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Separation, factorization and finite sheaves on Nash manifolds

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0. Introduction

Nash functions are those real analytic functions which are algebraic over the polynomials. So it is natural to think that they can separate the analytic components of real algebraic varieties. A first step in this direction was the separation theorem of Mostowski [Mo], which implies that Nash functions separate the connected components of real algebraic varieties.

We must be a little careful in the formulation of the separation problem, since real algebraic varieties may have analytic components which are not given by global analytic equations. We will state a general algebraic form of the separation problem, which amounts to consider the analytic components of the germ at the real part of a complexification. Let $M \subset \mathbf{R}^m$ be a Nash manifold, $\mathcal{N}(M)$ the ring of Nash functions on M and $\mathcal{O}(M)$ the ring of analytic functions on M.

Separation problem. *Let* $\mathfrak{p}$ *be a prime ideal of* $\mathcal{N}(M)$. *Is* $\mathfrak{p}\mathcal{O}(M)$ *a prime ideal ?*

We will denote by $\mathbf{Sep}(M)$ the property that the separation problem has a positive answer for any prime ideal $\mathfrak{p}$ of $\mathcal{N}(M)$. We will use the notation $\mathbf{Sep}_1(M)$ when we consider only prime ideals $\mathfrak{p}$ of height 1.

The separation problem for height one prime ideals is related to a problem about factorization of Nash functions.

Factorization problem. *Given a Nash function* f *on* M *and an analytic factorization* $f = f_1 f_2$, *do there exist Nash functions* g_1 *and* g_2 *on* M *and positive analytic functions* φ_1 *and* φ_2 *such that* $\varphi_1\varphi_2 = 1$, $f_1 = \varphi_1 g_1$ *and* $f_2 = \varphi_2 g_2$*?*

We will denote by $\mathbf{Fact}(M)$ the property that the factorization problem has a positive answer on M.

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The local separation problem has an affirmative answer. Let $\mathcal{N}_x$ (resp. $\mathcal{O}_x$) be the ring of germs of Nash (resp. analytic) functions at a point x of M. If $\mathfrak{p}$ is a prime ideal of $\mathcal{N}_x$, then $\mathfrak{p}\mathcal{O}_x$ is prime. This is an easy consequence of Artin's approximation theorem [Ar]. Unfortunately, there are difficulties to pass from local to global results, due to the bad cohomological properties of Nash functions. If $\mathcal{N}_\mathbf{R}$ is the sheaf of Nash functions on $\mathbf{R}$, then $H^1(\mathbf{R},\mathcal{N}_\mathbf{R}) \neq 0$ [Hd]. Despite of this lack of a good cohomology theory, some properties of Nash sheaves, related to the separation problem, are expected.

Let $\mathcal{N}$ be the sheaf of Nash functions on M (we write $\mathcal{N}_M$ if we need to emphasize M), and let $\mathcal{O}$ (or $\mathcal{O}_M$) denote the sheaf of analytic functions on M. We know that $\mathcal{N}$ is coherent as a sheaf of $\mathcal{N}$-modules [Sh2, I.6.6]. Hence, if a sheaf of $\mathcal{N}$-ideals $\mathcal{I}$ is locally generated by Nash functions, then $\mathcal{I}$ is coherent. Coherent sheaves of $\mathcal{N}$-ideals seem to be interesting, but the concept of a coherent sheaf of $\mathcal{N}$-ideals is too wide for real algebraic geometry. For example, consider $M = \mathbf{R}$ and $\mathfrak{I}(\mathbf{Z})$ the sheaf of germs of Nash functions vanishing on $\mathbf{Z}$. Then $\mathfrak{I}(\mathbf{Z})$ is not generated by its global sections (the constant zero is the only global section), and the global section of the quotient sheaf $\mathcal{N}/\mathfrak{I}(\mathbf{Z})$ which is 0 for even integers and 1 for odd ones does not lift to a global Nash function. Clearly, the sheaf $\mathfrak{I}(\mathbf{Z})$ lacks some finiteness property. We call a sheaf of ideals $\mathcal{I}$ of $\mathcal{N}$ *finite* if there exists a finite open semialgebraic covering $\{U_i\}$ of M such that for each i, $\mathcal{I}|_{U_i}$ is generated by Nash functions on U_i. Note that any finite sheaf of ideals of $\mathcal{N}$ is coherent, and that the two notions coincide when M is compact. Here are the main problems about finite sheaves of ideals of $\mathcal{N}$, which would play the role of Cartan's Theorems A and B to construct a good sheaf theory on Nash manifolds.

Global equations problem. *Is every finite sheaf $\mathcal{I}$ of ideals of $\mathcal{N}$ generated by global Nash functions?*

Extension problem. *For the same $\mathcal{I}$ as above, is the natural homomorphism*

$$H^0(M,\mathcal{N}) \to H^0(M,\mathcal{N}/\mathcal{I})$$

surjective?

We will denote by **Glob**(M) the property that the global equations problem has a positive answer for any finite sheaf $\mathcal{I}$ of ideals of $\mathcal{N}_M$. We will use the notation $\mathbf{Glob}_1(M)$ when we consider only *locally principal* finite sheaves $\mathcal{I}$ of ideals of $\mathcal{N}_M$, i.e. those for which every stalk $\mathcal{I}_x$ is principal. We will use the notation $\mathbf{Glob}^r(M)$ when we consider only finite sheaves of *radical* ideals of $\mathcal{N}_M$. Of course, we will use $\mathbf{Glob}_1^r(M)$ when we have both restrictions. For the extension problem we use the corresponding notations **Ext**(M), $\mathbf{Ext}_1(M)$ $\mathbf{Ext}^r(M)$ and $\mathbf{Ext}_1^r(M)$.

In the last 20 years, several partial results concerning the problems we have stated were obtained: see [Ef1, Ef2, Sh2, Sh3, MoRa, Pe, TaTo]. Recently, all the problems were given positive answers in the case of a *compact* Nash manifold [CoRzSh]. The key point is to use Artin's conjecture, which says that a regular morphism between noetherian rings is an inductive limit of smooth finite type algebras. This result is applied to the morphism $\mathcal{N}(M) \to \mathcal{O}(M)$. This does not apply to the *non compact* case, since $\mathcal{O}(M)$ is no more noetherian. For the moment, we are not able to give a positive answer to these problems in the general non compact case.

The aim of this paper is to prove, in the non compact case, the equivalences of all these problems. Precisely, we have:

THEOREM 0.1. *For any Nash manifold M, the properties* $\mathbf{Sep}(M)$, $\mathbf{Glob}^r(M)$, $\mathbf{Glob}(M)$, $\mathbf{Ext}^r(M)$, $\mathbf{Ext}(M)$ *are all equivalent.*

THEOREM 0.2. *For any Nash manifold M, the properties* $\mathbf{Fact}(M)$, $\mathbf{Sep}_1(M)$, $\mathbf{Glob}_1^r(M)$, $\mathbf{Glob}_1(M)$, $\mathbf{Ext}_1^r(M)$, $\mathbf{Ext}_1(M)$ *are all equivalent.*

Several partial results in the direction of this equivalence appear in [Sh2]; this kind of problem is also considered in [BaTo]. See also [RzSh] which relates the separation problem with the problem whether a semialgebraic subset of M described by global analytic inequalities may be described by the same number of Nash inequalities.

Recently, R. Quarez [Qu] has shown that if a property known as the 'idempotency of the real spectrum' holds, then it is possible to apply Artin's conjecture in the non compact case to the morphism $\mathcal{N}(M)/I \to H^0(M, \mathcal{N}/I\mathcal{N})$ where I is a radical ideal of $\mathcal{N}(M)$, getting a positive answer to the extension problem in this particular case. The results of our paper show that this particular case imply all properties of Theorem 0.1. Unfortunately, it appeared that there is up to now no valid proof of this idempotency: see [Qu] for a clarification of the situation and a proof under an assumption of normality.

The first section of the paper contains preliminary results about finite sheaves of ideals of $\mathcal{N}$, and their zero-sets considered as complex germs. The second and third sections are devoted respectively to the proofs of Theorems 0.1 and 0.2. The fourth section contains some new partial positive answers to our problems.

