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On the residue of the Eisenstein series and the Siegel–Weil formula

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Abstract. Some residue of a Siegel Eisenstein series is expressed as a theta integral in some cases. This formula is a refinement of the Siegel–Weil formula for the residue of the Eisenstein series given by Kudla and Rallis. The proof of the formula is carried out by comparing the Fourier–Jacobi coefficients of the Eisenstein series and the theta integral.

Key words: Eisenstein series, Siegel–Weil formula, Fourier–Jacobi coefficient.

Introduction

The Siegel–Weil formula [24] is an identity between a value of an Eisenstein series and an integral of a theta function. Consider a quadratic form (Q, U) of rank m and a Schwartz function Φ on $U^n(\mathbb{A})$. Then one can form a theta integral and an Eisenstein series on $\mathrm{Sp}_n(\mathbb{A})$ attached to Φ (see below). The Eisenstein series is absolutely convergent if $m > 2n+2$, and the theta integral is absolutely convergent if $r_0 = 0$ or $m - r_0 > n + 1$ ([24]). Here r_0 is the Witt index of the quadratic form. However the Eisenstein series is known to have an analytic continuation to the whole complex plane. Kudla and Rallis [3–4] proved that when m is even and the theta integral is absolutely convergent, the Eisenstein series is holomorphic at the point in question, and the Siegel–Weil formula holds. In [8], Kudla and Rallis introduced a regularized theta integral and proved that when m is even and the Eisenstein series is holomorphic at the point in question, then the Eisenstein series can be expressed by the regularized theta integral. They also proved that when the Eisenstein series has a pole at the point in question, the residue can be expressed as a regularized theta integral for the 'complementary' quadratic form. This regularized theta integral is characterized as the image of the intertwining operator.

In this paper, we are going to calculate the residue of the Eisenstein series and give an explicit form of the theta integral under the assumption that the 'complementary' quadratic form is anisotropic. Note that the result of Kudla and Rallis implies that our formula holds up to constant if m is even, but we can calculate the constant explicitly.

We shall prove our formula by comparing the Fourier–Jacobi coefficients of the both sides. The Fourier–Jacobi coefficients of the Eisenstein series was considered in [2]. Here we apply the results of [2] and prove the formula by the induction with respect to the rank of the 'complementary' quadratic form Q'. In particular, our method works for metaplectic cases (i.e., when m is odd) as well.

Let us explain our result more explicitly. Let k be a global field with char. $k \neq 2$ and (Q, U) be a non-degenerate quadratic form of rank m over k. We fix a non-trivial character ψ of the adele group $\mathbb{A}$ trivial on k. Let $H = O_Q$ be the orthogonal group for Q and $\widetilde{G(\mathbb{A})} = \widetilde{\mathrm{Sp}_n(\mathbb{A})}$ be the metaplectic cover of the symplectic group $G(\mathbb{A}) = \mathrm{Sp}_n(\mathbb{A})$ of rank n. Then $\widetilde{\mathrm{Sp}_n(\mathbb{A})} \times H(\mathbb{A})$ acts on the Schwartz space $\mathcal{S}(U^n(\mathbb{A}))$ via the Weil representation ω_Q. For $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$, $g \in \widetilde{G(\mathbb{A})}$, $h \in H(\mathbb{A})$, we define the theta function

$$\Theta^\Phi(g, h) = \sum_{l \in U^n(k)} \omega_Q(g)\Phi(h^{-1}l)$$

and consider the integral

$$I_Q(g; \Phi) = \int_{H(k)\backslash H(\mathbb{A})} \Theta^\Phi(g, h)\, \mathrm{d}h.$$

It is well-known [24] that this integral is absolutely convergent for any $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ if either $r_0 = 0$ or $m - r_0 > n + 1$.

Let P_G be the Siegel parabolic subgroup of G and $\widetilde{K_G}$ be the standard maximal compact subgroup of $\widetilde{G(\mathbb{A})}$. As in [3], [4], we put

$$f_\Phi^{(s)}(g) = |a(p)|^{s-s_0}\omega_Q(g)\Phi(0)$$

for $g = pk$, $p \in \widetilde{P_G(\mathbb{A})}$, $k \in \widetilde{K_G}$. Here $s_0 = \frac{m}{2} - \frac{n+1}{2}$ and

$$a\left(\begin{pmatrix} A & * \\ \mathbf{0}_n & {}^tA^{-1} \end{pmatrix}, \zeta\right) = \det A.$$

We consider the Eisenstein series

$$E(g; f_\Phi^{(s)}) = \sum_{\gamma \in P_G\backslash G} f_\Phi^{(s)}(\gamma g).$$

It is absolutely convergent for $\mathrm{Re}(s) > \frac{n+1}{2}$ and can be meromorphically continued to the whole s-plane if Φ is $\widetilde{K_G}$-finite. (In fact, we consider a slightly larger class of $f^{(s)}$.) The behavior of $E(g; f_\Phi^{(s)})$ at $s = s_0$ is of our interest. In the 'critical' range $n + 1 < m \leqslant 2n + 2$, $E(g; f_\Phi^{(s)})$ may have a pole if $0 < r_0 \leqslant m - n - 1$, Moreover, it is known that the pole is at most simple.

In this paper, we calculate the residue in the case $r_0 = m - n - 1$. This is the only case that we do not need the 'regularization' of the theta integral (cf. [8]). In this case, $Q = Q' \oplus \mathcal{H}^{r_0}$, where $\mathcal{H}$ is the hyperbolic plane and (Q', U') is an anisotropic quadratic form. Take two totally isotropic subspaces Y and X

of $\mathcal{H}^{r_0}$ such that $\mathcal{H}^{r_0} = Y \oplus X$. Then $U = U' \oplus Y \oplus X$. We define an operator $\pi_Q^{Q'} : \mathcal{S}(U^n(\mathbb{A})) \to \mathcal{S}(U'^n(\mathbb{A}))$ by

$$\pi_Q^{Q'}\Phi(u') = \int_{X^n(\mathbb{A})} \Phi\begin{pmatrix} u' \\ 0 \\ x \end{pmatrix} \mathrm{d}x.$$

We fix a maximal compact subgroup K of $H(\mathbb{A})$. Then we shall prove that

$$\mathrm{Res}_{s=s_0} E(g; f_\Phi^{(s)}) = c_K I_{Q'}(g; \pi_Q^{Q'} \pi_K \Phi),$$

where c_K is a certain constant which appears in a normalization of the Haar measure and

$$\pi_K\Phi(u) = \int_K \Phi(ku)\,\mathrm{d}k.$$

The value of c_K will be calculated explicitly in Section 9. (Theorem 9.6 and Theorem 9.7).

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Notation

The space of $n \times n$ and $m \times n$ matrices over k is denoted by $\mathrm{M}_n(k)$, and $\mathrm{M}_{m\,n}(k)$, respectively. The space of $n \times n$ symmetric and alternative matrices are denoted by $\mathrm{Sym}_n(k)$ and $\mathrm{Alt}_n(k)$, respectively. The $n \times n$ zero and identity matrices are denoted by $\mathbf{0}_n$ and $\mathbf{1}_n$, respectively. If X is a square matrix, $\mathrm{tr}\, X$ and $\det X$ stand for its trace and determinant, respectively. We consider a symplectic vector space as a row vector space, and a quadratic vector space as a column vector space. Suppose a group G acts on a space X from the right (resp. left). For a function f on X and $g \in G$, we denote by $\rho(g)f$ (resp. $\lambda(g)f$) the right translation (resp. the left inverse translation) of f by g, i.e., $\rho(g)f(x) = f(xg)$ (resp. $\lambda(g)f(x) = f(g^{-1}x)$). If G is an algebraic group defined over a field k, the group of k-valued points of G is denoted by $G(k)$ or G. For each place v of a global field k, the group of k_v-valued points of G is denoted by $G(k_v)$ or G_v. The modulus character of G is denoted by δ_G. If π is a representation of G, its contragredient is denoted by $\tilde{\pi}$. If k is a global field, the adele ring (resp. the idele group) of k is denoted by $\mathbb{A}_k$ or $\mathbb{A}$ (resp. $\mathbb{A}_k^\times$ or $\mathbb{A}^\times$). The volume of an adele $\alpha \in \mathbb{A}^\times$ is denoted by $|\alpha|$. For each non-archimedean place v of k, the maximal order is denoted by $\mathfrak{o}_v$, and the maximal ideal of $\mathfrak{o}_v$ is denoted by $\mathfrak{p}_v$. We denote a prime element of k_v by ϖ_v. For a unipotent algebraic group U, we normalize Haar measure $\mathrm{d}u$ on $U(\mathbb{A})$ so that $\mathrm{Vol}(U(k)\backslash U(\mathbb{A})) = 1$. We fix a non-trivial additive character ψ of $\mathbb{A}/k$. For each finite or infinite place v of k, we denote the local factor of the Dedekind zeta function by $\zeta_v(s)$. We put $\zeta_k(s) = \prod_{v \le \infty} \zeta_v(s)$ and $\xi_k(s) = |D_k|^{s/2}\zeta_k(s)$. Here D_k is the discriminant of k.

The residue of $\xi_k(s)$ at $s = 1$ is denoted by ρ_k. Similarly, if χ is a Hecke character of $\mathbb{A}^\times/k^\times$, we denote the local factor for the Hecke L-function by $L_v(s,\chi)$. We put $L(s,\chi) = \prod_{v\leq\infty} L_v(s,\chi)$.

1. Weil representations and theta functions

Let G be the symplectic group of rank n and P_G be the Siegel parabolic subgroup of G:

$$\begin{aligned} G(k) = \mathrm{Sp}_n(k) &= \left\{ g \in \mathrm{GL}_{2n}(k) \,\middle|\, g \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix} {}^t g = \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \,\middle|\, A,B,C,D \in \mathrm{M}_n(k), \right. \\ &\qquad \left. A\,{}^tB = B\,{}^tA, C\,{}^tD = D\,{}^tC, A\,{}^tD - B\,{}^tC = \mathbf{1}_n \right\}, \end{aligned}$$

$$P_G(k) = \left\{ \begin{pmatrix} A & B \\ \mathbf{0}_n & {}^tA^{-1} \end{pmatrix} \,\middle|\, A \in \mathrm{GL}_n(k),\ A^{-1}B \in \mathrm{Sym}_n(k) \right\}.$$

For each place v of k, we define 2-cocycle $c(g_1,g_2)$ on $G(k_v)$ with values in $\{\pm1\}$ as in [12]. The metaplectic group $\widetilde{G(k_v)}$ is by definition the 2-fold covering group of $G(k_v)$ determined by $c(g_1,g_2)$: An element of $\widetilde{G(k_v)}$ is an pair (g,ζ), $g \in G(k_v)$, $\zeta \in \{\pm1\}$, and the multiplication law is given by $(g_1,\zeta_1)(g_2,\zeta_2) = (g_1g_2,\ c(g_1,g_2)\zeta_1\zeta_2)$.

Let (Q,U) be a non-degenerate quadratic form of rank m. We sometimes regard U as a space of column vectors k^m and Q as an $m\times m$ symmetric matrix. The Weil representation ω_{Q_v} of $\widetilde{G(k_v)}$ associated to Q_v is defined on the Schwartz space of $\mathcal{S}(U^n(k_v))$. ω_{Q_v} is characterized by the following properties: (see e.g., [17])

$$\begin{aligned} &\omega_{Q_v}\left(\left(\begin{pmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & {}^tA^{-1} \end{pmatrix}, \zeta\right)\right)\Phi(X) \\ &= \zeta^m \frac{\gamma_{Q_v}(1)}{\gamma_{Q_v}(\det A)} |\det A|_v^{m/2}\Phi(XA), \end{aligned} \tag{1.1}$$

$$\omega_{Q_v}\left(\left(\begin{pmatrix} \mathbf{1}_n & B \\ \mathbf{0}_n & \mathbf{1}_n \end{pmatrix}, \zeta\right)\right)\Phi(X) = \zeta^m\psi_v(\tfrac{1}{2}\mathrm{tr}(Q_vXB\,{}^tX))\Phi(X), \tag{1.2}$$

$$\omega_{Q_v}\left(\left(\begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix}, \zeta\right)\right)\Phi(X) = \zeta^m\gamma_{Q_v}(1)^{-n}\mathcal{F}_{Q_v}\Phi(-X), \tag{1.3}$$

$X \in U^n(k_v)$, $A \in \mathrm{GL}_n(k_v)$, $B \in \mathrm{Sym}_n(k_v)$. Here $\mathcal{F}_{Q_v}\Phi$ is the Fourier transform of Φ with respect to Q:

$$\mathcal{F}_{Q_v}\Phi(X) = \int_{X(k_v)} \Phi(Y)\psi(\mathrm{tr}(Q_v X\,{}^tY))\,\mathrm{d}Y.$$

