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Reducibility of certain representations for symplectic and odd-orthogonal groups

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Abstract. In this paper, we give a reducibility criterion for certain induced representations of p-adic symplectic and odd-orthogonal groups.

Key words: p-adic field, induced representation, symplectic group, odd-orthogonal group.

1. Introduction

In this paper, we give a necessary and sufficient condition for reducibility of a certain class of induced representations of p-adic symplectic or odd-orthogonal groups. This class includes representations obtained by inducing a one-dimensional representation from an arbitrary parabolic subgroup, as well as the representations obtained by inducing a Steinberg representation of an arbitrary parabolic twisted by a one-dimensional representation of that parabolic.

In the next section, we review some notation and background that will be needed. In the third section, we give the main theorem in the paper (cf. Theorem 3.3). It gives a necessary and sufficient condition for the reducibility of a representation obtained by inducing a square-integrable representation twisted by a one-dimensional representation (for $G = \mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$). In the fourth section, we determine when these conditions hold if the square-integrable representation is a Steinberg representation. By the involution of Aubert, the representation obtained by inducing the corresponding one-dimensional representation has the same reducibility points. (The results are a little more general than indicated above – e.g., one can replace 'one-dimensional representation' with 'Zelevinsky segment representation' and likewise for Steinbergs.)

2. Notation and preliminaries

In this section, we introduce notation and recall some facts that will be needed in the rest of this paper. This largely follows the setup in (Tadić, 1993). Throughout this paper, we work in the setting of admissible representations; so in what follows, representation will always mean admissible representation.

Let F be a p-adic field with char $F = 0$. Let $|\cdot|$ denote absolute value on F. Set $\nu = |\det|$ on $\mathrm{GL}_n(F)$. Define $\times$ on $\mathrm{GL}_n(F)$ as in (Zelevinsky, 1980): if $\rho_1, \dots, \rho_k$ are representations of $\mathrm{GL}_{n_1}(F), \dots, \mathrm{GL}_{n_k}(F)$, let $\rho_1 \times \cdots \times \rho_k$ denote the representation of $\mathrm{GL}_{n_1+\cdots+n_k}(F)$ obtained by inducing $\rho_1 \otimes \cdots \otimes \rho_k$ from the parabolic subgroup of $\mathrm{GL}_{n_1+\cdots+n_k}(F)$ with Levi factor $GL_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_k}(F)$.

We now turn to symplectic and odd-orthogonal groups. Let

$$J_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix},$$

denote the $n \times n$ antidiagonal matrix above. Then,

$$\mathrm{SO}_{2n+1}(F) = \{X \in \mathrm{SL}_{2n+1}(F) | {}^T X J_{2n+1} X = J_{2n+1}\},$$

$$\mathrm{Sp}_{2n}(F) = \left\{X \in \mathrm{GL}_{2n}(F) | {}^T X \begin{pmatrix} & -J \\ J & \end{pmatrix} X = \begin{pmatrix} & -J \\ J & \end{pmatrix}\right\}.$$

We use S_n to denote either $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$. In either case, the Weyl group is $W = \{\text{permutations and sign changes on } n \text{ letters}\}$.

In S_n, one may take as minimal parabolic subgroup the subgroup consisting of upper triangular matrices in S_n. In general, the Levi factor of a parabolic subgroup of S_n has the form

$$\mathrm{GL}_{m_1}(F) \times \cdots \times \mathrm{GL}_{m_k}(F) \times S_m,$$

with $m_1 + \cdots + m_k + m = n$ (with S_0 the trivial group). If $\rho_1, \dots, \rho_k$ are representations of $\mathrm{GL}_{m_1}(F), \dots, \mathrm{GL}_{m_k}(F)$ and σ a representation of S_m, let

$$\rho_1 \times \cdots \times \rho_k \rtimes \sigma,$$

denote the representation of S_n obtained by inducing $\rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma$ from the parabolic subgroup with Levi factor $\mathrm{GL}_{m_1}(F) \times \cdots \times \mathrm{GL}_{m_k}(F) \times S_m$. Note that if $\tilde{}$ denotes contragredient,

$$(\rho_1 \times \cdots \times \rho_k \rtimes \sigma)\tilde{} \cong \tilde{\rho}_1 \times \cdots \times \tilde{\rho}_k \rtimes \tilde{\sigma}.$$

Note that for clarity, we use $=$ when defining something or working in the Grothendieck group; $\cong$ when indicating an actual equivalence of two representations.