1. Finite sheaves of $\mathcal{N}$-ideals and their zero-sets

We want to associate to a finite ideal sheaf of $\mathcal{N}$ a zero-set which will be some germ of complex analytic set at M, in a complexification of M, and characterize these zero-sets.

A complexification of M may be constructed in the following way. Up to a Nash diffeomorphism, we may suppose that M is a connected component of a non singular real algebraic set $V \subset \mathbf{R}^m$ ([Sh2, I.5.3], [BoCoRo, 8.4.6]). Let $W^{\mathbf{C}}$ be the Zariski closure of M in $\mathbf{C}^m$, and take for $M^{\mathbf{C}}$ some open semialgebraic subset of the regular points of $W^{\mathbf{C}}$ (we consider $\mathbf{C}^m = \mathbf{R}^{2m}$ to define semialgebraic subsets of $\mathbf{C}^m$), invariant under conjugation, and such that $M^{\mathbf{C}} \cap \mathbf{R}^m = M$. This $M^{\mathbf{C}}$ is a complex analytic manifold, and its dimension as complex analytic manifold is the same as the dimension of M.

We will consider germs at M of complex analytic sets in $M^{\mathbf{C}}$ (see for instance [WhBr]), which we will call *M-germs* for short. We will mainly consider such germs which are invariant under complex conjugation, which we will call *invariant M-germs*. We will denote by ε the complex conjugation. An *irreducible invariant M-germ X* is an M-germ such that there exists an irreducible M-germ Y with $X = Y \cup \varepsilon(Y)$. If X is an irreducible invariant M-germ of dimension p, and if Y is an invariant M-germ contained in X, then either $Y = X$ or $\dim Y < p$ (cf. [WhBr, Prop. 5]). If X is an invariant M-germ, there exists a unique locally finite family (X_i) of irreducible invariant M-germs such that $X = \bigcup X_i$ and $X_i \not\subset X_j$ for $i \neq j$; the X_i are the *irreducible invariant components* of X (cf. [WhBr, Prop. 9]). We call an invariant M-germ X *finite* if it has a finite number of irreducible invariant components, and *locally semialgebraic* if the germ X_x at each point $x \in M$ is semialgebraic.

Let f be a Nash function on M. Then f extends uniquely to a semialgebraic open neighborhood U of M in $M^{\mathbf{C}}$. Denote by $\mathfrak{X}(f)$ the invariant M-germ of the zero set of f in U at M. We define in the same way $\mathfrak{X}(f_1, \dots, f_p)$ for a finite number of elements of $\mathcal{N}(M)$; this invariant M-germ depends only on the ideal $I = (f_1, \dots, f_p)$ in $\mathcal{N}(M)$, and we denote it by $\mathfrak{X}(I)$. We will call **C**-*Nash M-germ* an invariant M-germ of the form $\mathfrak{X}(I)$ for an ideal I of $\mathcal{N}(M)$, i.e. an invariant M-germ defined by global Nash equations.

Now let $\mathcal{I}$ be a finite sheaf of ideals of $\mathcal{N}$. There is a finite semialgebraic covering $M = \bigcup_{i=1}^{k} U_i$, and for each i an ideal I_i of $\mathcal{N}(U_i)$ such that $\mathcal{I}|_{U_i}$ is generated by I_i. The germs $\mathfrak{X}(I_i)$ at U_i and $\mathfrak{X}(I_j)$ at U_j coincide along $U_i \cap U_j$, and hence these germs may be glued together to give an invariant M-germ $\mathfrak{X}(\mathcal{I})$. Actually, there is a semialgebraic neighborhood U of M in $M^{\mathbf{C}}$, and an invariant semialgebraic complex analytic set X^* in U, whose germ at M is $\mathfrak{X}(\mathcal{I})$. We call X^* a *semialgebraic realization* of the M-germ $\mathfrak{X}(\mathcal{I})$ (in U). An invariant M-germ having such a semialgebraic realization will be called an *invariant semialgebraic M-germ*. We can also define $\mathfrak{X}(\mathcal{I})$ if $\mathcal{I}$ is a coherent sheaf of ideals of $\mathcal{N}$; this M-germ is *locally semialgebraic*, but in this case there may be no semialgebraic realization (remember **Z** in **R**).

We will have also to consider the zero sets not only as M-germs, but also as plain subsets of M. So we have to introduce another notation for these. If I is an ideal of $\mathcal{N}(M)$, we denote by $I^{-1}(0)$ the set of those $x \in M$ such that $f(x) = 0$ for all $f \in I$. We call $I^{-1}(0)$ a *Nash set*. Correspondingly, if $\mathcal{I}$ is a coherent sheaf of ideals of $\mathcal{N}$, we denote by $\mathcal{I}^{-1}(0)$ the set of those $x \in M$ such that $\mathcal{I}_x$ is different from $\mathcal{N}_x$. We have of course $\mathcal{I}^{-1}(0) = \mathfrak{X}(\mathcal{I}) \cap M$.

A Nash function $f \in \mathcal{N}(M)$ vanishes on an M-germ X if $X \subset \mathfrak{X}(f)$, and we define $\mathfrak{I}(X)$ to be the sheaf of germs of Nash functions vanishing on X. It is clearly a sheaf of radical ideals of $\mathcal{N}$.

LEMMA 1.1 *Let $\mathcal{I}$ be a finite sheaf of ideals of $\mathcal{N}$. Then $\sqrt{\mathcal{I}}$ is a finite sheaf of ideals of $\mathcal{N}$, and $\mathfrak{I}(\mathfrak{X}(\mathcal{I})) = \sqrt{\mathcal{I}}$.*

If X is a **C***-Nash M-germ, then $\mathfrak{I}(X)$ is generated by its global sections.*

Proof. There is a finite covering of M by semialgebraic open subsets U_i such that $\mathcal{I}|_{U_i}$ is generated by an ideal I_i of $\mathcal{N}(U_i)$. Then, since $\mathcal{N}_x$ is ind-étale over $\mathcal{N}(U_i)$ for any x in U_i, we know that $\sqrt{I_i}\mathcal{N}_x = \sqrt{I_i\mathcal{N}_x} = \sqrt{\mathcal{I}_x}$. So $\sqrt{I_i}$ generates $\sqrt{\mathcal{I}}|_{U_i}$. This proves that $\sqrt{\mathcal{I}}$ is finite. Using Rückert's Nullstellensatz [GuRo], and the fact that $\mathcal{O}_x$ is faithfully flat over $\mathcal{N}_x$ for any $x \in M$, we get that $\mathfrak{I}(\mathfrak{X}(\mathcal{I})) = \sqrt{\mathcal{I}}$.

If $X = \mathfrak{X}(I)$ where I is an ideal of $\mathcal{N}(M)$, then by the preceding arguments $\mathfrak{I}(X) = \sqrt{I\mathcal{N}}$, which shows the last part of the Lemma. □

LEMMA 1.2 *Let $\mathcal{I}$ be a finite sheaf of ideals of $\mathcal{N}$. Then $\mathfrak{X}(\mathcal{I})$ has a finite number of irreducible invariant components, which have semialgebraic realizations.*

Proof. Let X^* be a semialgebraic realization of $\mathfrak{X}(\mathcal{I})$. Let U be a sufficiently small open invariant semialgebraic neighborhood of M in $M^{\mathbf{C}}$. Let $\{X'_j\}$ denote the family of germs at M of the closures of the connected components of $\operatorname{Reg} X^* \cap U$, where $\operatorname{Reg} X^*$ denotes the regular point set of the complex analytic set X^*. Note that $\operatorname{Reg} X^*$ is semialgebraic. Then $\{X_i\} = \{X'_j \cup \varepsilon(X'_j)\}$ is the family of irreducible invariant components of $\mathfrak{X}(\mathcal{I})$. This assertion proves the Lemma, and now we are going to prove the assertion.

