(k-2))$ (modulo the choice of a isomorphism $K_Q \cong \mathcal{O}_Q(-2,-2)$ and of an equation for C_0) and the computation of the limits in the proof of the Proposition 3.3 shows

LEMMA 3.5. *Let ϕ be a local holomorphic section of $R^0\pi_{M*}(\pi_{\mathbb{P}^1}^*\mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_B) \otimes \mathcal{O}_{M_k}(1)$ near (C_0); then*

$$\lim_{(C)\to(C_0)} \langle s_\sigma, \phi \rangle = \langle s_\sigma((C_0)), \phi((C_0)) \rangle. \tag{3.31}$$

Now we can show the following Proposition 3.6, which shows the first part of the Theorem 3.2; for each sequence $d. = (d_1, \dots, d_k)$ of integers, with $\Sigma_i \, \mathrm{i} d_i \leqslant k$, we denote by $M_k^{d.}$ the smooth locally closed subvariety of M_k consisting of curves $C = C' \cup V_C$, such that C' is a smooth member of $|\mathcal{O}_Q(k - \Sigma_i \, \mathrm{i}d_i, 1)$ and $V_C = pr_1^{-1}(D_C)$ where D_C has d_i points of multiplicity i for each i. The $M_k^{d.}$'s form a stratification of M_k and $M_k^0 = M_k^{(0,\dots,0)}$. On each $M_k^{d.}$, σ gives a section of the bundle $E^{d.}$ with fiber at C the space $H^1(C', pr_2^*(\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{I}_{D''_C})$, that we will denote by $s_\sigma^{d.}$. As in Lemma 3.1, it is immediate to prove that $s_\sigma^{d.}$ is of class $\mathcal{C}^\infty$ on $M_k^{d.}$. We have

PROPOSITION 3.6. *Let σ_1, σ_2 be two $\mathcal{C}^\infty$ sections of $pr_1^*\Omega^{0,1}(\mathbb{P}^1) \otimes pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)$ on Q. Then $\overline{V}_{\sigma_1,\sigma_2}$ is contained in $\sqcup_{d.} V(s^{d.}_{\sigma_1}, s^{d.}_{\sigma_2})$; if σ_i are general, for each $d.$, $V(s^{d.}_{\sigma_1}, s^{d.}_{\sigma_2})$ is smooth of real dimension $6 - 2\Sigma_i d_i$.*

Proof. Let $(C) \in M_k^{d.}$, and let $D_C = \Sigma_i n_i p_i$, $B_C = \Sigma_i (n_i - 1)p_i$. Consider $H^0(\mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_{B_C}) \otimes \mathcal{O}_{M_k}(1)_{(C)} \subset E^*_{(C)}$. In a neighbourhood U of (C), we can find a holomorphic subbundle F of E^* whose sheaf of sections is contained in $R^0\pi_{M*}(\pi_{\mathbb{P}^1}^*\mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_B) \otimes \mathcal{O}_{M_k}(1)$ and such that $F_{(C)} = H^0(\mathcal{O}_{\mathbb{P}^1}(k - 2) \otimes \mathcal{I}_{B_C}) \otimes \mathcal{O}_{M_k}(1)_{(C)}$. Let $E/(F^\perp) \cong F^*$ be the corresponding quotient; the Proposition 3.3 shows that the projection $p_F(s_\sigma)$ of s_σ in F^* extends continuously. Furthermore, by definition of F and by the Lemma 3.5, we have $F^*_{|M_k^{d.} \cap U} = E_k^{d.}$

and we have the equality in $U \cap M_k^{d.}$

$$p_F(s_\sigma)_{|M_k^{d.}} = s_\sigma^{d.} \tag{3.32}$$

Now we have on $U \cap M_k^0$, $V(s_{\sigma_1}, s_{\sigma_2}) \subset V(p_F(s_{\sigma_1}), p_F(s_{\sigma_2}))$ for σ_1, σ_2 as above and by continuity of $p_F(s_{\sigma_i})$, we get

$$\overline{V}_{\sigma_1,\sigma_2} \cap U \subset V(p_F(s_{\sigma_1}), p_F(s_{\sigma_2})) \tag{3.33}$$

Finally, the equality (3.32) gives

$$\overline{V}_{\sigma_1,\sigma_2} \cap U \cap M_k^{d.} \subset V(s_{\sigma_1}^{d.}, s_{\sigma_2}^{d.}) \cap U \tag{3.34}$$

which shows the first part of the proposition.

