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# A Borel–Weil theorem for holomorphic forms

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Let $X = G/P$ be a homogeneous space of a semisimple complex Lie group $G$, $P$ being a parabolic subgroup, and let $E$ be a homogeneous vector bundle on $X$. Assume that $E$ satisfies the following positivity condition: the highest weights of the associated $P$-module are dominant. It is then a standard consequence of Bott's theorem [3, 5, 19], that $E$ is spanned by global sections and has no cohomology in positive degrees, $H^q(X, E) = 0$ for $q > 0$. The aim of this paper is to prove extensions of this vanishing property to Dolbeault cohomology, $H^{p,q}(X, E) = H^q(X, \Omega_X^p \otimes E)$. One of the motivations for understanding Dolbeault cohomology groups of homogeneous bundles is that they play an essential role in the standard proofs of vanishing theorems for general ample vector bundles [4, 10, 14, 15].

Although the bundle $\Omega_X^p$ of holomorphic $p$-forms on $X$ certainly does not have dominant highest weights, we discovered the following unexpected phenomenon: if $X = G/P \to Y = G/Q$ is a homogeneous fibration with fiber $Z$, and if $E$ has dominant highest weights, then the relative Dolbeault cohomology groups $H^{p,q}(Z, E|_Z)$ still have dominant highest weights as $Q$-modules, see Proposition 2.6. This is reminiscent of the well-known fact that the action of $Q$ on $H^{p,p}(Z)$ is trivial. This observation allows reduction to quotients $G/P$ with $P$ maximal.

The most favorable case is that of compact Hermitian symmetric spaces, which was investigated by one of us [17, 18] with the help of some deep results of Kostant [9]. We say that $X$ is a symmetric space tower if there exist fibrations $X \to Y_1 \to \cdots \to Y_s \to \{0\}$ whose fibers are compact Hermitian symmetric spaces. If $E$ is a homogeneous vector bundle on such a space $X$ with dominant highest weights, then $H^{p,q}(X, E) = 0$ whenever $q > p$. Note that this is sharp, since $H^{p,p}(X) \neq 0$ for all $p$, and allows us to obtain a refined version of the Nakano vanishing theorem for ample line bundles.

Symmetric space towers include products of ordinary flag manifolds, but not, for example, Grassmannians of isotropic subspaces of dimension greater than one in a given symplectic vector space. Nevertheless, if $G$ is a product of classical groups and $E$ has dominant highest weights, we prove that $H^{p,q}(X, E) = 0$ whenever $q > 2p$. The main point is to understand the highest weights of $\Omega_X^p$ which are not given by Kostant's results: we use simple geometric descriptions of the tangent bundle and standard results from classical representation theory to extend these descriptions to $p$-forms.

In the final section we give results for the case of the exceptional groups and classical groups of low rank where direct computation can be applied to yield to the desired vanishing theorems.

## 1. Preliminaries

We recall some well-know facts and establish the notation used in later sections. General references are [1, 7].

Let $G$ denote a simply-connected semi-simple complex Lie group and fix a maximal torus $H \subset G$. Let $\Lambda$ be the lattice of weights with respect to $H$ and let $\Phi \subset \Lambda$ denote the set of roots of $G$. Let $B \supset H$ be a Borel subgroup of $G$ and let the positive roots $\Phi^+$ of $G$ be chosen so that $B$ is generated by the root groups corresponding to the *negative* roots, $\Phi^- = -\Phi^+$. Let $\alpha_1, \dots, \alpha_\ell$, $\ell = \operatorname{rank} G$, be the simple roots of $G$ and let $\lambda_1, \dots, \lambda_\ell$ be the basis of $\Lambda$ dual to the simple roots with respect to the Killing form: $\langle \lambda_i, \alpha_j \rangle = 2(\lambda_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$. The set of dominant weights is denoted by $\Lambda^+$ and consists of $\lambda \in \Lambda$ such that $\langle \lambda, \alpha_i \rangle \geqslant 0$ for $1 \leqslant i \leqslant \ell$. A weight is singular if $\langle \lambda, \alpha \rangle = 0$ for some root $\alpha \in \Phi$. Let $\delta = \frac{1}{2}\Sigma_{\alpha \in \Phi^+} \alpha = \lambda_1 + \cdots + \lambda_\ell$ denote the minimal non-singular dominant weight.

Let $W$ be the Weyl group of $G$, the finite group of reflections of the weight lattice $\Lambda$ generated by simple reflections $\sigma_i(\mu) = \mu - \langle \mu, \alpha_i \rangle \alpha_i$, $\mu \in \Lambda$, $i = 1, \dots, \ell$. For every weight $\lambda \in \Lambda$ there is a $\sigma \in W$ such that $\sigma(\lambda) \in \Lambda^+$; $\sigma(\lambda)$ is called the dominant conjugate of $\lambda$ and the index of $\lambda$, ind $(\lambda)$, is the minimal number of reflections needed to take $\lambda$ to $\sigma(\lambda)$. It is given by ind $(\lambda) = \#\{\beta \in \Phi^+ \mid (\lambda, \beta) < 0\}$. When $\lambda$ is non-singular, then $\sigma$ is unique and ind $(\lambda) =$ len $(\sigma)$, where len $(\sigma)$ is the minimum number of simple reflections required to express $\sigma$ as a product of simple reflections.

Let $P$ be a parabolic subgroup of $G$. We may assume that $P$ contains $B$ so that $P$ is generated by $B$ and the root groups corresponding to the positive roots, $\Phi_P^+$, of $P$. Let $I_P \subset \{1, \dots, \ell\}$ denote the set of indexes of the simple roots $\alpha_i$, $i \in I_P$, that generate $\Phi_P^+$. We say that a weight $\lambda = n_1\lambda_1 + \cdots + n_\ell\lambda_\ell$ is $P$-dominant if $n_i = \langle \lambda, \alpha_i \rangle \geqslant 0$ for all $i \in I_P$. The set of all $P$-dominant weights is denoted by $\Lambda_P^+$.

Let $E$ be a $P$-module, i.e., $E$ is a finite-dimensional complex vector space and $P$ acts on $E$ via a holomorphic representation $P \to GL(E)$. The weights (resp. $P$-dominant weights) of $E$ are denoted by $\Lambda_P(E)$ (resp. $\Lambda_P^+(E)$). The irreducible $P$-module of highest weight $\lambda \in \Lambda_P^+$ is denoted by $V_P^\lambda$; the set of weights (respectively dominant weights) of $V_P^\lambda$ is denoted by $\Lambda_P(\lambda)$ (resp. $\Lambda_P^+(\lambda)$). When $P = G$ the subscript is sometimes dropped. The weights of $P$ are naturally imbedded in the weights of $G$, $\Lambda_P \subset \Lambda$, and the group $W_P$ of reflections of $\Lambda_P$ (generated by simple reflections $\sigma_i$ for $i \in I_P$) is naturally a subgroup of $W$.

