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Heights and reductions of semi-stable varieties

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Introduction

Using arithmetic geometric invariant theory developed in [Bu] [Z2] and a result of Soulé and Philippon ([So1], [P], [B–G–S]) on Chow variety, in this paper, we define the heights of semistable varieties in a projective space and study reductions at Archimedean places of such varieties.

In Section 1, we use the Deligne pairing ([D], [E1], [E2], [Fr1], [Fr2]) to obtain local versions of a theorem of Soulé and Philippon ([So1], [P], [B–G–S]) which links the Faltings heights of projective varieties and the Philippon heights for corresponding Chow points. In particular a metric (which I called Chow metric) on the $\mathcal{O}(1)$ bundle of Chow variety is defined which is different from the usual Fubini–Study metric.

In Section 2, we prove a general stable reduction theorem at Archimedean places which generalizes a theorem of Kempf–Ness [K–N], they worked on projective space with Fubini–Study metrics. The stable reduction theorem at non-Archimedean theorem follows from the work of Seshadri [Se] and Burnol [Bu].

In Section 3, we prove that the Chow metrics defined in Section 1 are positive. The stable reduction theorem in Section 2 in this case gives special metrics (which I called critical metrics) on the $\mathcal{O}(1)$ bundles which characterize the semistabilities of the varieties. The critical metrics can be described as follows. Let $X \subset \mathbb{P}(V)$ be a projective variety which is not contained in any hyperplane. Let $\mathcal{L} = O(1)|_X$ and consider V as a linear system of $\mathcal{L}$. For any positive Hermitian metric $\|\cdot\|$ on $\mathcal{L}$ we define the detorsion function $b_{\|\cdot\|}$ as follows. Let $\{s_0, s_1, \ldots, s_N\}$ be an orthonormal basis with respect to the Hermitian structure induced by the metric and the curvature of $(\mathcal{L}, \|\cdot\|)$. Then

$$b_{\|\cdot\|}(x) = \frac{1}{N+1}\sum_{i=0}^{N} \|s_i\|^2(x).$$

It is easy to see that $b_{\|\cdot\|}$ does not depend on the choice of basis. We say $\|\cdot\|$ is critical with respect to V if $b_{\|\cdot\|}$ is a constant function. We show that the existence of a critical metric implies the semistability, and stability implies the existence and the uniqueness of critical metrics. The limits of these critical metrics are conjectured to be Kähler–Einstein metrics when $\mathcal{L}$ is the canonical bundle. The idea to obtain Kähler–Einstein metrics on a projective complex variety as a limit of Fubini–Study metrics is due to Yau and Tian ([Y2] p. 35, [T], [D–T]).

In Section 4, we define the height $\hat{h}(X)$ of a semistable variety $X \in \mathbb{P}^N$ to be the height of the corresponding point on the quotient variety of Chow variety. This height can also be defined by the following more direct way (inspired by a question of B. Mazur). For any Hermitian vector bundle $\mathcal{E}$ over $\mathrm{Spec}(\mathcal{O}_K)$ which extends the trivial bundle K^N, let $\mathcal{L}_{\mathcal{E}}$ be the restriction of $\mathcal{O}(1)$ bundle of $\mathbb{P}(\mathcal{E})$ on the Zariski closure of X in $\mathbb{P}(\mathcal{E})$, then we define the height $h_{\mathcal{E}}(X)$ with respect to $\mathcal{E}$ by the formula

$$h_{\mathcal{E}}(X) = \frac{c_1(\mathcal{L}_{\mathcal{E}})^{\dim X+1}}{(\dim X + 1)\deg X[K : \mathbb{Q}]} - \frac{\deg \mathcal{E}}{N+1}.$$

Then we show that $h_{\mathcal{E}}(X)$ is bounded below as a function of $\mathcal{E}$ if and only if X is semistable (so this gives converses of results of Cornalba–Harris [C–H] and Bost [Bo],) and if X is semistable then the minimal value of $h_{\mathcal{E}}(X)$ is $\hat{h}(X)$. The stable reduction theorem can even tell us such minimal value can be obtained by some $\mathcal{E}$ (which is unique in some sense) such that X has stable reduction in $\mathbb{P}(\mathcal{E})$ and the Hermitian metric on $\mathcal{E}$ induces a critical metric on $\mathcal{L}_{\mathcal{E}}$.

The height $\hat{h}(X)$ should be positive as it is true in function field case [C–H]. In this paper we can only prove that

$$\hat{h}(X) \geqslant - \hat{h}(\mathbb{P}^N),$$

where $\hat{h}(\mathbb{P}^N) = h(\mathbb{P}^N)/(N+1)$ and

$$h(\mathbb{P}^N) = \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{n}\frac{1}{m},$$

is the Faltings height of the projective space [Fa]. This improves a result of Bost [Bo] where bounds are not explicitly given and a result of Soulé [So2] where bounds are given for $h_{\mathcal{E}}(X)$ in terms of the definition field of $\mathcal{E}$ and X. Our proof is based on Soulé's arguments with some improvements.

In 1991, B. Mazur asked me a question about the minimized Faltings heights $h(X)$ in a projective space $\mathbb{P}^N$ under the action of $\mathrm{SL}(N+1, \bar{\mathbb{Q}})$. Our height here is its adelic version: $\hat{h}(X)$ is the minimized height under the action of $\mathrm{SL}(N+1, \mathbb{A} \otimes \bar{\mathbb{Q}})$ where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$.

The importance of the quantity $h_{\mathcal{E}}(X)$ already appears in [C–H] and [Bo]. Moreover the importance of some combination of geometric invariant theory and Arakelov geometry to define heights on moduli spaces has been first advocated by Burnol [Bu].

1. Deligne pairings and Chow sections

1.1. DELIGNE'S PAIRINGS

Let π: $X \to S$ be a flat and projective morphism of integral schemes of pure relative dimension n. Let $\mathcal{L}_0, \dots, \mathcal{L}_n$ be line bundles on X and let $\langle \mathcal{L}_0, \dots, \mathcal{L}_n\rangle(X/S)$ be the Deligne pairing ([D Sect. 8.1], [E1], [E2], [Fr1], [Fr2]). More precisely, $\langle \mathcal{L}_0, \dots, \mathcal{L}_n\rangle$ is locally in Zariski topology generated by symbols $\langle l_0, \dots, l_n\rangle$ with a relation, where l_i's are sections of $\mathcal{L}_i$'s such that their divisors have no intersection. The relation is as follows. For some i between 0 and n and a function f on X, if the intersection $\prod_{j\neq i} \operatorname{div}(l_j) = \Sigma_i n_i Y_i$ is finite over S and has empty intersection with $\operatorname{div}(f)$, then

$$\langle l_0, \dots, f l_i, \dots, l_n\rangle = \prod_i \operatorname{Norm}_{Y_i/S}(f)^{n_i} \langle l_0, \dots, l_n\rangle. \tag{1.1.1}$$

From definition, it is easy to see that the functor $\langle \cdots\rangle(X/S)$ is multilinear and symmetric from $\prod_0^n \operatorname{Pic}(X)$ to $\operatorname{Pic}(S)$, and commutes with base change of S. For our purpose we list two properties here.

The first one is the projection formula. Let ϕ: $X \to Y$ be a morphism of projective and flat integral schemes over S, $m = \dim(Y/S)$, $n = \dim(X/Y)$. Let $\mathcal{K}_0, \dots, \mathcal{K}_m$ be line bundles on X, and $\mathcal{L}_1, \dots, \mathcal{L}_n$ be line bundles on Y. Then there is a canonical isomorphism

$$\begin{aligned}&\langle \mathcal{K}_0, \dots, \mathcal{K}_m, \phi^*\mathcal{L}_1, \dots, \phi^*\mathcal{L}_n\rangle(X/S)\\ &\simeq \langle\langle \mathcal{K}_0, \dots, \mathcal{K}_m\rangle(X/Y), \mathcal{L}_1, \dots, \mathcal{L}_n\rangle\,(Y/S).\end{aligned} \tag{1.1.2}$$

The isomorphism locally on S sends $\langle k_0, \dots, k_m, \phi^* l_1, \dots, \phi^* l_n\rangle$ to $\langle\langle k_0, \dots, k_m\rangle$, $l_1, \dots, l_n\rangle$, where k_i's and l_j's are sections of $\mathcal{K}_i$'s and $\mathcal{L}_j$'s respectively, such that (i) divisors of k_i's have no intersection in the generic fiber of ϕ, so $\langle k_0, \dots, k_m\rangle$ define a rational section of $\langle \mathcal{K}_0, \dots, \mathcal{K}_m\rangle$, and (ii) divisors of $\langle k_0, \dots, k_m\rangle$ and l_j's have no intersection.

Setting $\mathcal{K}_0 = \phi^*\mathcal{L}_0$ for a line bundle $\mathcal{L}_0$ on Y in (1.1.2), we will obtain

$$\begin{aligned}&\langle \mathcal{K}_1, \dots, \mathcal{K}_m, \phi^*\mathcal{L}_0, \dots, \phi^*\mathcal{L}_n\rangle(X/S)\\ &\simeq \langle \mathcal{L}_0, \dots, \mathcal{L}_n\rangle(Y/S)^{\otimes \deg[c_1(\mathcal{K}_{1\eta})\cdots c_1(\mathcal{K}_{n\eta})]},\end{aligned} \tag{1.1.3}$$

where $\mathcal{K}_{i\eta}$'s are restrictions of $\mathcal{K}_i$'s on the fiber of a generic point η of Y. After applying (1.1.2) to the left-hand side of (1.1.3), it suffices to construct an isomorphism

$$\langle \mathcal{K}_1, \dots, \mathcal{K}_m, \phi^*\mathcal{L}_0\rangle(X/Y) \simeq \mathcal{L}_0^{\otimes \deg[c_1(\mathcal{K}_{1\eta})\cdots c_1(\mathcal{K}_{n\eta})]}.$$

Locally on Y, choosing an invertible section l_0 of $\mathcal{L}_0$ and nonzero sections k_i's of $\mathcal{K}_i$'s will give trivializations of both sides of the above formula. It is easy to show that this trivializations of both sides compatible with relation (1.1.1).

Setting $\mathcal{K}_m = \phi^*\mathcal{L}_{n+1}$ in (1.1.3), we also obtain

$$\langle \mathcal{K}_1, \ldots, \mathcal{K}_{m-1}, \phi^*\mathcal{L}_0, \ldots, \phi^*\mathcal{L}_{n+1}\rangle(X/S) \simeq \mathcal{O}_S, \tag{1.1.4}$$

since the product of $c_1's$ in (1.1.3) is 0.

The second one is an induction formula. Let $\pi\colon X \to S$ and $\mathcal{L}_0, \ldots, \mathcal{L}_n$ be as before. Let l be a rational section of $\mathcal{L}_n$. Assume all components of $\operatorname{div} l$ are flat over S. Then we have a canonical isomorphism

$$\langle \mathcal{L}_0, \ldots, \mathcal{L}_n\rangle(X/S) \simeq \langle \mathcal{L}_0, \ldots, \mathcal{L}_{n-1}\rangle\left(\frac{\operatorname{div} l}{S}\right), \tag{1.1.5}$$

where if $Z = \Sigma n_i Y_i$ is a cycle over S with Y_i flat, projective, and of dimension n over S then

$$\langle \mathcal{L}_0, \ldots, \mathcal{L}_{n-1}\rangle(Z/S) := \prod_i \langle \mathcal{L}_0, \ldots, \mathcal{L}_n\rangle^{\otimes n_i}.$$

1.2. Deligne's pairings with metrics

Now assume that both X and S are defined over complex numbers, and $\mathcal{L}_0, \ldots, \mathcal{L}_n$ have smooth Hermitian metrics. By a smooth metric on a vector bundle $\mathcal{E}$ over X we mean that for any holomorphic map f from a complex manifold Y to X, the pull-back metric on $f^*\mathcal{E}$ is smooth on Y. One can define a metric on $\langle \mathcal{L}_0, \ldots, \mathcal{L}_n\rangle$ as follows. For each $0 \leqslant i \leqslant n$, let $c_1'(\mathcal{L}_i)$ be the curvature of $\mathcal{L}_i$ which is locally defined as $\partial\bar{\partial}/\pi i \log \|l\|$ for an invertible section l of $\mathcal{L}_i$. Then the metric on $\langle \mathcal{L}_0, \ldots, \mathcal{L}_n\rangle$ can be defined by induction on n

$$\log \|\langle l_0, \ldots, l_n\rangle\| = \log \|\langle l_0, \ldots, l_{n-1}\rangle(\operatorname{div} l_n)\| + \int_{X/S} \log \|l_n\| \bigwedge_{i=0}^{n-1} c_1'(\mathcal{L}_i).$$

It is not difficult to show that isomorphisms (1.1.2), (1.1.3), and (1.1.4) are isometric, and (1.1.5) gives the following isometry

$$\begin{aligned} &\langle \mathcal{L}_0, \ldots, \mathcal{L}_n\rangle(X/S) \\ &\simeq \langle \mathcal{L}_0, \ldots, \mathcal{L}_{n-1}\rangle(\operatorname{div} l_n/S) \otimes \mathcal{O}\left(\int_{X/S} -\log \|l\| \bigwedge_{i=0}^{n-1} c_1'(\mathcal{L}_i)\right), \end{aligned} \tag{1.2.1}$$

where for a function $f(s)$ on S, $\mathcal{O}(f)$ denotes the trivial line bundle $\mathcal{O}$ with metric $\|1\| = \exp(-f)$.