We now recall Langlands classification for S_n. Suppose δ is an essentially square-integrable representation of $\mathrm{GL}_n(F)$. Then, there is an $\varepsilon(\delta) \in \mathbb{R}$ such that $\nu^{-\varepsilon(\delta)}\delta$ is unitarizable. Let $\delta_1, \dots, \delta_k$ be irreducible essentially square-integrable representations of $\mathrm{GL}_{m_1}(F), \dots, \mathrm{GL}_{m_k}(F)$ satisfying $\varepsilon(\delta_1) \geqslant \cdots \geqslant \varepsilon(\delta_k) >$

0 and τ an irreducible tempered representation of S_m. Then, $\delta_1 \times \cdots \delta_k \rtimes \tau$ has a unique irreducible quotient which we denote by $L(\delta_1, \ldots, \delta_k; \tau)$. Note that $L(\delta_1, \ldots, \delta_k; \tau)$ occurs in $\delta_1 \times \cdots \times \delta_k \rtimes \tau$ with multiplicity one.

At this point, we introduce a little shorthand. Let ρ be a unitarizable supercuspidal representation of $\mathrm{GL}_r(F)$. Then, $\nu^{-k+1/2}\rho \times \nu^{-k+1/2}\rho \times \cdots \times \nu^{k-1/2}\rho$ has a unique irreducible subrepresentation which we denote $\zeta(\rho, k)$ and a unique irreducible quotient which we denote by $\delta(\rho, k)$ (n.b. $\delta(\rho, k)$ is square-integrable). Similarly, suppose that σ is a supercuspidal representation of S_m and $\nu^\alpha\rho \rtimes \sigma$ reduces for some $\alpha > 0$ (note that this implies $\tilde{\rho} \cong \rho$). Then

$$\nu^{-\ell+1-\alpha}\rho \times \nu^{-\ell+2-\alpha}\rho \times \cdots \times \nu^{-\alpha}\rho \rtimes \sigma,$$

has a unique irreducible subrepresentation which we denote $\zeta(\rho, \ell; \sigma)$ and a unique irreducible quotient which we denote $\delta(\rho, \ell; \sigma)$ (n.b. $\delta(\rho, \ell; \sigma)$ is square-integrable).

Let ρ be an irreducible unitarizable supercuspidal representation of $\mathrm{GL}_r(F)$ and σ an irreducible unitarizable supercuspidal representation of S_m. For $\alpha \geqslant 0$, let us say (ρ, σ) satisfies (Cα), if it satisfies

(Cα) $\nu^\alpha\rho \rtimes \sigma$ is reducible and $\nu^\beta\rho \rtimes \sigma$

is irreducible for all $\beta \in \mathbb{R}$ with $|\beta| \neq \alpha$.

Let $1_{F^\times}$ denote the trivial representation of $F^\times$, sgn an order two character of $F^\times$, and 1 the trivial representation of S_0. Then, for $\mathrm{Sp}_{2n}(F)$, $(1_{F^\times}, 1)$ satisfies (C1) and (sgn, 1) satisfies (C0). For $\mathrm{SO}_{2n+1}(F)$, both $(1_{F^\times}, 1)$ and (sgn,1) satisfy (C1/2).

We close with the following observation (cf. Sect. 2, (Tadić, 1993)): a conjecture of Shahidi (Shahidi, 1990) implies that for ρ, σ as above, if $\nu^\alpha\rho \rtimes \sigma$ is reducible for some $\alpha \in \mathbb{R}$, then (ρ, σ) satisfies one of (C0), (C1/2), (C1). This explains why we restrict our attention to those conditions in the rest of this paper.