A $P$-module $E_0$ determines a homogeneous vector bundle $E$ on $X = G/P$: $E = G \times E_0/P$ where the action of $P$ on the product $G \times E_0$ is the usual diagonal action. Conversely, any homogeneous vector bundle on $X$ can be written in this form. To avoid excess notation, we sometimes use the same letter to denote both the $P$-module and the associated homogeneous vector bundle. We shall also use $E$ to denote the sheaf of germs of sections of $E$.

Every $P$-module $E$ has a filtration by $P$-submodules, $E = E_0 \supset E_1 \supset \cdots \supset E_t \supset E_{t+1} = 0$, such that the quotient modules $\hat{E}_i = E_i/E_{i+1}$ are irreducible $P$-modules of highest weight $\mu_i$. We call these weights the highest weights of $E$ and denote the set of them by $\Lambda_P^{++}(E) = \{\mu_1, \ldots, \mu_t\}$. Although such filtrations of $E$ are not unique, the set of highest weights $\Lambda_P^{++}(E)$ is uniquely determined by the decomposition of $E$ into irreducible components $\hat{E}_i$ with respect to the reductive group $H \cdot S_P$ where $S_P$ is a semisimple Levi factor (generated by the roots $\Phi_P^+ \cup -\Phi_P^+$) of $P$.

Let $T_X$ be the $P$-module defined by the isotropy representation of $P$ on the tangent space of $X = G/P$ at the identity coset. Then the weights of $T_X$ are called the roots of $X$, $\Phi_X^+ = \Lambda_P(T_X) = \Phi^+ \setminus \Phi_P^+$. The vector bundle $\Omega_X^p = \wedge^p T_X$ of holomorphic $p$-forms on $X$ is naturally homogeneous and the weights of the associated $P$-module are $\Lambda_P(\Omega_X^p) = \{-\Sigma_{\beta \in S}\beta \mid S \subset \Phi_X^+, \#S = p\}$.

A fundamental tool for calculating the cohomology of a homogeneous vector bundle is Bott's Theorem [3]. We shall use the following version of this theorem, see for example [5, 19].

LEMMA 1.1 *Let $E$ be a homogeneous vector bundle on $X = G/P$. If $\lambda$ is a highest weight of the $G$-module $H^q(X, E)$, $\lambda \in \Lambda^{++}(H^q(X, E))$, then $\lambda = \sigma(\mu + \delta) - \delta$ where $\mu \in \Lambda_P^{++}(E)$, $\sigma(\mu + \delta)$ is the non-singular dominant conjugate of $\mu + \delta$, $\sigma \in W$, and $q = \operatorname{ind}(\mu + \delta) = \operatorname{len}(\sigma)$. In particular, if $I(E)$ is the set of indexes, $\operatorname{ind}(\mu+\delta)$, of non-singular weights $\mu+\delta$ where $\mu \in \Lambda_P^{++}(E)$, then $H^q(X, E) = 0$, if $q \notin I(E)$.*

## 2. Dolbeault cohomology

We retain the notation and conventions from the previous section. Let $G$ be a semisimple complex Lie group and $P$ a parabolic subgroup of $G$ defined by a set of indexes $I_P$. In this section we prove some general propositions about the Dolbeault cohomology of a homogeneous vector bundle $E$ on $X = G/P$. We assume that the highest weights of $E$ are dominant. In particular, $E$ is generated by global sections and $H^q(X, E) = 0$ for $q > 0$, see [20, Lemma 3.4]. The first goal is to find an integer function $u_X$, independent of $E$, for which $H^{p,q}(X, E) := H^q(X, \Omega_X^p \otimes E) = 0$ whenever $q > u_X(p)$. Since this notion appears frequently, we make the following definition.

DEFINITION 2.1 An integer function $u_X$ is called an upper bound for the homogeneous Dolbeault cohomology on $X$ if, for any homogeneous bundle $E$ on $X$ with dominant highest weights, $H^{p,q}(X,E)=0$ whenever $q>u_X(p)$.

The basic technique for finding such an upper bound is to apply Lemma 1.1 to $\Omega^p_X\otimes E$ by estimating the index of appropriate weights.

LEMMA 2.2 *Let $E$ and $F$ be $P$-modules. Any weight $\omega\in\Lambda_P^{++}(E\otimes F)$ can be expressed as a sum $\omega=\mu+\nu\in\Lambda_P^+$ where $\mu\in\Lambda_P^{++}(E)$ and $\nu\in\Lambda_P(\lambda)$ for some $\lambda\in\Lambda_P^{++}(F)$.*

*Proof.* The weights $\Lambda_P^{++}(E\otimes F)$ are determined by the decomposition of $E\otimes F$ into irreducible components with respect to a semisimple Levi factor $S_P$ of $P$. As an $S_P$-module, $E\otimes F\cong\bigoplus_{\mu,\lambda}V_P^\mu\otimes V_P^\lambda$ where the sum is over $\mu\in\Lambda_P^{++}(E)$, $\lambda\in\Lambda_P^{++}(E)$. It follows from Steinberg's formula that the highest weights $\omega$ that appear in the decomposition $V_P^\mu\otimes V_P^\lambda\cong\bigoplus_\omega n_{\mu,\lambda}(\omega)V_P^\omega$ have the form $\omega=\mu+\nu$ where $\nu\in\Lambda_P(\lambda)$, see, e.g., [7, 24.4]. □

LEMMA 2.3 *Let $\lambda\in\Lambda^+$, $\mu\in\Lambda_P^+$ and $\nu\in\Lambda_P(\lambda)$. If $\mu+\nu\in\Lambda_P^+$, then* $\operatorname{ind}(\mu+\nu)\leqslant\operatorname{ind}(\mu)$.