When X is smooth over S, Deligne's pairing with metrics are constructed in [D, Sect. 8.3] and [E2]. In this case one can show that the metric is smooth and has curvature $\int_{X/S} \bigwedge_{i=0}^{n} c_1'(\mathcal{L}_i)$. For a general X, the metric is not necessarily smooth. However, we expect it to be continuous. When $\dim S = 1$, the continuity follows from the induction on n and Proposition 1.5.1 in [B–G–S]. When X is embedded in a projective space $\mathbb{P}^N \times S$ over S and $\mathcal{L}_i$'s are the pullback on X of the $\mathcal{O}(1)$ line bundle of $\mathbb{P}^N$ with the standard Fubini–Study metric, then the continuity will follow from Theorem 1.4 and Theorem 3.6.

1.3. Chow sections

We summarize some constructions and results in [F–M], 5.3.4 and [B–G–S] 4.3.1 and 4.3.2(i). Let S be an integral scheme, $\mathcal{E}$ be a vector bundle on S of rank $N+1$, and X be an effective cycle of $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\operatorname{Sym}^* \mathcal{E})$ whose components are flat and of dimension n over S. Let $\mathcal{L}$ and $\mathcal{M}$ denote the $\mathcal{O}(1)$ bundles of $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E}^\vee)$ respectively. Then the canonical section of $\mathcal{E} \otimes \mathcal{E}^\vee$, which is dual to the canonical pairing $\mathcal{E}^\vee \otimes \mathcal{E} \to \mathcal{O}_S$, gives a section w of $\mathcal{L} \otimes \mathcal{M}$ on $\mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}^\vee)$. Let $\mathcal{M}_i$ denote the pullback on $\prod_{i=0}^{n} \mathbb{P}(\mathcal{E}^\vee) = \mathbb{P}(\mathcal{E}^\vee)^{n+1}$ via the ith projection, and let w_i's be the corresponding sections of $\mathcal{L} \otimes \mathcal{M}_i$ on $\mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}^\vee)^{n+1}$. Let Γ denote the intersection of divisors of w_i's. If we consider the points of $\mathbb{P}(\mathcal{E}^\vee)$ as hyperplanes of $\mathbb{P}(\mathcal{E})$, then the points $(x, H_0, \dots, H_n)$ of Γ are such that $x \in \cap_{i=0}^{n} H_i$. Consider Γ as a correspondence from $\mathbb{P}(\mathcal{E})$ to $\mathbb{P}(\mathcal{E}^\vee)^{n+1}$. Then $Y = \Gamma_*(X)$ will be a divisor of degree $(d, \dots, d)$ of $\mathbb{P}(\mathcal{E}^\vee)^{n+1}$ whose components are flat over S, where d is the degree of X over S.

Let $\mathcal{N}$ be the $\mathcal{O}(1)$ bundle of $\mathbb{P}[(\operatorname{Sym}^d \mathcal{E})^{\otimes(n+1)}]$. Then the canonical pairing gives a section w' of $(\operatorname{Sym}^d \mathcal{E})^{\otimes(n+1)} \otimes (\operatorname{Sym}^d \mathcal{E}^\vee)^{\otimes(n+1)}$ which in turn gives a section of the bundle $\mathcal{N} \otimes \mathcal{M}_0^d \otimes \cdots \otimes \mathcal{M}_n^d$ on $\mathbb{P}[(\operatorname{Sym}^d \mathcal{E})^{\otimes(n+1)}] \times \mathbb{P}(\mathcal{E}^\vee)^{n+1}$. Let Γ' denote the divisor of w'. If we consider points of $\mathbb{P}[(\operatorname{Sym}^d \mathcal{E})^{\otimes(n+1)}]$ as hypersurfaces of $\mathbb{P}(\mathcal{E}^\vee)^{n+1}$ of degree $(d, \dots, d)$ then Γ' has points $(H, y_0, \dots, y_n)$ such that $(y_0, \dots, y_n) \in H$. The section Z of $\mathbb{P}[(\operatorname{Sym}^d \mathcal{E})^{\otimes(n+1)}]$ over S corresponding to Y defined as above is called the Chow section for X. We have $Y = \Gamma'_*(Z)$, where Γ' is considered as a correspondence. The following result is a local version of a result of Philippon and Soulé ([So1], [P], [B–G–S]).

THEOREM 1.4. *There is a canonical isomorphism on* S

$$\langle \mathcal{L}, \dots, \mathcal{L} \rangle (X/S) \simeq Z^*(\mathcal{N}).$$

Proof. We use the following notations

$$\begin{aligned}
\langle \mathcal{L}_1^{\cdot n_1}, \dots \rangle &:= \left\langle \underbrace{\mathcal{L}_1, \dots, \mathcal{L}_1}_{n_1 \text{ times}}, \dots, \right\rangle \\
\mathcal{L}^{\langle n \rangle} &:= \underbrace{\langle \mathcal{L}, \dots, \mathcal{L} \rangle}_{n \text{ times}}, \\
\mathcal{L}^n &= \mathcal{L}^{\otimes n}.
\end{aligned}$$

Let p denote the projection $\mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}^\vee)^{n+1} \to \mathbb{P}(\mathcal{E})$, then

$$\begin{aligned}
\mathcal{L}^{\langle n+1 \rangle}(X/S) &\simeq \langle \mathcal{L}^{\cdot (n+1)}, \mathcal{M}_0^{\cdot N}, \dots, \mathcal{M}_n^{\cdot N} \rangle (p^* X/S) \\
&\quad \text{by (1.1.3) for } p^* X \to X \\
&\simeq \langle \mathcal{L} \otimes \mathcal{M}_0, \dots, \mathcal{L} \otimes \mathcal{M}_n, \mathcal{M}_0^{\cdot N}, \dots, \mathcal{M}_n^{\cdot N} \rangle (p^* X/S) \\
&\quad \otimes (\mathcal{M}_0^{\langle N+1 \rangle})^{-d} \otimes \cdots \otimes (\mathcal{M}_n^{\langle N+1 \rangle})^{-d} \\
&\quad \text{by linearity of pairing and (1.1.4)} \\
&\simeq \langle \mathcal{M}_0^{\cdot N}, \dots, \mathcal{M}_n^{\cdot N} \rangle (p^* X \cdot \Gamma / S) \\
&\quad \otimes (\mathcal{M}_0^{\langle N+1 \rangle})^{-d} \otimes \cdots \otimes (\mathcal{M}_n^{\langle N+1 \rangle})^{-d} \\
&\quad \text{by (1.1.5) for the sections } w_i{}'s \\
&\simeq \langle \mathcal{M}_0^{\cdot N}, \dots, \mathcal{M}_n^{\cdot N} \rangle (Y) \\
&\quad \otimes (\mathcal{M}_0^{\langle N+1 \rangle})^{-d} \otimes \cdots \otimes (\mathcal{M}_n^{\langle N+1 \rangle})^{-d} \\
&\quad \text{by (1.1.2) applied to morphism } p^* X \cdot \Gamma \to Y. \qquad (1.4.1)
\end{aligned}$$

Similarly let p' denote the projection $\mathbb{P}[(\mathrm{Sym}^d\, \mathcal{E})^{\otimes(n+1)}] \times \mathbb{P}(\mathcal{E}^\vee)^{n+1} \to \mathbb{P}[(\mathrm{Sym}^d\, \mathcal{E})^{\otimes(n+1)}]$, then

$$\begin{aligned}
Z^*(\mathcal{N}) &\simeq \langle \mathcal{M}_0^{\cdot N}, \dots, \mathcal{M}_n^{\cdot N}, \mathcal{N} \rangle (p'^* Z) \\
&\quad \text{by (1.1.3) for } p'^* Z \to Z. \\
&\simeq \left\langle \mathcal{M}_0^{\cdot N}, \dots, \mathcal{M}_n^{\cdot N}, \mathcal{N} \otimes \bigotimes_{i=0}^{n} \mathcal{M}_i^d \right\rangle (p'^* Z) \\
&\quad \otimes (\mathcal{M}_0^{\langle N+1 \rangle})^{-d} \otimes \cdots \otimes (\mathcal{M}_n^{\langle N+1 \rangle})^{-d} \\
&\quad \text{by linearility of pairing and (1.1.4)} \\
&\simeq \langle \mathcal{M}_0^{\cdot N}, \dots, \mathcal{M}_n^{\cdot N} \rangle (p'^* Z \cdot \Gamma') \\
&\quad \otimes (\mathcal{M}_0^{\langle N+1 \rangle})^{-d} \otimes \cdots \otimes (\mathcal{M}_n^{\langle N+1 \rangle})^{-d}
\end{aligned}$$

$$\begin{aligned}
&\text{by (1.1.5) for the section } w' \\
\simeq\ & \langle \mathcal{M}_0^{\cdot N}, \ldots, \mathcal{M}_n^{\cdot N}\rangle(Y/S) \\
&\otimes(\mathcal{M}_0^{\langle N+1\rangle})^{-d} \otimes \cdots \otimes (\mathcal{M}_n^{\langle N+1\rangle})^{-d} \\
&\text{by (1.1.2) for the morphism } p'^{*}Z\cdot\Gamma' \to Y.
\end{aligned} \tag{1.4.2}$$

Now the theorem follows from (1.4.1) and (1.4.2).

1.5. Chow metrics

Now we assume that S is a complex variety and $\mathcal{E}$ has a smooth Hermitian metric $\|\cdot\|$. It induces Fubini–Study metrics on $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$. All these metrics and their products are denoted by $\|\cdot\|$. We define a new metric $\|\cdot\|_{\mathrm{Ch}}$ on $\mathcal{N}$ which I call Chow metric as follows. Let w' be the canonical section of $\mathcal{N}\otimes\otimes_{i=0}^{n}\mathcal{M}_i^d$ corresponding to the canonical section of $(\mathrm{Sym}^d\,\mathcal{E})^{\otimes(n+1)}\otimes(\mathrm{Sym}^d\,\mathcal{E}^{\vee})^{\otimes(n+1)}$. Then $\|w'\|$ is a function on $\mathbb{P}[(\mathrm{Sym}^d\,\mathcal{E})^{\otimes(n+1)}]\times\mathbb{P}(\mathcal{E}^{\vee})^{n+1}$. For a section s of $\mathcal{N}$ we define

$$\log\|s\|_{\mathrm{Ch}} = \log\|s\| - \tfrac{1}{2}(n+1)d\sum_{j=1}^{N}\frac{1}{j} - \int_{\mathbb{P}(\mathcal{E}^{\vee})^{n+1}}\log\|w'\|\bigwedge_{i=0}^{n}c_1'(\mathcal{M}_i)^N.$$

If S is a point and $\mathcal{E}=\mathbb{C}^{N+1}$ with the standard Hermitian structure, then $\mathbb{P}[(\mathrm{Sym}^d\,\mathcal{E})^{\otimes(n+1)}]$ has a homogeneous coordinates (z_α), where α are multi-indices for monomials of degree $(d,\ldots,d)$ on $\mathbb{P}^N$: $\alpha=(\alpha_{i,j})$, $0\leqslant i\leqslant n$, $0\leqslant j\leqslant N$, $\alpha_{i,j}\geqslant 0$ are integers and $\Sigma_j\alpha_{i,j}=d$ for each i. A section of s of N can be written as $\Sigma_\alpha a_\alpha s_\alpha$. For a point (z_α) of $\mathbb{P}[(\mathrm{Sym}^d\,\mathcal{E})^{\otimes(n+1)}]$, we have

$$\begin{aligned}
\log\|s\|_{\mathrm{Ch}}(z_\alpha) &= \log\left|\sum_\alpha a_\alpha z_\alpha\right| - \tfrac{1}{2}(n+1)d\sum_{j=1}^{N}\frac{1}{j} \\
&\quad - \int_{S(\mathbb{C}^{N+1})^{n+1}}\log\left|\sum_\alpha z_\alpha x_\alpha\right|\mathrm{d}x,
\end{aligned}$$

where $x_\alpha=\prod_{i,j}x_{i,j}^{\alpha_{i,j}}$ are monomials on $\mathbb{C}^{(N+1)(n+1)}$, $S(\mathbb{C}^{N+1})$ is the unit sphere of $\mathbb{C}^{N+1}$, and $\mathrm{d}x$ is the invariant measure on $S(\mathbb{C}^{N+1})^{n+1}$ with Volume 1.