We obtain U in the following way. Let K be a finite simplicial complex and π a semialgebraic homeomorphism from a union Q of some open simplices of K to $M^{\mathbf{C}}$ such that $\pi^{-1}(M)$ and $\pi^{-1}(\operatorname{Reg} X^*)$ are unions of some open simplices of K. By subdividing K, we may assume that if the vertices of a closed simplex $\sigma \in K$ are all contained in the adherence of $\pi^{-1}(M)$ then so is σ. Let V denote the union of the open simplices σ of K such that $\sigma \subset Q$ and $\overline{\sigma} \cap \pi^{-1}(M) \neq \emptyset$, and set

$$Z = V \cap \pi^{-1}(\operatorname{Reg} X^*) \quad \text{and} \quad U = \pi(V).$$

Then U is an open semialgebraic neighborhood of M in $M^{\mathbf{C}}$, and $U \cap \overline{\pi(Z)} = U \cap X^*$ is a semialgebraic complex analytic set in U. Let $Z_1, \ldots, Z_k$ denote the connected components of Z, and set

$$X'^*_j = U \cap \overline{\pi(Z_j)}, \quad j = 1, \ldots, k.$$

Then $U \cap X^* = \bigcup_{j=1}^k X'^*_j$ and each X'^*_j is a semialgebraic complex analytic set in U. It remains to prove that the germ X'_j of X'^*_j at M is irreducible as an M-germ.

Assume the M-germ X'_j were not irreducible. Then there would be an open neighborhood $U' \subset U$ of M in $M^{\mathbf{C}}$ and at least two connected components Y_1 and Y_2 of $U' \cap \operatorname{Reg} X'^*_j$ such that $\overline{Y_i} \cap M \neq \emptyset$, $i = 1, 2$. In other words, there would be no path in $U' \cap \operatorname{Reg} X'^*_j$ joining a point of Y_1 with a point of Y_2. We can assume that $U' \cap \pi(\sigma)$ is connected for every open simplex σ. Since Z_j is connected, we have a path $\varphi \subset \operatorname{Reg} X'^*_j$ joining a point $y_1 \in Y_1$ with a point $y_2 \in Y_2$. The path φ

decomposes into finitely many others $\varphi_1, \dots, \varphi_s$ such that for some open simplices $\sigma_1, \dots, \sigma_s$ we have $\varphi_i \subset \pi(\sigma_i) \subset \operatorname{Reg} X_j'^*$. Now, since every $\pi(\sigma_i)$ is adherent to M, we can deform the path φ_i into a path

$$\varphi_i' \subset U' \cap \pi(\sigma_i) \subset U' \cap \operatorname{Reg} X_j'^*.$$

Clearly, this can be done so that $\varphi_1', \dots, \varphi_s'$ build up a new path $\varphi' \subset U' \cap \operatorname{Reg} X_j'^*$ still joining y_1 and y_2, which is a contradiction. □

We record here a result using the proof of the preceding Lemma that will be useful in the proof of Proposition 3.4.

LEMMA 1.3 *Let $\mathcal{I}$ be a finite sheaf of radical ideals of $\mathcal{N}$ such that $\mathfrak{X}(\mathcal{I})$ is an irreducible invariant M-germ. Let (Ω_j) be a finite covering of M by semialgebraic open sets, such that for each j the restricted sheaf $\mathcal{I}|_{\Omega_j}$ is generated by an ideal I_j of $\mathcal{N}(\Omega_j)$. Let $\mathfrak{p}_j \subset \mathcal{N}(\Omega_j)$ and $\mathfrak{p}_{j'} \subset \mathcal{N}(\Omega_{j'})$ be minimal prime divisors of I_j and $I_{j'}$ respectively. Then there is a finite sequence $\mathfrak{p}_j = \mathfrak{p}_{j_0}, \mathfrak{p}_{j_1}, \dots, \mathfrak{p}_{j_s} = \mathfrak{p}_{j'}$, where each $\mathfrak{p}_{j_r}$ is a minimal prime divisor of I_{j_r}, such that, for $r = 1, \dots, s$, the extensions of the ideals $\mathfrak{p}_{j_{r-1}}$ and $\mathfrak{p}_{j_r}$ to $\mathcal{N}(\Omega_{j_{r-1}} \cap \Omega_{j_r})$ have a common minimal prime divisor.*

Proof. We perform the construction of the proof of Lemma 1.2, from which we borrow the notations. We can ask moreover that the triangulation K is compatible with the covering Ω_j. Let V_j denote the union of open simplices σ of K such that $\sigma \subset Q$ and $\overline{\sigma} \cap \pi^{-1}(\Omega_j) \neq \emptyset$, and let $Z_j = V_j \cap \pi^{-1}(\operatorname{Reg} X^*)$. These Z_j's cover Z. If σ is a simplex contained in Z_j, then the germ of $\pi(\sigma)$ at Ω_j is contained in the Ω_j-germ of zeroes of a unique minimal prime divisor of I_j, which we denote by $\mathfrak{p}_j(\sigma)$. Also, if σ is contained in $V_j \cap V_{j'}$, then the germ of $\pi(\sigma)$ at $\Omega_j \cap \Omega_{j'}$ is contained in the $\Omega_j \cap \Omega_{j'}$-germ of zeroes of a unique minimal prime divisor $\mathfrak{p}_{j,j'}(\sigma)$ of the extension to $\mathcal{N}(\Omega_j \cap \Omega_{j'})$ of I_j (which is the same as the extension of $I_{j'}$); moreover $\mathfrak{p}_{j,j'}(\sigma)$ is a common minimal prime divisor of the extensions of $\mathfrak{p}_j(\sigma)$ and $\mathfrak{p}_{j'}(\sigma)$. Now choose two simplices $\sigma \subset Z_j$ and $\sigma' \subset Z_{j'}$ such that $\mathfrak{p}_j(\sigma) = \mathfrak{p}_j$ and $\mathfrak{p}_{j'}(\sigma') = \mathfrak{p}_{j'}$, and which are contained in the same connected component of Z; this is possible because, by the irreducibility of $\mathfrak{X}(\mathcal{I})$, Z is either connected or the union of two connected components exchanged by conjugation. Then there is a piecewise linear path joining σ to σ' inside Z and going successively through the simplices $\sigma = \sigma_0, \sigma_1, \dots, \sigma_s = \sigma'$. Choose Ω_{j_r} such that σ_r is contained in Z_{j_r}, and set $\mathfrak{p}_{j_r} = \mathfrak{p}_{j_r}(\sigma_r)$. Now it is enough to understand that since σ_{r-1} is a face of σ_r or vice-versa, we have $\mathfrak{p}_{j_{r-1},j_r}(\sigma_{r-1}) = \mathfrak{p}_{j_{r-1},j_r}(\sigma_r)$. □

LEMMA 1.4 *Let X be an irreducible invariant M-germ, which is locally semialgebraic. Then there is an ideal I of $\mathcal{N}(M)$ such that X is an irreducible invariant component of $\mathfrak{X}(I)$, and* $\dim X = \dim \mathfrak{X}(I)$. *Hence X has a semialgebraic realization.*

Proof. We can suppose that M is a connected component of a non singular real algebraic set $W \subset \mathbf{R}^m$, and that $M^{\mathbf{C}}$ is an open semialgebraic set of the Zariski closure $W^{\mathbf{C}}$ of W in $\mathbf{C}^m$. Let X^* be a locally semialgebraic complex analytic set in an open neighborhood of M in $M^{\mathbf{C}}$ whose germ at M is X. We can assume that $\operatorname{Reg} X^*$ is either connected or the union of two connected components exchanged by the conjugation ε. Let Z^* denote the Zariski closure of $\operatorname{Reg} X^*$ in $\mathbf{C}^m$; it is an algebraic set, defined and irreducible over $\mathbf{R}$, contained in W^C. We first prove that the dimension of Z^* is equal to $\dim X^* = m'$ (these dimensions are complex dimensions).