Now note that the real dimension of $M_k^{d.}$ is equal to $2(2(k - \Sigma_i\, id_i + 1) - 1 + \Sigma_i\, d_i)$, and the rank over R of $E^{d.} \times E^{d.}$ is equal to $4(k - 1 - \Sigma_i(i-1)d_i)$. Since $s_{\sigma_i}^{d.}$ are of class $\mathcal{C}^\infty$ over $M_k^{d.}$, the fact that $V(s_{\sigma_1}^{d.}, s_{\sigma_2}^{d.})$ is smooth of real dimension $6 - 2(\Sigma_i\, d_i)$ for general σ_1, σ_2 follows from the following

LEMMA 3.7. *There exists a finite number of $\mathcal{C}^\infty$ sections σ_i of $pr_1^*\Omega^{0,1}(\mathbb{P}^1) \otimes pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)$ on Q such that the corresponding sections $s_{\sigma_i}^{d.}$ generate $E^{d.}$ on $M_k^{d.}$ for any sequence d..*

Proof. Since M_k is compact, it suffices to check it locally on M_k. Now let $(C) \in M_k$; for σ supported away from Sing C, one shows exactly as in 3.1 that s_σ extends as $\mathcal{C}^\infty$ section of E at (C). Next, using Lemma 3.4, one checks easily that the values at (C) of such sections s_σ generate the fiber $E_{(C)}$. So they generate E in a neighbourhood U of (C) and its quotients $E^{d.}$ in $U \cap M_k^{d.}$.

It follows from this proposition that for general (σ_1, σ_2), $\overline{V}_{\sigma_1,\sigma_2}$ has a homology class $[\overline{V}_{\sigma_1,\sigma_2}] \in H_6(M_k, \mathbb{Z})$, which is defined using the natural orientation of V_{σ_1,σ_2} coming from the complex structure on M_k and $E \times E$. Now we have

PROPOSITION 3.8. *$[\overline{V}_{\sigma_1,\sigma_2}]$ is Poincaré dual to the top Chern class of $E \times E$.*

Proof. We show first the existence of a continuous section (s_1', s_2') of $E \times E$ with zero locus equal to $\sqcup_{d.} V((s_{\sigma_1}^{d.}, s_{\sigma_2}^{d.}))$: consider the coherent subsheaf $(E^*)' = R^0\pi_{M*}(\pi_{\mathbb{P}^1}^*\mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_B) \otimes \mathcal{O}_{M_k}(1) \subset E^*$; let F be a holomorphic vector bundle on M_k such that there exists a surjective morphism $\phi'\colon F \to (E^*)'$. We denote by ϕ the composition of ϕ' with the inclusion $(E^*)' \subset E^*$. Putting hermitian metrics on F and E^*, we construct a $\mathcal{C}^\infty$ complex linear endomorphism $\Phi = \phi \circ {}^t\phi\colon E^* \to E^*$, which has the property: $\forall (C) \in M_k$, $\operatorname{Im} \Phi_{(C)} = \operatorname{Im} \phi_{(C)} = H^0(\mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_{B_C}) \otimes \mathcal{O}_{M_k}(1)_{(C)}$. Also, by construction, for any $\mathcal{C}^\infty$ section τ of E^*, $\Phi(P)$ can be written locally as $\Sigma_j\, f_j\tau_j$ where f_j are $\mathcal{C}^\infty$ complex functions and τ_j are sections of $(E^*)'$. It follows from the Proposition 3.3 that for any such τ, the function $\langle s_\sigma, \tau\rangle$ is continuous on M_k, which means that $s' = \Phi^*(s_\sigma)$ is a continuous section of E. Furthermore, for $(C) \in M_k^{d.}$, s' vanishes at (C) if and only if $s_\sigma^{d.}$ vanishes at