3. Reducibility criterion

In this section, we give the main theorem in this paper (cf. Theorem 3.3); a necessary and sufficient condition for the reducibility of a representation obtained by inducing a square-integrable representation twisted by a one-dimensional representation. This is just a generalization of a result from (Tadić, 1994).

LEMMA 3.1 *Let $\delta_1, \ldots, \delta_k$ be irreducible square-integrable representations of $\mathrm{GL}_{n_1}(F), \ldots, \mathrm{GL}_{n_k}(F)$ and δ an irreducible square-integrable representation of S_n. Set $\pi = \delta_1 \times \cdots \times \delta_k \rtimes \delta$. Then π is reducible if and only if at least one of $\delta_i \rtimes \delta$ reduces.*

Proof. First, suppose $\delta_i \rtimes \delta$ reduces for some i. In the Grothendieck group, we have

$$\pi = \delta_1 \times \cdots \times \delta_{i-1} \times \delta_{i+1} \times \cdots \times \delta_k \rtimes (\delta_i \rtimes \delta),$$

so π must be reducible.

Now, suppose $\delta_i \rtimes \delta$ is irreducible for all i. By (Silberger, 1978), to show π is irreducible, it is enough to show the R-group for π is trivial. Without loss of generality, write

$$\pi = \delta_1^{(1)} \times \cdots \times \delta_{j_1}^{(1)} \times \delta_1^{(2)} \times \cdots \times \delta_{j_2}^{(2)} \times \cdots \times \delta_1^{(m)} \times \cdots \times \delta_{j_m}^{(m)} \rtimes \delta,$$

where $\delta_1^{(i)}, \ldots, \delta_{j_i}^{(i)}$ are representations of $\mathrm{GL}_{k_i}(F)$ with the k_i all distinct. Let R_i denote the R-group for $\delta_1^{(i)} \times \cdots \times \delta_{j_i}^{(i)} \rtimes \delta$. If R denotes the R-group for π, Theorems 4.9 and 4.18 in (Goldberg, 1994) tell us

$$R = R_1 \times \cdots \times R_m.$$

In particular, R is trivial if and only if each R_i is trivial. By Theorems 6.4 and 6.5(a) in (Goldberg, 1994), R_i is trivial if and only if $\delta_1^{(i)} \rtimes \delta, \ldots, \delta_{j_i}^{(i)} \rtimes \delta$ are all irreducible. (More generally, $R_i = \mathbb{Z}_2^d$ where d is the number of inequivalent $\delta_j^{(i)}$ having $\delta_j^{(i)} \rtimes \delta$ reducible.) The lemma follows. □

LEMMA 3.2 *If* $\rho_1, \ldots, \rho_k$ *are irreducible essentially square-integrable representations of* $\mathrm{GL}_{m_1}(F), \ldots, \mathrm{GL}_{m_k}(F)$ *with* $\varepsilon(\rho_1) \geqslant \cdots \geqslant \varepsilon(\rho_k) > 0$ *and* σ *is a tempered representation of* S_m*, then*

$$L(\rho_1, \ldots, \rho_k; \sigma)\tilde{} = L(\rho_1, \ldots, \rho_k; \tilde{\sigma}).$$

Proof. See Chapter 6 (p. 147) of (Tadić, 1994). □

THEOREM 3.3 *Let* $\delta_1, \ldots, \delta_\ell$ *be irreducible square-integrable representations of* $\mathrm{GL}_{n_1}(F), \ldots, \mathrm{GL}_{n_\ell}(F)$ *and* δ *an irreducible square-integrable representation of* S_n*. For* $\alpha_1, \ldots, \alpha_\ell \in \mathbb{R}$*, set*

$$\pi = \nu^{\alpha_1}\delta_1 \times \cdots \times \nu^{\alpha_\ell}\delta_\ell \rtimes \delta.$$

Then π *is reducible if and only if (at least) one of the following reduces:*

(1) $\nu^{\alpha_i}\delta_i \times \nu^{\alpha_j}\delta_j$, $i \neq j$.
(2) $\nu^{\alpha_i}\delta_i \rtimes \delta$.
(3) $\nu^{\alpha_i}\delta_i \times \nu^{-\alpha_j}\tilde{\delta}_j$, $i \neq j$.