Here the Haar measure $\mathrm{d}Y$ is the self-dual measure for the Fourier transform $\mathcal{F}_{Q_v}$ and $\gamma_{Q_v}(a)$ is the Weil constant associated to Q_v. It is defined as follows. When Q_v is equivalent to $\mathrm{diag}(q_1, \dots, q_m)$, then $\gamma_{Q_v}(a) = \prod_{i=1}^m \gamma_v(q_i a)$, and $\gamma_v(a)$ is determined by the following equation:

$$\int_{k_v} \psi_v(\tfrac{1}{2}ax^2)\phi(x)\,\mathrm{d}x = \gamma_v(a)|a|_v^{-(1/2)} \int_{k_v} \psi(-\tfrac{1}{2}a^{-1}x^2)\hat{\phi}(x)\,\mathrm{d}x,$$

$$\hat{\phi}(x) = \int_{k_v} \phi(y)\psi(xy)\,\mathrm{d}y.$$

Here $\mathrm{d}x$, $\mathrm{d}y$ are the self-dual measure for the Fourier transform. If $v < \infty$ and $v \not| 2$, then there is a canonical splitting over the standard maximal compact subgroup K_{G_v}. The image of the splitting, which we also denote by K_{G_v}, is the stabilizer of the characteristic function of $\mathfrak{o}_v^m$ for almost all v. The global metaplectic group $\widetilde{G(\mathbb{A})}$ is the quotient of the restricted direct product of $\widetilde{G(k_v)}$ with respect to $\{K_{G_v}\}$ divided by $\{(\zeta_v) \in \oplus_v\{\pm 1\} \mid \prod_v \zeta_v = 1\}$. Then the global Weil representation ω_Q of $\widetilde{G(\mathbb{A})}$ on $\mathcal{S}(U^n(\mathbb{A}))$ is well-defined. It is well-known that there is a unique splitting over $G(k)$, whose image is identified with $G(k)$. Since $c(g_1, g_2)$ is identically 1 on $(P_{G_v} \cap K_{G_v}) \times (P_{G_v} \cap K_{G_v})$ for almost all v, the inverse image $\widetilde{P_G(\mathbb{A})}$ of $P_G(\mathbb{A})$ is identified with the covering group defined by the 2-cocycle $\prod_v c(g_1, g_2)$, $g_1, g_2 \in P(\mathbb{A})$. Then by (1.1) and (1.2),

$$\omega_Q\left(\left(\begin{pmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & {}^tA^{-1} \end{pmatrix}, \zeta\right)\right)\Phi(X) = \zeta^m \frac{1}{\gamma_Q(\det A)}|\det A|^{m/2}\Phi(XA),$$

$$\omega_Q\left(\left(\begin{pmatrix} \mathbf{1}_n & B \\ \mathbf{0}_n & \mathbf{1}_n \end{pmatrix}, \zeta\right)\right)\Phi(X) = \zeta^m \psi(\tfrac{1}{2}\mathrm{tr}(QXB\,{}^tX))\Phi(X),$$

$X \in U^n(\mathbb{A})$, $A \in \mathrm{GL}_n(\mathbb{A})$, $B \in \mathrm{Sym}_n(\mathbb{A})$, $\gamma_Q(a) = \prod_v \gamma_{Q_v}(a_v)$. Put $w_n = \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix}$. Then

$$\omega_Q(w_n)\Phi(X) = \mathcal{F}_Q\Phi(X).$$

Let $H = O_Q$ be the orthogonal group associated to Q:

$$O_Q(k) = \{g \in \mathrm{GL}(U) \mid {}^tgQg = Q\}.$$

Then $H(\mathbb{A})$ acts on $\mathcal{S}(U(\mathbb{A}))$ by the left inverse translation $\lambda : \lambda(h)\Phi(X) = \Phi(h^{-1}X)$. λ is compatible with ω_Q.

For any $\Phi \in \mathcal{S}(U(\mathbb{A}))$, we define the theta function associated to Φ as follows:

$$\Theta^{\Phi}(g,h) = \sum_{l \in U^n(k)} \omega_Q(g)\Phi(h^{-1}l),$$

$g \in \widetilde{G(\mathbb{A})}$, $h \in H(\mathbb{A})$.

Then Θ^{Φ} is a slowly increasing function on $(G(k)\backslash\widetilde{G(\mathbb{A})}) \times (H(k)\backslash H(\mathbb{A}))$. Put

$$I_Q(g;\Phi) = \int_{H(k)\backslash H(\mathbb{A})} \Theta^{\Phi}(g,h)\, \mathrm{d}h,$$

if the integral is absolutely convergent. Here the Haar measure $\mathrm{d}h$ of $H(\mathbb{A})$ is normalized by the condition $\mathrm{Vol}(H(k)\backslash H(\mathbb{A})) = 1$. By [24], it is absolutely convergent if and only if $r_0 = 0$ or $m - r_0 > n+1$. where r_0 is the dimension of a maximal totally isotropic subspace for Q.

2. Fourier–Jacobi coefficients

In this section, we shall review the theory of Jacobi forms and Fourier–Jacobi coefficients [2].

Let $L = k^{n-1}$ be the space of row vectors. We define some subgroups of G :

$$Z(k) = \left\{ \left(\begin{array}{c|cc} \mathbf{1}_n & \begin{matrix} z & 0 \\ 0 & \mathbf{0}_{n-1} \end{matrix} \\ \hline \mathbf{0}_n & \mathbf{1}_n \end{array} \right) \middle| z \in k \right\},$$

$$V(k) = \left\{ \left(\begin{array}{cc|cc} 1 & x & z & y \\ 0 & \mathbf{1}_{n-1} & {}^t y & \mathbf{0}_{n-1} \\ \hline & & 1 & 0 \\ \multicolumn{2}{c|}{\mathbf{0}_n} & -{}^t x & \mathbf{1}_{n-1} \end{array} \right) \middle| x, y \in L, z \in k \right\},$$

$$G_1(k) = \left\{ \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ \hline 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{array} \right) \middle| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{n-1}(k) \right\}.$$

G_1 can be naturally identified with Sp_{n-1}. We set $J = VG_1$, $\widetilde{J(\mathbb{A})} = V(\mathbb{A})\widetilde{G_1(\mathbb{A})}$.

We use the notation

$$v(x,y;z) = \left(\begin{array}{cc|cc} 1 & x & z & y \\ 0 & \mathbf{1}_{n-1} & {}^t y & \mathbf{0}_{n-1} \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}_n$}} & 1 & 0 \\ & & -{}^t x & \mathbf{1}_{n-1} \end{array}\right)$$

for the elements of V. V is a Heisenberg group with center Z. The Schrödinger representation ω_ψ of $V(\mathbb{A})$ on $\mathcal{S}(L(\mathbb{A}))$ is given by

$$\omega_\psi(v)\phi(t) = \phi(t+x)\psi(\tfrac{1}{2}z + t\,{}^t y + \tfrac{1}{2}x\,{}^t y),$$

for $v = v(x,y;z)$ and $\phi \in \mathcal{S}(L(\mathbb{A}))$. By the Stone von-Neumann theorem, ω_ψ is, up to isomorphism, the unique irreducible representation of $V(\mathbb{A})$ on which $Z(\mathbb{A})$ acts by $z \mapsto \psi(\frac{1}{2}z)$. The Schrödinger representation of $V(\mathbb{A})$ extends to the representation of $\widetilde{J(\mathbb{A})}$, the Weil representation ω_ψ, in a unique way by

$$\omega_\psi\left(\left(\begin{pmatrix} A & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1} & {}^t A^{-1} \end{pmatrix}, \zeta\right)\right)\phi(t) = \zeta\frac{\gamma(1)}{\gamma(\det A)}|\det A|^{1/2}\phi(tA),$$

$$\omega_\psi\left(\left(\begin{pmatrix} \mathbf{1}_{n-1} & B \\ \mathbf{0}_{n-1} & \mathbf{1}_{n-1} \end{pmatrix}, \zeta\right)\right)\phi(t) = \zeta\psi(\tfrac{1}{2}tB\,{}^t t)\phi(t),$$

$$\omega_\psi(w_{n-1})\phi(t) = \mathcal{F}\phi(t).$$

$t \in L(\mathbb{A})$, $A \in \mathrm{GL}_{n-1}(\mathbb{A})$, $B \in \mathrm{Sym}_{n-1}(\mathbb{A})$. Here γ is the Weil constant with respect to ψ and $\mathcal{F}\phi$ is the Fourier transform of ϕ with respect to ψ:

$$\mathcal{F}\phi(t) = \int_{L(\mathbb{A})} \phi(x)\psi(t\,{}^t x)\,\mathrm{d}x.$$

The restriction of ω_ψ to $\widetilde{G_1(\mathbb{A})}$ will be also denoted by ω_ψ. For each $\phi \in \mathcal{S}(L(\mathbb{A}))$, the theta function $\vartheta^\phi(vg_1)$ is given by

$$\begin{aligned} \vartheta^\phi(vg_1) &= \sum_{l \in L(k)} \omega_\psi(vg_1)\phi(l) \\ &= \sum_{l \in L(k)} \omega_\psi(g_1)\phi(l+x)\psi(\tfrac{1}{2}z + l\,{}^t y + \tfrac{1}{2}x\,{}^t y), \end{aligned}$$

for $v \in V(\mathbb{A})$, $g_1 \in \widetilde{G_1(\mathbb{A})}$.

Let $C_\psi^\infty(J(k)\backslash\widetilde{J(\mathbb{A})})$ be the space of smooth functions f on $J(k)\backslash\widetilde{J(\mathbb{A})}$ such that

$$f(zvg_1) = \psi(\tfrac{1}{2}z)f(vg_1)$$

for $z \in Z(\mathbb{A})$.

LEMMA 2.1 *Let* $\varphi \in C_{\psi}^{\infty}(J(k)\backslash\widetilde{J(\mathbb{A})})$. *Then* $\varphi = 0$, *if and only if, for all* $g_1 \in \widetilde{G_1(\mathbb{A})}$ *and for all* $\phi \in \mathcal{S}(L(\mathbb{A}))$,

$$\int_{V(k)\backslash V(\mathbb{A})} \varphi(vg_1)\overline{\vartheta^{\phi}(vg_1)}\,\mathrm{d}v = 0.$$

In fact, it is enough to consider $\widetilde{K}_{G_1}$*-finite* ϕ*'s.*

Proof. It will suffice to prove $\varphi(e) = 0$. This is an immediate consequence of the fact that any C^{∞}-function φ on $V(k)\backslash V(\mathbb{A})$ such that $\varphi(zv) = \psi(\frac{1}{2}z)\varphi(v)$ is equal to ϑ^{ϕ} for some $\phi \in \mathcal{S}(L(\mathbb{A}))$. The last assertion follows since $\widetilde{K}_{G_1}$-finite vectors in $\mathcal{S}(L(\mathbb{A}))$ generate a dense subspace.

For any automorphic form $A(g)$ on $G(k)\backslash\widetilde{G(\mathbb{A})}$ and any $\phi \in \mathcal{S}(L(\mathbb{A}))$, we define a function $\mathrm{FJ}_{\psi}^{\phi}(g_1; A)$ on $G_1(k)\backslash\widetilde{G_1(\mathbb{A})}$ by

$$\mathrm{FJ}_{\psi}^{\phi}(g_1; A) = \int_{V(k)\backslash V(\mathbb{A})} A(vg_1)\overline{\vartheta^{\phi}(vg_1)}\,\mathrm{d}v.$$

When ψ is clear from the context, we omit ψ from the notation.

LEMMA 2.2 *Let* A *be an automorphic form on* $\widetilde{G(\mathbb{A})}$. *Let* $\mathcal{H}(\widetilde{G(\mathbb{A})})$ *be the space of compactly supported bi* $\widetilde{K}_G$*-finite* C^{∞} *function on* $\widetilde{G(\mathbb{A})}$. *If* $\mathrm{FJ}_{\psi}^{\phi}(g_1; \rho(f)A) = 0$ *for any non-trivial* ψ, *any* $\widetilde{K}_{G_1}$*-finite* $\phi \in \mathcal{S}(L(\mathbb{A}))$, *and any Hecke operator* $\rho(f)$, *with* $f \in \mathcal{H}(\widetilde{G(\mathbb{A})})$, *then* A *is a constant function.*

Proof. It follows that $\rho(f)A$ is left $Z(\mathbb{A})$ invariant for any $f \in \mathcal{H}(\widetilde{G(\mathbb{A})})$. In particular, A is left $gZ(\mathbb{A})g^{-1}$ invariant for any $g \in \widetilde{G(\mathbb{A})}$, since one can take a sequence f_i which converges to the Dirac distribution at g. Since the conjugates of $Z(\mathbb{A})$ generates dense subgroup of $\widetilde{G(\mathbb{A})}$, A is $\widetilde{G(\mathbb{A})}$ invariant.

3. Eisenstein series

We define some subgroups of G as follows:

$$P_G = \left\{ \begin{pmatrix} A & B \\ \mathbf{0}_n & {}^tA^{-1} \end{pmatrix} \middle| A \in \mathrm{GL}_n, A^{-1}B \in \mathrm{Sym}_n(k) \right\},$$

$$M = \left\{ \begin{pmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & {}^tA^{-1} \end{pmatrix} \middle| A \in \mathrm{GL}_n \right\},$$

$$N = \left\{ \begin{pmatrix} \mathbf{1}_n & B \\ \mathbf{0}_n & \mathbf{1}_n \end{pmatrix} \middle| B \in \mathrm{Sym}_n(k) \right\}$$

The pullbacks of $P_G(\mathbb{A})$ and $M(\mathbb{A})$ in $\widetilde{G(\mathbb{A})}$ are denoted by $\widetilde{P_G(\mathbb{A})}$ and $\widetilde{M(\mathbb{A})}$, respectively. On $N(\mathbb{A})$, there is a canonical splitting $n \longmapsto (n, 1)$, and the image of this splitting is also denoted by $N(\mathbb{A})$. Then we have $\widetilde{P_G(\mathbb{A})} = \widetilde{M(\mathbb{A})}N(\mathbb{A})$.

We define a character χ_Q of $\widetilde{M(\mathbb{A})}$ by:

$$\chi_Q\left(\left(\begin{pmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & {}^tA^{-1} \end{pmatrix}, \zeta\right)\right) = \zeta^m \frac{\gamma_Q(1)}{\gamma_Q(\det A)}.$$

If $m = \operatorname{rk} Q$ is even,

$$\frac{\gamma_Q(1)}{\gamma_Q(\det A)} = \langle \det A, (-1)^{m/2}\det Q\rangle.$$

Here $\langle\cdot,\cdot\rangle$ is the Hilbert symbol. In this case χ_Q is a character of $M(\mathbb{A})$. On the other hand, if m is odd

$$\frac{\gamma_Q(1)}{\gamma_Q(\det A)} = \langle \det A, (-1)^{(m-1)/2}\det Q\rangle \frac{\gamma(1)}{\gamma(\det A)}.$$

In this case, χ_Q is a 'genuine' character of $\widetilde{M(\mathbb{A})}$. The natural extension of χ_Q to $\widetilde{P_G(\mathbb{A})}$ will be also denoted by χ_Q.