(C), by Lemma 3.5. Applying this construction to the couple (σ_1, σ_2) we get a continuous section (s'_1, s'_2) of $E \times E$ which vanishes exactly on $\sqcup_{d.} V((s^{d.}_{\sigma_1}, s^{d.}_{\sigma_2})$.

Notice that (s'_1, s'_2) is smooth when $(s_{\sigma_1}, s_{\sigma_2})$ is, so (s'_1, s'_2) is smooth on M^0_k; furthermore, since the map Φ^* is $\mathbb{C}$-linear the orientation of $V(s_{\sigma_1}, s_{\sigma_2})$ corresponding to the section (s'_1, s'_2) coincides with the one given by the section $(s_{\sigma_1}, s_{\sigma_2})$.

Now, using approximation by smooth sections, we can construct a $\mathcal{C}^\infty$ section (s''_1, s''_2) of $E \times E$, which is equal to (s'_1, s'_2) outside an arbitrarily small neighbourhood of $M_k - M^0_k$, and such that the zero locus $V(s''_1, s''_2)$ is contained in the union of $V(s_{\sigma_1}, s_{\sigma_2})$ and of an arbitrarily small neighbourhood of $\sqcup_{d.\neq(0\ldots,0)} V((s^{d.}_{\sigma_1}, s^{d.}_{\sigma_2}))$. Using the fact that $\dim V((s^{d.}_{\sigma_1}, s^{d.}_{\sigma_2})) \leqslant 4$ for $d. \neq (0, \ldots, 0)$, by Proposition 3.6, any homology class of dimension $2 \dim M_k - 6$ can be represented by a subvariety W of M_k which does not meet a small neighbourhood of $\sqcup_{d.\neq(0\ldots,0)} V((s^{d.}_{\sigma_1}, s^{d.}_{\sigma_2}))$. So W may be choosen to meet $V(s_{\sigma_1}, s_{\sigma_2})$ transversally and only in the open set where $(s_{\sigma_1}, s_{\sigma_2})$ and (s''_1, s''_2) coincide, and then the intersection number $W \cdot \overline{V}_{\sigma_1,\sigma_2} = W \cdot V(s''_1, s''_2)$ is simply the top Chern class of $E \times E$ evaluated on W, which proves the Proposition 3.8, hence also the Theorem 3.2.

4. Proof of the Theorem 1.1

The homology class that we want to compute is defined as follows: let (J_ϵ, ν) be a small general deformation of $(J, 0)$, where J is the original complex structure; there is a component $W^V_{kA,J_\epsilon,\nu}$ of $W_{kA,J_\epsilon,\nu}$ made of curves contained in a given small neighbourhood V of $\mathbb{P}^1 \subset X$ (cf. Introduction); one can construct a compactification $\overline{W}^V_{kA,J_\epsilon,\nu}$ of $W^V_{kA,J_\epsilon,\nu}$, such that the points of the boundary parametrize curves in $\mathbb{P}^1 \times X$, which are limits of graphs of functions $\psi \in W^V_{kA,J_\epsilon,\nu}$. One has then a family of curves

$$\begin{array}{ccc} D & \xrightarrow{(p_2,p_3)} & \mathbb{P}^1 \times V \subset \mathbb{P}^1 \times X \\ {\scriptstyle p_1}\downarrow & & \\ \overline{W}^V_{kA,J_\epsilon,\nu}, & & \end{array} \tag{4.35}$$

which induces the family of threefolds

$$\begin{array}{ccc} D \times_{p_1} D \times_{p_1} D & \xrightarrow{(p_2^3,p_3^3)} & \mathbb{P}^{1^3} \times V^3 \subset \mathbb{P}^{1^3} \times X^3 \\ {\scriptstyle p_1^3}\downarrow & & \\ \overline{W}^V_{kA,J_\epsilon,\nu}. & & \end{array} \tag{4.36}$$