Suppose that $\dim Z^* = d > m'$. Let $0 \in \operatorname{Reg} X^*$. After a linear change of coordinates, the restriction $q|_{Z^*}$ of the projection

$$q: \mathbf{C}^m \ni (z_1, \dots, z_m) \mapsto (z_1, \dots, z_{m'+1}) \in \mathbf{C}^{m'+1}$$

is surjective. Indeed, after a linear change the projection

$$Z^* \ni (z_1, \dots, z_m) \mapsto (z_1, \dots, z_d) \in \mathbf{C}^d$$

is finite, and consequently surjective. Since $q|_{Z^*}$ is the composition of this surjection with

$$\mathbf{C}^d \ni (z_1, \dots, z_d) \mapsto (z_1, \dots, z_{m'+1}) \in \mathbf{C}^{m'+1},$$

it is surjective. Moreover, the linear change can be chosen to have a small open semialgebraic neighborhood U of 0 in $\mathbf{C}^m$ such that $q|_{X^* \cap U}: X^* \cap U \to q(U)$ is proper, and moreover $X^* \cap U$ is semialgebraic. Then $q(X^* \cap U)$ is a semialgebraic complex analytic set in $q(U)$ of dimension m'. We will prove in the following Lemma 1.5 that the Zariski closure of $q(X^* \cap U)$ in $\mathbf{C}^{m'+1}$ has dimension m'. Then we have a nonzero complex polynomial function Q on $\mathbf{C}^{m'+1}$ vanishing on $q(X^* \cap U)$. Clearly $P = Q \circ q$ vanishes on $X^* \cap U$ and, hence, on $\operatorname{Reg} X^*$ and also on Z^*. Since $q(Z^*) = \mathbf{C}^{m'+1}$, then Q must be zero: here is the contradiction.

Let Z be the invariant M-germ of Z^*, and let $I \subset \mathcal{N}(M)$ be the ideal of Nash functions vanishing on Z. It contains the restrictions to M of the real polynomial equations of Z^*, and hence $Z = \mathfrak{X}(I)$. Let $\{Z_j\}$ be the finite family of irreducible invariant components of Z. Since X is irreducible invariant, contained in Z and of the same dimension as Z, it must be equal to one of the Z_j's. □

Finally, we prove the Lemma quoted above. Although this is not new ([An], [Kn], [FoŁoRa]), we include the (easy) proof for the convenience of the reader.

LEMMA 1.5 *Let Y be a semialgebraic complex analytic subset, of complex dimension m', of an open subset of $\mathbf{C}^{m'+1}$. Then the dimension of the Zariski closure of Y in $\mathbf{C}^{m'+1}$ is m'.*

Proof. The set $\operatorname{Reg} Y$ is semialgebraic, has a finite number of connected components which are semialgebraic, and its adherence is Y. Hence we can retreat to the case where Y is smooth and connected, and from that to the case where Y is the graph of a complex analytic function $\varphi : W \to \mathbf{C}$ where W is an open neighborhood of 0 in $\mathbf{C}^{m'}$, and moreover Y is semialgebraic.

Writing $z = (z_1, \ldots, z_{m'}) = (x_1 + iy_1, \ldots, x_{m'} + iy_{m'})$, we may describe φ by a pair of real functions

$$\varphi(z) = \varphi_1(x_1, y_1, \ldots, x_{m'}, y_{m'}) + i\varphi_2(x_1, y_1, \ldots, x_{m'}, y_{m'}).$$

Then φ_1 and φ_2 are Nash functions on W. Set $W^r = W \cap (\mathbf{R} \times \{0\} \times \mathbf{R} \times \{0\} \times \cdots)$, i.e. W^r is the real part of W. Clearly $\varphi_i|_{W^r}$, $i = 1, 2$, are Nash functions. Let W' be a small open semialgebraic neighborhood of W^r in $\mathbf{C}^{m'}$. Then we have complexifications $(\varphi_i|_{W^r})^{\mathbf{C}}$ of $\varphi_i|_{W^r}$ on W', which are the restrictions to W' of branches of algebraic functions. Since

$$\varphi|_{W'} = (\varphi_1|_{W^r})^{\mathbf{C}} + i(\varphi_2|_{W^r})^{\mathbf{C}},$$

$\varphi|_{W'}$ is the restriction to W' of one branch of an algebraic function, and hence the dimension of the Zariski closure of the graph of φ is m'. □

Let us sum up what we know: if $\mathcal{I}$ is a finite ideal sheaf of $\mathcal{N}$, then $\mathfrak{X}(\mathcal{I})$ is a finite semialgebraic invariant M-germ; if X is an irreducible locally semialgebraic invariant M-germ, then it is an irreducible invariant component of $\mathfrak{X}(I)$ for some ideal I of $\mathcal{N}(M)$ and it is semialgebraic. What we have to show now is clear: that an irreducible invariant component of $\mathfrak{X}(I)$ is of the form $\mathfrak{X}(\mathcal{I})$ for some finite ideal sheaf $\mathcal{I}$ of $\mathcal{N}$. The finiteness is the point here, since we do not suppose M compact. The following Proposition 1.7 will be the tool to get finiteness. It is very close in spirit to [EkTn, Prop. 5.2.2], and we will follow the lines of the proof of this proposition. Since the preprint [EkTn] has not yet been published, we will repeat for the convenience of the reader some of its arguments. The first proof of this result was given to us by R. Huber, using his work on isoalgebraic functions [Hr].

The formulation and the proof of 1.7 make use of the real spectrum; this seems difficult to avoid. We recall a few facts about the real spectrum, which will be useful in the sequel. We associate to M a space $\widetilde{M}$, the real spectrum of $\mathcal{N}(M)$, which is a compactification of M. A point α of $\widetilde{M}$ is identified with a ultrafilter of semialgebraic subsets of M, which we denote by $\widehat{\alpha}$. We may consider M as a subset of $\widetilde{M}$, identifying a point $x \in M$ with the principal ultrafilter of semialgebraic subsets of M generated by x. To a semialgebraic subset S of M is associated the subset $\widetilde{S} \subset \widetilde{M}$ of those α's such that $S \in \widehat{\alpha}$. A point α of the real spectrum defines a prime ideal $\mathfrak{p}$ of $\mathcal{N}(M)$ whose elements are the Nash functions $f \in \mathcal{N}(M)$ such that $\{x \in M;\ f(x) = 0\} \in \widehat{\alpha}$, and an ordering of the quotient field of $\mathcal{N}(M)/\mathfrak{p}$ such that the image of $f \in \mathcal{N}(M)$ is strictly positive if and only if $\{x \in M;\ f(x) > 0\} \in \widehat{\alpha}$; these two data determine α. We denote by $\kappa(\alpha)$ the real closure of this ordered field, and by supp(α) (the *support* of α) the zero set of $\mathfrak{p}$ in M. The topology on $\widetilde{M}$ is generated by the $\widetilde{U}$'s, for U open semialgebraic subset of M. It induces the usual topology on M. With this topology, $\widetilde{M}$ is compact (not Hausdorff).

Let d be the dimension of the support of α. We can choose a Nash chart with domain a semialgebraic open subset U of M, and Nash coordinates in this chart $(x_1, \ldots, x_d, t_1, \ldots, t_e)$ which are restrictions of global Nash functions on M, such that $\alpha \in \widetilde{U}$, supp$(\alpha) \cap U$ consists of non singular points of supp(α) and is given by $t_1 = \cdots = t_e = 0$. The ultrafilter $\widehat{\alpha}$ is generated by the semialgebraic open subsets S of supp$(\alpha) \cap U$ such that $\alpha \in \widetilde{S}$ [BoCoRo, 9.6.10]. The field $\kappa(\alpha)$ is canonically isomorphic (as $\mathcal{N}(M)$-algebra) to the inductive limit of the rings of Nash functions $\mathcal{N}_{\text{supp}(\alpha)\cap U}(S)$ on such S.

We can define the ring $\mathcal{N}_{M,\alpha}$ of germs of Nash functions on M at α: it is the inductive limit of the rings $\mathcal{N}_M(\Omega)$ for Ω semialgebraic open subset of M such that $\alpha \in \widetilde{\Omega}$. It is an henselian local ring with residue field $\kappa(\alpha)$, and ind-étale over $\mathcal{N}(M)$ [BoCoRo, 8.8.3]. The choice of a Nash chart as above gives an isomorphism from $\mathcal{N}_{M,\alpha}$ onto the ring $\kappa(\alpha)[[t_1, \ldots, t_e]]_{\text{alg}}$ of series which are algebraic over the polynomials; this is the henselisation of the localisation of $\kappa(\alpha)[t_1, \ldots, t_e]$ at the maximal ideal $(t_1, \ldots, t_e)$. For this ring we have the preparation and division theorems [BoCoRo, 8.2.7 and 8.2.9], and also Artin's approximation theorem [Ar].