*Proof.* It is enough to prove that if $\beta\in\Phi^+$ and $(\mu+\nu,\beta)<0$ then $(\nu,\beta)\geqslant 0$, because this implies

$$\operatorname{ind}(\mu+\nu)=\#\{\beta>0\,|\,(\mu+\nu,\beta)<0\}\leqslant\#\{\beta>0\,|\,(\mu,\beta)<0\}=\operatorname{ind}\mu.$$

If $\beta\in\Phi_P^+$ then $(\mu+\nu,\beta)\geqslant 0$ by assumption. So let $\beta=\Sigma_{i=1}^\ell m_i(\beta)\alpha_i\in\Phi_X^+$. Write $\lambda=\Sigma_{i=1}^\ell\langle\lambda,\alpha_i\rangle\lambda_i$ with $\langle\lambda,\alpha_i\rangle\geqslant 0$ for all $i$, and $\nu=\Sigma_{j=1}^\ell\langle\nu,\alpha_j\rangle\lambda_j=\lambda-\Sigma_{i\in I_P}m_i(\nu)\alpha_i$ where $m_i(\nu)\geqslant 0$ for $i\in I_P$. Then $\langle\nu,\alpha_j\rangle=\langle\lambda,\alpha_j\rangle-\Sigma_{i\in I_P}m_i(\nu)c_{i,j}$ where $c_{i,j}=\langle\alpha_i,\alpha_j\rangle$ are the entries in the Cartan matrix for $G$.

Assume for the moment that $\nu\in\Lambda_P^+(\lambda)$ so that $\langle\nu,\alpha_i\rangle\geqslant 0$ for $i\in I_P$. Let $\check{m}_j(\beta)=m_j(\beta)(\alpha_j,\alpha_j)/2$ for all $j$. Then

$$\begin{aligned}(\nu,\beta)&=\sum_{j=1}^{\ell}\check{m}_j(\beta)\langle\nu,\alpha_j\rangle\geqslant\sum_{j\notin I_P}\check{m}_j(\beta)\langle\nu,\alpha_j\rangle\\&=\sum_{j\notin I_P}\check{m}_j(\beta)[\langle\lambda,\alpha_j\rangle-\sum_{i\in I_P}m_i(\nu)c_{i,j}]\geqslant 0,\end{aligned}$$

because $c_{i,j}\leqslant 0$ when $i\neq j$. Now, if $\nu\notin\Lambda_P^+$, then there is a $\sigma\in W_P$ and $\nu_0\in\Lambda_P^+(\lambda)$ such that $\nu=\sigma\nu_0$. Since $\sigma\Phi_X^+=\Phi_X^+$, we see as above that $(\nu,\beta)=(\nu_0,\sigma^{-1}\beta)\geqslant 0$. □

DEFINITION 2.4 Let $X = G/P$. For each $0 \leqslant p \leqslant \dim X$ define

$$m_X(p) = \max\{\mathrm{ind}\,(\mu + \delta) \mid \mu \in \Lambda_P^{++}(\Omega_X^p)\}.$$

Little is known about the function $m_X$ for general $X$, although it is easy to check that $m_X(1) = 1$. In the next section we show that $m_X(p) \geqslant p$, and for compact hermitian symmetric spaces $m_X(p) = p$, for $0 \leqslant p \leqslant \dim X$. There are other homogeneous spaces, however, for which $m_X(p)$ is strictly greater than some $p$, although in all the examples we computed, $m_X(p)$ remains 'close' to $p$, see Section 5. In any case, we now show that $m_X$ is always an upper bound for the homogeneous Dolbeault cohomology on $X$.

PROPOSITION 2.5 *Let $X = G/P$ and let $E$ be a homogeneous vector bundle on $X$. Assume the highest weights of $E$ are dominant, $\Lambda_P^{++}(E) \subset \Lambda^+$. Then $H^{p,q}(X, E) = 0$ whenever $q > m_X(p)$.*

*Proof.* By Lemma 2.2, $\Lambda_P^{++}(\Omega_X^p \otimes E)$ consists of weights of the form $\mu + \nu \in \Lambda_P^+$ where $\mu \in \Lambda_P^{++}(\Omega_X^p)$ and $\nu \in \Lambda_P(\lambda)$ for some $\lambda \in \Lambda_P^{++}(E)$. By Lemma 2.3, $\mathrm{ind}\,(\mu + \nu + \delta) \leqslant \mathrm{ind}\,(\mu + \delta) \leqslant m_X(p)$. Therefore, by Lemma 1.1, $H^q(X, \Omega^p \otimes E) = 0$ if $q > m_X(p)$. □

The Borel-Le Potier spectral sequence can be used to 'lift' vanishing theorems for the Dolbeault cohomology of vector bundles on certain homogeneous spaces to spaces that fiber over them. An important part of the process is understanding the relative cohomology of the fiber. The next proposition addresses this point.

PROPOSITION 2.6 *Let $X = G/P \to Y = G/Q$ be a homogeneous fibration with fiber $Z = Q/P$ where $P \subset Q$ are parabolic subgroups of $G$. Let $E$ be a homogeneous vector bundle on $X$ and assume that the highest weights of the associated $P$-module are dominant, $\Lambda_P^{++}(E) \subset \Lambda^+$. Then the highest weights of the $Q$-module $H^{p,q}(Z, E|_Z)$ are also dominant, $\Lambda_Q^{++}(H^{p,q}(Z, E|_Z)) \subset \Lambda^+$.*

*Proof.* By Bott's Theorem (Lemma 1.1), a weight $\xi \in \Lambda_Q^{++}(H^q(Z, \Omega_Z^p \otimes E|_Z))$ has the form $\xi = \sigma(\eta + \delta) - \delta$ where $\sigma \in W_Q$ and $\eta \in \Lambda_P^{++}(\Omega_Z^p \otimes E)$. We know that $\langle \xi, \alpha_i \rangle \geqslant 0$ for $i \in I_Q$, so we must show that $\langle \xi, \alpha_j \rangle \geq 0$ for $j \notin I_Q$. By Lemma 2.2, $\eta = \mu + \nu$ for some $\mu \in \Lambda_P^{++}(E)$ and $\nu \in \Lambda_P(\Omega_Z^p)$. By [9, Lemma 5.9], $\Lambda_Q(\delta) = \{\delta - \Sigma_{\beta \in S} \beta \mid S \subset \Phi_Q^+\}$. Since $\nu = -\Sigma_{\beta \in S} \beta$ for some subset $S \subset \Phi_Z^+ \subset \Phi_Q^+$, it follows that $\eta + \delta = \mu + \delta - \Sigma_{\beta \in S} \beta$ is a weight of the $Q$-representation $V_Q^\mu \otimes V_Q^\delta$. Therefore, its $Q$-dominant conjugate, $\sigma(\eta + \delta)$, must have the form $\mu + \delta - \Sigma_{\beta \in \Phi_Q^+} n_\beta \beta$ with $n_\beta \geqslant 0$, by Lemma 2.2. Let $\beta = \Sigma_{i \in I_Q} m_i(\beta) \alpha_i$ with $m_i \geqslant 0$. Then, if $j \notin I_Q$, $\langle \xi, \alpha_j \rangle = \langle \mu, \alpha_j \rangle - \Sigma_{\beta \in \Phi_Q^+, i \in I_Q} n_\beta m_i(\beta) \langle \alpha_i, \alpha_j \rangle \geqslant 0$, since $\langle \alpha_i, \alpha_j \rangle \leqslant 0$ when $i \neq j$. □