The class that we want to compute is the class of $p_3^3((p_2^3)^{-1}((x_1, x_2, x_3)))$, for x_1, x_2, x_3 three distinct generic points of $\mathbb{P}^1$. Now we do this computation with $W^V_{kA,J_\epsilon,\nu}$ replaced by $V(s_{\sigma_1}, s_{\sigma_2})$, that we have identified set theoretically to a component of $W^V_{kA,J,\nu}$ for special ν in Section 2; as before we identify V to a neighbourhood of the zero section of the bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, and call $\pi: V \to \mathbb{P}^1$ the projection; we may assume that π induces an isomorphism $\pi_*: H_*(V) \cong H_*(\mathbb{P}^1)$ hence an isomorphism $\pi^3_*: H_*(V^3) \to H_*(\mathbb{P}^{1^3})$. Now, by construction, for $(C) \in V(s_{\sigma_1}, s_{\sigma_2})$, the associated map $\psi: \mathbb{P}^1 \to V$ solution of the equation (2.9), satisfies $\pi \circ \psi = \psi_{(C)}$, where $\psi_{(C)}: \mathbb{P}^1 \to \mathbb{P}^1$ is the map determined by $C \subset Q = \mathbb{P}^1 \times \mathbb{P}^1$. It follows that the image under (Id, π) of the family (4.35) is simply the restriction to $V(s_{\sigma_1}, s_{\sigma_2}) \subset M_k^0$ of the divisor D of Section 2.

$$\begin{array}{ccc} D_{|V(s_{\sigma_1}, s_{\sigma_2})} & \xrightarrow{(p_2, \pi\circ p_3)} & \mathbb{P}^1 \times \mathbb{P}^1 \\ {\scriptstyle p_1}\downarrow & & \\ V(s_{\sigma_1}, s_{\sigma_2}). & & \end{array} \qquad (4.37)$$

Since we know that $\overline{V}_{\sigma_1,\sigma_2} \subset M_k$ has for homology class the Poincaré dual of the top Chern class of $E \times E$, with $E \cong \mathcal{O}_{M_k}^{k-1} \otimes \mathcal{O}_{M_k}(1)$, we find as in [1] that $[\overline{V}_{\sigma_1,\sigma_2}]$ is the homology class of a $\mathbb{P}^3 \subset M_k \cong \mathbb{P}^{2k+1}$. It is then immediate to conclude that $(\pi \circ p_3)^3_*([p_2^{3^{-1}}((x_1, x_2, x_3))])$ is equal to the fundamental homology class of $\mathbb{P}^{1^3}$.

In order to complete the proof of the Theorem 1.1, it remains to verify that the computation of the class of $p_3^3((p_2^3)^{-1}((x_1, x_2, x_3)))$ (for generic J_ϵ, ν) can be done using $V(s_{\sigma_1}, s_{\sigma_2})$, that is we have to verify the following points

LEMMA 4.1. *$W^0_{kA,J,\nu}$ is smooth along $V(s_{\sigma_1}, s_{\sigma_2})$, for ν as in Section 2 and generic σ_i.*

In other words we have to identify 'schematically' $W^V_{kA,J,\nu}$ and $V(s_{\sigma_1}, s_{\sigma_2})$.