The following theorem is a local version of a result of Soulé and Philippon ([So1], [P], [B–G–S]) at the Archimedean places.

THEOREM 1.6. *With the metric on Deligne pairing $\langle\mathcal{L},\ldots,\mathcal{L}\rangle$ and Chow metric defined on $\mathcal{N}$, the isomorphism in* 1.4 *is isometric.*

Proof. By 1.2, the 1st, 2nd, and 4th isomorphisms of (1.4.1) are isometric. By (1.2.1), the 3rd isomorphism induces the following isometries

$$\begin{aligned}
&\langle \mathcal{L}\otimes\mathcal{M}_0,\ldots,\mathcal{L}\otimes\mathcal{M}_n,\mathcal{M}_0^{\cdot N},\ldots,\mathcal{M}_n^{\cdot N}\rangle(p^*X/S)\\
&\simeq \langle \mathcal{L}\otimes\mathcal{M}_1,\ldots,\mathcal{L}\otimes\mathcal{M}_n,\mathcal{M}_0^{\cdot N},\ldots,\mathcal{M}_n^{\cdot N}\rangle(p^*X\cdot\operatorname{div}(w_0))\\
&\quad\otimes\mathcal{O}\left(-\int_{p^*X}\log\|w_0\|\left[\bigwedge_{j>0}c_1'(\mathcal{L}\otimes\mathcal{M}_j)\right]\cdot\left[\bigwedge_{i=0}^{n}c_1'(\mathcal{M}_i)^N\right]\right)\\
&\simeq\cdots\\
&\simeq \langle\mathcal{M}_0^{\cdot N},\ldots,\mathcal{M}_n^{\cdot N}\rangle(p^*X\cdot\Gamma)\\
&\quad\otimes\mathcal{O}\left(\sum_{i=0}^{n}\int_{p^*X\cdot\Pi_{j<i}\operatorname{div}(w_j)/S}\right.\\
&\left.-\log\|w_i\|\left[\bigwedge_{j>i}c_1'(\mathcal{L}_j\otimes\mathcal{M}_j)\right]\cdot\left[\bigwedge_{i=0}^{n}c_i'(\mathcal{M}_i)^N\right]\right).
\end{aligned}$$

Notice that the function inside $\mathcal{O}$ which we denote as $f(s)$ commutes with base change of S. We may compute f by assuming that S is a point and $\mathcal{E}=\mathbb{C}^{N+1}$ with standard Hermitian structure. Then $w=\Sigma_{i=0}^{N}x_i\otimes y_i$ where x_i's and y_i's are homogeneous coordinates of $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E}^\vee)$. Notice that the action of the unitary group $U(N+1)$ on ith factor of $\prod_{i=0}^{n}\mathbb{P}(\mathcal{E}^\vee)$ induces an action on $p^*X\cdot\prod_{j<i}\operatorname{div}(w_j)$ and the measure $[\bigwedge_{j>i}c_1'(\mathcal{L}_j\otimes\mathcal{M}_j)]\cdot[\bigwedge_{i=0}^{n}c_i'(\mathcal{M}_i)^N]$ is invariant under this action. So we may replace $\log\|w_i\|$ by its integral on $\mathbb{P}(\mathcal{E}^\vee)$. Now for any point $x=(x_0,\ldots,x_n)\in\mathbb{P}(\mathcal{E})$

$$\begin{aligned}
&-\int_{\mathbb{P}(\mathcal{E}^\vee)}\log\|w\|c_1'(\mathcal{M})^N\\
&=-\int_{\mathbb{P}(\mathcal{E}^\vee)}\tfrac{1}{2}\log\frac{\sum_{i=0}^{N}|x_iy_i|^2}{\sum_{i=0}^{N}|x_i|^2\sum_{j=0}^{N}|y_j|^2}c_1'(\mathcal{M})^N.
\end{aligned}$$

Changing coordinates suitably, we have

$$-\int_{\mathbb{P}(\mathcal{E}^\vee)}\log\|w\|c_1(\mathcal{M})^N=\int_{\mathbb{P}(\mathcal{E}^\vee)}\tfrac{1}{2}\log\left(\sum_{i=0}^{N}\left|\frac{y_i}{y_0}\right|^2\right)c_1'(\mathcal{M})^N.$$

This is $\frac{1}{2}\Sigma_{j=1}^{N} 1/j$ by a computation of Stoll. It follows that

$$\begin{aligned} f(s) &= \left(\frac{1}{2}\sum_{j=1}^{N}\frac{1}{j}\right) \\ &\quad\times\left(\sum_{i=0}^{n}\int_{p^*X\cdot\Pi_{j<i}\mathrm{div}(w_j)/S}\left[\bigwedge_{j>i} c_1'(\mathcal{L}_j\otimes\mathcal{M}_j)\right]\cdot\left[\bigwedge_{i=0}^{n} c_i'(\mathcal{M}_i)^N\right]\right) \\ &= \tfrac{1}{2}(n+1)d\sum_{j=1}^{N}\frac{1}{j}. \end{aligned}$$

One has an isometry

$$\begin{aligned} \mathcal{L}^{\langle n+1\rangle}(X/S) &\simeq \langle\mathcal{M}_0^{\cdot N},\ldots,\mathcal{M}_n^{\cdot N}\rangle(Y/S)\otimes\mathcal{O}\left(\tfrac{1}{2}(n+1)d\sum_{j=1}^{N}\frac{1}{j}\right) \\ &\qquad\otimes(\mathcal{M}_0^{\langle N+1\rangle})^{-d}\otimes\cdots\otimes(\mathcal{M}_n^{\langle N+1\rangle})^{-d}. \end{aligned} \tag{1.5.1}$$

Similarly, the 1st, 2nd, and 4th isomorphisms of (1.4.2) are isometric. By (1.2.1) the 3rd isomorphism gives the following isometry

$$\begin{aligned} &\left\langle\mathcal{M}_0^{\cdot N},\ldots,\mathcal{M}_n^{\cdot N},\mathcal{N}\otimes\bigotimes_{i=0}^{n}\mathcal{M}_i^d\right\rangle(p'^*Z) \\ &\simeq\langle\mathcal{M}_0^{\cdot N},\ldots,\mathcal{M}_n^{\cdot N}\rangle\left(\frac{p'^*Z\cdot\Gamma'}{S}\right) \\ &\quad\otimes\mathcal{O}\left(\int_{\mathbb{P}(\mathcal{E}^\vee)^{n+1}/S}-\log\|w'\|\bigwedge_{j=0}^{n}c_1'(\mathcal{M}_j)^N\right). \end{aligned}$$

This gives isometry

$$\begin{aligned} Z^*(\mathcal{N},\|\cdot\|) &\simeq\langle\mathcal{M}_0^{\cdot N},\ldots,\mathcal{M}_n^{\cdot N}\rangle(Y/S) \\ &\qquad\otimes\left(\int_{\mathbb{P}(\mathcal{E}^\vee)^{n+1}/S}-\log\|w'\|\bigwedge c_1'(\mathcal{M}_j')^N\right) \\ &\qquad\otimes(\mathcal{M}_0^{\langle N+1\rangle})^{-d}\otimes\cdots\otimes(\mathcal{M}_n^{\langle N+1\rangle})^{-d}. \end{aligned} \tag{1.5.2}$$

The theorem follows from (1.5.1) and (1.5.2).

REMARK 1.7. As pointed by the referee, maybe one can prove Theorem 1.4 and Theorem 1.6 by using the formalism of Franke and the proof of Theorem 4.3.2 in [B–G–S]. We leave this task to interested readers.

2. Semistable reduction theorem at Archimedean places

2.1. DEFINITIONS. Let X be a projective complex variety, $\mathcal{L}$ be an ample line bundle on X, and $\|\cdot\|$ be a continuous metric on $\mathcal{L}$. We give the following working definition for positivity and semipositivity of the metric $\|\cdot\|$. We say $\|\cdot\|$ is positive (resp. semipositive) if for any holomorphic morphism $f\colon \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \to X$, the curvature $c_1'(\bar{\mathcal{M}}) = \partial\bar{\partial}/\pi i \log \|m\|$ of the pullback metrized line bundle $\mathcal{M} = f^*(\mathcal{L}, \|\cdot\|)$ is positive (resp. semipositive), where m is an invertible section of $\mathcal{M}$. This means that for any smooth function ϕ on $\mathbb{D}$ with compact support, the integral

$$\int_{\mathbb{D}} \phi c_1'(\bar{\mathcal{M}}) := \int_{\mathbb{D}} \log \|m\| \frac{\partial\bar{\partial}}{\pi i}\phi,$$

is positive (resp. semipositive) if ϕ is semipositive and nonzero.

Now fix a metric $\|\cdot\|$ on $\mathcal{L}$ in this section. Let $\bar{\mathcal{L}}$ denote the metrized line bundle $(\mathcal{L}, \|\cdot\|)$. Let G be a complex connected reductive group, and let U be a maximal compact subgroup of G. By a Hermitian action of (G, U) on $(X, \bar{\mathcal{L}})$ we mean a linear action

$$\sigma\colon G \times X \to X, \qquad \phi\colon \sigma^*\mathcal{L} \simeq p_2^*\mathcal{L},$$

of G on $(X, \mathcal{L})$ such that $\phi|_{U\times X}$ is an isometry of Hermitian line bundles.

We fix a Hermitian action of $\bar{G} = (G, U)$ on a $(X, \bar{\mathcal{L}})$. Then ϕ induces an isometry $\phi\colon \sigma^*\bar{\mathcal{L}} \simeq p_2^*\bar{\mathcal{L}} \otimes \mathcal{O}(\mu)$, where μ is a function on $G \times X$, and $\mathcal{O}(\mu)$ denotes the trivial bundle with metric $\|1\| = e^{-\mu}$. Let $x \in X$, $l \in \mathcal{L}(x) - \{0\}$. Via ϕ we may consider l as a section of $\mathcal{L}$ on the orbit Gx. Then we have a formula for μ

$$\mu(g, x) = -\log \frac{\|l(gx)\|}{\|l\|}.$$

For a fixed x, $\mu(g, x)$ can be considered as a function on G or on the orbit Gx.

By an one-dimensional parameter Hermitian subgroup of $\bar{G}$ we mean an injective homomorphism $\lambda\colon \bar{\mathbb{G}}_m = (\mathbb{G}_m, \mathbb{U}_m) \to \bar{G} = (G, U)$, namely, an injective morphism from $\mathbb{G}_m$ to G such that the image of unitary elements $\mathbb{U}_m = \{z \in \mathbb{C}^* : |z| = 1\}$ is contained in U. We abbreviate this to 'λ is a 1-phs of $\bar{G}$'.

A point of x is called critical with respect to the Hermitian action, if for any 1-phs λ of $\bar{G}$, the function $f(t) = \mu(\lambda(e^t), x)$ is critical at $t = 0$

$$\liminf_{(t,s)\to(0^+,0^+)} \frac{f(t) - f(0)}{t}\frac{f(-s) - f(0)}{s} \geqslant 0.$$

Our main result of this section is the following generalization of a result of Kempf and Ness [K–N], they worked on the projective space with a Fubini–Study metric.

THEOREM 2.2. *Let $(X, \bar{\mathcal{L}}, \bar{G})$ be assumed as above, and let x be any point of X.*

(1) *The function $\mu(g,x)$ is bounded below as a function on the orbit Gx if and only if x is semistable with respect to the linear action of G. Moreover, if x is stable then the infimum value of this function is reachable on the orbit Gx.*
(2) *If the metric $\|\cdot\|$ on $\mathcal{L}$ is semipositive, then $\inf_{g\in G}\mu(g,x) = \mu(e,x)$ if and only if x is a critical point of X where e is the unit element of G. Moreover, the set of critical points is connected in Gx, and this set is nonempty if x is stable.*
(3) *If the metric $\|\cdot\|$ is positive, then the set of critical points in Gx contains at most one orbit of U.*

Proof of the 1st Part of 2.2. For the first statement we notice that the boundedness of μ will not be changed if we change the metric on $\mathcal{L}$ or replace $\mathcal{L}$ by some positive power. By these it is easy to reduce the problem to case that $X = \mathbb{P}^n$, $\mathcal{L} = \mathcal{O}(1)$ with the Fubini–Study metric, and the action is induced by a Hermitian action of (G,U) on $\mathbb{C}^{N+1}$ with the standard Hermitian structure. Let $x^* \in \mathbb{C}^{N+1}$ be a point which has the image x on $\mathbb{P}^N$. Then

$$\mu(g,x) = \log\frac{\|gx^*\|}{\|x^*\|}.$$

Now $\mu(g,x)$ is bounded below on Gx if and only if $0 \notin \overline{Gx^*}$, or if and only if x is semistable.