Proof. The proof parallels that used in Chapter 7 of (Tadić, 1994).

First, consider $\nu^{\alpha_1}\delta_1 \otimes \cdots \otimes \nu^{\alpha_\ell}\delta_\ell \otimes \delta$. If some $\alpha_i < 0$, replace $\nu^{\alpha_i}\delta_i$ with $\nu^{-\alpha_i}\tilde{\delta}_i$. This corresponds to a block of sign changes – a Weyl group action – so the result of inducing this representation is the same as π in the Grothendieck group (by Lemma 5.4 (Bernstein *et. al.*, 1986)). In particular, it is irreducible if and only if π is. Thus we may without loss of generality assume that $\alpha_i \geqslant 0$ for all i. Then, take a permutation of the $\nu^{\alpha_i}\delta_i$ so that $\alpha_1 \geqslant \cdots \geqslant \alpha_\ell \geqslant 0$. Again, this corresponds to a Weyl group action, so in the Grothendieck group nothing has changed. Thus, without loss of generality, we assume

$$\pi = \nu^{\alpha_1}\delta_1 \times \cdots \times \nu^{\alpha_k}\delta_k \times \delta_{k+1} \times \cdots \times \delta_\ell \rtimes \delta,$$

with $\alpha_1 \geqslant \cdots \geqslant \alpha_k > 0$.

Next, suppose the representations in (1), (2), (3) above are all irreducible. For convenience, let $\sigma = \delta_{k+1} \times \cdots \times \delta_\ell \rtimes \delta$. By Lemma 3.1, σ is irreducible and tempered. Note that by definition, $\pi_1 = L(\nu^{\alpha_1}\delta_1, \ldots, \nu^{\alpha_k}\delta_k; \sigma)$ is the unique irreducible quotient of π.

The irreducibility in (1), (2), (3) gives the equivalences $\nu^{\alpha_i}\delta_i \times \nu^{\alpha_j}\delta_j \cong \nu^{\alpha_j}\delta_j \times \nu^{\alpha_i}\delta_i$, $\nu^{\alpha_i}\delta_i \rtimes \delta \cong \nu^{-\alpha_i}\tilde{\delta}_i \rtimes \delta$, $\nu^{\alpha_i}\delta_i \times \nu^{-\alpha_j}\tilde{\delta}_j \cong \nu^{-\alpha_j}\tilde{\delta}_j \times \nu^{\alpha_i}\delta_i$, and $\nu^{-\alpha_i}\tilde{\delta}_i \times \nu^{-\alpha_j}\tilde{\delta}_j \cong \nu^{-\alpha_j}\tilde{\delta}_j \times \nu^{-\alpha_i}\tilde{\delta}_i$. Then we have the following equivalences