LEMMA 4.2. *The orientation of $V(s_{\sigma_1}, s_{\sigma_2})$ as the zero set of a section of a complex vector bundle on M_k coincide with the natural orientation of $W^V_{kA,J,\nu}$ (defined in* [9], *Chapter* 3*).*

LEMMA 4.3. *For (J_n, ν_n) a sequence of generic deformations of $(J, 0)$ converging to (J, ν), $\overline{W}^V_{kA,J_n,\nu_n}$ converges to $\overline{V}_{\sigma_1,\sigma_2}$.*

(That is we have to exclude the existence of a limit component which would be made of curves in $\mathbb{P}^1 \times X$ with a vertical component).

Proof of Lemma 4.1. We want to show that for $(C) \in V(s_{\sigma_1}, s_{\sigma_2})$ defining $\psi_{(C)}: \mathbb{P}^1 \to \mathbb{P}^1$ such that $(\mathrm{Id} \times \psi_{(C)})^*((\sigma_1, \sigma_2) = (\overline{\partial}\psi_1, \overline{\partial}\psi_2)$, $\psi_i \in \mathcal{C}^\infty(\psi^*_{(C)}(\mathcal{O}_{\mathbb{P}^1}(-1))$, and $\psi: \mathbb{P}^1 \to V$, $\psi = (\psi_{(C)}, \psi_1, \psi_2)$, where V is identified to an open set of N_ϕ as in Section 2, the tangent space at (C) of $V(s_{\sigma_1}, s_{\sigma_2})$ and at ψ of $W^V_{kA,J,\nu}$ coincide. But the last space is the kernel of the linearized equation

$$D_\psi := D(\overline{\partial} - (\mathrm{Id}, \psi)^*\nu) : \mathcal{C}^\infty(\psi^*T_X) \to A^{0,1}_{\mathbb{P}^1}(\psi^*T_X). \tag{4.38}$$

The bundle $T_{X|V}$ fits into the exact sequence

$$0 \to \pi^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \to T_{X|V} \to \pi^*(T_{\mathbb{P}^1}) \to 0 \tag{4.39}$$

and $\nu = ((\mathrm{Id} \times \pi)^*\sigma_1, (\mathrm{Id} \times \pi)^*\sigma_2)$. Since $\pi \circ \psi = \psi_C$ is holomorphic, it is immediate to verify that $D_{\psi|\mathcal{C}^\infty(\psi^*_{(C)}(\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1))}$ is simply the $\overline{\partial}$ operator, and that the induced quotient map $\overline{D}_\psi: \mathcal{C}^\infty(\psi^*_{(C)}(T_{\mathbb{P}^1})) \to A^{0,1}(\psi^*_{(C)}(T_{\mathbb{P}^1}))$ is also the $\overline{\partial}$-operator. Since $\overline{\partial}: \mathcal{C}^\infty(\psi^*_{(C)}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))) \to A^{0,1}(\psi^*_{(C)}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)))$ is injective, and $\overline{\partial}: \mathcal{C}^\infty(\psi^*_{(C)}(T_{\mathbb{P}^1})) \to A^{0,1}(\psi^*_{(C)}(T_{\mathbb{P}^1}))$ is surjective, we get an exact sequence

$$0 \to \mathrm{Ker}\, D_\psi \to \mathrm{Ker}\, \overline{\partial}_{\psi^*_C T_{\mathbb{P}^1}} \xrightarrow{\beta} \mathrm{Coker}\, \overline{\partial}_{(\psi^*_{(C)}(\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1))} \to 0 \tag{4.40}$$

and identifying the second term to $T_{M_k(C)}$ and the last term to $H^1(\psi^*_{(C)}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))) = (E \times E)_{(C)}$, it is immediate to verify that β is equal to the linearization of $(s_{\sigma_1}, s_{\sigma_2})$ at (C), which proves Lemma 4.1.