LEMMA 1.6 *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{N}_{M,\alpha}$. Then there exists an open semialgebraic subset U of M, $\alpha \in \widetilde{U}$, such that, if $\mathfrak{q} = \mathfrak{p} \cap \mathcal{N}(U)$, we have $\mathfrak{q}\mathcal{N}_{M,\alpha} = \mathfrak{p}$. Moreover, if U_1 is an open semialgebraic subset of M, $\widetilde{U_1} \ni \alpha$, and if I is an ideal of $\mathcal{N}(U_1)$ such that $I\mathcal{N}_{M,\alpha} = \mathfrak{p}$, then there is a smaller open semialgebraic subset $U_2 \subset U \cap U_1$, $\widetilde{U_2} \ni \alpha$, such that $I\mathcal{N}(U_2) = \mathfrak{q}\mathcal{N}(U_2) = \mathfrak{p} \cap \mathcal{N}(U_2)$.*

Proof. Set $\mathfrak{q}_0 = \mathfrak{p} \cap \mathcal{N}(M)$. We know that $\mathfrak{q}_0\mathcal{N}_{M,\alpha} = \mathfrak{p} \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r$, where $\mathfrak{p}_2, \ldots, \mathfrak{p}_r$ are prime ideals which have all the same height as $\mathfrak{p}$ and $\mathfrak{q}_0$. Now choose f_i in $\mathfrak{p}\backslash\mathfrak{p}_i$, for $i = 2, \ldots, r$. These $f_2, \ldots, f_r$ are Nash functions on some open semialgebraic subset U of M, $\widetilde{U} \ni \alpha$. We set $\mathfrak{q} = \mathfrak{p} \cap \mathcal{N}(U)$. Then $\mathfrak{q}\mathcal{N}_{M,\alpha}$ is the intersection of $\mathfrak{p}$ and some prime ideals among $\mathfrak{p}_2, \ldots, \mathfrak{p}_r$. Since $f_i \in \mathfrak{q}\mathcal{N}_{M,\alpha}$, it is impossible that $\mathfrak{q}\mathcal{N}_{M,\alpha} \subset \mathfrak{p}_i$. So we have $\mathfrak{q}\mathcal{N}_{M,\alpha} = \mathfrak{p}$.

Now let U_1 and I be as in the statement of the Lemma. Let $g_1, \dots, g_s$ be generators of $\mathfrak{q}$, and $h_1, \dots, h_t$ be generators of I. We can find $\phi_{i,j}$ and $\psi_{j,i}$ in $\mathcal{N}_{M,\alpha}$ for $i = 1, \dots, s$ and $j = 1, \dots, t$ such that

$$g_i = \sum_{j=1}^{t} \phi_{i,j} h_j; \qquad h_j = \sum_{i=1}^{s} \psi_{j,i} g_i.$$

There is an open semialgebraic set $U_2 \subset U \cap U_1$, $\widetilde{U_2} \ni \alpha$, such that all the $\phi_{i,j}$ and $\psi_{j,i}$ are Nash functions on U_2. Then we have $I\mathcal{N}(U_2) = \mathfrak{q}\mathcal{N}(U_2)$.

The ideal $\mathfrak{q}\mathcal{N}(U_2)$ is the intersection of a finite number of prime ideals of the same height as $\mathfrak{p}$ and $\mathfrak{q}$. Let $\mathfrak{r}$ be such a prime ideal associated to $\mathfrak{q}\mathcal{N}(U_2)$. Since $\mathfrak{q}\mathcal{N}_{M,\alpha} = \mathfrak{p}$, we have also $\mathfrak{r}\mathcal{N}_{M,\alpha} = \mathfrak{p}$ Hence $\mathfrak{r} = \mathfrak{p} \cap \mathcal{N}(U_2)$. This proves $\mathfrak{q}\mathcal{N}(U_2) = \mathfrak{p} \cap \mathcal{N}(U_2)$. □

PROPOSITION 1.7 *Let* $\mathfrak{p}$ *be a prime ideal of* $\mathcal{N}_{M,\alpha}$. *We can find an open semialgebraic subset* $U \subset M$ *and a Nash submanifold* S *of* U, $\alpha \in \widetilde{S}$, *such that, if* $\mathfrak{q} = \mathfrak{p} \cap \mathcal{N}(U)$, *then for every* $O \in S$, $\mathfrak{q}\mathcal{N}_{M,O}$ *is a prime ideal.*

Proof. Since we are only interested in a neighborhood of α, we can suppose, using a chart as above, that M is an open semialgebraic subset of $\mathbf{R}^d \times \mathbf{R}^e$, and that the support of α is $\mathbf{R}^d \times \{0\}$. We note $x = (x_1, \dots, x_d)$ the variables in $\mathbf{R}^d$, and $t = (t_1, \dots, t_e)$ the variables in $\mathbf{R}^e$. We identify $\mathcal{N}_{M,\alpha}$ with the ring $\kappa(\alpha)[[t_1, \dots, t_e]]_{\text{alg}}$. Let $e-k$ be the height of the prime ideal $\mathfrak{p}$ in $\mathcal{N}_{M,\alpha}$. Following [GuRo, Chap. 3, A], we can perform a linear change of coordinates in $t_1, \dots, t_e$, to be in the situation we will now describe. Remark that this change of coordinates has a finite number of coefficients in $\kappa(\alpha)$, so that this may be viewed as a linear change of coordinates t with coefficients Nash functions in x, on an open semialgebraic subset of $\mathbf{R}^d \times \{0\}$ whose tilda contains α. Set $t' = (t_1, \dots, t_k)$ and $\mathcal{N}'_\alpha = \kappa(\alpha)[[t']]_{\text{alg}}$ (here $\mathcal{N}'$ is the sheaf of Nash functions on $\mathbf{R}^d \times \mathbf{R}^k \times \{0\}$). Then:

1. $\mathfrak{p} \cap \mathcal{N}'_\alpha = \{0\}$.
2. $\mathcal{N}_\alpha/\mathfrak{p}$ is integral over $\mathcal{N}'_\alpha$.
3. The field of fractions of $\mathcal{N}_\alpha/\mathfrak{p}$ is generated over the field of fractions of $\mathcal{N}'_\alpha$ by the image of t_{k+1}.

As in [GuRo], it follows that:

4. $\mathfrak{p}$ contains distinguished polynomials $P = P_1$ in $\mathcal{N}'_\alpha[t_{k+1}]$, irreducible of degree s, P_2 in $\mathcal{N}'_\alpha[t_{k+2}], \dots, P_{e-k}$ in $\mathcal{N}'_\alpha[t_e]$. These polynomials are the minimal polynomials, over the field of fractions of $\mathcal{N}'_\alpha$, of the classes modulo $\mathfrak{p}$ of $t_{k+1}, t_{k+2}, \dots, t_e$ respectively.
5. Let ξ be the discriminant of P; it is an element of $\mathcal{N}'_\alpha$, different from 0.
6. The ideal $\mathfrak{p}$ contains also $Q_2 = \xi t_{k+2} - R_2, \dots, Q_{e-k} = \xi t_e - R_{e-k}$ where, for $i = 2, \dots, e-k$, R_i belongs to $\mathcal{N}'_\alpha[t_{k+1}]$ and $\deg R_i < s$.