Let K_G be the standard maximal compact subgroup of $G(\mathbb{A})$, and $\widetilde{K_G}$ be the pullback of K_G in $\widetilde{G(\mathbb{A})}$. For $s \in \mathbb{C}$, we define $I(\chi_Q, s)$ to be the space of $\widetilde{K_G}$-finite functions f on $\widetilde{G(\mathbb{A})}$ such that

$$f(pg) = \chi_Q(p)|a(p)|^{s+((n+1)/2)}f(g), \quad \forall p \in \widetilde{P_G(\mathbb{A})}, \forall g \in \widetilde{G(\mathbb{A})}.$$

Here $a(p) = \det A$, for $p = \left(\begin{pmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & {}^tA^{-1} \end{pmatrix}, \zeta\right)$. For each place v of k, we define the local analogue $I_v(\chi_{Q_v}, s)$ of $I(\chi_Q, s)$, i.e., $I_v(\chi_{Q_v}, s)$ is the space of $\widetilde{K_{G_v}}$-finite functions f_v on $\widetilde{G_v}$ such that

$$f_v(pg) = \chi_{Q_v}(p)|a(p)|^{s+((n+1)/2)}f_v(g), \quad \forall p \in \widetilde{P_G(k_v)}, \forall g \in \widetilde{G_v}.$$

If $v < \infty$ and $v \nmid 2$, then there is a canonical splitting $K_{G_v} \longmapsto \widetilde{K_{G_v}}$. If $v < \infty$, $v \nmid 2$, ψ_v is of order 0, and Q_v is unramified, then $I_v(\chi_{Q_v}, s)$ has a distinguished vector $f_{v,0}$, which is identically 1 on the image of K_G in $\widetilde{K_G}$. $I(\chi_Q, s)$ is the restricted tensor product $\otimes'_v I_v(\chi_{Q_v}, s)$ with respect to $\{f_{v,0}\}$.

For each v, a holomorphic section of $I(\chi_{Q_v}, s)$ is a function $f_v^{(s)}(g)$ on $\mathbb{C} \times \widetilde{G_v}$ which satisfies the following conditions:

(1) $f_v^{(s)}(g)$ is holomorphic with respect to $s \in \mathbb{C}$.
(2) As a function of $g \in \widetilde{G(\mathbb{A})}$, $f^{(s)}(g) \in I(\chi_{Q_v}, s)$ for any $s \in \mathbb{C}$.

(3) $f_v^{(s)}$ is $\widetilde{K}_v$-finite.

As before, for almost all v, there exists a distinguished holomorphic section $f_{v,0}^{(s)}$. We shall say that $f^{(s)}$ is a (global) holomorphic section of $I(\chi_Q, s)$ if $f^{(s)}(g)$ is a finite sum of functions of the form $\prod_v f_v^{(s)}(g)$, where $f_v^{(s)}$ is a local holomorphic section of $I(\chi_{Q_v}, s)$ for all v and $f_v^{(s)} = f_{v,0}^{(s)}$ for almost all v.

For a holomorphic section $f^{(s)}$ of $I(\chi_Q, s)$, we define the Eisenstein series $E(g; f^{(s)})$ by

$$E(g; f^{(s)}) = \sum_{\gamma \in P_G \backslash G} f^{(s)}(\gamma g).$$

$E(g; f^{(s)})$ is absolutely convergent for $\mathrm{Re}(s) > \frac{n+1}{2}$ and can be analytically continued to the whole s-plane. For $\mathrm{Re}(s) \geqslant 0$, the set of poles of $E(g; f^{(s)})$ is contained in the following set:

If m:even, and $\chi_Q = 1$: $\left\{\frac{n+1}{2} - s \,\middle|\, s \in \mathbb{Z}, 0 \leqslant s < \frac{n+1}{2}\right\}$

If m:even, and $\chi_Q \neq 1$: $\left\{\frac{n-1}{2} - s \,\middle|\, s \in \mathbb{Z}, 0 \leqslant s < \frac{n-1}{2}\right\}$

If m:odd: $\{\frac{n}{2} - s \,|\, s \in \mathbb{Z}, 0 \leqslant s < \frac{n}{2}\}$

Moreover, these poles are at most simple. (For m even, see [1], [5]. The case m is odd will be proved later. See Proposition 7.2.) It follows that if s_0 belongs to this set, then $\mathrm{Res}_{s=s_0} E(g; f^{(s)})$ depends only on $f^{(s_0)}$ and the map:

$$f^{(s_0)} \longmapsto \mathrm{Res}_{s=s_0} E(g; f^{(s)})$$

respects $\widetilde{G(\mathbb{A})}$ action. (At archimedean places, it just means $(\mathfrak{g}_\infty, K_{G_\infty})$-action. Here $\mathfrak{g}_\infty$ is the Lie algebra of the infinite part of $G(\mathbb{A})$.)

DEFINITION 3.1 For a $\widetilde{K}_{G_v}$-finite $\Phi_v \in \mathcal{S}(U_v^n)$, define

$$f_{\Phi_v}^{(s)}(g) = \chi_{Q_v}(p)|a(p)|^{s+(n+1)/2}\omega_Q(k)\Phi_v(0),$$

for $g = pk$, $p \in \widetilde{P_G(k_v)}$, $k \in \widetilde{K}_{G_v}$. We shall say that $f_{\Phi_v}^{(s)}$ is the SW section associated to Φ_v.

DEFINITION 3.2 Let $f_v(s)$ be a local holomorphic section of $I_v(\chi_{Q_v}, s)$. We shall say that $f_v^{(s)}$ is a weak SW section associated to $\Phi_v \in \mathcal{S}(U_v^n)$ if $f_v^{(s_0)}(g) = \omega_{Q_v}(g)\Phi_v(0)$. Here $s_0 = \frac{m}{2} - \frac{n+1}{2}$, and $\Phi_v \in \mathcal{S}(U_v^n)$ is $\widetilde{K}_{G_v}$-finite function. We shall say that $f_v^{(s)}$ is a weak local SW section belonging to Q_v if $f_v^{(s)}$ is a weak local SW section associated to some $\widetilde{K}_{G_v}$-finite $\Phi_v \in \mathcal{S}(U_v^n)$.

DEFINITION 3.3 For a $\widetilde{K_G}$-finite $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$, define

$$f_\Phi^{(s)}(g) = \chi_Q(p)|a(p)|^{s+(n+1)/2}\omega_Q(k)\Phi(0),$$

for $g = pk, p \in \widetilde{P_G(\mathbb{A})}, k \in \widetilde{K_G}$. We shall say that $f_\Phi^{(s)}$ is the SW section associated to Φ.

DEFINITION 3.4 Let $f^{(s)}$ be a global holomorphic section of $I(\chi_Q, s)$. We shall say that $f^{(s)}$ is a weak global SW section associated to $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ if $f^{(s)}$ and Φ can be expressed as a finite sum

$$f^{(s)} = \sum_{i\in I}\prod_v f_{v,i}^{(s)}, \quad \Phi = \sum_{i\in I}\prod_v \Phi_{v,i}$$

such that each $f_{v,i}^{(s)}$ is a weak global SW section associated to $\Phi_{v,i}$. We shall say that $f^{(s)}$ is a weak global SW section belonging to Q if $f^{(s)}$ is a weak global SW section associated to some $\widetilde{K_G}$-finite $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$.

4. Siegel–Weil formula for $n = 1$

In this section, we study the behavior of Eisenstein series associated to SW sections for $n = 1$. Throughout this section, we assume $n = 1$.

If $m \geqslant 5$, then the Eisenstein series is absolutely convergent at $s = s_0 = \frac{m}{2} - 1$ and the Siegel–Weil formula holds:

$$E(g; f_\Phi^{(s)})|_{s=s_0} = \int_{H(k)\backslash H(\mathbb{A})} \Theta^\Phi(g,h)\,\mathrm{d}h.$$

We shall consider the cases $(m, r_0) = (4,0), (4,1), (3,0), (2,0)$. In the cases $(m, r_0) = (4,0), (4,1), (2,0)$, Kudla and Rallis proved the Siegel–Weil formula for SW sections:

$$E(g; f_\Phi^{(s)})|_{s=s_0} = \kappa \int_{H(k)\backslash H(\mathbb{A})} \Theta^\Phi(g,h)\,\mathrm{d}h.$$

Here $\kappa = 1$ if $(m, r_0) = (4,0), (4,1)$, and $\kappa = 2$ if $(m, r_0) = (2,0)$. In fact, these Siegel–Weil formulas hold for weak SW sections. If $(m, r_0) = (4,1)$ or $(2,0)$ then $E(g; f^{(s)})$ is holomorphic at $s = s_0$ for any holomorphic section. It follows that $E(g; f^{(s)})|_{s=s_0}$ depends only on $f^{(s_0)}$ and respects $\widetilde{G(\mathbb{A})}$ action. Therefore in these cases the Siegel–Weil formula is valid for any weak SW section. Now we shall prove the Siegel–Weil formula holds for $(m, r_0) = (3,0)$ with $\kappa = 1$. It is enough to prove that the constant term of $E(g; f_\Phi^{(s)})|_{s=s_0}$ is equal to $f_\Phi^{(s_0)}$. Let M_w be the intertwining operator. We have to prove that

$$M_w f_\Phi^{(s)}|_{s=(1/2)} = 0.$$

We may assume Φ is decomposable.

$$M_w f_\Phi^{(s)} = \frac{\zeta_S(2s)}{\zeta_S(2s+1)} \prod_{v \notin S} f_{v,0}^{(s)} \times \prod_{v \in S} M_w f_{\Phi_v}^{(s)}.$$

As Q_v is anisotropic at at least two places, $\prod_{v\in S} M_w f_{\Phi_v}^{(s)}$ has a zero of order at least 2 at $s = \frac{1}{2}$. Since $\frac{\zeta_S(2s)}{\zeta_S(2s+1)}$ has at most simple pole at $s = \frac{1}{2}$, $M_w f_\Phi^{(s)}$ has a zero at $s = \frac{1}{2}$. The case $(m, r_0) = (4, 0)$ is similar.

LEMMA 4.1 *Let (Q, U) be a quadratic form of rank $m \geqslant 2$. Let r_0 be the dimension of a maximal totally isotropic subspace for Q. We assume:*

$$(m, r_0) \neq (4,2), (3,1), (2,1)$$

Let $f_\Phi^{(s)}$ be a weak global SW section associated to $\Phi \in \mathcal{S}(U(\mathbb{A}))$. Put $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then

$$\int_{\mathbb{A}} f_\Phi^{(s)} \left(w \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \right) \overline{\psi(\tfrac{z}{2})}\, \mathrm{d}z \tag{4.1}$$

can be meromorphically continued to the whole s-plane and is holomorphic at $s = s_0 = \frac{m}{2} - 1$. Its value at $s = s_0$ is zero unless Q expresses 1. If $Q = \begin{pmatrix} 1 & \\ & Q_1 \end{pmatrix}$, its value at $s = s_0$ is equal to the absolutely convergent integral:

$$\kappa \int_{H_1(\mathbb{A}) \backslash H(\mathbb{A})} \Phi \left(h^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \mathrm{d}h. \tag{4.2}$$

Here $H = O_Q$ and $H_1 = O_{Q_1}$ are orthogonal group for Q and Q_1, respectively. The measures of $H(\mathbb{A})$ and $H_1(\mathbb{A})$ are normalized by Vol$(H(k)\backslash H(\mathbb{A}))$ = Vol$(H_1(k)\backslash H_1(\mathbb{A})) = 1$. $\kappa = 2$ if $(m, r_0) = (2, 0)$, and $\kappa = 1$, otherwise. If $m \geqslant 5$, then (4.1) is absolutely convergent at $s = s_0$ and is equal to

$$\int_{\mathbb{A}} \left[\int_{U(\mathbb{A})} \Phi(u) \psi(({}^t u Q u - 1) z/2)\, \mathrm{d}u \right] \mathrm{d}z. \tag{4.3}$$

Proof. The ψth Fourier coefficient of $E(g; f_\Phi^{(s)})$ is equal to (4.1). On the other hand, the ψth Fourier coefficient of

$$\int_{H(k)\backslash H(\mathbb{A})} \Theta^\Phi(g, h)\, \mathrm{d}h$$

is equal to

$$\int_{k\backslash \mathbb{A}} \int_{H(k)\backslash H(\mathbb{A})} \left[\sum_{l \in U(k)} \Phi(h^{-1} l) \psi(({}^t l Q l - 1) z/2) \right] \mathrm{d}h\, \mathrm{d}z.$$

The integral with respect to z is zero unless ${}^t\!lQl = 1$. If ${}^t\!lQl = 1$, then there exists $h \in H(k)$ such that $h^{-1}\begin{pmatrix}1\\0\end{pmatrix} = l$. Thus the integral is equal to

$$\int_{H_1(k)\backslash H(\mathbb{A})} \Phi\left(h^{-1}\begin{pmatrix}1\\0\end{pmatrix}\right) \mathrm{d}h$$

$$= \int_{H_1(\mathbb{A})\backslash H(\mathbb{A})} \Phi\left(h^{-1}\begin{pmatrix}1\\0\end{pmatrix}\right) \mathrm{d}h.$$

If $m \geqslant 5$, then (4.1) is absolutely convergent at $s = s_0$ and is easily seen to be equal to (4.3).