Proof of Lemma 4.2. The orientation of the variety $W^V_{kA,J,\nu}$ at the point ψ corresponding to (C) is described as follows (cf. [9]): Replacing $\mathcal{C}^\infty$ sections of the bundles $\psi^*T_X, \Omega^{0,1}(\psi^*T_X)$ by sections with L^1 derivatives up to order k, the operator D_ψ gives a Fredholm operator (surjective at a smooth point)

$$D_\psi: W^{k,1}(\psi^*T_X) \to W^{k-1,1}(\Omega^{0,1}(\psi^*T_X)). \tag{4.41}$$

The observation is that both spaces have natural (continuous) complex structures and that the $\mathbb{C}$-antilinear part of D_ψ is of order 0, hence is compact. So there is a natural (linear) homotopy from D_ψ to its $\mathbb{C}$-linear part D^L_ψ in the space of Fredholm operators from $W^{k,1}(\psi^*T_X)$ to $W^{k-1,1}(\Omega^{0,1}(\psi^*T_X))$. The orientation on $T_{W^V_{kA,J,\nu}}$ at the point ψ is obtained by using the real line bundle $\mathrm{Det}_t := \bigwedge^{\max}_{\mathbb{R}} \mathrm{Ker}\, D_t \otimes (\bigwedge^{\max}_{\mathbb{R}} \mathrm{Coker}\, D_t)^*$ on $[0, 1]$, where $D_t = (1 - t)D_\psi + tD^L_\psi$. Since for $t = 1$, $D_1 = D^L_\psi$ is complex linear Det_1 is naturally oriented, hence $\mathrm{Det}_0 = \bigwedge^{\max}_{\mathbb{R}} T_{W^V_{kA,J,\nu}}$ is also naturally oriented.

Now as mentioned above, the operator D_ψ induces the complex linear operators

$$\overline{\partial}: W^{k,1}(\psi^*_{(C)}(\mathcal{O}_{\mathbb{P}^1}(-1)^2)) \to W^{k-1,1}(\Omega^{0,1}(\psi^*_{(C)}(\mathcal{O}_{\mathbb{P}^1}(-1)^2))) \tag{4.42}$$

and

$$\overline{\partial}: W^{k,1}(\psi^*_{(C)} T_{\mathbb{P}^1}) \to W^{k-1,1}(\Omega^{0,1}(\psi^*_{(C)} T_{\mathbb{P}^1})). \tag{4.43}$$

So its complex linear part satisfies the same property, as do all the operators D_t. It follows that for each t we have an exact sequence

$$0 \to \operatorname{Ker} D_t \to \operatorname{Ker} \overline{\partial}_{\psi_C^* T_{\mathbb{P}^1}} \xrightarrow{\beta_t} \operatorname{Coker} \overline{\partial}_{\psi_{(C)}^* (\mathcal{O}_{\mathbb{P}^1}(-1)^2)} \\ \to \operatorname{Coker} D_t \to 0, \tag{4.44}$$

hence a canonical isomorphism

$$\operatorname{Det}_t \cong \bigwedge_{\mathbb{R}}^{\max} \operatorname{Ker} \overline{\partial}_{\psi_C^* T_{\mathbb{P}^1}} \otimes \left(\bigwedge_{\mathbb{R}}^{\max} \operatorname{Coker} \overline{\partial}_{\psi_{(C)}^* (\mathcal{O}_{\mathbb{P}^1}(-1)^2)} \right)^*, \tag{4.45}$$

which is easily seen to be continuous. The right hand side has a natural orientation coming from the complex structure on $\operatorname{Ker} \overline{\partial}$ and $\operatorname{Coker} \overline{\partial}$. But for $t = 1$, the exact sequence (4.44) is an exact sequence of complex vector spaces and complex linear maps, so the isomorphism (4.45) for $t = 1$ is compatible with the complex orientation. On the other hand, for $t = 0$, the isomorphism (4.45) induces on the left hand side (which is equal to $\bigwedge_{\mathbb{R}}^{\max} T_{W_{kA,J,\nu}^V}$ at ψ) the orientation of $V(s_{\sigma_1}, s_{\sigma_1})$, given by the complex structure on M_k and the complex structure on $E \times E$. So Lemma 4.2 is proved.