$$\begin{aligned}
\pi &\cong \nu^{\alpha_1}\delta_1 \times \cdots \times (\nu^{\alpha_k}\delta_k \times \delta_{k+1}) \times \delta_{k+2} \times \cdots \times \delta_\ell \rtimes \delta \\
&\cong \nu^{\alpha_1}\delta_1 \times \cdots \times \delta_{k+1} \times \nu^{\alpha_k}\delta_k \times \delta_{k+2} \times \cdots \times \delta_\ell \rtimes \delta \\
&\vdots \text{(commuting } \nu^{\alpha_k}\delta_k \text{ around } \delta_{k+2}, \ldots, \delta_\ell) \\
&\cong \nu^{\alpha_1}\delta_1 \times \ldots \nu^{\alpha_{k-1}}\delta_{k-1} \times \delta_{k+1} \times \cdots \times \delta_\ell \rtimes (\nu^{\alpha_k}\delta_k \rtimes \delta) \\
&\cong \nu^{\alpha_1}\delta_1 \times \cdots \nu^{\alpha_{k-1}}\delta_{k-1} \times \delta_{k+1} \times \cdots \times \delta_\ell \times \nu^{-\alpha_k}\tilde{\delta}_k \rtimes \delta \\
&\cong \nu^{\alpha_1}\delta_1 \times \cdots \nu^{\alpha_{k-1}}\delta_{k-1} \times \delta_{k+1} \times \cdots \times \nu^{-\alpha_k}\tilde{\delta}_k \times \delta_\ell \rtimes \delta \\
&\vdots \text{(commuting } \nu^{-\alpha_k}\tilde{\delta}_k \text{ around } \delta_{\ell-1}, \ldots, \delta_{k+1}) \\
&\cong \nu^{\alpha_1}\delta_1 \times \cdots \times \nu^{\alpha_{k-1}}\delta_{k-1} \times \nu^{-\alpha_k}\tilde{\delta}_k \times \delta_{k+1} \times \cdots \times \delta_\ell \rtimes \delta \\
&\cong \nu^{\alpha_1}\delta_1 \times \cdots \times \nu^{\alpha_{k-2}}\delta_{k-2} \times \nu^{-\alpha_k}\tilde{\delta}_k \times \nu^{\alpha_{k-1}}\delta_{k-1} \\
&\quad \times \delta_{k+1} \times \cdots \times \delta_\ell \rtimes \delta \\
&\vdots \text{(arguing as above)} \\
&\cong \nu^{\alpha_1}\delta_1 \times \cdots \times \nu^{\alpha_{k-2}}\delta_{k-2} \times \nu^{-\alpha_k}\tilde{\delta}_k \times \nu^{-\alpha_{k-1}}\tilde{\delta}_{k-1} \\
&\quad \times \delta_{k+1} \times \cdots \times \delta_\ell \rtimes \delta \\
&\cong \nu^{\alpha_1}\delta_1 \times \cdots \times \nu^{\alpha_{k-2}}\delta_{k-2} \times \nu^{-\alpha_{k-1}}\tilde{\delta}_{k-1} \times \nu^{-\alpha_k}\tilde{\delta}_k \\
&\quad \times \delta_{k+1} \times \cdots \times \delta_\ell \rtimes \delta \\
&\vdots \text{(continuing this process)} \\
&\cong \nu^{-\alpha_1}\tilde{\delta}_1 \times \nu^{-\alpha_2}\tilde{\delta}_2 \times \cdots \times \nu^{-\alpha_k}\tilde{\delta}_k \times \delta_{k+1} \times \cdots \times \delta_\ell \rtimes \delta.
\end{aligned}$$

Now, consider $\pi^* = \nu^{\alpha_1}\delta_1 \times \cdots \times \nu^{\alpha_k}\delta_k \times \tilde{\delta}_{k+1} \times \cdots \times \tilde{\delta}_\ell \rtimes \tilde{\delta}$. The representation π^* has $\pi_1^* = L(\nu^{\alpha_1}\delta_1, \ldots, \nu^{\alpha_k}\delta_k; \tilde{\sigma})$ as unique irreducible quotient. Taking contragredients, we see that $\widetilde{\pi^*} = \nu^{-\alpha_1}\tilde{\delta}_1 \times \cdots \times \nu^{-\alpha_k}\tilde{\delta}_k \times \delta_{k+1} \times \cdots \delta_\ell \rtimes \delta$ has unique irreducible subrepresentation $\widetilde{\pi_1^*}$. From the equations above, $\widetilde{\pi_1^*}$ is the unique irreducible subrepresentation of π. By Lemma 3.2, $\widetilde{\pi_1^*} = \pi_1$. Thus, if π were reducible, it would contain π_1 with multiplicity two – once as (unique) subrepresentation, once as (unique) quotient – contradicting multiplicity one in Langlands classification.