5. Statement of the main theorem

Assume that $Q = Q' \oplus \mathcal{H}^r$, where (Q', U') is a nondegenerate quadratic form of rank m' and $\mathcal{H}^r$ is a direct sum of r copies of hyperbolic planes. Let X and Y be maximal totally isotropic subspaces of $\mathcal{H}^r$, complementary to each other: $X \oplus Y = \mathcal{H}^r$. We put $H' = O_{Q'}$. We identify H' with the pointwise stabilizer of $\mathcal{H}^r$ in H. We will denote elements of U^n by column vectors

$$\begin{pmatrix}u'\\y\\x\end{pmatrix}, \quad u' \in U'^n,\ x \in X^n,\ y \in Y^n.$$

We define an operator

$$\pi_Q^{Q'} : \mathcal{S}(U^n(\mathbb{A})) \longrightarrow \mathcal{S}(U'^n(\mathbb{A}))$$

by

$$\pi_Q^{Q'}\Phi(u') = \int_{X^n(\mathbb{A})} \Phi\begin{pmatrix}u'\\0\\x\end{pmatrix} \mathrm{d}x.$$

Then it is easy to check $\pi_Q^{Q'}\omega_Q(g)\Phi = \omega_{Q'}(g)\pi_Q^{Q'}\Phi$, for $g \in \widetilde{G(\mathbb{A})}$. Moreover $\pi_Q^{Q'}\lambda(h')\Phi = \lambda(h')\pi_Q^{Q'}\Phi$, for $h' \in H'(\mathbb{A})$.

We fix Haar measures on various groups. On $H(\mathbb{A}) = O_Q(\mathbb{A})$, we take the Haar measure $\mathrm{d}h$ such that $\mathrm{Vol}(O_Q(k)\backslash O_Q(\mathbb{A})) = 1$. Let $P = P_X$ be the stabilizer of X. The Levi factor of P is isomorphic to $O_{Q'} \times \mathrm{GL}_r$. On $H'(\mathbb{A}) = O_{Q'}(\mathbb{A})$, we take the Haar measure $\mathrm{d}h'$ such that $\mathrm{Vol}(O_{Q'}(k)\backslash O_{Q'}(\mathbb{A})) = 1$. We take the global Tamagawa measure $\mathrm{d}m$ on $\mathrm{GL}_r(\mathbb{A})$. On the unipotent radical $U_P(\mathbb{A})$ of $P(\mathbb{A})$, we take the Haar measure $\mathrm{d}u$ such that $\mathrm{Vol}(U_P(k)\backslash U_P(\mathbb{A})) = 1$. Then on $P(\mathbb{A})$, we take the left Haar measure $\mathrm{d}_l p = \mathrm{d}h'\,\mathrm{d}m\,\mathrm{d}u$. Let K be a maximal compact subgroup of $O_Q(\mathbb{A})$ such that $O_Q(\mathbb{A}) = P(\mathbb{A})K$. We take the Haar measure $\mathrm{d}k$ on

K such that $\mathrm{Vol}(K) = 1$. We define a constant $c_K = c(X, K)$ as follows. Since the integral

$$\int_K \int_{P(\mathbb{A})} f(pk)\, \mathrm{d}_l p\, \mathrm{d}k, \quad f \in L^1(O_Q(\mathbb{A}))$$

is $O_Q(\mathbb{A})$-invariant, there exists a constant c_K such that the above integral is equal to

$$c_K{}^{-1} \int_{O_Q(\mathbb{A})} f(h)\, \mathrm{d}h.$$

When $Q = \mathcal{H}^r$, and $Q' = (0)$, the Levi factor of P is isomorphic to GL_r, in this case we just ignore $O_{Q'}$. The explicit calculation of c_K will be carried out in Section 9.

Now we state our main theorem.

THEOREM 5.1 *Let (Q, U) be a quadratic form of rank m over k. We assume*

(A.1): $n + 1 < m \leqslant 2n + 2$.
(A.2): *The dimension r_0 of a maximal isotropic subspace for Q is equal to $m - n - 1$.*

Let Q' be the quadratic form such that $Q = Q' \oplus \mathcal{H}^{r_0}$. Let $f^{(s)}$ be a weak SW section of $I(\chi_Q, s)$ associated to $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$. Then

$$\mathrm{Res}_{s=s_0} E(g, f^{(s)}) = c_K I_{Q'}(g, \pi_Q^{Q'} \pi_K \Phi). \tag{5.1}$$

Here

$$\pi_K \Phi(u) = \int_K \Phi(ku)\, \mathrm{d}k.$$

Observe that the right hand side of (5.1) does not depend on the choice of K and is $H(\mathbb{A})$-invariant.

COROLLARY 5.2 *Let (Q, U) be as in Theorem 5.1. Then for any holomorphic section $f^{(s)}$ of $I(\chi_Q, s)$ such that $f^{(s_0)} = \omega_Q(g)\Phi(0)$, $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$, the equation* (5.1) *holds.*

Proof. In fact, the left hand side of (5.1) depends only on $f^{(s_0)}$, as we have seen in Section 3.

6. Fourier–Jacobi coefficients of theta integrals

Recall that we have defined a function $\mathrm{FJ}^\phi(g_1; A)$ on $G_1(k)\backslash\widetilde{G_1(\mathbb{A})}$ by

$$\mathrm{FJ}^\phi(g_1; A) = \int_{V(k)\backslash V(\mathbb{A})} A(vg_1)\overline{\vartheta^\phi(vg_1)}\, \mathrm{d}v$$

for an automorphic form $A(g)$ on $G(k)\backslash\widetilde{G(\mathbb{A})}$ and $\phi \in \mathcal{S}(L(\mathbb{A}))$.

Let (Q, U) be a quadratic form of rank m over k. First we assume $r_0 = 0$ or $m - r_0 > n+1$, so that the integral $I_Q(g, \Phi)$ is absolutely convergent. Now we shall consider the following integral.

$$\mathrm{FJ}^{\phi}(g_1; I_Q(\Phi)) = \int_{V(k)\backslash V(\mathbb{A})} I_Q(vg_1, \Phi)\overline{\vartheta^{\phi}(vg_1)}\,\mathrm{d}v. \tag{6.1}$$

Here $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$, $\phi \in \mathcal{S}(L(\mathbb{A}))$. Obviously this integral vanishes unless Q expresses 1. So we may assume $Q = \begin{pmatrix} 1 & \\ & Q_1 \end{pmatrix}$ without loss of generality. The corresponding direct decomposition of U will be denoted by $U = k \oplus U_1$.

LEMMA 6.1 *Given* $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$ *and* $\phi \in \mathcal{S}(L(\mathbb{A}))$, *let* $\Psi(\Phi, \phi) \in \mathcal{S}({U_1}^{n-1}(\mathbb{A}))$ *be*

$$\Psi(\Phi, \phi; u) = \int_{x \in L(\mathbb{A})} \Phi\begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix} \overline{\phi(x)}\,\mathrm{d}x.$$

Then Ψ *satisfies the following equation:*

$$\omega_{Q_1}(g_1)\Psi(\Phi, \phi) = \Psi(\omega_Q(g_1)\Phi, \omega_\psi(g_1)\phi).$$

Moreover, the integral (6.1) *is equal to*

$$\int_{x \in L(\mathbb{A})} \int_{H_1(k)\backslash H(\mathbb{A})} \sum_{t \in U^{n-1}} \omega_Q(g_1)\Phi\left(h^{-1}\begin{pmatrix} 1 & x \\ 0 & t \end{pmatrix}\right) \overline{\omega_\psi(g_1)\phi(x)}\,\mathrm{d}x\,\mathrm{d}h$$

$$= \int_{H_1(\mathbb{A})\backslash H(\mathbb{A})} I_{Q_1}(g_1, \Psi(\lambda(h)\Phi, \phi))\,\mathrm{d}h.$$

Proof. If $v = v(x, 0; 0) \cdot v(0, y; z) = v(x, y; z + x\,{}^t\!y)$,

$$\begin{aligned}\Theta^{\Phi}(vg_1, h) = & \sum_{t_1, t_2, t_3, t_4} \omega_Q(g_1)\Phi\left(h^{-1}\begin{pmatrix} t_1 & t_2 + t_1 x \\ t_3 & t_4 + t_3 x \end{pmatrix}\right) \\ & \times\psi\left(\left(\frac{z}{2} + x\,{}^t\!y\right)\operatorname{tr}\left({}^t\begin{pmatrix} t_1 \\ t_3 \end{pmatrix}\begin{pmatrix} 1 & \\ & Q_1 \end{pmatrix}\begin{pmatrix} t_1 \\ t_3 \end{pmatrix}\right)\right) \\ & \times\psi\left(\operatorname{tr}\left({}^t\begin{pmatrix} t_1 \\ t_3 \end{pmatrix}\begin{pmatrix} 1 & \\ & Q_1 \end{pmatrix}\begin{pmatrix} t_2 \\ t_4 \end{pmatrix}{}^t\!y\right)\right).\end{aligned}$$

Here $t_1 \in k$, $t_2 \in k^{n-1}$, $t_3 \in U_1(k)$, and $t_4 \in U_1^{n-1}(k)$. On the other hand,

$$\vartheta^{\phi}(vg_1) = \sum_{t \in L(k)} \omega_\psi(g_1)\phi(t + x)\psi\left(\tfrac{1}{2}(z + 2x\,{}^t\!y + 2t\,{}^t\!y)\right).$$

The integration with respect to z vanishes unless ${}^t\begin{pmatrix} t_1 \\ t_3 \end{pmatrix} \begin{pmatrix} 1 & \\ & Q_1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_3 \end{pmatrix} = 1$, in that case $\begin{pmatrix} t_1 \\ t_3 \end{pmatrix} = h^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for some $h \in H(k)$ by Witt's theorem. Substituting $\begin{pmatrix} t_2 \\ t_4 \end{pmatrix}$ by $h^{-1}\begin{pmatrix} t_2 \\ t_4 \end{pmatrix}$, we have

$$\begin{aligned}(6.1) = & \int_{x\in L(k)\backslash L(\mathbb{A})} \int_{y\in L(k)\backslash L(\mathbb{A})} \int_{H_1(k)\backslash H(\mathbb{A})} \\ & \times \sum_{t\in L(k)} \sum_{t_2,t_4} \omega_Q(g_1)\Phi\left(h^{-1}\begin{pmatrix} 1 & t_2+x \\ 0 & t_4 \end{pmatrix}\right) \\ & \times \overline{\omega_\psi(g_1)\phi(t+x)}\psi((t_2-t)\,{}^t y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}h.\end{aligned}$$

Since the integration with respect to y vanishes unless $t = t_2$, we have

$$\begin{aligned}(6.1) = & \int_{x\in L(\mathbb{A})} \int_{H_1(k)\backslash H(\mathbb{A})} \sum_{t_4} \omega_Q(g_1)\Phi\left(h^{-1}\begin{pmatrix} 1 & x \\ 0 & t_4 \end{pmatrix}\right) \\ & \times \overline{\omega_\psi(g_1)\phi(x)}\,\mathrm{d}x\,\mathrm{d}h \\ = & \int_{H_1(\mathbb{A})\backslash H(\mathbb{A})} I_{Q_1}(g_1, \Psi(\lambda(h)\Phi,\phi))\,\mathrm{d}h.\end{aligned}$$

Hence Lemma 6.1.

Now we shall calculate the following integral:

$$\mathrm{FJ}^\phi(g_1; I_{Q'}(\pi_Q^{Q'}\pi_K\Phi)) = \int_{V(k)\backslash V(\mathbb{A})} I_{Q'}(vg_1, \pi_Q^{Q'}\pi_K\Phi)\overline{\vartheta^\phi(vg_1)}\,\mathrm{d}v, \tag{6.2}$$

for $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$, $\phi \in \mathcal{S}(L(\mathbb{A}))$, $g_1 \in \widetilde{G(\mathbb{A})}$.

PROPOSITION 6.2 *Assume* (A.1) *and* (A.2). *If* Q' *expresses* 1, *then*

$$\begin{aligned}& \int_{V(k)\backslash V(\mathbb{A})} I_{Q'}(vg_1, \pi_Q^{Q'}\pi_K\Phi)\overline{\vartheta^\phi(vg_1)}\,\mathrm{d}v \\ & = c_K^{-1}c_{K_1}\int_{H_1(\mathbb{A})\backslash H(\mathbb{A})} I_{Q_1'}(g_1, \pi_{Q_1}^{Q_1'}\pi_{K_1}\Psi(\lambda(h)\Phi,\phi))\,\mathrm{d}h.\end{aligned}$$

for any $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$, $\phi \in \mathcal{S}(L(\mathbb{A}))$, $g_1 \in \widetilde{G(\mathbb{A})}$.