The following consequences are also essentially in [GuRo]. Nevertheless we give the arguments. Set

$$r = \sup\left(\sum_{i=2}^{e-k}(\deg P_i - 1), \deg P_2, \dots, \deg P_{e-k}\right),$$

$$I = (P, Q_2, \dots, Q_{e-k}) \subset \mathcal{N}'_\alpha[t_{k+1}, \dots, t_e].$$

Then:

7. For every $i = 2, \dots, e-k$ we have $\xi^r P_i \in I$. Indeed the polynomial $\xi^r P_i(R_i/\xi)$ in $\mathcal{N}'_\alpha[t_{k+1}]$ is divisible by P since its image in $\mathcal{N}_\alpha/\mathfrak{p}$ is null, and then the Taylor expansion
$$\xi^r P_i(t_{k+i}) = \xi^r P_i\left(\frac{R_i}{\xi}\right) + \sum_{l\geqslant 1}\frac{1}{l!}\xi^{r-l}\frac{\partial^l P_i}{\partial t_{k+i}^l}\left(\frac{R_i}{\xi}\right) Q_i^l,$$
shows the assertion.
8. For any $r' \geqslant r$, we have
$$\mathfrak{p} = (I\mathcal{N}_{M,\alpha} : \xi^{r'}).$$
The inclusion from right to left is clear. If f is in $\mathfrak{p}$, it can be divided successively by $P, P_2, \dots, P_{e-k}$, and so f is congruent modulo $(P, P_2, \dots, P_{e-k})\mathcal{N}_{M,\alpha}$ to a polynomial of total degree not greater than r in $t_{k+2}, \dots, t_e$, with coefficients in $\mathcal{N}'_\alpha[t_{k+1}]$. By 7, to prove that $\xi^r f \in I$, we can suppose that f is this polynomial. Then $\xi^r f$ is congruent modulo $(Q_2, \dots, Q_k)$ to a g in $\mathfrak{p} \cap \mathcal{N}'_\alpha[t_{k+1}]$. This g is divisible by P.

Since P is irreducible, there is no equality $P = AB$ in $\mathcal{N}'_\alpha[t_{k+1}]$ where A and B are monic polynomials of degree q and $s - q$ respectively, with $0 < q < s$. Set

$$P = t_{k+1}^s + p_{s-1}t_{k+1}^{s-1} + \dots + p_0,$$

$$A = t_{k+1}^q + a_{q-1}t_{k+1}^{q-1} + \dots + a_0, \quad a_q = 1,$$

$$B = t_{k+1}^{s-q} + b_{s-q-1}t_{k+1}^{s-q-1} + \dots + b_0, \quad b_{s-q} = 1.$$

The coefficients p_j are in $\mathcal{N}'_\alpha$, hence they are roots of polynomials $E_j(t_1, \dots, t_k, u_j)$ in the indeterminate u_j, with coefficients in $\kappa(\alpha)[t_1, \dots, t_k]$. Since E_j has a finite number of roots in the fraction field of $\mathcal{N}'_\alpha$, there is a μ such that any root in $\mathcal{N}'_\alpha$ of E_j which coincides with p_j till order μ (i.e. modulo $(t_1, \dots, t_k)^{\mu+1}$) is equal to p_j. Consider the system of $2s$ polynomial equations in the $2s$ variables $u_{s-1}, \dots, u_0, a_{q-1}, \dots, a_0, b_{s-q-1}, \dots, b_0$, with coefficients in $\kappa(\alpha)[t_1, \dots, t_k]$:

$$E_j(t_1, \dots, t_k, u_j) = 0 \quad \text{and} \quad u_j = \sum_i a_i b_{j-i}, \quad \text{for } j = 0, \dots, s-1.$$

We know that it has no solution in $\mathcal{N}'_\alpha$ with u_j coinciding with p_j till order μ. Hence, by Artin's approximation theorem [Ar, Th. 6.1], there is an integer $\nu > \mu$ such that the truncated system at order ν (i.e. considered modulo $(t_1, \ldots, t_k)^{\nu+1}$) has no solution with u_j coinciding with p_j till order μ. This translates into the fact that a polynomial system with coefficients in $\kappa(\alpha)$ and indeterminates $\underline{u}$ (the coefficients of the series u_j for the terms of order $> \mu$ and $\leqslant \nu$), $\underline{a}$, $\underline{b}$ (the coefficients of the series a_i and b_j for the terms of order $\leqslant \nu$) has no solution in $\kappa(\alpha)$. Let us call $\Phi_\lambda(\underline{u}, \underline{a}, \underline{b})$ the equations of this system. The real Nullstellensatz translates the fact that the system $(\Phi_\lambda(\underline{u}, \underline{a}, \underline{b}) = 0)_\lambda$ has no solution in the real closed field $\kappa(\alpha)$ to an identity

$$1 + \Lambda_1(\underline{u}, \underline{a}, \underline{b})^2 + \cdots + \Lambda_m(\underline{u}, \underline{a}, \underline{b})^2 = \sum_\lambda \Delta_\lambda(\underline{u}, \underline{a}, \underline{b})\Phi_\lambda(\underline{u}, \underline{a}, \underline{b}), \qquad (*_q)$$

where Λ_j and Δ_λ are polynomials with coefficients in $\kappa(\alpha)$. There is such an identity $(*_q)$ for each q with $0 < q < s$.

Choose an open semialgebraic subset U of M with $\widetilde{U} \ni \alpha$, such that all the coefficients of the polynomials $P, P_2, \ldots, P_{e-k}, Q_2, \ldots, Q_{e-k}$ (hence also ξ) are Nash functions defined on $U' = U \cap (\mathbf{R}^d \times \mathbf{R}^k \times \{0\})$. Let $I' \subset \mathcal{N}'(U')[t_{k+1}, \ldots, t_e]$ be the ideal generated by $(P, Q_2, \ldots, Q_{e-k})$. By 7, we can choose U so small that $\xi^r P_2, \ldots, \xi^r P_{e-k}$ are all in I'. Set $\mathfrak{q} = (I'\mathcal{N}(U) : \xi^{2r})$ in $\mathcal{N}(U)$. By flatness of $\mathcal{N}_{M,\alpha}$ over $\mathcal{N}(U)$ and by 8, we know that $\mathfrak{q}\mathcal{N}_{M,\alpha} = \mathfrak{p}$, and by Lemma 1.6 we can suppose, possibly shrinking U, that $\mathfrak{q} = \mathfrak{p} \cap \mathcal{N}(U)$.

Choose S, an open semialgebraic subset of $U \cap (\mathbf{R}^d \times \{0\})$ such that $\widetilde{S} \ni \alpha$, in order that all the coefficients of the polynomials Λ_j, Δ_λ and Φ_λ for all identities $(*_q)$ are Nash functions defined over S. We can also suppose that the non-leading coefficients of $P, P_2, \ldots, P_{e-k}$ vanish on S.

Let O be a point of S; we can suppose that O is the origin of coordinates. Then $\mathcal{N}_{M,O}$ may be identified with $\mathbf{R}[[x, t]]_{\text{alg}}$. We want to show that $\mathfrak{q}\mathcal{N}_{M,O}$ is a prime ideal. Set $\mathcal{N}'_O = \mathbf{R}[[x, t']]_{\text{alg}}$. We still call by the same names the images of P_i in $\mathcal{N}'_O[t_{k+i}]$ for $i = 1, \ldots, e-k$, of ξ in $\mathcal{N}'_O$, of R_i in $\mathcal{N}'_O[t_{k+1}]$ for $i = 2, \ldots, e-k$, and of $Q_i = \xi t_{k+i} - R_i$ for $i = 2, \ldots, e-k$. Since the identities $(*_q)$ hold in $\mathbf{R}[[x]]_{\text{alg}}[\underline{u}, \underline{a}, \underline{b}]$, it follows (going backwards in the arguments above) that $P = P_1$ is irreducible in $\mathcal{N}'_O[t_{k+1}]$. Let us sum up the situation:

9. $P_i \in \mathcal{N}'_O[t_{k+i}]$ are all distinguished polynomials for $i = 1, \ldots, e-k$.
10. $P = P_1$ is irreducible of degree s.
11. ξ is an element of $\mathcal{N}'_O$, different from 0.
12. $Q_i = \xi t_{k+i} - R_i$ where $R_i \in \mathcal{N}'_O[t_{k+i}]$ has degree $< s$, for $i = 2, \ldots, k$.
13. $\xi^r P_2, \ldots, \xi^r P_{e-k}$ are in the ideal $I'\mathcal{N}'_O[t_{k+1}, \ldots, t_e]$

To continue we need:

LEMMA 1.8 *The properties* 9, 11, 12, 13 *imply that the canonical homomorphism*

$$\mathcal{N}'_O[t_{k+1}]/(P) \to \mathcal{N}_{M,O}/(P, Q_2, \ldots, Q_{e-k})$$

is an injection and that, for any $f \in \mathcal{N}_{M,O}$, $\xi^r f$ *is congruent to an element of* $\mathcal{N}'_O[t_{k+1}]$ *modulo* $(P, Q_2, \dots, Q_{e-k})$.