By the same argument as above we will obtain that for a stable point x and a positive number M the set $\{gx\colon \mu(x,g) \leqslant M\}$ is closed in X. The second statement of Part 1 follows.

To prove other two parts we need to show the positivity of the second variation of μ.

LEMMA 2.3. *Let λ be an 1-phs of $\bar{G} = (G,U)$ and x be a point of X. Let $f(t) = \mu(\lambda(e^t),x)$ as a function on $\mathbb{R}$.*

(1) *If the metric on $\mathcal{L}$ is semipositive then $f(t)$ is concave up on $\mathbb{R}$: for any two real numbers a and b*
$$\frac{f(a)+f(b)}{2} \geqslant f\left(\frac{a+b}{2}\right).$$
(2) *If the metric on $\mathcal{L}$ is positive then $f(t)$ is strictly concave up on $\mathbb{R}$: for any two distinct real numbers a and b*
$$\frac{f(a)+f(b)}{2} > f\left(\frac{a+b}{2}\right).$$

Proof. Let $g\colon \mathbb{C} \to X$ be the holomorphic map such that $g(z) = \lambda(e^z)x$ and $\mathcal{M} = g^*\mathcal{L}$ with pullback metric which we still denote as $\|\cdot\|$. It is positive (resp. semipositive) if the metric on $\mathcal{L}$ is positive (resp. semipositive). Let l be a nonzero section of $\mathcal{L}$ at x with norm 1 and $m = g^*l$. Then $f(t) = -\log\|m\|(t)$. Since the metric on $\mathcal{L}$ is invariant under the action of U, the function $-\log\|m\|(z)$ as on $\mathbb{C}$ is invariant under transformation $z \to z + bi$ for any $b \in \mathbb{R}$. In other words it is determined by its restriction f to real numbers

$$-\log\|m\|(z) = -\log\|m\|\left(\frac{z+\bar{z}}{2}\right) = f\left(\frac{z+\bar{z}}{2}\right).$$

We claim that f'' as a distribution on smooth functions with compact support on $\mathbb{R}$ is positive (resp. semipositive) if $\|\cdot\|$ on $\mathcal{M}$ is positive (resp. semipositive). This means that for any bounded open subset V of $\mathbb{R}$ and any smooth function g with compact support on V, the integral

$$\int_{\mathbb{R}} f'' g \, \mathrm{d}x := \int_{\mathbb{R}} f g'' \, \mathrm{d}x,$$

is positive (resp. semipositive) if g is semipositive and nonzero. Our lemma will follow this claim and a well known fact that a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is convex (resp. strictly convex) if and only if the distribution (resp. positive with support $\mathbb{R}$). See for instance Theorem 4.1.6 in [H]. (I am grateful to the referee for telling me this reference.)

Let us prove our claim on f''. Assume $(\mathcal{M}, \|\cdot\|)$ is positive (resp. semipositive). Fix an open subset U of $\mathbb{R}$ and a positive smooth function h on U with compact support and integral 1. For any smooth function g with compact support on U, setting $\phi(x+yi) = g(x)h(y)$ and noticing that

$$\frac{\partial\bar{\partial}}{\pi i}(g(x)h(y)) = \frac{-1}{2\pi}(g''(x)h(y) + g(x)h''(y))\, \mathrm{d}x\, \mathrm{d}y,$$

we have

$$\begin{aligned} \int_{\mathbb{C}} \phi c_1'(\mathcal{M}, \|\cdot\|) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x) g''(x) h(y)\, \mathrm{d}x\, \mathrm{d}y \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} f(x) g''(x)\, \mathrm{d}x. \end{aligned}$$

It follows that the integral $\int_R f g''\, \mathrm{d}x$ is positive (resp. semipositive) if g is semipositive and nonzero. Our claim and therefore the lemma follow.

We also need the Cartan decomposition theorem:

LEMMA 2.4. *Let S be the set of* 1*-phs of* (G, U) *then*

$$G = \bigcup_{\lambda \in S} U\lambda(\mathbb{R}_+^*).$$

2.5. Proof of 2nd and 3rd parts of (2.2)

Let $y = gx \neq x$ be any point in the orbit containing x. By the Cartan decomposition theorem we have an 1-phs λ, a real number t_0, and a unitary element $u \in U$ such that $g = u\lambda(e^{t_0})$. Let $f(t) = \mu(\lambda(e^t), x)$ then

$$\mu(e, x) = f(0), \qquad \mu(g, x) = \mu(e, y) = f(t_0).$$

If $\mathcal{L}$ is semipositive then f is concave up by (2.3), the only critical points of f are minimal points, and the set of minimal points must be a closed interval. This implies that x is a critical point if and only if at x the function μ has minimal value. If both x and y are critical points then all points in the curve $y(t) = u\lambda(e^t)x$ $(0 \leqslant t \leqslant t_0)$ are all critical points. The first two statements of Part 2 follow immediately. The third statement of Part 2 follows from these two and Part 1.

If $\mathcal{L}$ is positive then f is strictly concave up, so f has at most one critical point. If x and y are both critical points then $t_0 = 0$. It follows that y is in the orbit of U containing x. This proves the third part of (2.2).

The proof of (2.2) is complete.

2.6. Semistable reductions at non-archimedean places

Instead of complex variety we may also consider a variety defined over an algebraically closed nonarchimedean field. Let K be a discrete valuation field with valuation ring R. Let $\tilde{X}$ be a projective and flat variety over Spec R and $\tilde{\mathcal{L}}$ be an ample line bundle on X. Let $\tilde{G}$ be a geometrically connected group scheme over Spec R with a linear action on $(\tilde{X}, \tilde{\mathcal{L}})$. Denote by $X, \mathcal{L}, G$ the geometric fibers of $\tilde{X}, \tilde{\mathcal{L}}, \tilde{G}$ on Spec $\bar{K}$, where $\bar{K}$ is an algebraic closure of K with valuation ring $\bar{R}$. Let $\|\cdot\|$ be the $\bar{K}$ norm on $\mathcal{L}$ induced by the integral structure $\tilde{L}$, and let U denote $G(\bar{R})$. Then G acts on $(X, \mathcal{L})$ and the norm $\|\cdot\|$ is invariant under the induced action of U.

By an one-dimensional parameter Hermitian subgroup of $\bar{G}$ we mean an injective homomorphism $\lambda\colon \bar{\mathbb{G}}_m = (\mathbb{G}_m, \mathbb{U}_m) \to \bar{G} = (G, U)$ which is induced by an injective homomorphism $\lambda\colon \mathbb{G}_m \to G$ over Spec $\bar{R}$ of group schemes. As before we abbreviate this to 'λ is a 1-phs of $\bar{G}$'.

We define $\mu(g, x)$ as in Section 2.1. We fix a uniformizer π of K and a set of roots $\{\pi^{1/n}\colon n = 1, 2, \ldots,\}$ such that $(\pi^{1/mn})^m = \pi^{1/n}$. For any rational number $t = a/b$, $((a, b) = 1, b > 0)$ we define π^t to be $(\pi^{1/b})^a$. Then for any 1-phs λ of $\bar{G}$ we define $f(t) = \mu(\lambda(\pi^t), x)$ over $\mathbb{Q}$. It is easy to see that the function $f(t)$ does not depend on the choices of π and its roots. We call a point x in X critical with respect to the Hermitian action, if for any 1-phs λ, the function $f(t) = \mu(\lambda(\pi^t), x)$ is critical at $t = 0$ as in (2.1).

THEOREM 2.7. *Let $(X, \bar{\mathcal{L}}, \bar{G})$ be assumed as above, and let x be a point of X.*

(1) *The function* $\mu(g,x)$ *is bounded below as a function on the orbit* Gx *iff* x *is semistable with respect to the linear action of* G*. Moreover if* x *is semistable, then the infimum value of this function is reachable on the orbit* Gx*.*
(2) *The equality* $\inf_{g\in G}\mu(g,x)=\mu(e,x)$ *holds iff* x *is a critical point of* X *where* e *is the unit element of* G*. Moreover, if* x *and* $y=gx$ *are both critical points, writing* $g=u_0\lambda(\pi^{t_0})$ *with* u *in* U *and* t *in* $\mathbb{Q}$ *by the Cartan decomposition theorem, then for any* $u\in U$ *and any* t *between* 0 *and* t_0*, the point* $u\lambda(\pi^t)$ *is critical.*
(3) *The set of critical points of* X *is the set of points of* X *which have semistable reductions over* $\operatorname{Spec}\bar{R}$*. Moreover any stable point has a semistable reduction.*

Proof. (1) Replacing $\mathcal{L}$ by a power and embedding X to projective spaces by sections of $\mathcal{L}$, we may assume that $X=\mathbb{P}^n$ and $\mathcal{L}=\mathcal{O}(1)$ with metric induced from the standard metric on $\bar{K}^{n+1}$

$$\|(x_0,\ldots,x_n)\|=\max_i|x_i|.$$

For $x^*\in\bar{K}^{n+1}$ be a point which has image x in $\mathbb{P}^n$, then

$$\mu(g,x)=\log\frac{\|gx^*\|}{\|x^*\|}.$$

The function $\mu(g,x)$ is bounded below on Gx if and only if $\|gx^*\|$ is bounded below by a positive constant. The first statement of Part 1 of the theorem follows from the definition of stability, while the second statement follows from Proposition 2 in [Bu].

(2) Let $y=gx$ be any point in the orbit of x. By the Cartan decomposition theorem we have an 1-phs λ, a rational number t_0, and a element $u_0\in U$ such that $g=u_0\lambda(\pi^{t_0})$. Let $f(t)=\mu(\lambda(\pi^t),x)$ then

$$\mu(e,x)=f(0),\qquad \mu(g,x)=\mu(e,y)=f(t_0).$$

Diagonalizing $\bar{K}^{n+1}$ according to the action of λ, we may assume that

$$\lambda(a)(a_0,\ldots,a_n)=(a^{\alpha_0}a_0,\ldots,a^{\alpha_n}a_n),$$

where $\alpha_0,\ldots,\alpha_n$ are integers. If $x^*=(x_0,\ldots,x_n)$ has image x then

$$f(t)=\log\frac{\|\lambda(\pi^t)x^*\|}{\|x^*\|}=\max_i\left(t\alpha_i\log|\pi|+\log\left(\frac{|x_i|}{\|x^*\|}\right)\right).$$

So $f(t)$ is a concave up function of $\mathbb{Q}$. The only critical points are minimal points, and the set of minimal points must be a closed interval of $\mathbb{Q}$. This implies that x is a critical point iff at x the function $\mu(g,x)$ has minimal value. If both x and y are critical points then they are minimal, so are points with form $u\lambda(\pi^t)$ for $u\in U$ and t between 0 and t_0. This proves Part 2.

(3) This follows from Part 1 and 2, and Proposition 1 in [Bu].

3. Reductions of stable cycles at Archimedean places

3.1. DEFINITION OF CRITICAL METRICS

By an effective cycle of dimension n, we mean a cycle of a codimension 0 on a reduced scheme $X_{\text{red}} = \bigcup X_i$ whose irreducible components are of dimension n: $X = \Sigma_i m_i X_i$, $m_i \geqslant 0$. Let $\mathcal{L}$ be a line bundle on X_{red}, and let $V \subset \Gamma(X, \mathcal{L})$ be a linear system of dimension $N + 1$ which defines an embedding $X_{\text{red}} \subset \mathbb{P}(V)$. We say X is stable (resp. semistable) with respect to V if the corresponding Chow point in $\mathbb{P}[\text{Sym}^d(V)^{\otimes(n+1)}]$ is stable (resp. semistable), where $d = \Sigma_i m_i \deg_{\mathcal{L}}(X_i)$ is the degree of X. Now we assume that X_{red} is a complex projective variety. Let $\| \cdot \|$ be a semipositive and smooth metric on $\mathcal{L}$. We can define the distortion function $b_{\|\cdot\|}(x)$ on X_{red} as follows. Let $\langle , \rangle$ be the L^2-Hermitian structure defined by the metric $\| \cdot \|$ on $\mathcal{L}$ as follows

$$\langle s_1, s_2 \rangle = \sum_k m_k \int_{X_k} \langle s_1, s_2 \rangle(x)\, \mathrm{d}x,$$

where $\mathrm{d}x = c_1'(\mathcal{L}, \| \cdot \|)^n / \deg X$ is the induced measure on X. Let $\{s_0, \ldots, s_N\}$ be an orthonormal basis of V with respect to this Hermitian structure. Then

$$b_{\|\cdot\|}(x) = \frac{1}{N+1} \sum_{i=0}^{N} \|s_i\|^2(x).$$

It is easy to see that $b_{\|\cdot\|}$ is independent of choice of the s_i's and has integral 1. We say that $\| \cdot \|$ is critical with respect to V if $b_{\|\cdot\|} = 1$. The main result of this section is the following theorem.