In the context of homogeneous fibrations, this result extends to Dolbeault cohomology certain general positivity properties of direct images of dualizing sheaves: if $f\colon X \to Y$ is a surjective map between smooth projective varieties, then $f_*\omega_{X/Y}$ is semi-positive as soon as a normal crossing hypothesis is fulfilled, see [8, Corollary 3.7]. Moreover, if $L$ is an ample line bundle on $Y$, then $H^i(Y, \omega_Y \otimes R^j f_*\omega_{X/Y} \otimes L) = 0$ whenever $i > 0$, [8, Theorem 2.1].

We now give a general lifting property that is useful for the inductive proofs of Sections 3 and 4, as well as for calculations with the tables of Section 5.

PROPOSITION 2.7 *Let $\pi\colon X = G/P \to Y = G/Q$ be a homogeneous fibration with fiber $Z$. Let $u_Y$ and $u_Z$ be upper bounds for the homogeneous Dolbeault cohomology on $Y$ and $Z$, respectively. Then the function $u_X(p) = \max_t\{u_Y(t) + u_Z(p-t)\}$ is an upper bound for the homogeneous Dolbeault cohomology on $X$.*

*Proof.* Let $E$ be a homogeneous bundle on $X$ with dominant highest weights. The Borel–Le Potier spectral sequence ([2, 10, 11]) associated to $E$ and $\pi$ is defined by the filtration of $\Omega^p_X \otimes E$, $F^{t,p} = \Omega^{p-t}_X \wedge \pi^*\Omega^t_Y \otimes E$, with quotients $G^{t,p} = F^{t,p}/F^{t+1,p} = \Omega^{p-t}_Z \otimes \pi^*\Omega^t_Y \otimes E$. The terms of order one are ${}^pE_1^{t,q-t} = H^q(X, G^{t,p})$ and the sequence converges to $H^{p,q}(X,E)$. Now, ${}^pE_1^{t,q-t}$ is itself the abutment of the Leray spectral sequence associated to $R^j\pi_*G^{t,p} = H^{p-t,j}(Z, E|_Z)\otimes\Omega^t_Y$ whose terms of order two are $E_2^{i,j} = H^{t,i}(Y, H^{p-t,j}(Z,E|_Z))$. By assumption, $H^{p-t,j}(Z,E|_Z) = 0$ whenever $j > u_Z(p-t)$. Also, by Proposition 2.6, the highest weights of $H^{p-t,j}(Z,E|_Z)$ as a $Q$-module are dominant. Therefore, $E_2^{i,j} = 0$ for $i > u_Y(t)$. If $q = i + j > \max_t\{u_Y(t) + u_Z(p-t)\}$ then for each $t$ either $j > u_Z(p-t)$ or $i > u_Y(t)$ and so $\bigoplus_{i+j=q} E_2^{i,j} = 0$. This implies that ${}^pE_1^{t,q-t} = 0$ and hence that $H^{p,q}(X,E) = 0$ whenever $q > u_X(p)$. □

## 3. Symmetric space towers

We now turn our attention to compact hermitian symmetric spaces and spaces built from them which we call symmetric space towers. A compact hermitian symmetric space $X$ is a direct product, $X = X_1 \times \cdots \times X_t$, of irreducible hermitian symmetric spaces $X_i$ that are quotients of simple complex Lie groups by maximal parabolic subgroups. The possibilities for the type of simple group and for the complement $I'_P = \{m\}$ are as follows (the simple roots are numbered as in [22]): type $A_\ell$, $m = 1, \dots, \ell$ (Grassmann); type $B_\ell$, $m = 1$ (quadric); type $C_\ell$, $m = \ell$; type $D_\ell$, $m = 1$ (quadric), $\ell - 1$, or $\ell$; type $E_6$, $E_7$, $m = 1$.

For a general homogeneous space, $X = G/P$, where $G$ is semisimple and $P$ is parabolic, it is well-known that $H^{p,q}(X) \cong H^q(X, \Omega^p_X) \neq 0$ if and only if $p = q$. One way to see this is to use Bott's Theorem (Lemma 1.1) and the following fact proved in [9]

$$\mu \in \Lambda^{++}_P(\Omega^p_X) \quad \text{and } \mu + \delta$$

is non-singular if and only if

$$\mu = \sigma(\delta) - \delta \quad \text{for } \sigma \in W^1(p) = \{\sigma \in W \mid \sigma^{-1}\Phi_P^+ \subset \Phi^+, \ \mathrm{len}\,(\sigma) = p\}.$$

It follows directly from this statement that $m_X(p) \geqslant p$. In general, there are weights $\mu \in \Lambda_P^{++}(\Omega_X^p)$ such that $\mu + \delta$ is singular. While such weights do not contribute to the Dolbeault cohomology groups $H^{p,q}(X)$ they can play a role in the Dolbeault cohomology of a homogeneous vector bundle $E$ on $X$. For compact hermitian symmetric spaces, the situation is much simpler since the vector bundle of holomorphic $p$-forms decomposes into a direct sum of irreducible $P$-modules, see [9]:

$$\Omega_X^p = \bigoplus_{\sigma \in W^1(p)} V_P^{\sigma(\delta)-\delta}.$$

In other words, for these spaces $\Lambda_P^{++}(\Omega_P^p) = \{\sigma(\delta) - \delta \mid \sigma \in W^1(p)\}$ so that $m_X(p) = p$. As a corollary of Proposition 2.5, we then obtain the following vanishing theorem which generalizes results for line bundles in [17, 18] and for Grassmann manifolds in [14]

COROLLARY 3.1 *Let $X$ be a compact hermitian symmetric space and let $E$ be a homogeneous vector bundle on $X$. Assume the highest weights of $E$ are dominant. Then $H^{p,q}(X, E) = 0$ whenever $q > p$.*

This statement can be extended to a wider class of homogeneous spaces by applying Proposition 2.7 to fibrations $X \to Y$, where $Y$ is a compact hermitian symmetric space, and using induction on the fiber $Z$. For this purpose we make the following definition.