We now address reducibility. Suppose, e.g., $\nu^{\alpha_i}\delta_i \times \nu^{-\alpha_j}\tilde{\delta}_j$ were reducible. Then, in the Grothendieck group,

$$\begin{aligned}\pi &= \nu^{\alpha_1}\delta_1 \times \cdots \nu^{\alpha_i}\delta_i \times \nu^{-\alpha_j}\tilde{\delta}_j \times \nu^{\alpha_{i+1}}\delta_{i+1} \times \cdots \times \nu^{\alpha_{j-1}}\delta_{j-1} \\ &\quad \times \nu^{\alpha_{j+1}}\delta_{j+1} \times \cdots \times \nu^{\alpha_k}\delta_k \times \delta_{k+1} \times \cdots \times \delta_\ell \rtimes \delta.\end{aligned}$$

This clearly reduces. □

Remark 3.4. The same argument gives a necessary and sufficient condition for the reducibility of $\nu^{\alpha_1}\delta_1 \times \cdots \times \nu^{\alpha_k}\delta_k \rtimes \tau$ for τ any tempered representation of S_n. It reduces if and only if one of $\nu^{\alpha_i}\delta_i \times \nu^{\alpha_j}\delta_j$, $\nu^{\alpha_i}\delta_i \rtimes \tau$, $\nu^{\alpha_i}\delta_i \times \nu^{-\alpha_j}\tilde{\delta}_j$ $(i \neq j)$ reduces. (Here, we are assuming $\alpha_1 \neq 0$ for all i.)

4. Special case

In this section, we determine when the conditions in Theorem 3.3 are satisfied for the special case where $\delta_i = \delta(\rho_i, k_i)$ and $\delta = \delta(\rho, \ell; \sigma)$. We use the involution of Aubert to relate reducibility points for $\nu^{\alpha}\delta(\rho_i, k_i) \rtimes \delta(\rho, \ell; \sigma)$ to those for $\nu^{\alpha}\zeta(\rho_i, k_i) \rtimes \zeta(\rho, \ell; \sigma)$, which are known. The reducibility points for $\nu^{\alpha_i}\delta(\rho_i, k_i) \times \nu^{\alpha_j}\delta(\rho_j, k_j)$ are also known (n.b., $(\nu^{\alpha}\delta(\rho, k))\tilde{} \cong \nu^{-\alpha}\delta(\tilde{\rho}, k)$, so all the conditions in Theorem 3.3 are covered). In combination, these give the reducibility points for the special case mentioned. Aubert's involution implies the representation obtained by substituting $\nu^{\alpha_i}\zeta(\rho_i, k_i)$ for $\nu^{\alpha_i}\delta(\rho_i, k_i)$ and $\zeta(\rho, \ell; \sigma)$ for $\delta(\rho, \ell; \sigma)$ has the same reducibility points (cf. Corollary 4.3).

We start by recalling Aubert's involution. If L is the Levi factor of a standard parabolic $P_L \subset G$, let i_{GL} denote induction from P_L to G; r_{LG} the functor taking the Jacquet module with respect to P_L.

THEOREM 4.1 (Aubert). *Define the operator D_G on the Grothendieck group $\mathcal{R}(G)$ by*

$$D_G = \sum_{\Phi \subset \Pi} (-1)^{|\Phi|} i_{\mathrm{GL}_\Phi} \circ r_{L_\Phi G},$$

where Π denotes the set of simple roots and for $\Phi \subset \Pi$, L_Φ is the Levi of the standard parabolic obtained by adjoining the simple reflections from Φ to the minimal parabolic. D_G has the following properties:

(1) $D_G \circ \tilde{} = \tilde{} \circ D_G$.
(2) $D_G \circ i_{\mathrm{GL}_\Phi} = i_{\mathrm{GL}_\Phi} \circ D_{L_\Phi}$.
(3) $r_{L_\Phi G} \circ D_G = \mathrm{Ad}(w_\Phi) \circ D_{L_{\Phi'}} \circ r_{L_{\Phi'} G}$, *where* w_Φ *is the longest element of* $W^{AL_\Phi} = \{w \in W | w^{-1}(P_{\min} \cap L_\Phi) \subset P_{\min}\}$ *and* $\Phi' = w_\Phi^{-1}(\Phi)$.
(4) $D_G^2 =$ *identity.*
(5) D_G *takes irreducible representations to irreducible representations.*

Proof. 1–4 are in Théorème 1.7 of (Aubert, 1995). 5 is in Corollaire 3.9 of (Aubert, 1995). □

COROLLARY 4.2 *Under Aubert's involution, up to* $\pm$,

$$\nu^\alpha \zeta(\rho, n) \longleftrightarrow \nu^\alpha \delta(\rho, n),$$

and

$$\zeta(\rho, n; \sigma) \longleftrightarrow \delta(\rho, n; \sigma).$$

Proof. Suppose (ρ, σ) satisfies (C1). Then $\zeta(\rho, n; \sigma)$ and $\delta(\rho, n; \sigma)$ may be characterized by

$$r_{MG}\zeta(\rho, n; \sigma) = \nu^{-n}\rho \otimes \cdots \otimes \nu^{-1}\rho \otimes \sigma,$$

$$r_{MG}\delta(\rho, n; \sigma) = \nu^{n}\rho \otimes \cdots \otimes \nu\rho \otimes \sigma,$$

where $M \cong \mathrm{GL}_r(F) \times \cdots \times \mathrm{GL}_r(F) \times S_m$. The claim then follows immediately from 3 in the theorem above. The other cases are similar. □

COROLLARY 4.3 *Under Aubert's involution, up to* $\pm$

$$\begin{gathered}\nu^{\alpha_1}\zeta(\rho_1, m_1) \times \cdots \times \nu^{\alpha_k}\zeta(\rho_k, m_k) \rtimes \zeta(\rho, \ell; \sigma) \\ \updownarrow \\ \nu^{\alpha_1}\delta(\rho_1, m_1) \times \cdots \times \nu^{\alpha_k}\delta(\rho_k, m_k) \rtimes \delta(\rho, \ell; \sigma).\end{gathered}$$

Proof. This follows immediately from Theorem 4.1 and Corollary 4.2. □

Recall the segment notation of Zelevinsky

$$[\nu^\beta\rho, \nu^{\beta+m}\rho] = \nu^\beta\rho, \nu^{\beta+1}\rho, \ldots, \nu^{\beta+m}\rho.$$

THEOREM 4.4 *Let* ρ_1, ρ_2 *be irreducible unitarizable supercuspidal representations of* $\mathrm{GL}_{m_1}(F)$, $\mathrm{GL}_{m_2}(F)$ *and* $\alpha_1, \alpha_2 \in \mathbb{R}$. *Then* $\nu^{\alpha_1}\zeta(\rho_1, m_1) \times \nu^{\alpha_2}\zeta(\rho_2, m_2)$ *is reducible if and only if* $\rho_1 \cong \rho_2$ *and* $[\nu^{\alpha_1+(-m_1+1)/2}\rho_1, \nu^{\alpha_1+(m_1-1)/2}\rho_1] \cup [\nu^{\alpha_2+(-m_2+1)/2}\rho_1, \nu^{\alpha_2+(m_2-1)/2}\rho_1]$ *is also a segment and is strictly larger than both* $[\nu^{\alpha_1+(-m_1+1)/2}\rho_1, \nu^{\alpha_1+(m_1-1)/2}\rho_1]$ *and* $[\nu^{\alpha_2+(-m_2+1)/2}\rho_1, \nu^{\alpha_2+(m_2-1)/2}\rho_1]$.