THEOREM 3.2. *Let $X, \mathcal{L}$ be given as above.*

(1) *If $\| \cdot \|$ is a critical metric on $\mathcal{L}$ with respect to V then X is semistable with respect to V.*
(2) *If X is stable with respect to V then there exists a unique critical metric on $\mathcal{L}$ with respect to V.*

3.3. A CHARACTERIZATION OF CRITICAL METRICS

We say a metric $\| \cdot \|_{\text{FS}}$ on $\mathcal{L}$ is the Fubini–Study metric induced by a Hermitian structure on V if for an orthonormal basis $\{s_0, \ldots, s_N\}$ of V

$$\|s\|_{\text{FS}} = \left(\sum_{i=0}^{N} \left| \frac{s_i}{s} \right|^2 \right)^{-1/2}.$$

First of all we claim that a metric $\| \cdot \|$ on $\mathcal{L}$ is critical with respect to V if and only if $\| \cdot \|^2 = (N + 1) \| \cdot \|_{\text{FS}}^2$, where $\| \cdot \|_{\text{FS}}$ is the Fubini–Study metric induced by

L^2-Hermitian structure induced by $\|\cdot\|$. In fact, let $\{s_0, \ldots, s_N\}$ be an orthonormal basis of V induced the norm $\|\cdot\|$ on $\mathcal{L}$. The induced Fubini–Study metric $\|\cdot\|_{\mathrm{FS}}$ is

$$\|s\|_{\mathrm{FS}}^2(x) = \left(\sum_{i=0}^{N} \frac{\|s_i\|^2(x)}{\|s\|^2(x)}\right)^{-1},$$

which is $\|s\|^2(x)/(N+1)$ if and only if $\Sigma\|s_i\|^2(x) = N+1$, and by definition if and only if $\|\cdot\|$ is critical.

By this for the proof of (3.2), we may restrict our discussions on Fubini–Study metrics induced by the set M of Hermitian structures on V. For any $m \in M$, let $\mathcal{L}_m = (\mathcal{L}, \|\cdot\|_m)$ be the line bundle $\mathcal{L}$ with the corresponding Fubini–Study metric. Then the Deligne pairing $\mathcal{L}_m^{\langle n+1\rangle}$ is the one-dimensional vector space $\mathcal{L}^{\langle n+1\rangle}$ with a metric depending on m. Fix one m in M. For any g in $\mathrm{SL}(V)$ one defines a new Hermitian structure gm as follows: for any $v \in V$, $\|v\|_{gm} := \|gv\|_m$. Let $\nu(g)$ be a real function on $\mathrm{SL}(V)$ defined by the isometry $\mathcal{L}_{gm}^{\langle n+1\rangle} \simeq \mathcal{L}_m^{\langle n+1\rangle} \otimes \mathcal{O}(\nu(g))$. Then we have the following characterization of critical metrics.

THEOREM 3.4. *The Fubini–Study metric induced by the Hermitian structure m on V is critical with respect to V if and only if, for any 1-parameter subgroup $\lambda\colon \mathbb{C}^* \to \mathrm{SL}(V)$ which sends elements with absolute value 1 to unitary elements of $\mathrm{SL}(V)$, the real function $f(t) = \nu(\lambda(e^t))$ is critical at $t = 0$.*

Proof. To give an 1-parameter subgroup λ as in the theorem is equivalent to give an isometry $V \simeq \mathbb{C}^{N+1}$ and integers a_i $(0 \leqslant i \leqslant N)$ such that $\Sigma a_i = 0$. The corresponding action $\lambda(g)$ has the form

$$\lambda(g)(x_0, \ldots, x_N) = (g^{a_0}x_0, \ldots, g^{a_N}x_N).$$

Fix a λ and choose an isometry $V \simeq \mathbb{C}^{N+1}$ and a_i's as above. Let ϕ_t be the real function defined by the isometry

$$\mathcal{L}_{\lambda(e^t)m} = \mathcal{L}_m \otimes \mathcal{O}(\phi_t),$$

then

$$\begin{aligned}\mathcal{O}(f(t)) &= (\mathcal{L}_m \otimes \mathcal{O}(\phi_t))^{\langle n+1\rangle} \otimes (\mathcal{L}_m^{\langle n+1\rangle})^{-1} \\ &= \mathcal{O}\left(\sum_{i=1}^{n} \int_X \phi_t \left[-\frac{\partial\bar{\partial}}{\pi i}\phi_t\right]^{n-i} c_1'(\mathcal{L}_m)^i\right). \end{aligned} \qquad (3.4.1)$$

To prove the theorem, we need to study the leading terms of the Taylor expansions of ϕ_t and $(\partial\bar{\partial}/\pi i)\phi_t$. Let s be a section of $\mathcal{O}(1)$, then

$$\phi_t(x_0, \ldots, x_N) = -\log \frac{\|s\|_{\lambda(e^t)m,\mathrm{FS}}}{\|s\|_{m,\mathrm{FS}}}$$

$$
\begin{aligned}
&= -\tfrac{1}{2}\log\frac{\sum_i |x_i|^2}{\sum_i \mathrm{e}^{2a_i t}|x_i|^2} \\
&= \tfrac{1}{2}\log\sum_i \mathrm{e}^{2a_i t}\|x_i\|^2,
\end{aligned}
$$

where $\|s\|_{m,\mathrm{FS}}$ be the Fubini–Study metric induced by the Hermitian structure m of V, and $\|x_i\| = |x_i|/\sqrt{\Sigma_i |x_i|^2}$ is the norm of x_i as a section of $\mathcal{L}$. Using the identity $\Sigma_i \|x_i\|^2 = 1$, one has

$$
\begin{aligned}
\phi_t &= \tfrac{1}{2}\log\left[\sum_i (1+2a_i t)\|x_i\|^2 + O(t^2)\right] \\
&= \tfrac{1}{2}\log\left[1 + 2t\sum_i a_i\|x_i\|^2 + O(t^2)\right] \\
&= t\left(\sum_i a_i\|x_i\|^2\right) + O(t^2),
\end{aligned}
$$

where $O(t^2)$'s are all smooth functions on $\mathbb{P}^N$ for small t. Similarly

$$
\begin{aligned}
\frac{\partial\bar{\partial}}{\pi i}\phi_t &= \frac{\partial}{2\pi i}\frac{\sum_i \mathrm{e}^{2a_i t}\bar{\partial}\|x_i\|^2}{\sum_i \mathrm{e}^{2a_i t}\|x_i\|^2} \\
&= \frac{1}{2\pi}\left[\frac{\sum_i \mathrm{e}^{2a_i t}\partial\bar{\partial}\|x_i\|^2}{\sum_i \mathrm{e}^{2a_i t}\|x_i\|^2} - \frac{(\sum_i \mathrm{e}^{2a_i t}\partial\|x_i\|^2)(\sum_i \mathrm{e}^{2a_i t}\bar{\partial}\|x_i\|^2)}{(\sum_i \mathrm{e}^{2a_i t}\|x_i\|^2)^2}\right] \\
&= \frac{1}{2\pi}\frac{2t\sum_i a_i\partial\bar{\partial}\|x_i\|^2 + O(t^2)}{1+O(t)} \\
&= t\sum_i a_i\frac{\partial\bar{\partial}}{\pi i}\|x_i\|^2 + O(t^2),
\end{aligned}
$$

where $O(t)$'s and $O(t^2)$'s are smooth functions and smooth forms on $\mathbb{P}^N$ for small t. Bring these to (3.4.1), one has

$$
\begin{aligned}
f(t) &= \sum_{j=1}^{n}\int_X \left[t\sum_i a_i\|x_i\|^2 + O(t^2)\right] \\
&\quad \times \left[-t\sum_i a_i\frac{\partial\bar{\partial}}{\pi i}\|x_i\|^2 + O(t^2)\right]^{n-j} c_1'(\mathcal{L}_m)^j \\
&= \int_X t\sum_i a_i\|x_i\|^2 c_1'(\mathcal{L}_m)^n + O(t^2)
\end{aligned}
$$

$$= t \deg X \sum_i a_i \|x_i\|_{L^2}^2 + O(t^2), \tag{3.4.2}$$

where $\|\cdot\|_{L^2}$ is the Hermitian structure on V induced by the Hermitian metric on $\mathcal{L}_m$ and the measure $\mathrm{d}x = c_1'(\mathcal{L}_m)^{n+1}/\deg X$. From (3.4.2), one sees that $f(t)$ is critical at $t = 0$ for all 1-parameter subgroup of $\mathrm{SL}(V)$ as in the theorem if and only if for any orthonormal basis $\{s_0, \ldots, s_N\}$ of V and any integers a_i $(0 \leqslant i \leqslant N)$ with $\Sigma a_i = 0$, the following equation holds

$$\sum_i a_i \|s_i\|_{L^2}^2 = 0.$$

Since $\Sigma_i \|x_i\|_{L^2}^2 = N + 1$, $f(t)$ is critical at $t = 0$ for all 1-parameter subgroup as in the theorem if and only if $\{s_0, \ldots, s_N\}$ is an orthonormal basis of V with respect to the Hermitian structure induced by the metric on $\mathcal{L}_m$. Now the theorem follows from the discussion in (3.3).

3.5. Proof of (3.2)

Fix a Hermitian structure m on V. Let $z \in \mathbb{P}[(\mathrm{Sym}^d\ V)^{\otimes(n+1)}]$ be the Chow point corresponding to X. Then (1.6) gives a canonical isometry $\mathcal{L}_m^{\langle n+1\rangle} \simeq \mathcal{N}(z)$, where $\mathcal{N}$ is the $O(1)$ bundle of $\mathbb{P}[(\mathrm{Sym}^d\ V)^{\otimes(n+1)}]$ with the Chow metric defined in (1.5). It follows that $\nu(g) = \mu(g, z)$, where μ is defined in (2.1). Now (3.2) follows from (3.4), (2.2), and the positivity of the Chow metric on $\mathcal{N}$:

THEOREM 3.6. *For the standard Hermitian structure on* $\mathcal{E} = \mathbb{C}^{N+1}$, *consider the induced Chow metric* $\|\cdot\|_{\mathrm{Ch}}$ *of the* $\mathcal{O}(1)$ *bundle* $\mathcal{N}$ *on* $\mathbb{P}[(\mathrm{Sym}^d\ \mathbb{C}^{N+1})^{\otimes(n+1)}]$ *defined in* (1.5).

(1) *The metric* $\|\cdot\|_{\mathrm{Ch}}$ *is continuous and positive in the sense of* (2.1).
(2) *The metric* $\|\cdot\|_{\mathrm{Ch}}$ *is ample. This means that for any point* x *in* X *and any* $\varepsilon > 0$, *there exists a nonzero section* l *of a positive power* $\mathcal{N}^m$ *such that* $\|l\|_{\sup} \leqslant \mathrm{e}^{m\varepsilon}\|l\|(x)$.