Proof. We assume that Q' expresses 1. Put $Q' = \begin{pmatrix} 1 & \\ & Q_1' \end{pmatrix}$, $Q_1 = Q_1' \oplus \mathcal{H}^{r_0}$. By Lemma 6.1, (6.2) is equal to

$$\int_{H_1'(\mathbb{A})\backslash H'(\mathbb{A})} I_{Q_1'}(g_1, \Psi(\lambda(h')\pi_Q^{Q'}\pi_K\Phi,\phi))\,\mathrm{d}h'$$

$$= \int_{h' \in H_1'(\mathbb{A})\backslash H'(\mathbb{A})} \int_{h_1 \in H_1'(k)\backslash H_1'(\mathbb{A})} \int_{X^n(\mathbb{A})} \int_{y \in L(\mathbb{A})}$$

$$\times \sum_{t' \in U_1'^{n-1}} \omega_Q(g_1)\pi_K \Phi \begin{pmatrix} h'^{-1}\begin{pmatrix} 1 & y \\ 0 & h_1^{-1}t' \end{pmatrix} \\ 0 \\ x \end{pmatrix} \overline{\omega_\psi(g_1)\phi(y)}\, dy\, dx\, dh_1\, dh'$$

$$= \int_{H_1'(\mathbb{A})\backslash H'(\mathbb{A})} \int_{H_1'(k)\backslash H_1'(\mathbb{A})} \int_{x_1 \in X(\mathbb{A})} \int_{x_2 \in X^{n-1}(\mathbb{A})} \int_{y \in L(\mathbb{A})}$$

$$\times \sum_{t' \in U_1'^{n-1}} \omega_Q(g_1)\pi_K \Phi \left(h'^{-1} \begin{pmatrix} 1 & y \\ 0 & h_1^{-1}t' \\ 0 & 0 \\ x_1 & x_2 \end{pmatrix} \right)$$

$$\times \overline{\omega_\psi(g_1)\phi(y)}\, dy\, dx_1\, dx_2\, dh_1\, dh'.$$

Let $U_{(X)}$ be the unipotent subgroup of $H(\mathbb{A})$ defined by

$$U_{(X)} = \left\{ u_x = \begin{pmatrix} 1 & 0 & {}^t x & 0 \\ 0 & \mathbf{1}_{m'-1} & 0 & 0 \\ 0 & 0 & \mathbf{1}_{r_0} & 0 \\ -x & 0 & -x\,{}^t x/2 & \mathbf{1}_{r_0} \end{pmatrix} \middle|\, x \in X \right\}.$$

Then

$$(6.2) = \int_{H_1'(\mathbb{A})\backslash H'(\mathbb{A})} \int_{H_1'(k)\backslash H_1'(\mathbb{A})} \int_{u_x \in U_{(X)}(\mathbb{A})} \int_{x_2 \in X^{n-1}(\mathbb{A})} \int_{y \in L(\mathbb{A})}$$

$$\times \sum_{t' \in U_1'^{n-1}} \omega_Q(g_1)\pi_K \Phi \left(h'^{-1} u_x^{-1} \begin{pmatrix} 1 & y \\ 0 & h_1^{-1}t' \\ 0 & 0 \\ 0 & x_2 - yx \end{pmatrix} \right)$$

$$\times \overline{\omega_\psi(g_1)\phi(y)}\, dy\, du_x\, dx_2\, dh_1\, dh'$$

$$= \int_{H_1'(\mathbb{A})\backslash H'(\mathbb{A})} \int_{H_1'(k)\backslash H_1'(\mathbb{A})} \int_{u \in U_{(X)}(\mathbb{A})} \int_{x_2 \in X^{n-1}(\mathbb{A})} \int_{y \in L(\mathbb{A})} \int_K$$

$$\times \sum_{t' \in U_1'^{n-1}} \omega_Q(g_1)\Phi \left(k^{-1}h'^{-1}u^{-1} \begin{pmatrix} 1 & y \\ 0 & h_1^{-1}t' \\ 0 & 0 \\ 0 & x_2 \end{pmatrix} \right)$$

$$\times\, \overline{\omega_\psi(g_1)\phi(y)}\, \mathrm{d}k\, \mathrm{d}y\, \mathrm{d}u\, \mathrm{d}x_2\, \mathrm{d}h_1\, \mathrm{d}h'.$$

We get Proposition 6.2 by applying the following lemma.

LEMMA 6.3 *Put $P_1 = P \cap H_1$. Let K_1 be a good maximal compact subgroup of $H_1(\mathbb{A})$. Let f be a function on $H(\mathbb{A})$ such that $f(hp_1) = \delta_{P_1}(p_1) f(h)$. Then if*

$$c_K \int_{h' \in H_1'(\mathbb{A}) \backslash H'(\mathbb{A})} \int_{U_{(X)}(\mathbb{A})} \int_K f(k^{-1} h'^{-1} u^{-1})\, \mathrm{d}k\, \mathrm{d}u\, \mathrm{d}h'$$

is absolutely convergent, then so is the following, and they are equal:

$$c_{K_1} \int_{h \in H_1(\mathbb{A}) \backslash H(\mathbb{A})} \int_{K_1} f(h^{-1} k_1^{-1})\, \mathrm{d}k_1\, \mathrm{d}h.$$

Proof. We can take $F \in L^1(H(\mathbb{A}))$ such that $f(h^{-1}) = \int_{P_1(\mathbb{A})} F(p_1 h)\, \mathrm{d}p_1$. Then they are both equal to

$$\int_{H(\mathbb{A})} F(h)\, \mathrm{d}h.$$

7. Fourier–Jacobi coefficients of Eisenstein series

We recall some results on Fourier–Jacobi coefficients of the Eisenstein series [2]. Let (Q, U) be as before. Let $f^{(s)}$ be a holomorphic section of $I(\chi_Q, s)$. We consider the following integral:

$$\mathrm{FJ}^\phi(g_1; E(f^{(s)})) = \int_{V(k) \backslash V(\mathbb{A})} E(vg_1; f^{(s)}) \overline{\vartheta^\phi(vg_1)}\, \mathrm{d}v, \tag{7.1}$$

$g_1 \in G_1(\mathbb{A})$, $\phi \in \mathcal{S}(L(\mathbb{A}))$. Put

$$R(g_1; f^{(s)}, \phi) = \int_{y \in L(\mathbb{A})} \int_{\mathbb{A}} f^{(s)} \left(w_n \left(\begin{array}{c|cc} \mathbf{1}_n & z & y \\ & {}^t y & \mathbf{0}_{n-1} \\ \hline \mathbf{0}_n & & \mathbf{1}_n \end{array} \right) w_{n-1} g_1 \right) \times \overline{\omega_\psi(g_1)\phi(-y)\psi(z/2)}\, \mathrm{d}z\, \mathrm{d}y.$$

Here $w_n = \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix}$, $w_{n-1} = \begin{pmatrix} \mathbf{0}_{n-1} & \mathbf{1}_{n-1} \\ -\mathbf{1}_{n-1} & \mathbf{0}_{n-1} \end{pmatrix} \in \mathrm{Sp}_{n-1} \subset \mathrm{Sp}_n$.

PROPOSITION 7.1 *Assume $\phi \in \mathcal{S}(L(\mathbb{A}))$ is $\widetilde{K}_{G_1}$-finite. Then*

(1) *For $Re(s) > -(n-2)/2$, the integral $R(g_1; f^{(s)}, \phi)$ is absolutely convergent and defines a holomorphic section of $I(\chi_{Q_1}, s)$.*

$$\text{(2)} \qquad \mathrm{FJ}^{\phi}(g_1; E(f^{(s)})) = \int_{V(k)\backslash V(\mathbb{A})} E(vg_1; f^{(s)})\overline{\vartheta^{\phi}(vg_1)}\,\mathrm{d}v$$

is an Eisenstein series associated to $R(g_1; f^{(s)}, \phi)$, *i.e.,*

$$\int_{V(k)\backslash V(\mathbb{A})} E(vg_1; f^{(s)})\overline{\vartheta^{\phi}(vg_1)}\,\mathrm{d}v = E(g_1; R(f^{(s)}, \phi)) = \sum_{\gamma \in P_1\backslash G_1} R(\gamma g_1; f^{(s)}, \phi).$$

where $P_1 = P_G \cap G_1$ *is the Siegel parabolic subgroup of* G_1.

Proof. See [2] Section 3.

PROPOSITION 7.2 *Assume that* χ_Q *is 'genuine' (i.e.,* m *is odd). Let* $f^{(s)}$ *be a holomorphic section of* $I(\chi_Q, s)$. *Then the set of possible poles of* $E(g; f^{(s)})$ *which lie in the half plane* $\mathrm{Re}(s) \geqslant 0$ *is*

$$\left\{ \frac{n}{2} - s \,\middle|\, s \in \mathbb{Z}, 0 \leqslant s < \frac{n}{2} \right\}.$$

Moreover, all these poles are at most simple.

Proof. When $n = 1$, this is well-known. We may assume $n \geqslant 2$. It is well-known [10] that $E(g; f^{(s)})$ is holomorphic on the line $\mathrm{Re}(s) = 0$. Assume $\mathrm{Re}(s_0) > 0$ and s_0 does not belong to this set. Let k be the order of the pole of $E(g; f^{(s)})$ at $s = s_0$. If $k \geqslant 1$, then the assumption of Lemma 2.2 is satisfied for $\lim_{s \to s_0}(s - s_0)^k E(g; f^{(s)})$. It follows that $\lim_{s \to s_0}(s - s_0)^k E(g; f^{(s)})$ is a constant function. This is impossible because of the assumption on χ_Q. Thus $k = 0$. Similarly, if s_0 belongs to the above set, $E(g; f^{(s)})$ has at most simple pole.

PROPOSITION 7.3 *Let* (Q, U) *be as in Lemma 4.1. If* Q *does not express* 1, *then*

$$R(g_1; f^{(s)}, \phi)|_{s=s_0} = 0.$$

If $Q = \begin{pmatrix} 1 & \\ & Q_1 \end{pmatrix}$, *then for any weak SW section* $f^{(s)}$ *associated to* $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$,

$$R(g_1; f^{(s)}, \phi)|_{s=s_0} = \int_{H_1(\mathbb{A})\backslash H(\mathbb{A})} \int_{y \in L(\mathbb{A})} \omega_Q(g_1)\Phi \times \left(h^{-1}\begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}\right) \overline{\omega_{\psi}(g_1)\phi(y)}\,\mathrm{d}y\,\mathrm{d}h$$

$$= \int_{H_1(\mathbb{A})\backslash H(\mathbb{A})} \omega_{Q_1}(g_1)\Psi(\lambda(h)\Phi, \phi; 0)\,\mathrm{d}h.$$

Proof. We embed Sp_1 into $G = \mathrm{Sp}_n$ by

$$g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & & b & \\ & \mathbf{1}_{n-1} & & \mathbf{0}_{n-1} \\ c & & d & \\ & \mathbf{0}_{n-1} & & \mathbf{1}_{n-1} \end{pmatrix}.$$

We denote this embedding by ι. The lift $\widetilde{\mathrm{Sp}_1(\mathbb{A})} \to \widetilde{\mathrm{Sp}_n(\mathbb{A})}$ of ι is also denoted by ι. We consider

$$f^{(s)}\left(\iota(g_0)w_{n-1}\left(\begin{array}{c|cc} \mathbf{1}_n & \begin{matrix} 0 & y \\ {}^t y & \mathbf{0}_{n-1} \end{matrix} \\ \hline \mathbf{0}_n & \mathbf{1}_n \end{array}\right)w_{n-1}g_1\right).$$

As a function of $g_0 \in \widetilde{\mathrm{Sp}_1(\mathbb{A})}$, this is a weak SW section associated with

$$u \longmapsto \omega_Q\left(w_{n-1}\left(\begin{array}{c|cc} \mathbf{1}_n & \begin{matrix} 0 & y \\ {}^t y & \mathbf{0}_{n-1} \end{matrix} \\ \hline \mathbf{0}_n & \mathbf{1}_n \end{array}\right)w_{n-1}g_1\right)\Phi(u,0).$$

By Lemma 4.1, if Q does not expresses 1, then $R(g_1; f^{(s)}, \phi)$ vanishes at $s = s_0$. Again by Lemma 4.1, If $Q = \begin{pmatrix} 1 & \\ & Q_1 \end{pmatrix}$, then

$$\begin{aligned} & R(g_1; f^{(s)}, \phi)|_{s=s_0} \\ = & \int_{H_1(\mathbb{A})\backslash H(\mathbb{A})}\int_{L(\mathbb{A})} \omega_Q\left(w_{n-1}\left(\begin{array}{c|cc} \mathbf{1}_n & \begin{matrix} 0 & y \\ {}^t y & \mathbf{0}_{n-1} \end{matrix} \\ \hline \mathbf{0}_n & \mathbf{1}_n \end{array}\right)w_{n-1}g_1\right) \\ & \times \Phi\left(h^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\overline{\omega_\psi(g_1)\phi(-y)}\,\mathrm{d}y\,\mathrm{d}h. \end{aligned}$$

Here

$$\begin{aligned} & \omega_Q\left(w_{n-1}\left(\begin{array}{c|cc} \mathbf{1}_n & \begin{matrix} 0 & y \\ {}^t y & \mathbf{0}_{n-1} \end{matrix} \\ \hline \mathbf{0}_n & \mathbf{1}_n \end{array}\right)w_{n-1}g_1\right)\Phi\left(h^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\ = & \int_{L(\mathbb{A})}\int_{U_1^{n-1}(\mathbb{A})} \omega_Q\left(\left(\begin{array}{c|cc} \mathbf{1}_n & \begin{matrix} 0 & y \\ {}^t y & \mathbf{0}_{n-1} \end{matrix} \\ \hline \mathbf{0}_n & \mathbf{1}_n \end{array}\right)w_{n-1}g_1\right) \\ & \times \Phi\left(h^{-1}\begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix}\right)\,\mathrm{d}u\,\mathrm{d}x \end{aligned}$$

$$= \int_{L(\mathbb{A})} \int_{U_1^{n-1}(\mathbb{A})} \omega_Q (w_{n-1} g_1) \Phi \left(h^{-1} \begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix} \right)$$

$$\times \psi \left(\frac{1}{2} \mathrm{tr} \left(\begin{pmatrix} 1 & \\ & Q_1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix} \begin{pmatrix} 0 & y \\ {}^t y & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ {}^t x & {}^t u \end{pmatrix} \right) \right) \mathrm{d}u \, \mathrm{d}x$$

$$= \int_{L(\mathbb{A})} \int_{U_1^{n-1}(\mathbb{A})} \omega_Q (w_{n-1} g_1) \Phi \left(h^{-1} \begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix} \right) \psi(x {}^t y) \, \mathrm{d}u \, \mathrm{d}x$$

$$= \omega_Q(g_1) \Phi \left(h^{-1} \begin{pmatrix} 1 & -y \\ 0 & 0 \end{pmatrix} \right).$$

Hence the proposition.

8. Proof of the main theorem

LEMMA 8.1 *We denote the trivial representation of* $\widetilde{G(\mathbb{A})} \times H(\mathbb{A})$ *by* $\mathbb{C}$.