Proof. We first want to prove that if g is a polynomial in $\mathcal{N}'_O[t_{k+1}]$ such that

$$g = \Theta_1 P + \Theta_2 Q_2 + \cdots + \Theta_{e-k} Q_{e-k},$$

with $\Theta_i \in \mathcal{N}_{M,O}$, then P divides g in $\mathcal{N}'_O[t_{k+1}]$. Dividing successively by $P_{e-k}, \dots, P_2, P$ (thanks to 9), we can write

$$g = G_1 P + G_2 P_2 + \cdots + G_{e-k} P_{e-k} + H_2 Q_2 + \cdots + H_{e-k} Q_{e-k},$$

where $H_i \in \mathcal{N}'_O[t_{k+1}, \dots, t_e]$, and by uniqueness of division we can moreover suppose that $G_i \in \mathcal{N}'_O[t_{k+1}, \dots, t_e]$. By 13, we get

$$\xi^r g = \Gamma_1 P + \Gamma_2 Q_2 + \cdots + \Gamma_{e-k} Q_{e-k},$$

where $\Gamma_i \in \mathcal{N}'_O[t_{k+1}, \dots, t_e]$. Substituting R_i/ξ for t_{k+i} and multiplying by an appropriate power of ξ, we find that P divides $\xi^{r'} g$ in $\mathcal{N}'_O[t_{k+1}]$ with $r' \geqslant r$, and since P is monic it divides g.

Now we prove the second assertion. Dividing f successively by $P_{e-k}, \dots, P_2, P$ we find that f is congruent modulo $(P, P_2, \dots, P_{e-k})$ to a polynomial in $t_{k+2}, \dots, t_e$ of total degree $\leqslant r$, with coefficients in $\mathcal{N}'_O[t_{k+1}]$. Hence, using 13, we may suppose that f is this polynomial. Then, by 12, $\xi^r f$ is congruent modulo $(Q_2, \dots, Q_k)$ to an element of $\mathcal{N}'_O[t_{k+1}]$. □

Now we finish the proof that $\mathfrak{q}\mathcal{N}_{M,O}$ is a prime ideal. By flatness of $\mathcal{N}_{M,O}$ over $\mathcal{N}'(U')[t_{k+1}, \dots, t_e]$, we know that $h \in \mathfrak{q}\mathcal{N}_{M,O}$ if and only if

$$\xi^{2r} h \in (P, Q_2, \dots, Q_{e-k})\mathcal{N}_{M,O}.$$

Now, let $fg \in \mathfrak{q}\mathcal{N}_{M,O}$. By the Lemma above, we may suppose that $\xi^r f$ and $\xi^r g$ are in $\mathcal{N}'_O[t_{k+1}]$, and that their product is divisible by P in this ring. Since P is irreducible and $\mathcal{N}'_O[t_{k+1}]$ factorial, one of $\xi^r f$ and $\xi^r g$ is divisible by P. Hence either f or g belongs to $\mathfrak{q}\mathcal{N}_{M,O}$. □

LEMMA 1.9 *Let I be an ideal of $\mathcal{N}(M)$, and let Y be an irreducible invariant component of $\mathfrak{X}(I)$. Let $\mathcal{I} = \mathfrak{J}(Y)$ be the sheaf of germs of Nash functions vanishing on Y. Then $\mathcal{I}$ is a finite sheaf of radical ideals, and $\mathfrak{X}(\mathcal{I}) = Y$.*

If I is a prime ideal of $\mathcal{N}(M)$ and $Y_1, \dots, Y_l$ are the irreducible invariant components of $\mathfrak{X}(I)$, then $I\mathcal{N} = \mathfrak{J}(Y_1) \cap \cdots \cap \mathfrak{J}(Y_l)$.

Proof. Without loss of generality we can suppose that I is a prime ideal. We want to show that $\mathcal{I}$ is finite. Taking into account the compactness of the real spectrum $\widetilde{M}$, it is sufficient to show that for every $\alpha \in \widetilde{M}$, there is a semialgebraic open subset U of M with $\alpha \in \tilde{U}$, such that $\mathcal{I}|_U$ is generated by Nash functions on U. Let

$$I\mathcal{N}_{M,\alpha} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$$

be the decomposition of $I\mathcal{N}_{M,\alpha}$ into prime ideals. Let $X^* \subset M^{\mathbf{C}}$ be a semialgebraic realization of $\mathfrak{X}(I)$. We can suppose, as in the proof of 1.2 that we have a finite simplicial complex K, a semialgebraic homeomorphism π from the union V of some open simplices of K onto an open semialgebraic subset of $M^{\mathbf{C}}$, and an open simplex σ contained in V such that:

1. V is the union of all open simplices τ of K such that σ is contained in the closure of τ.
2. The intersections of $\pi(V)$ with M, X^*, $\mathrm{Reg}(X^*)$ are all unions of images under π of open simplices contained in V.
3. Set $S = \pi(\sigma)$ and $U = \pi(V) \cap M$. Then $\alpha \in \tilde{S}$ and S and U satisfy the property of Proposition 1.7, for $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$.

Now set $\mathfrak{q}_j = \mathfrak{p}_j \cap \mathcal{N}(U)$. Since $\mathfrak{q}_j\mathcal{N}_x$ is prime for any $x \in S$, so is $\mathfrak{q}_j\mathcal{O}_x$, and the germ $\mathfrak{X}(\mathfrak{q}_j)_x$ is irreducible as invariant germ at x. Let us prove that $\mathfrak{X}(\mathfrak{q}_j)$ is irreducible as invariant germ at U. Suppose that Y_1 and Y_2, are two distinct irreducible invariant components of $\mathfrak{X}(\mathfrak{q}_j)$. The dimension of Y_1 and Y_2 is the same as the dimension of $\mathfrak{X}(\mathfrak{q}_j)$. Since σ is in the closure of any open simplex of V, both Y_1 and Y_2 contain S. Hence, for any x in S, we must have $X(\mathfrak{q}_j)_x = Y_{1,x} = Y_{2,x}$, against the assumption that Y_1 and Y_2 are different.

So the germ of Y at U is the union of some of the $X(\mathfrak{q}_j)$'s, and $\mathcal{I}|_U$ is generated by the intersection J of the corresponding $\mathfrak{q}_j$'s. Indeed, clearly J generates a subsheaf of radical ideals of $\mathcal{N}|_U$ and by the complex Nullstellensatz (Lemma 1.1), $J\mathcal{N}|_U = \mathcal{I}|_U$.

The fact that $\mathfrak{X}(\mathcal{I}) = Y$ is clear from the proof, as well as the last assertion of the lemma. □

We gather what we know in the following result.

THEOREM 1.10 *The assignment $\mathcal{I} \mapsto \mathfrak{X}(\mathcal{I})$ defines a bijection from the set of finite sheaves of radical ideals of $\mathcal{N}$ onto the set of finite semialgebraic invariant M-germs (which are the same as finite locally semialgebraic invariant M-germs).*

Proof. Lemmas 1.2, 1.4, 1.9 show that the assignment $\mathcal{I} \mapsto \mathfrak{X}(\mathcal{I})$ is a well-defined surjection from the set of finite sheaves of radical ideals of $\mathcal{N}$ onto the set of finite locally semialgebraic invariant M-germs. The injectivity comes from the complex Nullstellensatz 1.1. □

We close the section with a remark about another possible choice for the notion of finiteness for a sheaf of ideals of $\mathcal{N}$. Say that a coherent sheaf of ideals $\mathcal{I}$ of $\mathcal{N}$ is *weakly finite* when there is no nontrivial infinite decomposition $\mathcal{I} = \bigcap_i \mathcal{I}_i$ into coherent sheaves of ideals of $\mathcal{N}$ (nontrivial means that for each i, $\mathcal{I} \neq \bigcap_{j \neq i} \mathcal{I}_j$). As the terminology suggests, any finite sheaf of ideals is weakly finite. Moreover, for a radical sheaf of ideals, the notions of finiteness and weak finiteness coincide. The proofs of these assertions require some work. We do not know whether it is always the case that a weakly finite sheaf of ideals of $\mathcal{N}$ is finite.