Proof. (1) We prove the continuity first. By definition in 1.5 we need only prove that the integral

$$\int_{\mathbb{P}(\mathcal{E}^\vee)^{n+1}} \log \|w'\| \bigwedge c_1'(\mathcal{M}_i)^N,$$

defines a continuous function on $\mathbb{P}[(\mathrm{Sym}^d\ \mathbb{C}^{N+1})^{\otimes(n+1)}]$. We prove this by using an argument in the proof of Proposition 1.5.1 in [B–G–S]. Let (p, q) be a point in $\mathbb{P}[(\mathrm{Sym}^d\ \mathbb{C}^{N+1})^{\otimes(n+1)}] \times \mathbb{P}(\mathcal{E}^\vee)^{n+1}$. Then in term of local coordinates $x = (x_1, x_2, \ldots, x_{N(n+1)})$ in a neighborhood of p, we can write

$$\log \|w'\| \bigwedge c_1'(\mathcal{M}_i)^N$$

$$= a(p,x)(\log|f(p,x)| + g(p,x)) \prod_{j=1}^{N(n+1)} \mathrm{d}x_j \wedge \mathrm{d}\bar{x}_j,$$

where $a(p,x)$ and $g(p,x)$ are smooth functions of p and x, and $f(p,x)$ is polynomial in x with coefficients holomorphic in p. By Weierstrass' preparation theorem, after a possible linear change of the local coordinates x, we may write in a neighborhood of q

$$f(p,x) = \left(x_1^h + \sum_{j=1}^{h} s_j(p, x_2, \dots, x_{N(n+1)}) x_1^{h_j} \right) u(p,x),$$

where s_j and u are holomorphic functions such that $u(p,0) \neq 0$. By Lemma 1.5.3 in [B–G–S], for a small neighborhood U of 0 in $\mathbb{C}$, a small neighborhood V of 0 in $\mathbb{C}^{N(n+1)-1}$, a small neighborhood W of p in $\mathbb{P}[(\mathrm{Sym}^d\ \mathbb{C}^{N+1})^{\otimes(n+1)}]$, and any smooth function $\rho(x_1, \dots x_{N(n+1)}, p)$ in $U \times V \times W$ with compact support, one has that the integral

$$\int_U \rho(x_1, \dots, x_{N(n+1)}, p) \log|f(p,x)|\, \mathrm{d}x_1 \wedge \mathrm{d}\bar{x}_1,$$

defines a continuous function $\psi(x_2, \dots, x_{N(n+1)}, p)$ on $V \times W$. By Fubini's Theorem we have

$$\int_{U \times V} \rho(x_1, \dots, x_{N(n+1)}, p) \log|f(p,x)|\, \mathrm{d}x_1 \wedge \mathrm{d}\bar{x}_1$$

$$= \int_V \psi(x_2, \dots x_{N(n+1)}) \prod_{j=2}^{N(n+1)} \mathrm{d}x_j \wedge \mathrm{d}\bar{x}_j.$$

It follows that the integral

$$\int_{U \times V} \rho(x_1, \dots, x_{N(n+1)}, p) \log|f(p,x)| \prod_{j=1}^{N(n+1)} \mathrm{d}x_j \wedge \mathrm{d}\bar{x}_j,$$

defines a continuous function on W. Using partitions of unity, this shows the continuity of

$$\int_{\mathbb{P}(\mathcal{E}^\vee)^{n+1}} \log\|w'\| \wedge c_1'(\mathcal{M}_i)^N.$$

Now we want to prove the positivity. Let $f\colon \mathbb{D} \to \mathbb{P}[(\mathrm{Sym}^d\ \mathbb{C}^{N+1})^{\otimes(n+1)}]$ be a finite morphism. We need to show that for any smooth function ϕ on $\mathbb{D}$ with compact support, the integral $\int_{\mathbb{D}} \phi c_1'(f^*(\mathcal{N}, \|\cdot\|_{\mathrm{Ch}}))$ is positive if ϕ is semipositive and nonzero. Let $\Gamma' \in \mathbb{P}[(\mathrm{Sym}^d\ \mathbb{C}^{N+1})^{\otimes(n+1)}] \times \mathbb{P}(\mathbb{C}^{N+1})^{n+1}$ be the universal hypersurface in $\mathbb{P}(\mathbb{C}^{N+1})^{n+1}$ of degree $(d, \ldots, d)$. By (1.5.2) we have an isometry

$$(\mathcal{N}, \|\cdot\|_{\mathrm{Ch}}) \simeq \langle \mathcal{M}_0^{\cdot N}, \ldots, M_n^{\cdot N} \rangle \left(\frac{\Gamma'}{\mathbb{P}[(\mathrm{Sym}^d\ \mathbb{C}^{N+1})^{\otimes(n+1)}]} \right) \otimes \mathcal{K},$$

where $\mathcal{M}_i$ endows with Fubini–Study metric and $\mathcal{K}$ is a constant line bundle with a constant metric. By [D, Sect. 8.5] we have

$$c_1'(\mathcal{N}, \|\cdot\|_{\mathrm{Ch}}) = \int_{\Gamma'/\mathbb{P}[(\mathrm{Sym}^d\ \mathbb{C}^{N+1})^{\otimes(n+1)}]} \bigwedge_{i=0}^{n} c_1'(\mathcal{M}_i)^N.$$

Let $p\colon \Gamma'_{\mathbb{D}} \to \mathbb{D}$ be the pull-back by f of $\Gamma' \to \mathbb{P}[(\mathrm{Sym}^d\ \mathbb{C}^{N+1})^{\otimes(n+1)}]$. Since f is finite, the morphism $q\colon \Gamma'_D \to \mathbb{P}[(\mathrm{Sym}^d\ \mathbb{C}^{N+1})^{\otimes(n+1)}]$ is generically finite. There is a Zariski open subset U of Γ'_D which is étale over $\mathbb{P}[(\mathrm{Sym}^d\ \mathbb{C}^{N+1})^{\otimes(n+1)}]$. The image $V = q(U)$ of U is a nonempty open subset (in the complex topology) of $\mathbb{P}[(\mathrm{Sym}^d\ \mathbb{C}^{N+1})^{\otimes(n+1)}]$. If ϕ is semipositive and nonzero, then

$$\begin{aligned}\int_{\mathbb{D}} c_1'(\mathcal{N}, \|\cdot\|_{\mathrm{Ch}})\phi &= \int_{\Gamma'_D} (p^*\phi) q^* \bigwedge_{i=0}^{n} c_1'(\mathcal{M}_i)^N \\ &= \int_V (q_* p^* \phi) \bigwedge_{i=0}^{n} c_1'(\mathcal{M}_i)^N \\ &> 0.\end{aligned}$$

This proves the Part 1 of the theorem.

(2) By Part 1 and Theorem 2.2 in [Z1] (Please notice that the ample metric here was called semiample metric in [Z1]), we need only show that for a projective space $\mathbb{P}^M$ and a positive continuous metric $\|\cdot\|$ on $\mathcal{O}(1)$, there is a sequence of smooth metrics $\|\cdot\|_n$ on $\mathcal{O}(1)$ convergent uniformly to $\|\cdot\|$ and such that the curvatures of $\|\cdot\|_n$ are positive. Let ϕ be a positive function on $U := \mathbb{U}(M+1)$ such that $\int_U \phi\, \mathrm{d}u = 1$ where $\mathrm{d}u$ is the invariant measure on U with volume 1. We define a new metric $\|\cdot\|_\phi$ on $\mathcal{O}(1)$ as follows. If s is a section of $\mathcal{O}(1)$, then

$$\log \|s\|_\phi(x) = \int_U \log \|u^{-1}s\|(u^{-1}x)\phi(u)\, \mathrm{d}u,$$

where U acts on both $\mathcal{O}(1)$ and $\mathbb{P}^M$ in the standard way. We claim at first that $\|\cdot\|_\phi$ is smooth. For this let $\|\cdot\|_0$ be the standard Fubini–Study metric which is smooth and invariant under the action of U. Consider functions $f(x) = \log(\|\cdot\|(x)/\|\cdot\|_0(x))$ and $f_\phi(x) = \log(\|\cdot\|_\phi(x)/\|\cdot\|_0(x)$ then

$$f_\phi(x) = \int_U f(u^{-1}x)\phi(u)\, \mathrm{d}u.$$

Fix a point x_0 on $\mathbb{P}^M$, and let $p\colon U \to \mathbb{P}^M$ be the map $p(g) = gx_0$. Then $p^*(f_\phi)(g) = \int_U p^*(f)(u^{-1}g)\phi(u)\,\mathrm{d}u$, now $p^*(f_\phi)$ is smooth by standard argument on convolution of distributions. Since p is surjective and smooth, it follows that f_ϕ is smooth, and so is $\|\cdot\|_\phi$. We claim at second that $\|\cdot\|_\phi$ has positive curvature. This follows from the following identity of distributions and the positivity of ϕ

$$c'(\mathcal{O}(1), \|\cdot\|_\phi) = \int_U c_1'(\mathcal{O}(1), \|\cdot\|)(u^{-1}x)\phi(u)\,\mathrm{d}u.$$

Finally we choose a sequence of positive smooth functions ϕ_n on U such that $\mathrm{supp}(\phi_n) \to \{e\}$ as $n \to \infty$, where e is the unit element of U. Then the metrics $\|\cdot\|_{\phi_n}$ converges to $\|\cdot\|$. This proves the second part of the theorem.

3.7. Questions

(1) Let X be a variety in $\mathbb{P}(V)$, where V is a finite dimensional vector space over a discrete valuation field K with valuation ring R. Assume that X is stable in $\mathbb{P}(V)$. Then (2.7) tells us that after replacing K by a finite extension, there is a free submodule $\tilde{V}$ over R such that the Zariski closure $\tilde{X}$ of X in $\mathbb{P}(\tilde{V})$ has a semistable special fiber. Motivated by (3.2), I would like to ask the following question: *Can one describe $\tilde{X}$ without the reference to group action?* For example, by Mumford [M, Sect. 5.1], a (moduli) stable curve X in sense of Deligne–Mumford [D–M] is stable in $\mathbb{P}(\Gamma(\omega_X^n))$ when $n \geqslant 5$.

(2) Let X be a smooth complex variety and let $\mathcal{L}$ be an ample line bundle such that X is stable in $\Gamma(\mathcal{L}^n)$ for $n \geqslant 1$. Fix a section l of $\mathcal{L}$ at a point x in X. Let $\|\cdot\|_n$ be the metrics on $\mathcal{L}$ such that $\|l\|_n = 1$ and $\|\cdot\|_n^n$ give critical metrics on $\mathcal{L}^n$ by (3.2). Motivated by a conjecture of Mumford on the existence of asymptotically stable limits [F–M, p. 187], I would like to ask the following question: *Does the critical metrics $\|\cdot\|_n$ converges to a continuous metric on $\mathcal{L}$?*

(3) Let X be a complex curve or surface with ample canonical line bundle ω_X. By Gieseker [G], Mumford [M] we know that X is stable in $\Gamma(\omega_X^n)$ for n sufficiently large. (For higher dimensional varieties, Viehweg [V] proves some stability on Hilbert points.) By Riemann's uniformization theorem and Yau's Theorem [Y1] we know the existence of Kähler–Einstein metric on X. *If the critical metrics on ω_X converges, is the limit equal to a Kähler–Einstein metric?* I should mention that the idea that projective stability is related to the existence of a Kähler–Einstein metric is due to Yau and Tian ([Y2], [T], [D–T]).

4. Heights of semistable cycles

4.1. Definition of heights

With $S = \mathrm{Spec}\,\mathbb{Z}$, $\mathcal{E} = \mathcal{O}^{N+1}$, and the standard Hermitian structure at $\mathcal{E} \otimes \mathbb{C} = \mathbb{C}^{N+1}$, constructions in Section 1 give a Chow metric $\|\cdot\|_{\mathrm{Ch}}$ on the $\mathcal{O}(1)$ bundle of $\mathbb{P}[(\mathrm{Sym}^d\ \mathcal{E})^{\otimes(n+1)}]$. Let $(P, \lambda, \|\cdot\|_{\mathrm{Ch}})$ denote the uniform categorical quotient of semistable points of $(\mathbb{P}[(\mathrm{Sym}^d\ \mathcal{E})^{\otimes(n+1)}], \mathcal{N}, \|\cdot\|_{\mathrm{Ch}})$ by the action of group scheme $\mathrm{SL}(N+1)$. Notice that for a section k of λ at $p \in P$, the norm $\|k\|$ is defined to be the supremum norm of k on the fiber of $\mathbb{P}[(\mathrm{Sym}^d\ \mathcal{E})^{\otimes(n+1)}]$ over p, where we also consider k as a rational section of $\mathcal{N}$ on this fiber [Bu, Z2]. We will show at the end of the section that the metric on λ is continuous and ample. For any semistable cycle X of dimension n and degree d in $\mathbb{P}^N_{\bar{\mathbb{Q}}}$, let $p \in P(\bar{\mathbb{Q}})$ be the quotient of the corresponding Chow point. We define the G.I.T height $\hat{h}(X)$ to be $h_{(\lambda,\|\cdot\|)}(x)/(n+1)d$. Notice that if x is defined over a number field K which extends to a finite morphism $\tilde{x}\colon \mathrm{Spec}\, \mathcal{O}_K \to P$ then $h_{\lambda,\|\cdot\|}(x) = \deg \tilde{x}^*(\lambda, \|\cdot\|)/[K\colon \mathbb{Q}]$.