DEFINITION 3.2 The homogeneous space $X = G/P$ is said to be a symmetric space tower if there exists a sequence of parabolic subgroups, $P = Q_0 \subset Q_1 \subset \cdots \subset Q_s \subset Q_{s+1} = G$ such that $Z_i = Q_i/Q_{i-1}$ is a compact hermitian symmetric space for $i = 1, \ldots, s+1$. Thus, $X$ is a tower of homogeneous bundles,

$$X \xrightarrow{Z_1} Y_1 \xrightarrow{Z_2} Y_2 \cdots \xrightarrow{Z_s} Y_s = Z_{s+1},$$

where $Y_i = G/Q_i$ and the fibers $Z_i$ are compact hermitian symmetric spaces for all $i$.

Note that if $G$ is a product of simple groups of type $A_\ell$ and $P$ is any parabolic subgroup then $X = G/P$ is a symmetric space tower. Also, a symmetric space tower fibers over a compact hermitian symmetric space with a fiber that is again a symmetric space tower. This permits the induction argument in the proof of the next theorem.

THEOREM 3.3 *Let $X$ be a symmetric space tower and let $E$ be a homogeneous vector bundle on $X$. Assume the highest weights of $E$ are dominant. Then $H^{p,q}(X,E)=0$ whenever $q>p$.*

*Proof.* If $X$ is already a compact hermitian symmetric space we may apply Corollary 3.1. Otherwise, let $\pi\colon X=G/P\to Y=G/Q$ be a non-trivial fibration of $X$ where $Y$ is a compact hermitian symmetric space and the fiber $Z=Q/P$ is again a symmetric space tower. By Corollary 3.1, an upper bound for the Dolbeault cohomology of $Y$ is given by $u_Y(p)=p$. By induction on dimension, we may assume that an upper bound for the homogeneous Dolbeault cohomology on $Z$ is also given by $u_Z(p)=p$. Therefore, by Proposition 2.7, an upper bound for the Dolbeault cohomology on $X$ is $u_X(p)=\max_t\{u_Y(t)+u_Z(p-t)\}=p$. □

## 4. Isotropic flag manifolds

We noticed in the last Section that when $G$ is a product of simple groups of type $A_\ell$, and $P$ any parabolic subgroup, then $X=G/P$ is a symmetric space tower. We now study the case where $G$ is a simply-connected simple group of type $B_\ell$, $C_\ell$ or $D_\ell$, $P$ is a maximal parabolic subgroup, and we exclude the cases for which $X=G/P$ is a compact hermitian symmetric space. Let $m$ be the index in the complement, $I'_P=\{m\}$. Geometrically, $X$ is the space of isotropic $m$-planes of a complex vector space $V$, endowed with a non-degenerate bilinear form. This form is symmetric when $G$ is of type $B_\ell$ or $D_\ell$ (the dimension of $V$ is then $2l+1$ and $2l$ respectively), and skew-symmetric when $G$ is of type $C_\ell$ ($V$ is then $2l$-dimensional). We denote by $\varepsilon_1,\dots,\varepsilon_\ell$ an orthonormal basis of the characters of the torus $H$ of $G$, and we follow the conventions of [22] for roots and weights.

PROPOSITION 4.1 *Suppose that $X=G/P$ is not symmetric. If $G$ is of type $B_\ell$ or $D_\ell$, then $\Lambda_P^{++}(\Omega_X^1)=\{-\varepsilon_{m-1}-\varepsilon_m,-\varepsilon_m+\varepsilon_{m+1}\}$, and if $G$ is of type $C_\ell$, then $\Lambda_P^{++}(\Omega_X^1)=\{-2\varepsilon_m,-\varepsilon_m+\varepsilon_{m+1}\}$.*

*Proof.* The linear action of $P$ on the tangent bundle is given by the adjoint representation of $P$ on the quotient of Lie algebras $\mathcal{G}/\mathcal{P}$. Hence, the action of $P$ on the cotangent bundle is given by the adjoint representation of $P$ on the orthogonal complement of $\mathcal{P}$ in $\mathcal{G}$ with respect to the Killing form. If $P$ is the stabilizer of the $m$-plane $W$ of $V$, a straightforward computation then establishes the isomorphism of $P$-modules $\Omega_X^1\simeq\{g\in\mathcal{G},\ g(V)\subset W^\perp,\ g(W^\perp)\subset W,\ g(W)=0\}$. This surjects onto $\mathrm{Hom}\,(W^\perp/W,W)$, with kernel $\{g\in\mathcal{G},\ g(V)\subset W,\ g(W^\perp)=0\}$ consisting of elements of $\mathrm{Hom}\,(V/W^\perp,W)\simeq W\otimes W$ coming from $\mathcal{G}$. When $G$ is an orthogonal group, such a two-tensor comes from $\mathcal{G}$ if and only if its is skew-symmetric, hence an exact sequence

$$0\to\wedge^2W\simeq V_P^{-\varepsilon_{m-1}-\varepsilon_m}\to\Omega_X^1\to\mathrm{Hom}\,(W^\perp/W,W)$$
$$\simeq V_P^{-\varepsilon_m+\varepsilon_{m+1}}\to 0$$

When $G$ is symplectic, tensors coming from $\mathcal{G}$ are symmetric and we get the sequence

$$0 \to S^2W \simeq V_P^{-2\varepsilon_m} \to \Omega^1_X \to \mathrm{Hom}\,(W^\perp/W, W)$$
$$\simeq V_P^{-\varepsilon_m+\varepsilon_{m+1}} \to 0.$$

Hence the highest weights of the cotangent bundle. □

Recall that if $S_P$ is a semi-simple Levi factor of $P$, irreducible $P$-modules and irreducible $S_P$-modules are the same, because the action of the unipotent radical of $P$ is necessarily trivial. This Levi factor is a product $S_P = M \times N$ with $M$ of type $A_{m-1}$ and $N$ of type $B_n$ (resp. $C_n$, $D_n$), with $m + n = l$, if $G$ is of type $B_\ell$ (resp. $C_\ell$, $D_\ell$). Moreover, a weight $\mu$ is $P$ dominant if and only if $V^\mu_{S_P} = V^\alpha_M \otimes V^\beta_N$ with $\alpha = \Sigma_{j\leqslant m}\mu_j\varepsilon_j$ and $\beta = \Sigma_{j>m}\mu_j\varepsilon_j$ dominants. In particular, $\beta$ is a non-increasing sequence of non-negative integers, that is, a partition, and we will denote by $|\beta|$ the sum of its parts.