Proof. See Theorem 4.2 of (Zelevinsky, 1980). □

THEOREM 4.5 *Let ρ_0, ρ be irreducible unitarizable supercuspidal representations of* $\mathrm{GL}_{r_0}(F)$, $\mathrm{GL}_r(F)$, *resp.; let σ an irreducible unitarizable supercuspidal representation of S_m.*

(1) *Suppose (ρ, σ) satisfies* (C1/2). *Then $\nu^\alpha\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$, $\alpha \in \mathbb{R}$ has the following reducibility points:*

(a) *if $\rho_0 \cong \rho$, then $\nu^\alpha\zeta(\rho, k) \rtimes \zeta(\rho, \ell; \sigma)$ reduces if and only if*

$$\alpha \in \left\{\pm\left(\ell+\frac{k}{2}\right), \pm\left(\ell+\frac{k}{2}-1\right), \dots, \pm\left(\ell-\frac{k}{2}+1\right)\right\} \cup \left\{\left\{-\frac{k}{2}, -\frac{k}{2}+1, \dots, \frac{k}{2}\right\} \Big\backslash \{0 \text{ if } k=2\ell\}\right\}.$$

(b) *if $\rho_0 \not\cong \rho$, then $\nu^\alpha\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$ reduces if and only if $\nu^\alpha\zeta(\rho_0, k) \rtimes \sigma$* reduces (*cf. Proposition* 4.6 *below*).

(2) *Suppose (ρ, σ) satisfies* (C1). *Then $\nu^\alpha\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$, $\alpha \in \mathbb{R}$ has the following reducibility points:*

(a) *if $\rho_0 \cong \rho$, then $\nu^\alpha\zeta(\rho, k) \rtimes \zeta(\rho, \ell; \sigma)$ reduces if and only if*

$$\alpha \in \left\{\pm\left(\ell+\frac{k+1}{2}\right), \pm\left(\ell+\frac{k+1}{2}-1\right), \dots, \pm\left(\ell+\frac{-k+3}{2}\right)\right\} \cup \left\{\left\{\frac{-k+1}{2}, \frac{-k+1}{2}+1, \dots, \frac{k-1}{2}\right\} \Big\backslash \{0 \text{ if } k=2\ell+1\}\right\}$$

(b) *if $\rho_0 \not\cong \rho$, then $\nu^\alpha\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$ reduces if and only if $\nu^\alpha\zeta(\rho_0, k) \rtimes \sigma$ reduces* (*cf. Proposition* 4.6 *below*).

Proof. See Theorems 4.1 and 4.3 in (Jantzen, 1995). □

PROPOSITION 4.6 *Suppose ρ_0 is an irreducible unitarizable supercuspidal representation of* $\mathrm{GL}_{r_0}(F)$ *and σ an irreducible unitarizable supercuspidal representation of S_m. Then $\nu^\alpha\zeta(\rho_0, k) \rtimes \sigma$ has the following reducibility points:*

(1) *if (ρ_0, σ) satisfies* (C0), *$\nu^\alpha\zeta(\rho_0, k) \rtimes \sigma$ is reducible if and only if*

$$\alpha \in \left\{\frac{-k+1}{2}, \frac{-k+1}{2}+1, \dots, \frac{k-1}{2}\right\}.$$

(2) *if (ρ_0, σ) satisfies* (C1/2), *$\nu^\alpha\zeta(\rho_0, k) \rtimes \sigma$ is reducible if and only if*

$$\alpha \in \left\{-\frac{k}{2}, -\frac{k}{2}+1, \dots, \frac{k}{2}\right\}.$$

(3) *if* (ρ_0, σ) *satisfies* (C1), $\nu^\alpha\zeta(\rho_0, k) \rtimes \sigma$ *is reducible if and only if*

$$\alpha \in \left\{\frac{-k-1}{2}, \frac{-k+1}{2}, \dots, \frac{k+1}{2}\right\} \Big\backslash \{0 \text{ if } k = 1\}.$$

(4) *if* $\rho_0 \not\cong \tilde{\rho}_0$, $\nu^\alpha\zeta(\rho_0, k) \rtimes \sigma$ *is irreducible for all* $\alpha \in \mathbb{R}$.

Proof. See Theorem 7.2 of (Tadić, 1993). □

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