For a (not necessary semistable) cycle $X \subset \mathbb{P}^N_{\bar{\mathbb{Q}}}$ defined over a number field K and a Hermitian vector bundle $\mathcal{E}$ on $\mathrm{Spec}\, \mathcal{O}_K$ with an isomorphism $\mathcal{E}_K \simeq K^{N+1}$, let $\tilde{X}$ be the Zariski closure of X in $\mathbb{P}(\mathcal{E})$ and $\mathcal{L}_\mathcal{E}$ be the restriction of the $\mathcal{O}(1)$ bundle of $\mathbb{P}(\mathcal{E})$ on $\tilde{X}$. We define the height $h_\mathcal{E}(X)$ of X with respect to $\mathcal{E}$ by the following formula

$$h_\mathcal{E}(X) = \frac{c_1(\mathcal{L}_\mathcal{E})^{n+1}}{(n+1)\deg X[K:\mathbb{Q}]} - \frac{\deg \mathcal{E}}{(N+1)[K:\mathbb{Q}]}.$$

It is easy to see that $h_\mathcal{E}(X)$ is invariant if we replace K by an extension and replace X and $\mathcal{E}$ by their base changes. The following proposition gives a characterization of G.I.T height without reference to group action:

PROPOSITION 4.2. *Let $\mathcal{V}$ be the direct limit of the set $\mathcal{V}_K$ of vector bundles on* $\mathrm{Spec}\, \mathcal{O}_K$ *with an identity $\mathcal{E} \otimes K = K^{N+1}$ as K varies in the set of number fields. Let X be a cycle of $\mathbb{P}^{N+1}$ over $\bar{\mathbb{Q}}$.*

(1) *The cycle X is semistable if and only if $h_\mathcal{E}(X)$ is bounded below as a function of $\mathcal{E} \in \mathcal{V}$.*
(2) *If X is semistable then $\hat{h}(X) = \inf_{\mathcal{E}\in\mathcal{V}} h_\mathcal{E}(X)$.*
(3) *If X is stable, there is a Hermitian vector bundle $\mathcal{E}$ in $\mathcal{V}$ defined over a number field K such that* (i) *the canonical isomorphism* $\det \mathcal{E} \otimes K \simeq \bar{K}$ *induces an isometry* $\det \mathcal{E} \simeq \mathcal{O}_K$, (ii) *the Zariski closure $\tilde{X}$ of X in $\mathbb{P}(\mathcal{E})$ has semistable fibers at all finite places, and* (iii) *the Hermitian structures at archimedean places of K induces the critical metrics on $\mathcal{O}(1)|_X$.*
(4) *If X is semistable, let X^* be a stable limit cycle of X in $\mathbb{P}^N$. Then $\hat{h}(X) = h_\mathcal{E}(X^*)$ where $\mathcal{E}$ is defined for X^* as in Part* 3.

Proof. For any $\mathcal{E} \in \mathcal{V}_K$, replacing K by an extension, we may assume that $\det \mathcal{E} = \mathcal{L}^{N+1}$ is a power of a line bundle $\mathcal{L}$ with an identity $\mathcal{L}_K = K$ which is compatible with the identity $\det \mathcal{E}_K = K$. Let $\mathcal{E}' = \mathcal{E} \otimes \mathcal{L}^{-1}$ we see that $h_\mathcal{E}(X) = h_{\mathcal{E}'}(X)$ and the identity $\mathcal{E}'_K = K^{N+1}$ induces the indentity $\det \mathcal{E}' = \mathcal{O}_K$. By replacing $\mathcal{E}$ by $\mathcal{E}'$, we assume that $\mathcal{E}$ already has the property of $\mathcal{E}'$.

For any place v of K, let g_v be an element of $\mathrm{SL}(N+1, K_v)$ such that (i) if v is finite, $\mathcal{E}_v = g_v \mathcal{O}_v^{N+1}$ where we consider both sides as subsets of K_v^{N+1} and (ii) if v is infinite, for any element $x \in K_v^{N+1}$, $\|x\|_{\mathcal{E}_v} = \|g_v x\|_{\mathcal{O}_v^{N+1}}$. It is easy to see that (g_v) is in $\mathrm{SL}(N+1, \mathbb{A} \otimes K)$ and $\mathcal{E}$ is uniquely determined by (g_v) in the coset $\mathbb{K} \setminus \mathrm{SL}(N+1, \mathbb{A} \otimes K)$ where $\mathbb{K}$ consists elements whose components at a place v are in $\mathrm{SL}(N+1, \mathcal{O}_v)$ if v is finite and in the unitary group $U(N+1, K_v)$ if v is infinite.

Let $\tilde{X}$ (resp. $\tilde{X}'$) be the Zariski closure of X in $\mathbb{P}(\mathcal{O}_K^{N+1})$ (resp. $\mathbb{P}(\mathcal{E})$). Let $\mathcal{L}$ (resp. $\mathcal{L}'$) be the restriction of the line bundle $\mathcal{O}(1)$ to $\tilde{X}$ (resp. $\tilde{X}'$). Let x be the Chow point in $\mathbb{P}[(\mathrm{Sym}^d\ K^{N+1})^{\otimes(n+1)}]$ corresponding to X. For each place v, let $\mu(g_v, x)$ be a real number defined by isometry as in (2.1) and (2.6)

$$\mathcal{N}(g_v x) = \mathcal{N}(x) \otimes \mathcal{O}(\mu(g_v, x)).$$

Notice the embedding $X_v = \tilde{X}' \otimes \mathcal{O}_v \to \mathbb{P}(\mathcal{E}_v)$ is isomorphic to the embedding of the Zariski closure of $g_v(X)$ in $\mathbb{P}(\mathcal{O}_v^{N+1})$. By (1.4) and (1.6), the isomorphism on $\mathrm{Spec}\, K$ induces an isometry of line bundles over $\mathrm{Spec}\, K_v$

$$\langle \mathcal{L}', \ldots, \mathcal{L}' \rangle \simeq \langle \mathcal{L}, \ldots, \mathcal{L} \rangle \otimes \mathcal{O}(\mu(g_v, x)).$$

This gives

$$h_{\mathcal{E}}(X) = h_{\mathcal{O}^{N+1}}(X) + \frac{1}{(n+1)d[K:\mathbb{Q}]} \sum_v \mu(g_v, x). \tag{4.2.1}$$

By (2.2) and (2.6), if X is not semistable then $\mu(g_v, \mu)$ is not bounded below as a function of g_v, the formula (4.2.1) shows that $h_{\mathcal{E}}(X)$ is not bounded below. If x is semistable, let p be the corresponding point on P. Then at each place v, $\lambda(p) \simeq \mathcal{N}(x) \otimes \mathcal{O}(\inf_{g_v} \mu(g_v, x))$, therefore $\hat{h}(X) = \inf_{\mathcal{E}} h_{\mathcal{E}}(X)$. This proves Part 1 and 2. If x is stable, then at each place v, by (2.1), (3.2), (2.7), there is a point $g_v^0 x$ in the orbit $\mathrm{SL}(\bar{K}_v, N+1)x$ such that $g_v^0 x$ has semistable reduction and $\inf_{g_v} \mu(g_v, x) = \mu(g_v^0, x)$. The vector bundle $\mathcal{E}$ corresponding to (g_v^0) will satisfy the conditions of Part 3. Part 4 follows from the fact that $\hat{h}(X) = \hat{h}(X^*)$ since they correspond to same point in P.

4.3. Positivity of $\hat{h}(X)$

The lower boundedness of $h_{\mathcal{E}}$ has already been proved by Cornalba–Harris [C–H] in the function field case and Bost [Bo] in the number field case. Part 1 of (4.2) therefore gives inverse of their results. In function field case we may say more: Let X be defined over $\overline{k(t)}$ over a constant field $k = \bar{k}$ in stead of $\bar{\mathbb{Q}}$, then P is replaced by $P_k = P \times \mathrm{Spec}\, k$. Let p be the corresponding point for X. Then $\hat{h}(X) = 0$ if and only if p is defined over k. In other words, every stable limits X^* of X is defined over k. The nonnegativity has already proved by Cornalba and Harris, and the implication $\hat{h}(X) = 0 \Rightarrow X$ defined k is already proved by Bost when X is stable. I would like to conjecture that this is still true in number field case. Here we want to prove the following lower bounds:

THEOREM 4.4. *If $X \subset \mathbb{P}^N_{\bar{\mathbb{Q}}}$ is semistable, then*

$$\hat{h}(X) \geqslant -\frac{h(\mathbb{P}^N)}{(N+1)},$$

where

$$h(\mathbb{P}^N) = \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{n}\frac{1}{m},$$

is the Faltings height of the projective space $\mathbb{P}^N$.

Our proof is based on a recent paper of Soulé [So2]. He proves that $h_{\mathcal{E}}(X) + \deg \mathcal{E}/(N+1)[K: \mathbb{Q}]$ is bounded below by the average of the successive minima. By the Bombieri–Vaaler version of the Minkowski Theorem, his result gives a lower bound of $h_{\mathcal{E}}$ depending on K over which $\mathcal{E}$ and X are defined. Our basic improvement is to eliminate this dependence. We start from the Gieseker–Mumford criterion on stability:

LEMMA 4.6. *Let $X = \Sigma_{i=1}^{s} m_i X_i$ be an effective and semistable cycle in $\mathbb{P}^N$ over a field k, let $r_0, \ldots, r_N$ be integers such that $\Sigma r_i = 0$, and let $\varepsilon > 0$ be any number. Then for $p \gg 0$, there are monomials of x_i of degree d, $\prod_{k=0}^{N} x_j^{\alpha_{ijk}}$ $(1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant h_{pi} := \dim \Gamma(X_i, \mathcal{O}(p)|_{X_i}))$, such that* (i) *for each i $(1 \leqslant i \leqslant s)$, monomials $\prod_{k=0}^{N} x_j^{\alpha_{ijk}}$ $(1 \leqslant j \leqslant h_{pi})$ generate $\Gamma(X_i, \mathcal{O}(p)|_{X_i}$, and* (ii)

$$\sum_{ijk} m_i r_k \alpha_{ijk} \leqslant \varepsilon \left(\sum m_i p h_{ip}\right).$$

Proof. By approximation, we may assume that r_i are rational numbers. By eliminating denominator we may assume that all r_i are integers. Let $z \in \mathbb{P}[(\mathrm{Sym}^d\ k^{N+1})^{\otimes(n+1)}]$ be the Chow point for X, let λ be the 1-parameter subgroup defined by r_i's then $\lambda(t)z$ be the Chow point for $\lambda(t)X$. Let $\mu(\lambda(t), z)$ be the integer such that the isomorphism over $k(t)$ gives isometry (i.e isomorphism over $k[t]$)

$$\mathcal{N}(\lambda(t)) = \mathcal{N}(z) \otimes \mathcal{O}(\mu(\lambda(t), z)).$$

Since z is semistable, we must have $\mu(\lambda(t), z) \geqslant 0$, see (2.6).

For each i, let z_i be the Chow point of X_i on $\mathbb{P}[\mathrm{Sym}^{d_i}(k^{N+1})^{n+1}]$ and $\mu(\lambda(t), z_i)$ be defined as above. By linearity of the Deligne pairing and Theorem (1.4) one has $\mu(\lambda(t), z) = \Sigma_i m_i \mu(\lambda(t), z_i)$. Now the lemma follows by noticing that $-\mu(\lambda(t), z_i)$ is the leading term of weights of $\Gamma(X, \mathcal{O}(p)|_{X_i})$ as p approaches to infinity.

The following lemma is also crucial in the proof of (4.4).

LEMMA 4.7. *Let $\mathcal{E}$ be a Hermitian vector bundle on* $\operatorname{Spec}\mathcal{O}_K$. *For any $\varepsilon > 0$, there is an extension K' of K and line subbundles $\mathcal{L}_0, \dots, \mathcal{L}_N$ of $\mathcal{E}$ such that $\mathcal{L}_i$'s generate $\mathcal{E}$ over* $\operatorname{Spec} K$ *and*

$$\deg \mathcal{E} \leqslant \sum_{i=0}^{N} \deg \mathcal{L}_i + [h(\mathbb{P}^N) + (N+1)\varepsilon][K' : \mathbb{Q}].$$

Proof. For an arithmetic variety X defined over $\operatorname{Spec}\mathcal{O}_K$ and an ample line bundle $\mathcal{L}$ with a semipositive Hermitian metric, we define the height function $h\colon X(\bar{\mathbb{Q}}) \to \mathbb{R}$ as follows. For any $x \in X(\bar{\mathbb{Q}})$, let $\tilde{x}\colon \operatorname{Spec}\mathcal{O}_{K(x)} \to X$ be an extension over $\operatorname{Spec}\mathcal{O}_K$, then $h(x) := \deg(\tilde{x}(\mathcal{L}))/[K(x) : \mathbb{Q}]$. For each i between 0 and $n = \dim X_K$, define

$$e_i = \sup_{\operatorname{codim} Y = i} \inf_{x \in X(\bar{\mathbb{Q}}) - Y(\bar{\mathbb{Q}})} h(x),$$

where Y runs throught the set of closed subvarieties of X_K. By [Z1, Sect. 5.2] we have

$$\frac{c_1(\mathcal{L})^{n+1}}{c_1(\mathcal{L}_K)^n} \geqslant [K : \mathbb{Q}] \sum_{i=0}^{n} e_i.$$

Applying this formula to $X = \mathbb{P}(\mathcal{E}^\vee)$, one obtains points $x_0, \dots, x_N$ in $X(\bar{\mathbb{Q}})$ such that $x_0, \dots, x_N$ are not in any hyperplane, and

$$\sum_{i=0}^{N} h(x_i) \leqslant \deg(\mathcal{E}^\vee) + h(\mathbb{P}^N) + \varepsilon(N+1).$$

Here we use the equality $c_1(\mathcal{L})^{n+1} = \deg \mathcal{E}^\vee + h(\mathbb{P}^N)$ proved in [B–G–S], Section 4.1.2, (4.1.4). Let $K' = K(x_0, \dots, x_N)$ and let $\mathcal{L}_i$ be the line subbundle of $\mathcal{E}$ corresponding to x_i. Then the above formula gives the estimate of the lemma.