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{G(\mathbb{A})} \times H(\mathbb{A})} (\mathcal{S}(U^n(\mathbb{A})), \mathbb{C}) = \begin{cases} 1 & \text{if } Q = (0) \text{ or } Q = \mathcal{H}^{n+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See [6–7]. It is not difficult to prove directly.

LEMMA 8.2 *Let Q be a quadratic form of* rank $m \geqslant n + 1$. *Assume* $r_0 = 0$ *or* $m - r_0 > n + 1$. *Then for any weak SW section* $f^{(s)}$ *belonging to* Q, $E(g; f^{(s)})$ *is holomorphic at* $s = s_0$.

Proof. We proceed by the induction with respect to n. When $n = 1$, we have seen in Section 4 that the Lemma is true. When $n > 1$, we will prove $\mathrm{Res}_{s=s_0} E(g; f^{(s)}) = 0$. In fact, Proposition 7.3 implies ψth Fourier–Jacobi coefficients of the residue is zero for any ψ. By Lemma 2.2, the residue is a constant function. By Lemma 8.1, it must be zero.

Now we shall prove Theorem 5.1 by an induction with respect to $m' = \mathrm{rk}\, Q'$. When $r_0 = 1$, the smallest value of m' is 1 and $n = 1$. In this case we make use of the results of [16] Chapter 1.

We may assume (Q, U) is the direct sum of one dimensional (Q', U') and the hyperbolic plane $\mathcal{H}$, where $Q'(u') = u'^2$. Fix a non-zero isotropic vector x_0 for $\mathcal{H}$. As in [16] Chapter 1, for $\Phi \in \mathcal{S}(U(\mathbb{A}))$, we put

$$\mathcal{R}(\Phi, g, h, s) = \int_{\mathbb{A}^\times} \omega_Q(g) \Phi(h^{-1} t x_0) |t|^s d^\times t.$$

Here $g \in \widetilde{\mathrm{Sp}_1(\mathbb{A})}$, $h \in O_Q(\mathbb{A})$, and $d^\times t$ is the global Tamagawa measure on $\mathbb{A}^\times$. Then $\mathcal{R}$ can be meromorphically continued to the whole s-plane and

$$\mathcal{R}\left(\Phi, \left(\begin{pmatrix} a & n \\ 0 & a^{-1} \end{pmatrix}, \zeta\right) g, h, s\right) = \zeta \frac{1}{\gamma_Q(a)} |a|^{(3/2)-s} \mathcal{R}(\Phi, g, h, s),$$

$$\mathrm{Res}_{s=0}\mathcal{R}(\Phi, g, h, s) = -\omega_Q(g)\Phi(0),$$

$$\mathrm{Res}_{s=1}\mathcal{R}(\Phi, g, h, s) = \int_{\mathbb{A}} \omega_Q(g)\Phi(h^{-1}tx_0)\, dt.$$

In particular, $(s - \frac{1}{2})\mathcal{R}(\Phi, g, h, \frac{1}{2} - s)$ is a weak SW section associated to Φ for any h. We may assume that K is the standard maximal compact subgroup of $O_Q(\mathbb{A}) \simeq \mathrm{PGL}_2(\mathbb{A}) \times \{\pm 1\}$. It is easy to see $c_K = \frac{\rho_k}{2\xi_k(2)}$. Then it suffices to prove that the constant terms of both sides of (5.1) are equal, i.e.,

$$\lim_{s\to(1/2)} (s - \tfrac{1}{2})^2 M_w \mathcal{R}(\Phi, g, h, \tfrac{1}{2} - s) = \frac{\rho_k}{2\xi_k(2)} \omega_{Q'}(g) \pi_Q^{Q'} \pi_K \Phi(0).$$

Here M_w is the intertwining operator for $\widetilde{\mathrm{Sp}_1(\mathbb{A})}$. We denote the intertwining operator for $\mathrm{SO}_Q(\mathbb{A}) \simeq \mathrm{PGL}_2(\mathbb{A})$ by $M_{\tilde{w}}$. Then by [16] p.13, Theorem 1.1,

$$\begin{aligned}&\lim_{s\to(1/2)}(s - \tfrac{1}{2})^2 M_w\, \mathcal{R}(\Phi, g, h, \tfrac{1}{2} - s)\\ &= \lim_{s\to(1/2)} (s - \tfrac{1}{2})^2\, M_{\tilde{w}}\, \mathcal{R}(\Phi, g, h, s + \tfrac{1}{2}).\end{aligned}$$

Observe that the right hand side is the residue of the Eisenstein series on $\mathrm{PGL}_2(\mathbb{A})$. Since it is a constant function on $\mathrm{SO}_Q(\mathbb{A})$, it is in fact $O_Q(\mathbb{A})$-invariant. In particular we may replace Φ by $\pi_K\Phi$. Then the residue is $\frac{\rho_k}{2\xi_k(2)}$ (= the residue of the Eisenstein series on $\mathrm{PGL}_2(\mathbb{A})$) times the value of $\lim_{s\to\frac{1}{2}}(s - \frac{1}{2})\mathcal{R}(\pi_K\Phi, g, h, s + \frac{1}{2})$ at $h = e$:

$$\begin{aligned}\lim_{s\to(1/2)} (s - \tfrac{1}{2})\, \mathcal{R}(\pi_K\Phi, g, e, s + \tfrac{1}{2}) &= \int_{\mathbb{A}} \omega_Q(g)\pi_K\Phi(tx_0)\, dt\\ &= \pi_Q^{Q'} \omega_Q(g)\pi_K\Phi(0)\\ &= \omega_{Q'}(g)\pi_Q^{Q'} \pi_K\Phi(0).\end{aligned}$$

When $r_0 \geqslant 2$, then the smallest value of m' is 0. In this case $r_0 = n + 1$ and $Q = \mathcal{H}^{r_0}$. Note that both sides of (5.1) are constant functions. By Lemma 8.1, it will suffice to prove the equality when Φ is unramified. We may assume K is the standard maximal compact subgroup of $H(\mathbb{A})$. Then

$$f_\Phi^{(s)}|_{K_G} \equiv 1, \qquad \pi_Q^{Q'} \pi_K \Phi = 1.$$

It will then suffice to prove that

$$\mathrm{Res}_{s=(n+1)/2} E(g; f_\Phi^{(s)}) = c_K.$$

In fact, we claim that both sides are equal to

$$\frac{\rho_k}{\xi_k(n+1)} \prod_{i=1}^{[\frac{n}{2}]} \frac{\xi_k(2i+1)}{\xi_k(2n+2-2i)}.$$

The calculation of the residue of $E(g; f_{\Phi}^{(s)})$ was carried out in [14], [2]. As for the calculation of c_K, let B_1 be a Borel subgroup of GL_r and put $B = B_1 U_P$, $K_1 = K \cap \mathrm{GL}_r(\mathbb{A})$, $K^+ = K \cap \mathrm{SO}_Q(\mathbb{A})$. Let $\mathrm{d}b$ and $\mathrm{d}b_1$ be the left invariant Tamagawa measure of $B(\mathbb{A})$ and $B_1(\mathbb{A})$, respectively. By definition, $\mathrm{d}b = \mathrm{d}b_1\,\mathrm{d}u$. Let $\mathrm{d}k_1$ and $\mathrm{d}k^+$ be the Haar measure of K_1 and K^+ with the total volume 1, respectively. Then by [11],

$$\frac{\rho_k^{r-1}}{\xi_k(2)\xi_k(3)\cdots\xi_k(r)}\mathrm{d}m = \mathrm{d}b_1\,\mathrm{d}k_1,$$

$$\frac{\rho_k^{r}}{\xi_k(2)\xi_k(4)\cdots\xi_k(2r-2)\cdot\xi(r)}\mathrm{d}h_1 = \mathrm{d}b\,\mathrm{d}k_0.$$

Here $\mathrm{d}h_1$ is the Tamagawa measure of $\mathrm{SO}_Q(\mathbb{A})$. It is well-known that the Tamagawa number of SO_Q is 2. It follows that $\mathrm{d}h = \mathrm{d}h_1\,\mathrm{d}\bar{k}$, where $\mathrm{d}\bar{k}$ is the Haar measure of $K^+\backslash K$ with the total volume 1. This proves the claim.

Now we assume Theorem 5.1 holds for any quadratic form of degree smaller than m. We consider $\mathrm{FJ}^{\phi}(g_1; A)$ where

$$A = \mathrm{Res}_{s=s_0}E(g, f^{(s)}) - c_K I_{Q'}(g, \pi_Q^{Q'}\pi_K\Phi).$$

If Q' expresses 1, then $\mathrm{FJ}^{\phi}(g_1; A) = 0$ by Proposition 6.2 and Proposition 7.3. If Q' does not express 1, then $\mathrm{FJ}^{\phi}(g_1; A) = 0$ by Proposition 7.3 and Lemma 8.2. Therefore by Lemma 2.2, A is a constant function. By Lemma 8.1, $A = 0$.

Similarly, one can prove the following theorem:

THEOREM 8.3. *Let (Q, U) be an anisotropic quadratic form of* rank $m = n + 1$. *Then for any holomorphic section $f^{(s)}$ of $I(\chi_Q, s)$ such that $f^{(0)}(g) = \omega_Q(g)\Phi(0)$, $\Phi \in \mathcal{S}(U^n(\mathbb{A}))$, the following Siegel–Weil formula holds:*

$$E(g; f^{(s)})|_{s=0} = 2I_Q(g; \Phi).$$

When m is even, this is a special case of [3].

9. Calculation of c_K

In this section, we shall explicitly calculate the value of c_K for some special choice of K. We assume $m \geqslant 3$ and $Q' \not\simeq \mathcal{H}$, but do not assume Q' is anisotropic. We take a maximally split torus $T_v \subset P_v$ of H_v and assume that K_v is T_v-good maximal compact subgroup of H_v if v is non-archimedean, and K_v is the fixed point set of a Cartan involution which stabilize T_v if v is archimedean.

First of all, we shall recall the definition of the Tamagawa measure. Let $\mathcal{G}$ be a connected reductive algebraic group define over k and $X(\mathcal{G})$ be the group of characters of $\mathcal{G}$. Let $L(s, \mathcal{G})$ be the Artin L-function corresponding to the $\mathrm{Gal}(\bar{k}/k)$-module $X(\mathcal{G}) \otimes \mathbb{Q}$, and let $L_v(s, \mathcal{G})$ be its v-component. Let $\mathrm{d}x_v$ be the Haar measure on k_v self-dual with respect to ψ_v. Let ω be a k-rational left-invariant nowhere vanishing exterior form of highest degree on $\mathcal{G}$. For each v, ω and dx_v defines a measure $|\omega|_v$ on $\mathcal{G}_v$. We put $\mathrm{d}g_v = L_v(1,\mathcal{G})|\omega|_v$. Then the Tamagawa measure dg on $\mathcal{G}(\mathbb{A})$ is the Haar measure on $\mathcal{G}(\mathbb{A})$ defined by

$$\mathrm{d}g = \lim_{s\to 1} \frac{1}{(s-1)^r L(s,\mathcal{G})} \prod_v \mathrm{d}g_v,$$

where r is the rank of the group of k-rational characters of $\mathcal{G}$. This measure is independent of the choice of ψ and ω.

Put $H^+ = \mathrm{SO}_Q$, $P^+ = P \cap H^+$, and $K^+ = K \cap H^+(\mathbb{A})$. Then the Levi factor of P^+ is isomorphic to $\mathrm{GL}_r \times \mathrm{SO}_{Q'}$. We consider the Tamagawa measure $\mathrm{d}h^+$, $\mathrm{d}m$ and $\mathrm{d}h'^+$ on $H^+(\mathbb{A})$, $\mathrm{GL}_r(\mathbb{A})$ and $\mathrm{SO}_{Q'}(\mathbb{A})$, respectively. We also take the Haar measure $\mathrm{d}k^+$ on K^+ such that $\mathrm{Vol}(K^+) = 1$. Then there is constant c_K^+ such that

$$\mathrm{d}h^+ = c_K^+ \mathrm{d}h^+ \mathrm{d}m\, \mathrm{d}u\, \mathrm{d}k^+.$$

LEMMA 9.1

$$c_K = \begin{cases} c_K^+, & \mathrm{rk}\, Q' = 0, \\ \frac{1}{2} c_K^+, & \mathrm{rk}\, Q' = 1, \\ c_K^+, & \mathrm{rk}\, Q' \geqslant 2,\ Q' \not\simeq \mathcal{H}. \end{cases}$$

Proof. Recall that the Tamagawa number of H^+ is 2. Let $\mathrm{d}\bar{k}$ (resp. $\mathrm{d}\bar{k}^+$) be the measure of $K^+\backslash K$ (resp. $(K^+ \cap H'(\mathbb{A}))\backslash(K \cap H'(\mathbb{A}))$) such that $\mathrm{Vol}(K^+\backslash K) = 1$. (resp. $\mathrm{Vol}((K^+ \cap H'(\mathbb{A}))\backslash(K \cap H'(\mathbb{A})) = 1$). Note that there is an exact sequence

$$1 \to H^+(k)\backslash H^+(\mathbb{A}) \to H(k)\backslash H(\mathbb{A}) \to K^+ \cdot (H(k) \cap K)\backslash K \to 1.$$

Since $[H(k) \cap K : H(k) \cap K^+] = 2$, we have $\mathrm{d}h = \mathrm{d}h^+ \, d\bar{k}$. Similarly $\mathrm{d}h' = \mathrm{d}h'^+ \, \mathrm{d}\bar{k}^+$ unless $m' = 1$. If $m' = 1$, then $H'^+ = 1$, so we have $\mathrm{d}h' = 2dh'^+ d\bar{k}^+$. Hence Lemma 9.1.