Rather than trying to bound $m_X(p)$, it will be easier to give a direct proof of our next vanishing theorem.

THEOREM 4.2 *Let $G$ be a classical simply-connected semi-simple group, and let $P$ be a parabolic subgroup. Let $E$ be a homogeneous vector bundle on $X = G/P$, and assume that the highest weights of $E$ are dominant. Then $H^{p,q}(X, E) = 0$ whenever $q > 2p$.*

*Proof.* Because of Lemma 1.1 and Propositions 2.6 and 2.7, we may suppose that $G$ is simple of type $B_\ell$, $C_\ell$ or $D_\ell$, that $P$ is a maximal parabolic subgroup, and that $E$ is an irreducible $P$-module. Moreover, it is enough to show that if $\lambda \in \Lambda^{++}_P(\Omega^p_X)$, $\mu \in \Lambda^+$, and $\rho$ is a highest weight of $V^\lambda_P \otimes V^\mu_P$, then $\mathrm{ind}\,(\rho+\delta) \leqslant 2p$.

Let us treat the case where $G$ is of type $B_\ell$. Proposition 4.1 implies that $\Lambda^{++}_P(\Omega^p_X)$ is the set of highest weights of the $p$-th wedge power of $\wedge^2 V^*_M \oplus V^*_M \otimes V_N$. In particular, $V^\lambda_{S_P} = (V^\alpha_M)^* \otimes V^\beta_N$, for some weights $\alpha$ and $\beta$, having non-negative components whose sum is $|\alpha| + |\beta| \leqslant 2p$. We may then write $V^\rho_{S_P} = (V^{\alpha-\delta\alpha}_M)^* \otimes V^{\beta+\delta\beta}_N$ for some weights $\delta\alpha$ and $\delta\beta$, and $\delta\alpha$ has non-negative components.

We divide the set of positive roots $\Phi^+$ into the subset $\Phi^+_+$ of roots $\varepsilon_s$, $1 \leqslant s \leqslant l$, and $\varepsilon_s + \varepsilon_t$, $1 \leqslant s < t \leqslant l$, and the subset $\Phi^+_-$ of roots $\varepsilon_s - \varepsilon_t$, $1 \leqslant s < t \leqslant l$. Hence a corresponding decomposition for the index, into the sum $\mathrm{ind}\,(\rho+\delta) = \mathrm{ind}_+(\rho+\delta) + \mathrm{ind}_-(\rho+\delta)$ with

$$\mathrm{ind}_+(\rho+\delta) = \#\{\gamma \in \Phi^+_+ \mid (\rho+\delta, \gamma) < 0\},$$
$$\mathrm{ind}_-(\rho+\delta) = \#\{\gamma \in \Phi^+_- \mid (\rho+\delta, \gamma) < 0\}.$$

LEMMA 4.3 *Suppose that $\rho+\delta$ is regular. Then*

$$\begin{aligned}\operatorname{ind}_+(\rho+\delta) &\leqslant \sum_j(\alpha_j-n-j+1)^+,\\ \operatorname{ind}_-(\rho+\delta) &\leqslant |\beta|+\sum_j\min(n,\alpha_j).\end{aligned}$$

*Proof.* Since $\delta=\Sigma_{j=1}^l(l+1/2-j)\varepsilon_j$, we have

$$\begin{aligned}\rho+\delta &= \sum_{j=1}^m(n-1/2-\alpha_j+\delta\alpha_j+j)\varepsilon_{m+1-j}\\ &\quad+\sum_{k=1}^n(n+1/2+\beta_k+\delta\beta_k-k)\varepsilon_{m+k}.\end{aligned}$$

Only the negative components of this weight, which are among the $m$ first ones (recall that $\beta+\delta\beta$ has non-negative components), may contribute to $\operatorname{ind}_+(\rho+\delta)$. Therefore, it is the sum of the integers $l_j$ defined as follows: if $n-1/2-\alpha_j+\delta\alpha_j+j>0$, then $l_j=0$; otherwise, $l_j$ is the number of indexes $i$ for which $(\rho+\delta,\varepsilon_i+\varepsilon_{m+1-j})<0$, with $i\leqslant m+1-j$ or $i>m$. But the regularity hypothesis implies that the corresponding components of $\rho+\delta$ have distinct absolute values. Hence the bound

$$l_j\leqslant\alpha_j-\delta\alpha_j-n-j+1\leqslant\alpha_j-n-j+1,$$

which implies our first claim. For the second one, note that we divided $\rho+\delta$ into two decreasing sequences. If $h_{j,k}=\alpha_j+\beta_k-j-k+1$ for $1\leqslant j\leqslant m$ and $1\leqslant k\leqslant n$, then

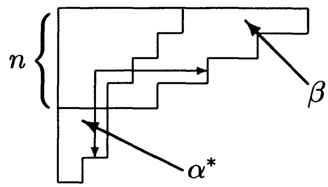
$$\operatorname{ind}_-(\rho+\delta)=\#\{\,(j,k),\quad h_{j,k}>\delta\alpha_j-\delta\beta_k\}.$$

Since $\mu\in\Lambda^+$, $V_{SP}^\mu=V_M^\sigma\otimes V_N^\tau$ with $\sigma_m=\mu_m\geqslant\tau_1=\mu_{m+1}$, so that $\delta\alpha_j\geqslant\sigma_m$ for every $j$. Moreover, $\delta\beta$, being a weight of $V_N^\tau$, is in the convex hull of the conjugates of $\tau$ by the Weyl group on $N$. This implies that $|\delta\beta_k|\leqslant\tau_1$ for every $k$, hence $\delta\alpha_j-\delta\beta_k\geqslant 0$ and

$$\operatorname{ind}_-(\rho+\delta)\leqslant\#\{\,(j,k),\quad h_{j,k}>0\}.$$

Recall that the conjugate partition $\alpha^*$ of $\alpha$ is the partition whose $j$th component is the number of components of $\alpha$ greater or equal to $j$: its Ferrer diagram is obtained from that of $\alpha$ by reflection through the main diagonal. When $\beta=\alpha^*$, the integers $h_{j,k}$ are the usual *hook lengths* of the partition $\alpha$. In general, they can be interpreted as *mixed hook lengths*, since $h_{j,k}$ is the sum of the horizontal distance to the frontier of the Ferrer diagram of $\beta$, and of the vertical distance to that of $\alpha^*$.