Proof of Lemma 4.3. We use the following version of the compacity theorem (cf. [4], [12])

THEOREM 4.4. *Assume (J_n, ν_n) converges to (J, ν) and let $\psi_n \in W_{kA,J_n,\nu_n}^V$; then one can extract a subsequence ψ_{n_k} such that the graph of ψ_{n_k} in $\mathbb{P}^1 \times X$ converges to the connected union of the graph of $\psi_0 \in W_{\eta,J,\nu}^V$, and of a vertical components $t_i \times C_i$, where $t_i \in \mathbb{P}^1$ and $C_i \subset U$ is holomorphic.*

Necessarily C_i must be equal to $\mathbb{P}^1 \subset X$ since its class may take only finitely values, and we may assume that there is no rational curve in V having one of these classes, excepted for $\mathbb{P}^1$. So we must have $\eta = lA$, $l \leqslant k$ and the "limit" ψ_0 corresponds to $(C_0) \in V_l(s_{\sigma_1}, s_{\sigma_2}) \subset M_l^0$. Now assume that there is a six dimensional family of limit graphs consisting of reducible curves; this would imply that for some $l < k$, there is an open set K of $V_l(s_{\sigma_1}, s_{\sigma_2})$ such that for $(C) \in K$, the corresponding map $\psi : \mathbb{P}^1 \to V$ meets $\mathbb{P}^1$; writing $\psi = (\psi_C, \psi_1, \psi_2)$ as above, this means that (ψ_1, ψ_2) vanishes at some point $t \in C$. But then, since by definition $\overline{\partial} \psi_i = (\mathrm{Id} \times \psi_C)^* \sigma_i$ we would have $(\mathrm{Id} \times \psi_C)^*(\sigma_1, \sigma_2) = 0$ in $H^1(C, \psi_C^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \psi_C^*(\mathcal{O}_{\mathbb{P}^1}(-1))(-t))$, and by Lemma 3.4 the curve $C \cup t \times \mathbb{P}^1$ would be in the zero set of the section $(s_{\sigma_1}, s_{\sigma_2})$ on M_{l+1}. (Notice that by the Proposition 3.3, $(s_{\sigma_1}, s_{\sigma_2})$ is continuous at reduced curves of M_{l+1}). On the other hand, $C \cup t \times \mathbb{P}^1$ belongs to the stratum $M_{l+1}^{(1,0,\dots,0)}$ of M_{l+1}, and we have proved that for general (σ_1, σ_2) the intersection $\overline{V}_{\sigma_1,\sigma_2} \cap M_{l+1}^{(1,0,\dots,0)}$ is at most four dimensional, which contradicts the fact that it would contain a 6 dimensional subvariety of M_{l+1}.

So we have proved the Theorem 1.1 for embedded rigid $\mathbb{P}^1 \subset X$. It remains to see what happens if $\mathbb{P}^1 \xrightarrow{j} X$ is only an immersion: but we can replace X by a neighbourhood V of $\mathbb{P}^1$ in its normal bundle, with the complex structure induced by an exponential map $V \to X$, which is a local diffeomorphism. The only thing that we have to verify is that we can choose the parameter ν on $\mathbb{P}^1 \times V$, of the form $((\mathrm{Id} \times \pi)^*(\sigma_1), (\mathrm{Id} \times \pi)^*(\sigma_2))$, as in section 2, satisfying the transversality conclusion of the Proposition 3.6, and coming from $\mathbb{P}^1 \times X$: but it suffices to choose σ_i on $\mathbb{P}^1 \times \mathbb{P}^1$ vanishing over $pr_2^{-1}(U_p)$ for an adequate (small) neighbourhood U_p in $\mathbb{P}^1$ of any $p \in \mathbb{P}^1$ such that $j^{-1}(j(p)) \neq \{p\}$. It is not difficult to show that the conclusion of the Proposition 3.6 still holds for a general couple (σ_1, σ_2) satisfying such a vanishing assumption.

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