For $s \in \mathbb{C}$, we define a function Φ_s on $H^+(\mathbb{A})$ by

$$\Phi_s(h^+) = |\det m|^{s+(m'+r-1/2)}$$

if $h^+ = h'^+ m u k^+$, $h'^+ \in \mathrm{SO}_{Q'}(\mathbb{A})$, $m \in \mathrm{GL}_r(\mathbb{A})$, $u \in U_P(\mathbb{A})$, and $k^+ \in K^+$. We put

$$M(s) = \int_{\bar{U}_P(\mathbb{A})} \Phi_s(u)\, \mathrm{d}u.$$

Here $\bar{U}_P$ is the unipotent radical of the parabolic subgroup opposite to P^+. As usual the Haar measure du is normalized so that $\mathrm{Vol}(\bar{U}_P(k)\backslash\bar{U}_P(\mathbb{A})) = 1$. Then c_K^+ can be calculated by

$$c_K^+ = |D_k|^{-(1/2)} \rho_k \prod_v \frac{M_v(\frac{m'+r-1}{2})}{\zeta_v(1)}. \tag{9.1}$$

See [11], p. 177. Although [11] treated only quasi-split groups, the calculation of c_K^+ works for any reductive group.

Now we are going to calculate $M(s)$. We may assume $\det Q' = 2(-1)^{(m'-1)/2} \bmod (k^\times)^2$ if m' is odd, since the orthogonal group for $Q' \oplus \mathcal{H}^r$ is isomorphic to that for $\alpha Q' \oplus \alpha\mathcal{H}^r \simeq \alpha Q' \oplus \mathcal{H}^r$ for any $\alpha \in k^\times$. We choose a basis $x_1, \ldots, x_r$ of X and its dual basis $y_1, \ldots, y_r$ of Y. Then the Lie algebra of $\bar{U}_P$ can be identified with $(U')^r \oplus \mathrm{Alt}_r(k)$. $M(s)$ is an Euler product of

$$M_v(s) = \int_{\bar{U}_P(k_v)} \Phi_{s,v}(u)\,\mathrm{d}u.$$

Here $\mathrm{d}u$ is normalized as follows. The Lie algebra of $\bar{U}_P(k_v)$ is canonically isomorphic to $(U'_v)^r \oplus k_v^{(r(r-1))/2}$. We give a self-dual measure on U'_v and k_v with respect to the Fourier transform $\phi \mapsto \mathcal{F}_{Q'}\phi$ and $\phi \mapsto \mathcal{F}\phi$, respectively. We may choose ψ so that, if v is archimedean, ψ_v is equal to the standard one: $\psi_v(x) = \exp(2\pi\sqrt{-1}\,\mathrm{tr}_{k_v/\mathbb{R}}x)$. Note that Q is isomorphic to

$$\begin{pmatrix} & & \underbrace{\begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix}}_{r} \\ & Q' & \\ \underbrace{\begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix}}_{r} & & \end{pmatrix}.$$

Here we think of the space of the first r vectors as X and the space of the last r vectors as Y. If $Q'' = {}^tAQ'A$, then

$$h \mapsto \begin{pmatrix} \mathbf{1}_r & & \\ & A^{-1} & \\ & & \mathbf{1}_r \end{pmatrix} h \begin{pmatrix} \mathbf{1}_r & & \\ & A & \\ & & \mathbf{1}_r \end{pmatrix}$$

gives an isomorphism from the orthogonal group for

$$\begin{pmatrix} & & \underbrace{\begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix}}_{r} \\ & Q' & \\ \underbrace{\begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix}}_{r} & & \end{pmatrix}.$$

to the orthogonal group for

$$\begin{pmatrix} & & \underbrace{\begin{matrix} & & 1\\ & \iddots & \\ 1 & & \end{matrix}}_{r} \\ & Q'' & \\ \underbrace{\begin{matrix} & & 1\\ & \iddots & \\ 1 & & \end{matrix}}_{r} & & \end{pmatrix}.$$

Note that this isomorphism sends the self dual measure of U'_v to the self dual measure of U''_v, where U''_v is the underlying vector space for Q''. Thus we can change Q'_v to an equivalent quadratic form for each v.

Now We recall the classification of anisotropic quadratic forms over a non-archimedean local field k_v. For each quadratic extension F/k_v, we define a quadratic form $(\mathcal{Q}_{2,F}, F)$ by $(\mathcal{Q}_{2,F}, F) = (N_F, F)$, where $N_F : F \to k_v$ is the norm form.

Let $\mathbb{D}$ be the division quaternion algebra over k_v and $N_{\mathbb{D}}$ be the reduced norm of $\mathbb{D}$. Let $\mathfrak{o}_{\mathbb{D}}$ be the maximal order of $\mathbb{D}$ and $\mathbb{D}_0$ be the set of elements of $\mathbb{D}$ with reduced trace 0. We set $(\mathcal{Q}_3, \mathbb{D}_0) = (-N_{\mathbb{D}}, \mathbb{D}_0)$ and $(\mathcal{Q}_4, \mathbb{D}) = (N_{\mathbb{D}}, \mathbb{D})$. Note that $\det \mathcal{Q}_3 = -1$.

LEMMA 9.2 *Let k_v be a non-archimedean local field. Then an anisotropic quadratic form $\mathcal{Q}$ of rank m over k_v is isomorphic to one of the following:*

$$\begin{cases} \varepsilon\cdot x^2,\ \varepsilon \in k_v^\times, & \textit{if } m = 1, \\ \varepsilon\cdot \mathcal{Q}_{2,F},\ [F : k_v] = 2,\ \varepsilon \in k_v^\times, & \textit{if } m = 2, \\ \varepsilon\cdot \mathcal{Q}_3, \varepsilon \in k_v^\times, & \textit{if } m = 3, \\ \mathcal{Q}_4, & \textit{if } m = 4, \end{cases}$$

Note that we may assume ε is a unit or a prime element of k_v. Moreover, if F/k_v is ramified we may assume ε is a unit.

Next, we will choose the maximal compact subgroups of $\mathrm{SO}_Q(k_v)$. In fact, our choice of the maximal compact subgroups exhausts the equivalence classes of special maximal compact subgroups under the action of $\mathrm{GO}_Q(k_v)$, (cf. [22]) and calculation of c_K is easily reduced to these cases. First assume k_v is non-archimedean.

When m is even and Q is split, Q is similar to

$$\begin{pmatrix} & & & & & 1 \\ & & & & \cdot^{\cdot^{\cdot}} & \\ & & & 1 & & \\ & & 1 & & & \\ & \cdot^{\cdot^{\cdot}} & & & & \\ 1 & & & & & \end{pmatrix}.$$

In this case we take K_v to be the stabilizer of $\mathfrak{o}_v^m$.

When m is odd and Q' is split, Q' is isomorphic to

$$\begin{pmatrix} & & & & & & 1 \\ & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & 1 & & \\ & & & 2 & & & \\ & & 1 & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & & \\ 1 & & & & & & \end{pmatrix}.$$

In this case we take K_v to be the stabilizer of $\mathfrak{o}_v^m$.

When Q' is isomorphic to

$$\begin{pmatrix} & & & & & & 1 \\ & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & 1 & & \\ & & & 2\varepsilon \cdot \mathcal{Q}_{2,F} & & & \\ & & 1 & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & & \\ 1 & & & & & & \end{pmatrix},$$

for $\varepsilon \in k^\times$, we take K_v to be the stabilizer of $\varepsilon \mathfrak{o}_v^{(m/2)-1} \oplus \mathfrak{o}_F \oplus \mathfrak{o}_v^{(m/2)-1}$. We assume ε is a unit or a prime element of $k_v^\times$, and if F_v/k_v is ramified we assume ε is a unit of $k_v^\times$.

When Q' is isomorphic to

$$\begin{pmatrix} & & & & & & 1 \\ & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & 1 & & \\ & & & 2 \cdot \mathcal{Q}_3 & & & \\ & & 1 & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & & \\ 1 & & & & & & \end{pmatrix},$$

then there are two different choices of good maximal compact subgroup. We take $K_v^{(1)}$ to be the stabilizer of $\mathfrak{o}_v^{(m-3)/2} \oplus (\mathfrak{o}_{\mathbb{D}} \cap \mathbb{D}_0) \oplus \mathfrak{o}_v^{(m-3)/2}$, and $K_v^{(2)}$ to be the stabilizer of $\mathfrak{p}_v^{(m-3)/2} \oplus (\mathfrak{p}_{\mathbb{D}} \cap \mathbb{D}_0) \oplus \mathfrak{o}_v^{(m-3)/2}$. Here $\mathfrak{p}_{\mathbb{D}}$ is the maximal ideal of $\mathfrak{o}_{\mathbb{D}}$. $K_v^{(1)}$ and $K_v^{(2)}$ are not conjugate to each other.

When Q' is isomorphic to

$$\begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & 2\cdot\mathcal{Q}_4 & & & \\ & 1 & & & & \\ \iddots & & & & & \\ 1 & & & & & \end{pmatrix},$$

we take K_v to be the stabilizer of $\mathfrak{o}_v^{(m-4)/2} \oplus \mathfrak{o}_{\mathbb{D}} \oplus \mathfrak{o}_v^{(m-4)/2}$.

When $k_v = \mathbb{R}$, we choose a maximal compact subgroup K as follows:

When $\operatorname{rk} Q' = m'$ and Q' is isomorphic to

$$\begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & \varepsilon\cdot\mathbf{1}_l & & & \\ & 1 & & & & \\ \iddots & & & & & \\ 1 & & & & & \end{pmatrix},$$

$0 \leqslant l \leqslant m'$, $\varepsilon = \pm 1$, then we take K_v to be $\mathrm{SO}_Q \cap \mathrm{SO}(m+2l)$.

When $k_v = \mathbb{C}$, we choose a maximal compact subgroup K as follows:

When $\operatorname{rk} Q' = m'$ and Q' is isomorphic to

$$\begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & \mathbf{1}_l & & & \\ & 1 & & & & \\ \iddots & & & & & \\ 1 & & & & & \end{pmatrix},$$

$0 \leqslant l \leqslant 1$, then we take K_v to be $\mathrm{SO}_Q(\mathbb{C}) \cap \mathrm{SU}(m)$.

We shall calculate $M(s)$ for each relative rank one group. Note that by Gindikin–Karperevich argument, the calculation of $M(s)$ is reduced to the relative rank 1 case. Such a rank one group is isomorphic to either SL_2 or SO_Q, $Q = \begin{pmatrix} & & 1 \\ & 2\cdot\mathcal{Q} & \\ 1 & & \end{pmatrix}$ with $\mathcal{Q}$ anisotropic.

LEMMA 9.3 *If* $G = \mathrm{SL}_2(k_v)$*,*

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a \in k_v^\times, b \in k_v \right\}, \quad \textit{and}$$

$$\chi_s\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = |a|^s,$$

then for archimedean k_v*, we have*

$$M(s) = \begin{cases} \pi^{(1/2)} \dfrac{\Gamma(\frac{s}{2})}{\Gamma(\frac{s+1}{2})}, & \textit{if } k_v = \mathbb{R}, \\ 2\pi \dfrac{\Gamma(s)}{\Gamma(s+1)} = \dfrac{2\pi}{s}, & \textit{if } k_v = \mathbb{C}. \end{cases}$$

If k_v *is non-archimedean, then*

$$M(s) = \begin{cases} q_v^{-(c/2)} \dfrac{1 - q_v^{-s-1}}{1 - q_v^{-s}}, & \textit{if } K = \mathrm{SL}_2(\mathfrak{o}_v), \\ q_v^{-1} q_v^{-(c/2)} \dfrac{1 - q_v^{-s-1}}{1 - q_v^{-s}}, & \textit{if } K = \begin{pmatrix} \varpi_v^{-1} & \\ & 1 \end{pmatrix} \mathrm{SL}_2(\mathfrak{o}_v) \begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix}, \end{cases}$$

Here c is the conductoral exponent of ψ_v*.*

The proof of Lemma 9.3 is well-known.

LEMMA 9.4 *Assume* k_v *is non-archimedean. If* $G = \mathrm{SO}_Q$*,* $Q = \begin{pmatrix} & & 1 \\ & 2\cdot \mathcal{Q} & \\ 1 & & \end{pmatrix}$ *with* $\mathcal{Q}$ *anisotropic,*

$$P = \left\{ \begin{pmatrix} a & * & * \\ 0 & \alpha & * \\ 0 & 0 & a^{-1} \end{pmatrix} \in \mathrm{SO}_Q, a \in k_v^\times, \alpha \in \mathrm{SO}_{\mathcal{Q}} \right\},$$

$$\chi_s\left(\begin{pmatrix} a & * & * \\ 0 & \alpha & * \\ 0 & 0 & a^{-1} \end{pmatrix}\right) = |a|^s,$$

then $M(s)$ is equal to

$$M(s)=\begin{cases} |2|_v^{1/2}q_v^{-(c/2)}\dfrac{1-q_v^{-2s-1}}{1-q_v^{-2s}}, & \text{if } Q=1,\\ |\varepsilon|_v q_v^{-c}\dfrac{1-q_v^{-2s-2}}{1-q_v^{-2s}}, & \text{if } Q=Q_{2,F} \text{ and } F/k_v \text{ is unramified},\\ q_v^{-c-(1/2)}\dfrac{1-q_v^{-s-1}}{1-q_v^{-s}}, & \text{if } Q=Q_{2,F} \text{ and } F/k_v \text{ is ramified},\\ |2|_v^{1/2}q_v^{-(3c/2)-1} \\ \quad\times\dfrac{(1+q_v^{-s+(1/2)})(1-q_v^{-s-(3/2)})}{1-q_v^{-2s}}, & \text{if } Q=Q_3 \text{ and } K=K_v^{(1)},\\ |2|_v^{1/2}q_v^{s-(3c/2)-2} \\ \quad\times\dfrac{(1+q_v^{-s-(1/2)})(1-q_v^{-s-(3/2)})}{1-q_v^{-2s}}, & \text{if } Q=Q_3 \text{ and } K=K_v^{(2)},\\ q_v^{-2c-1}\dfrac{1-q_v^{-s-2}}{1-q_v^{-s}}, & \text{if } Q=Q_4. \end{cases}$$

Proof. We will give a proof only for the case $Q = Q_3$ and $K = K_v^{(1)}$. It is easy to see that

$$\mathrm{Vol}(\mathfrak{p}_{\mathbb{D}}^n \cap \mathbb{D}_0) = \begin{cases} |2|_v^{1/2}q_v^{-(3n/2)-(1/2)}, & \text{if } n \text{ iseven},\\ |2|_v^{1/2}q_v^{-(3n/2)}, & \text{if } n \text{ isodd}. \end{cases}$$

By definition,

$$M(s) = \mathrm{Vol}(\mathfrak{o}_{\mathbb{D}} \cap \mathbb{D}_0) + \sum_{n=1}^{\infty} \mathrm{Vol}((\mathfrak{p}_{\mathbb{D}}^{-n} \setminus \mathfrak{p}_{\mathbb{D}}^{-n+1}) \cap \mathbb{D}_0) q^{-n(s+(3/2))},$$

and it is easy to prove the lemma for this case. The proof for the remaining cases are similar.