1. Mixed hook lengths.




The mixed hook lengths are obviously negative outside the union of these two Ferrer diagrams. Since we only take into account the $m$ first columns and the $n$ first rows, we get

$$\operatorname{ind}_{-}(\rho+\delta) \leqslant \sum_{j=1}^{m} \max(\beta_j^*, \min(n, \alpha_j)) \leqslant |\beta| + \sum_{j} \min(n, \alpha_j).$$

The lemma is proved. □

For every weight $\rho \in \Lambda_P^{++}(\Omega_X^p \otimes E)$ such that $\rho + \delta$ is regular, we get

$$\operatorname{ind}(\rho+\delta) \leq |\alpha| + |\beta| \leqslant 2p.$$

This proves the theorem for $G$ of type $B_\ell$. The other cases are quite similar. □

For an ample line bundle $L$ on a smooth projective variety $X$, the Nakano vanishing theorem states that $H^{p,q}(X, L) = 0$ whenever $q > \dim X - p$. If $X = G/P$, then any line bundle on $X$ is homogeneous and it is ample if an only if the corresponding weight is dominant. Combining this with Theorems 3.3 and 4.2, we obtain the following.

COROLLARY 4.4 *Let $L$ be an ample line bundle on $X = G/P$. If $X$ is a symmetric space tower, then $H^{p,q}(X, L) = 0$ whenever $q > \frac{1}{2}\dim X$. If $G$ is a classical semisimple complex Lie group, then $H^{p,q}(X, L) = 0$ whenever $q > \frac{2}{3}\dim X$.*

## 5. Special cases

In this final section we collect some results related to the Dolbeault cohomology of a homogeneous bundle $E$ on a space $X = G/P$ where $G$ has low rank or is an exceptional group, i.e., $G$ is of type $E_6$, $E_7$, $E_8$, $F_4$, or $G_2$. Again, we adopt the conventions of [22] for indexing roots. Our approach for this case is to directly compute the function $m_X$ when $P$ is a maximal parabolic subgroup. This establishes a vanishing theorem for $H^{p,q}(X, E)$ by Proposition 2.5. Similar vanishing theorems for a more general parabolic subgroup $P$ can be deduced from

*Table 1. Exceptional Groups*

| $E_6$ | 1 2 3 4 5 | 6 7 8 9 10 | 11 12 13 14 15 | 16 17 18 19 20 | 21 22 23 24 25 | 26 27 28 29 30 |
|---|---|---|---|---|---|---|
| 2,4 | 1 2 3 4 5 | 7 8 10 11 12 | 13 14 15 16 17 | 18 19 20 20 22 | 23 23 24 24 25 | |
| 3 | 1 2 3 4 6 | 8 9 12 12 13 | 17 17 18 20 21 | 22 25 25 25 25 | 26 26 28 28 27 | 28 28 28 29 |
| 6 | 1 2 3 4 5 | 6 7 8 9 10 | 11 12 14 14 17 | 17 19 19 20 20 | 21 | |

| $F_4$ | 1 2 3 4 5 | 6 7 8 9 10 | 11 12 13 14 15 | 16 17 18 19 20 |
|---|---|---|---|---|
| 1 | 1 2 3 4 5 | 6 7 8 10 10 | 13 13 14 14 15 | |
| 2 | 1 2 3 5 6 | 8 10 14 14 15 | 15 16 16 19 19 | 18 19 19 19 20 |
| 3 | 1 2 4 5 7 | 8 10 11 13 14 | 16 17 18 19 19 | 18 19 19 19 20 |
| 4 | 1 2 3 5 6 | 8 10 10 11 12 | 13 13 13 14 15 | |

| $G_2$ | 1 2 3 4 5 |
|---|---|
| 1 | 1 3 4 4 5 |
| 2 | 1 2 4 4 5 |

*Table 2. Type $B_\ell$*

| | 1 2 3 4 5 | 6 7 8 9 10 | 11 12 13 14 15 | 16 17 18 19 20 | 21 22 23 24 25 26 |
|---|---|---|---|---|---|
| $B_2$ 2 | 1 2 3 | | | | |
| $B_3$ 2 | 1 2 3 4 6 | 6 7 | | | |
| 3 | 1 2 4 4 5 | 6 | | | |
| $B_4$ 2 | 1 2 3 4 5 | 6 8 8 10 10 | 11 | | |
| 3 | 1 2 3 4 6 | 7 8 9 10 11 | 11 12 | | |
| 4 | 1 2 4 4 6 | 8 8 8 9 10 | | | |
| $B_5$ 2 | 1 2 3 4 5 | 6 7 8 10 11 | 12 12 14 14 15 | | |
| 3 | 1 2 3 4 5 | 7 9 9 10 11 | 12 13 15 15 16 | 17 17 18 | |
| 4 | 1 2 3 4 6 | 7 9 10 11 13 | 13 13 15 17 17 | 17 17 18 | |
| 5 | 1 2 4 4 6 | 8 8 9 11 13 | 13 13 13 14 15 | | |
| $B_6$ 2 | 1 2 3 4 5 | 6 7 8 9 10 | 12 13 14 15 16 | 16 18 18 19 | |
| 3 | 1 2 3 4 5 | 7 7 9 11 12 | 13 13 14 15 16 | 18 19 19 21 21 | 22 23 23 24 |
| 4 | 1 2 3 4 5 | 7 9 10 11 12 | 13 14 16 18 18 | 19 20 23 23 23 | 23 25 25 25 25 26 |
| 5 | 1 2 3 4 6 | 7 9 10 12 14 | 14 16 17 19 20 | 20 20 21 22 23 | 24 24 24 24 25 |
| 6 | 1 2 4 4 6 | 8 8 9 11 13 | 13 14 15 17 19 | 19 19 19 19 20 | 21 |

these by considering fibrations $X = G/P \to Y = G/Q$, where $Q$ is a maximal parabolic subgroup, and applying Proposition 2.7.

The steps involved in calculating $m_X$ are as follows. First, we collect all $P$-dominant weights of $\Omega_X^p$ by calculating $\mu = -\Sigma_{\beta\in S}\beta$ for all subsets $S \subset \Phi_X$, with $p = \#S$, saving only those that are $P$-dominant. This list $L$ of weights with multiplicities is then sorted according to the inner product with $\delta$. Starting with the first weight $\mu \in L$, i.e., $\langle\mu,\delta\rangle$ largest, the subdominant weights of $\Lambda^+(\mu)$ are computed along with their multiplicities using Freudenthal's formula. Then, for each $\nu \in \Lambda^+(\mu)$ the multiplicity of $\nu$ in $L$ is reduced by the multiplicity of $\nu$ in $V(\mu)$. The same procedure is repeated with the next weight in $L$ with positive multiplicity, and so on, until the end of $L$ is reached. The weights remaining in $L$ with positive multiplicity at that point are the highest weights, $\Lambda^{++}(\Omega_X^p)$. Finally, the index of $\mu + \delta$ is calculated for each $\mu \in \Lambda^{++}(\Omega_X^p)$ and the maximum for each $p$ is $m_X(p)$. The length of this entire calculation is roughly exponential in $\dim X$.