4.8. PROOF OF (4.4)

Write $X = \Sigma m_i X_i$ with X_i integral. By (4.2), it suffice to prove that $h_{\mathcal{E}}(X) \geqslant -(1/N+1)h(\mathbb{P}^N)$ for any Hermitian vector bundle $\mathcal{E}$ over some $\operatorname{Spec}\mathcal{O}_K$ such that $\deg \mathcal{E} = 0$. By Lemma 4.7 for any $\varepsilon > 0$, there are line subbundles $\mathcal{L}_0, \dots, \mathcal{L}_N$ of $\mathcal{E}$ which generate $\mathcal{E}$ generically and

$$\sum_i \deg \mathcal{L}_i \geqslant -[K : \mathbb{Q}][(N+1)\varepsilon + h(\mathbb{P}^N)]. \tag{4.8.1}$$

For each i, let x_i be a nonzero section of $\mathcal{L}_i$. These sections give an embedding $X \to \mathbb{P}^N$. Set $r_i = -\deg \mathcal{L}_i + 1/N + 1\Sigma_{i=0}^N \deg \mathcal{L}_i$. Then by (4.6) for $p \gg 0$, there are monomials of x_i of degree d, $\prod_{k=0}^N x_j^{\alpha_{ijk}}$ $(1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant h_{pi} :=$

$\dim \Gamma(X_i, \mathcal{O}(p)|_{X_i}))$, such that (i) for each i $(1 \leqslant i \leqslant s)$, monomials $\prod_{k=0}^{N} x_j^{\alpha_{ijk}}$ $(1 \leqslant j \leqslant h_{pi})$ generates $\Gamma(X_i, \mathcal{O}(p)|_{X_i})$, and (ii)

$$\sum_{ijk} m_i r_k \alpha_{ijk} \leqslant \varepsilon \left(\sum m_i p h_{ip}\right). \tag{4.8.2}$$

Let $\pi\colon X_i \to \operatorname{Spec} \mathcal{O}_K$ be the structure morphisms, and let $\pi_*(\mathcal{O}(p)|_{X_i})_q$ be $\pi_*(\mathcal{O}(p)|_{X_i})$ endowed with the quotient metric from $\operatorname{Sym}^p(\mathcal{E})$, then we have morphisms of vector bundles which are isomorphisms over $\operatorname{Spec} K$

$$\sum_{k=1}^{h_{ip}} \bigotimes_{j=0}^{N} \mathcal{L}_j^{\alpha_{ijk}} \to \pi_*(\mathcal{O}(p)|_{X_i})_q.$$

Since the norm of this morphism is $\leqslant 1$, we have

$$\begin{aligned}
&\sum_i m_i \deg \pi_*(\mathcal{O}(p)|_{X_i})_q \\
&\geqslant \sum_{ijk} m_i \alpha_{ijk} \deg \mathcal{L}_j \\
&= \sum_{ijk} m_i \alpha_{ijk} \left(-r_i + \frac{1}{N+1} \sum_{i=0}^{N} \deg \mathcal{L}_i\right) \\
&= -\left(\sum_{ijk} m_i \alpha_{ijk} r_j\right) + \frac{1}{N+1}\left(\sum_i \deg \mathcal{L}_i\right)\left(\sum_i m_i p h_{ip}\right).
\end{aligned}$$

By (4.8.1) and (4.8.2) we therefore have

$$\begin{aligned}
&\sum_i m_i \deg \pi_*(\mathcal{O}(p)|_{X_i})_q \\
&\geqslant -\left(2\varepsilon + \frac{1}{N+1} h(\mathbb{P}^N)\right)\left(\sum_i m_i h_{ip}\right).
\end{aligned} \tag{4.8.3}$$

Let $\pi_*(\mathcal{O}(p)|_{X_i})_{\text{sup}}$ be $\pi_*(\mathcal{O}(p)|_{X_i})$ endowed with supremum norm. Then $\|\cdot\|_q \geqslant \|\cdot\|_{\text{sup}}$, therefore

$$\begin{aligned}
\chi[\pi_*(\mathcal{O}(p)|_{X_i})_{\text{sup}}] &\geqslant \chi[\pi_*(\mathcal{O}(p)|_{X_i})_q] \\
&= \deg[\pi_*(\mathcal{O}(p)|_{X_i})_q] + o(p h_{ip}).
\end{aligned} \tag{4.8.4}$$

The arithmetic Hilbert–Samuel formula [Z1, Sect. 1.4] gives

$$\lim_{p\to\infty} \frac{\chi[\pi_*(\mathcal{O}(p)|_{X_i})_{\text{sup}}]}{p h_{ip}[K:\mathbb{Q}]} = \frac{c_1(\mathcal{O}(1)|_{X_i})^{n+1}}{(n+1)d[K:\mathbb{Q}]} = h_{\mathcal{E}}(X_i). \tag{4.8.5}$$

Now the theorem follows from (4.8.3), (4.8.4), and (4.8.5).

4.9. Continuity of Quotient Metrics

Now we want to show that continuity of the quotient metric for a Hermitian action. This fact is implicitly used in [Z2] without proof. I am grateful to the referee for pointing out this incompleteness in [Z2]. Combining Theorem 3.6 and the following Theorem 4.9, we have that the metric on quotient line bundle λ defined in 4.1 is continuous and ample.

With notation as 2.1. Let X be a complex variety, $\mathcal{L}$ be an ample line bundle with a continuous metric $\|\cdot\|$. We assume that this metric is ample. This means that for any $x \in X$ and any $\varepsilon > 0$, there is a nonzero section l a positive power $\mathcal{L}^n$ of $\mathcal{L}$ such that $\|l\|_{\sup} \leqslant e^{n\varepsilon}\|l\|(x)$. Let G be a complex reductive group, and U be a maximal compact subgroup. Fix a Hermitian action of (G, U) on $(X, \bar{\mathcal{L}})$.

Denote by $\pi\colon X^{ss} \to Y = X^{ss}//G$ the uniform category quotient of the set X^{ss} of the semistable points of X by G. As schemes, X is $\mathrm{Proj}(\oplus_{n\geqslant 0}\Gamma(X, \mathcal{L}^n))$ and Y is $\mathrm{Proj}(\oplus_{n\geqslant 0}\Gamma(X, \mathcal{L}^n)^G)$. Denote by $\mathcal{M}$ the $\mathcal{O}(1)$ line bundle on Y. Then π^*M is naturally identified with $\mathcal{L}_{X^{ss}}$. We define a metric $\|\cdot\|_q$ on $\mathcal{M}$ as follows: for any $y \in Y$ and $m \in \mathcal{M}(y)$

$$\|m\|_q(y) = \sup_{x\in\pi^{-1}(y)} \|\pi^* m\|(x).$$

THEOREM 4.10. *The metric $\|\cdot\|_q$ defined as above on $\mathcal{M}$ is continuous and ample.*

Proof. The ampleness is already proved in [Z2] 2.4. We are remain to prove the continuity. We will reduce the problem to case that $(X, \mathcal{O}(1)) = (\mathbb{P}^N, \mathcal{O}(1))$ with standard Fubini–Study metric. In this case the continuity is proved by Burnol [Bu] using a result of Neeman. We will use repeatedly the following principle.

Given a sequence of continuous and convex norm functions g_n on $\Gamma(X, \mathcal{L}^n)$ $(n \gg 1)$ which induces metrics $\|\cdot\|_n$ on $\mathcal{L}$ such that $\|\cdot\|_n^n$ are quotient metrics on $\mathcal{L}^n$ from g_n through the morphisms

$$\Gamma(X, \mathcal{L}^n) \otimes \mathcal{O}_X \to \mathcal{L}^n,$$

and also induces metrics $\|\cdot\|'_n$ on the $\mathcal{O}(1)$ bundles $\mathcal{L}_n$ of $P_n = \mathbb{P}(\Gamma(X, \mathcal{L}^n))$ through the morphisms

$$\Gamma(X, \mathcal{L}^n) \otimes \mathcal{O}_{P_n} \to \mathcal{L}_n.$$

Assume that g_n is U-invariant then we have induced Hermitian actions of (G, U) on $(P_n, \mathcal{L}_n)$. Assume that $\|\cdot\|_n$ converges to $\|\cdot\|$ on $\mathcal{L}$ (The convergence must be uniform, since metrics are continuous). Then our principle is that the continuity parts of the theorem for $(P_n, \mathcal{L}_n)$ $(n \gg 1)$ imply that for $(X, \mathcal{L})$. Actually on one hand let $\|\cdot\|_q$ and $\|\cdot\|_{q,n}$ be the metrics on $\mathcal{M}$ on $Y = X^{ss}//G$ induced from $\|\cdot\|$ and $\|\cdot\|_n$, then $\|\cdot\|_{q,n}$ uniformly converges to $\|\cdot\|_q$. On other hand let $\|\cdot\|'_{q,n}$ be the induced metrics on the quotient line bundles $\mathcal{M}_n$ on $Y_n = P_n^{ss}//G$ then there are embeddings $j_n: Y \to Y_n$ such that $j_n^* M_n = \mathcal{L}^n$ and $j_n^* \| \cdot \|'_{q,n} = \| \cdot \|_{q,n}^n$. Now the continuities of $\| \cdot \|'_{q,n}$ $(n \gg 1)$ imply the continuities of $\| \cdot \|_{q,n}$ and in turn imply the continuity of $\| \cdot \|_q$.

Since the norm $\| \cdot \|$ is the limit of the quotient norms $\| \cdot \|_{\sup,n}$ induced from the norm functions $\| \cdot \|_{\sup}$ on $(\Gamma(X, \mathcal{L}^n)$ $(n \gg 1)$ by continuity and ampleness of $\| \cdot \|$, it suffices to show the theorem for $(\mathbb{P}(\Gamma(X, \mathcal{L}^n), \mathcal{O}(1))$ with metric on $\mathcal{O}(1)$ induces from $\| \cdot \|_{\sup}$. We therefore reduced the theorem to $(X = \mathbb{P}^N, \mathcal{L} = \mathcal{O}(1))$ where the metric on $\mathcal{O}(1)$ is induced by a continuous convex norm function $g: \mathbb{C}^{N+1} - \{0\} \to \mathbb{R}_+$ which is invariant under the action of U. Approximate this norm by strictly convex smooth norm functions $g_n: \mathbb{C}^{N+1} - \{0\} \to \mathbb{R}_+$. Notice that $\| \cdot \|_n$ may not be invariant under U. Average them over U: $\bar{g}_n(x) = \int_U g_n(ux)\, du$. Then $\bar{g}_n$ are smooth, strictly convex, U-invariant, and convergent to g. Replacing g by $\bar{g}_n$, we may assume that X is smooth, and the metric on $\mathcal{L}$ is smooth and has strictly positive curvature. For each n let h_n be the Hermitian structure on $\Gamma(X, \mathcal{L}^n)$ induced by the metric on $\mathcal{L}$ and the curvature of $\mathcal{L}$. Then h_n is U-invariant. Let $\| \cdot \|_{h,n}$ be the metric on $\mathcal{L}$ whose nth power is the quotient metric induced from the Hermitian structure h_n. Then a theorem of Tian says that $\| \cdot \|_{h,n}$ converges to $\| \cdot \|$. So we reduced to the case that $X = \mathbb{P}^N$ and $\mathcal{L} = \mathcal{O}(1)$ with the Fubini–Study metric.

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