Similarly, when k_v is archimedean, we have the following lemma.

LEMMA 9.5 *Let $k_v = \mathbb{R}$. If $G = \mathrm{SO}_Q$, $Q = \begin{pmatrix} & & 1\\ & \varepsilon\cdot\mathbf{1}_l & \\ 1 & & \end{pmatrix}$,*

$$P = \left\{ \begin{pmatrix} a & * & * \\ 0 & \alpha & * \\ 0 & 0 & a^{-1} \end{pmatrix} \in \mathrm{SO}_Q, a \in \mathbb{R}^\times, \alpha \in \mathrm{SO}_l \right\},$$

and

$$\chi_s\left(\begin{pmatrix} a & * & * \\ 0 & \alpha & * \\ 0 & 0 & a^{-1} \end{pmatrix}\right) = |a|^s,$$

then

$$\begin{aligned} M(s) &= \int_{\mathbb{R}^l} \left(1 + \frac{x_1^2}{2} + \cdots + \frac{x_l^2}{2}\right)^{-s-(l/2)} \mathrm{d}x_1 \cdots \mathrm{d}x_l \\ &= (2\pi)^{l/2} \frac{\Gamma(s)}{\Gamma\left(s + \frac{l}{2}\right)}. \end{aligned}$$

Let $k_v = \mathbb{C}$*. If* $G = \mathrm{SO}_Q$, $Q = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$,

$$P = \left\{ \begin{pmatrix} a & * & * \\ 0 & 1 & * \\ 0 & 0 & a^{-1} \end{pmatrix} \in \mathrm{SO}_Q, a \in \mathbb{C}^\times \right\},$$

and

$$\chi_s\left(\begin{pmatrix} a & * & * \\ 0 & 1 & * \\ 0 & 0 & a^{-1} \end{pmatrix}\right) = |a|^s,$$

then

$$\begin{aligned} M(s) &= \int_{\mathbb{C}} \left(1 + \frac{|z|^2}{2}\right)^{-2s-1} |\mathrm{d}z \wedge \mathrm{d}\bar{z}| \\ &= 4\pi \frac{\Gamma(2s)}{\Gamma(2s+1)} = \frac{2\pi}{s}. \end{aligned}$$

Here $|z| = \sqrt{z\bar{z}}$ *means the usual absolute value.*

Now we can calculate c_K. First of all, we treat the case when $\operatorname{rk} Q'$ is odd. Recall that in this case we have assumed that $\det Q' = (-1)^{(m'-1)/2} 2 \bmod (k_v^\times)^2$. Let $\mathfrak{S}_f^{(1)}$ (resp. $\mathfrak{S}_f^{(2)}$) be the set of finite places v where Q'_v is isomorphic to

$$\begin{pmatrix} & & & & & & 1 \\ & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & 1 & & \\ & & & 2\cdot Q_3 & & & \\ & & 1 & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & & \\ 1 & & & & & & \end{pmatrix},$$

and $K_v = K_v^{(1)}$ (resp. $K_v = K_v^{(2)}$). Let $\mathfrak{S}_\infty$ be the set of real places and l_v an integer such that Q'_v is isomorphic to

$$\begin{pmatrix} & & & & & & 1 \\ & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & 1 & & \\ & & & \varepsilon\cdot \mathbf{1}_{l_v} & & & \\ & & 1 & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & & \\ 1 & & & & & & \end{pmatrix},$$

$\varepsilon = \pm 1$. Then by a calculation as in [14], one can show

$$M_v(s) = |2|_v^r M_v^0(s) \times \begin{cases} q_v^{cr(2m'+r+1)/4} & v<\infty, v \notin \mathfrak{S}_f^{(1)} \cup \mathfrak{S}_f^{(2)}, \\ q_v^{cr(2m'+r+1)/4} q_v^{-r} \dfrac{\zeta_v(2s+r)}{\zeta_v(2s-r)} & v \in \mathfrak{S}_f^{(1)}, \\ q_v^{cr(2m'+r+1)/4} q_v^{-(m'+r)r/2} \dfrac{\zeta_v(s+\frac{r}{2})}{\zeta_v(s-\frac{r}{2})} & v \in \mathfrak{S}_f^{(2)}, \\ \displaystyle\prod_{j=1}^{r} \prod_{i=1}^{[\frac{l_v-1}{4}]} \frac{2s-\frac{r}{2}-l_v+2j+4i-1}{2s-r+l_v+2j-4i-1} & v \in \mathfrak{S}_\infty, \\ 1 & v : \text{complex}. \end{cases} \tag{9.2}$$

Here

$$M^0(s) = \prod_{i=1}^{r} \frac{\zeta_v(s-\frac{r}{2}-\frac{m'}{2}+i+\frac{1}{2})}{\zeta_v(s-\frac{r}{2}+\frac{m'}{2}+i-\frac{1}{2})} \prod_{i=1}^{[\frac{r+1}{2}]} \frac{\zeta_v(2s-r+2i-1)}{\zeta_v(2s+r-2i+2)}.$$

By (9.1) and (9.2), we get the following theorem.

THEOREM 9.6 *Let (Q', U') be a quadratic form of odd rank m', and $Q = Q' \oplus \mathcal{H}^r$. $\mathrm{rk}\, Q = m = m' + 2r$. We choose a maximal compact subgroup of $\mathrm{SO}_Q(\mathbb{A})$ as above. If $m' \geqslant 3$, then*

$$c_K = c_K^+ = \frac{\rho_k}{\xi_k(m')} \prod_{i=2}^{r} \frac{\xi_k(i)}{\xi_k(m'+i-1)} \prod_{i=1}^{[\frac{r+1}{2}]} \frac{\xi_k(m'+2i-2)}{\xi_k(2r+m'-2i+1)}$$
$$\times \prod_{v \in \mathfrak{S}_f^{(1)}} q_v^{-r} \frac{\zeta_v(2r+m'-1)}{\zeta_v(m'-1)}$$
$$\times \prod_{v \in \mathfrak{S}_f^{(2)}} q_v^{-\frac{(m'+r)r}{2}} \frac{\zeta_v(r+\frac{m'-1}{2})}{\zeta_v(\frac{m'-1}{2})}$$
$$\times \prod_{v \in \mathfrak{S}_\infty} \prod_{j=1}^{r} \prod_{i=1}^{[\frac{l_v-1}{4}]} \frac{m'-l_v+2j+4i-2}{m'+l_v+2j-4i-2}.$$

If $m' = 1$, then

$$c_K = \tfrac{1}{2} c_K^+ = \frac{1}{2} \frac{\rho_k}{\xi_k(2r)} \prod_{i=1}^{[\frac{r-1}{2}]} \frac{\xi_k(2i+1)}{\xi_k(2r-2i)}.$$

Next we treat the case when $m' = \mathrm{rk} Q'$ is even. Let $F = k(\sqrt{(-1)^{m/2} \det Q})$, and χ_Q be the character of $\mathbb{A}_k^\times / k^\times$ corresponding to F/k by class field theory. (When $F = k$, we put $\chi_Q = 1$.) Let $\mathfrak{S}_f^u$ be the set of finite places v where F_v/k_v is an unramified quadratic extension and Q'_v is isomorphic to

$$\begin{pmatrix} & & & & & & 1 \\ & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & 1 & & \\ & & & 2\varpi_v \cdot \mathcal{Q}_{2,F} & & & \\ & & 1 & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & & \\ 1 & & & & & & \end{pmatrix},$$

Here ϖ_v is a prime element of k_v. Let $\mathfrak{S}_f^r$ be the set of finite places v where F_v/k_v is a ramified quadratic extension. Let $\mathfrak{S}_f^q$ be the set of finite places v where F/k is split and Q'_v is isomorphic to

$$\begin{pmatrix} & & & & & & 1 \\ & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & 1 & & \\ & & & 2\cdot Q_4 & & & \\ & & 1 & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & & \\ 1 & & & & & & \end{pmatrix},$$

Let $\mathfrak{S}_\infty^+$ (resp. $\mathfrak{S}_\infty^-$) be the set of real places v where Q'_v is isomorphic to

$$\begin{pmatrix} & & & & & & 1 \\ & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & 1 & & \\ & & & \varepsilon\cdot\mathbf{1}_{l_v} & & & \\ & & 1 & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & & \\ 1 & & & & & & \end{pmatrix},$$

$\varepsilon=\pm 1$, and $l_v \equiv 0 \bmod 4$ (resp. $l_v \equiv 2 \bmod 4$). Then

$$M_v(s)=M_v^0(s)\times\begin{cases} q_v^{cr(2m'+r+1)/4} & v<\infty,\\ & v\notin \mathfrak{S}_f^u\cup\mathfrak{S}_f^r\cup\mathfrak{S}_f^q,\\ q_v^{cr(2m'+r+1)/4}q_v^{-((m'+r-1)r/2)} & v\in\mathfrak{S}_f^u,\\ q_v^{cr(2m'+r+1)/4}q_v^{-(r/2)} & v\in\mathfrak{S}_f^r,\\ q_v^{cr(2m'+r+1)/4}q_v^{-r}\dfrac{\zeta_v(s+\frac{r-1}{2})\zeta_v(s+\frac{r+1}{2})}{\zeta_v(s-\frac{r-1}{2})\zeta_v(s-\frac{r+1}{2})} & v\in\mathfrak{S}_f^q,\\ \displaystyle\prod_{j=1}^{r}\prod_{i=1}^{\frac{l_v}{4}}\frac{2s-r+2j-4i+1}{2s-r+2j+4i-3} & v\in\mathfrak{S}_\infty^+,\\ \displaystyle\prod_{j=1}^{r}\prod_{i=1}^{\frac{l_v-2}{4}}\frac{2s-r+2j-4i-1}{2s-r+2j+4i-1} & v\in\mathfrak{S}_\infty^-,\\ 1 & v\ :\ \text{complex}.\end{cases} \tag{9.3}$$

Here

$$M_v^0(s) = \frac{L_v(s-\frac{r-1}{2},\chi_Q)}{L_v(s+\frac{r+1}{2},\chi_Q)} \prod_{i=1}^{r} \frac{\zeta_v(s-\frac{r}{2}-\frac{m'}{2}+i+\frac{1}{2})}{\zeta_v(s-\frac{r}{2}+\frac{m'}{2}+i-\frac{1}{2})} \times \prod_{i=1}^{[\frac{r}{2}]} \frac{\zeta_v(2s-r+2i)}{\zeta_v(2s+r+1-2i)}.$$

By (9.1) and (9.3), we get the following theorem.

THEOREM 9.7 *Let (Q',U') be a quadratic form of even* rank m', *and $Q=Q'\oplus\mathcal{H}^r$, $Q'\not\simeq\mathcal{H}$.* rk $Q=m=m'+2r$. *Put $\chi_Q(x)=\langle(-1)^{m/2}\det Q,x\rangle$ for $x\in\mathbb{A}^\times$. Let $\mathfrak{f}$ be the conductor of χ_Q. We choose a maximal compact subgroup of* $\mathrm{SO}_Q(\mathbb{A})$ *as above. If $m'\geqslant 2$, then*

$$c_K=c_K^+ = |\mathfrak{f}D_k|^{-\frac{r}{2}} \frac{\rho_k}{\xi_k(m')} \frac{L(\frac{m'}{2},\chi_Q)}{L(r+\frac{m'}{2},\chi_Q)} \prod_{i=2}^{r} \frac{\xi_k(i)}{\xi_k(m'+i-1)}$$

$$\times \prod_{i=1}^{[\frac{r}{2}]} \frac{\xi_k(m'+2i-1)}{\xi_k(2r+m'-2i)}$$

$$\times \prod_{v\in\mathfrak{S}_f^u} q_v^{-((m'+r-1)r/2)}$$

$$\times \prod_{v\in\mathfrak{S}_f^q} q_v^{-r} \frac{\zeta_v(r+\frac{m'}{2}-1)\zeta_v(r+\frac{m'}{2})}{\zeta_v(\frac{m'}{2}-1)\zeta_v(\frac{m'}{2})}$$

$$\times \prod_{v\in\mathfrak{S}_\infty^+} \prod_{j=1}^{r} \prod_{i=1}^{\frac{l_v}{4}} \frac{m'+2j-4i}{m'+2j+4i-4}$$

$$\times \prod_{v\in\mathfrak{S}_\infty^-} \prod_{j=1}^{r} \prod_{i=1}^{\frac{l_v-2}{4}} \frac{m'+2j-4i-2}{m'+2j+4i-2}.$$

If $m'=0$, then

$$c_K=c_K^+ = \frac{\rho_k}{\xi_k(r)} \prod_{i=1}^{[\frac{r-1}{2}]} \frac{\xi_k(2i+1)}{\xi_k(2r-2i)}.$$

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