*Table 3. Type $C_\ell$*

| | 1 2 3 4 5 | 6 7 8 9 10 | 11 12 13 14 15 | 16 17 18 19 20 | 21 22 23 24 25 26 |
|---|---|---|---|---|---|
| $C_3$ 1 | 1 2 3 4 5 | | | | |
| 2 | 1 2 4 4 5 | 6 7 | | | |
| $C_4$ 1 | 1 2 3 4 5 | 6 7 | | | |
| 2 | 1 2 3 4 6 | 7 8 8 9 10 | 11 | | |
| 3 | 1 2 4 5 6 | 9 9 9 9 10 | 11 12 | | |
| $C_5$ 1 | 1 2 3 4 5 | 6 7 8 9 | | | |
| 2 | 1 2 3 4 6 | 6 8 9 10 11 | 12 12 13 14 15 | | |
| 3 | 1 2 3 4 6 | 7 9 10 12 13 | 13 15 15 15 15 | 16 17 18 | |
| 4 | 1 2 4 5 6 | 9 9 10 12 15 | 15 15 15 15 15 | 16 17 18 | |
| $C_6$ 1 | 1 2 3 4 5 | 6 7 8 9 10 | 11 | | |
| 2 | 1 2 3 4 6 | 7 8 9 10 11 | 12 13 14 15 16 | 16 17 18 19 | |
| 3 | 1 2 3 4 6 | 7 9 10 12 12 | 13 15 16 17 18 | 19 19 21 21 21 | 21 22 23 24 |
| 4 | 1 2 3 4 6 | 7 9 10 12 14 | 14 16 17 19 19 | 21 21 23 23 23 | 23 23 23 24 25 26 |
| 5 | 1 2 4 5 6 | 9 9 10 12 15 | 15 15 17 19 22 | 22 22 22 22 22 | 22 22 23 24 25 |

*Table 4. Type $D_\ell$*

| | 1 2 3 4 5 | 6 7 8 9 10 | 11 12 13 14 15 | 16 17 18 19 20 | 21 22 23 24 25 | 26 27 28 29 30 |
|---|---|---|---|---|---|---|
| $D_4$ 2 | 1 2 3 4 5 | 6 8 8 9 | | | | |
| $D_5$ 2 | 1 2 3 4 5 | 6 7 8 10 10 | 12 12 13 | | | |
| 3 | 1 2 3 4 6 | 7 8 9 10 11 | 12 13 14 14 15 | | | |
| $D_6$ 2 | 1 2 3 4 5 | 6 7 8 9 10 | 12 13 14 14 16 | 16 17 | | |
| 3 | 1 2 3 4 5 | 7 8 9 11 11 | 12 13 14 15 16 | 18 18 19 20 20 | 21 | |
| 4 | 1 2 3 4 6 | 7 8 10 11 12 | 13 14 16 18 18 | 18 18 21 21 21 | 21 22 | |
| $D_7$ 2 | 1 2 3 4 5 | 6 7 8 9 10 | 11 12 14 15 16 | 17 18 18 20 20 | 21 | |
| 2 | 1 2 3 4 5 | 6 8 9 10 11 | 13 14 15 15 16 | 17 18 19 21 22 | 22 24 24 25 26 | 26 27 |
| 4 | 1 2 3 4 5 | 7 8 9 11 12 | 13 15 15 16 17 | 19 20 23 23 24 | 24 27 27 27 27 | 29 29 29 29 30 |
| 5 | 1 2 3 4 6 | 7 8 10 11 12 | 14 15 17 19 20 | 20 21 23 24 25 | 25 26 26 27 28 | 29 29 29 29 30 |

The results for the exceptional groups $E_6$, $F_4$ and $G_2$ are given in Table 1. The largest example we computed was $G$ of type $E_7$ and $P$ the sixth maximal parabolic subgroup so that $\dim X = 33$. The calculation of $m_X$ took about 20 hours on a SPARCstation-5. The values of $m_X$ in this case are as follows: $m_X(p) = p$ for $1 \leqslant p \leqslant 18$ and for $p = 32, 33$; $m_X(p) = p + 1$ for $p = 19, 20, 22, 30, 31$; $m_X(p) = p+2$ for $p = 21, 24, 26, 28, 29$; and $m_X(p) = p+3$ for $p = 23, 25, 27$. Most of the exact values of $m_X$ for the other maximal parabolic subgroups of $E_7$ and of $E_8$ are out of reach with this method, since $\dim X$ ranges from 42 up to 106. We did find, however, that for any maximal parabolic subgroup of $E_\ell$, $m_X(p) = p$ for $1 \leqslant p \leqslant 4$.

Tables 2–4 give the values of $m_X$ for maximal parabolic subgroups of classical Lie groups of low rank. Some of these are needed to apply Proposition 2.7 to the general case for exceptional groups, but they are also useful to obtain vanishing theorems sharper than Theorem 4.2 in the case of the orthogonal and symplectic groups of low rank. The tables do not include type $A_\ell$ or hermitian symmetric spaces, since Theorem 3.3 gives the best possible result for those cases.

The number $m$ in the first column identifies the maximal parabolic subgroup, i.e., $P = P_{I\setminus\{m\}}$. The values of $m_X(p)$ are listed sequentially until they reach $\dim X$. The value of $p$ can be read across the top row.

We close with a general vanishing theorem that can be deduced from the results of this and the previous sections.

THEOREM 5.1 *Let $G$ be a simply-connected semi-simple group with no factor of type $E_7$ or $E_8$, and let $P$ be a parabolic subgroup. Let $E$ be a homogeneous vector bundle on $X = G/P$, and assume that the highest weights of $E$ are dominant. Then $H^{p,q}(X, E) = 0$ whenever $q > 2p$.*

It is not clear whether the upper bound of $2p$ for the homogeneous Dolbeault cohomology on $X$ is the best general bound possible. The fact that $m_X(p)$ can be greater than $p$ does not in itself rule out the possibility that an upper bound may indeed be $p$.

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