2. Equivalence of separation, global equations, and extension

We prove in this section Theorem 0.1. We first devote some work to the extension problem, studying the global sections of a quotient of $\mathcal{N}$ by a finite ideal sheaf.

We will use in Proposition 2.2 and Lemmas 2.4, 2.5, 2.6, the C^k topology on spaces of C^k semialgebraic maps (for k a positive integer). See [Sh2, II.1] for the definition of this topology and its properties. We mainly use the facts that a C^k semialgebraic map between Nash manifolds may be approximated by Nash ones, and that the diffeomorphisms form an open subset for this topology. All this *without* compactness assumptions on the Nash manifolds involved.

We begin by a construction which will be used in the following results. Let $\mathcal{I}$ be a finite sheaf of ideals of $\mathcal{N}$, and let φ be a global section of the quotient sheaf $\mathcal{N}/\mathcal{I}$. Set $X = \mathfrak{X}(\mathcal{I})$, $M_1 = M \times \mathbf{R}$, and let $p: M_1 \to M$ be the projection. We associate to φ a coherent sheaf $\mathcal{I}(\varphi)$ of ideals of $\mathcal{N}_{M_1}$. Let (x_0, t_0) be a point in M_1, and let Φ_{x_0} be an element of $\mathcal{N}_{M,x_0}$ whose image in $\mathcal{N}_{M,x_0}/\mathcal{I}_{x_0}$ is φ_{x_0}. Take $\mathcal{I}(\varphi)_{(x_0,t_0)}$ to be the ideal of $\mathcal{N}_{M_1,(x_0,t_0)}$ generated by $t - \Phi_{x_0}$ and $\mathcal{I}_{x_0}$.

LEMMA 2.1 *The invariant M_1-germ $X_1 = \mathfrak{X}(\mathcal{I}(\varphi))$ is finite. If $\mathcal{I}$ is a finite sheaf of* radical *ideals, then $\mathcal{I}(\varphi)$ is a finite sheaf of radical ideals of $\mathcal{N}_{M_1}$.*

Proof. An analytic extension of φ to M is possible by Cartan's Theorem B. Let Φ be such an extension, and $\Phi^{\mathbf{C}}$ an analytic complexification of Φ defined on a neighborhood of M in $M^{\mathbf{C}}$. Then X_1 is the intersection of the invariant M_1-germ of the graph of $\Phi^{\mathbf{C}}$ and the invariant M_1-germ of $X \times \mathbf{C}$. Hence the numbers of irreducible invariant components of X and X_1 coincide, and X_1 is finite. It is also clearly locally semialgebraic. So by Theorem 1.10 there is an unique finite sheaf $\mathcal{I}_1$ of radical ideals of $\mathcal{N}$ such that $\mathfrak{X}(\mathcal{I}_1) = X_1$. If $\mathcal{I}$ is radical, then clearly $\mathcal{I}(\varphi)$ is radical, so by the complex Nullstellensatz we get $\mathcal{I}_1 = \mathcal{I}(\varphi)$. □

We do not know whether $\mathcal{I}(\varphi)$ is finite in general, without the hypothesis that $\mathcal{I}$ is radical. It can be proved when $\mathcal{I}^{-1}(0)$ is a finite number of points, or when $\mathfrak{X}(\mathcal{I})$ has complex dimension 1.

Now we see why the study of $\mathcal{I}(\varphi)$ is important for the problem of extending φ:

PROPOSITION 2.2 *If $\mathcal{I}(\varphi)$ is generated by its global sections, then there exists a global Nash function F on M whose image in $H^0(M, \mathcal{N}/\mathcal{I})$ is φ. Actually, it is sufficient that there exists a semialgebraic open neighborhood U of $X_1 \cap M_1$*

such that $\mathcal{I}(\varphi)|_U$ is generated by its sections on U. Moreover, $\mathcal{I}$ is generated by its global sections.

Proof. Suppose that $\mathcal{I}(\varphi)|_U$ is generated by the Nash functions $f_1, \dots, f_k$ in $\mathcal{N}(U)$. We claim that for every $(x_0, t_0) \in X_1 \cap M_1$ we can write

$$g_i f_{i(x_0,t_0)} = t - \Phi_{x_0}, \quad (\operatorname{mod} \mathcal{I}_{x_0}\mathcal{N}_{M_1,(x_0,t_0)}),$$

for some i and a unit g_i in $\mathcal{N}_{M_1,(x_0,t_0)}$. Fix (x_0, t_0). We have

$$f_{i,(x_0,t_0)} = h_i(t - \Phi_{x_0}), \quad (\operatorname{mod} \mathcal{I}_{x_0}\mathcal{N}_{M_1,(x_0,t_0)}), \quad 1 \leqslant i \leqslant k,$$

for some $h_i \in \mathcal{N}_{M_1,(x_0,t_0)}$. It suffices to prove that at least one of the h_i's is a unit. We have also

$$t - \Phi_{x_0} = \sum_j \mu_j f_{j,(x_0,t_0)},$$

for some $\mu_j \in \mathcal{N}_{M_1,(x_0,t_0)}$. Hence we get a homogeneous system mod $\mathcal{I}_{x_0}\mathcal{N}_{M_1,(x_0,t_0)}$

$$0 = (h_i\mu_1)f_{1,(x_0,t_0)} + \cdots + (h_i\mu_i - 1)f_{i,(x_0,t_0)} + \cdots + (h_i\mu_k)f_{k,(x_0,t_0)},$$

$$1 \leqslant i \leqslant k,$$

whose determinant has the form

$$(-1)^k + \lambda_1 h_1 + \cdots + \lambda_k h_k,$$

for some $\lambda_i \in \mathcal{N}_{M_1,(x_0,t_0)}$. Therefore, if no h_i were a unit, we would conclude

$$f_{i,(x_0,t_0)} = 0, \quad (\operatorname{mod} \mathcal{I}_{x_0}\mathcal{N}_{M_1,(x_0,t_0)}), \quad 1 \leqslant i \leqslant k,$$

and, consequently, $\mathcal{I}(\varphi)_{(x_0,t_0)} = \mathcal{I}_{x_0}\mathcal{N}_{M_1,(x_0,t_0)}$. This is impossible, which shows our claim.

It follows that the sets

$$\begin{aligned}\Omega_i &= \{(x,t) \in U;\ \mathcal{I}(\varphi)_{(x,t)} = (f_{i(x,t)}) + \mathcal{I}_x\mathcal{N}_{M_1,(x,t)}\} \\ &= \{(x,t) \in U;\ t - \Phi_x \in (f_{i(x,t)}) + \mathcal{I}_x\mathcal{N}_{M_1,(x,t)}\}, \quad i = 1, \dots, k,\end{aligned}$$

cover $X_1 \cap M_1$. Clearly each Ω_i is open in M_1. Moreover, it is semialgebraic because f_i is regular with respect to t at every point of $\Omega_i \cap X_1$, and, hence,

$$\Omega_i = (p^{-1}(M \setminus X) \cap U) \cup \left\{(x,t) \in X_1 \cap U;\ \frac{\partial f_i}{\partial t}(x,t) \neq 0\right\}$$

$$\cup \{(x,t) \in (p^{-1}(X \cap M) \cap U) \setminus X_1;\ f_i(x,t) \neq 0\}.$$

From this we see in addition that after shrinking Ω_i we can assume f_i regular with respect to t on Ω_i. Hence $f_i^{-1}(0) \cap \Omega_i$ is the graph of a Nash function F_i on $V_i = p(f_i^{-1}(0) \cap \Omega_i)$. These V_i cover $M \cap X_1$. On Ω_i, the function f_i is equal to the product of $t - F_i$ by an invertible function. Hence, for each x in V_i, the class of $F_{i,x}$ modulo $\mathcal{I}_x$ is φ_x. On $V_0 = M \backslash X$, we can take $F_0 = 0$.

Using a partition of unity we paste the F_i's as follows. Let (ρ_i) be a C^1 semialgebraic partition of unity subordinated to the covering (V_i). The function $h = \sum_i \rho_i(t - F_i)$ is C^1 semialgebraic on M_1, and it is is regular with respect to t. Since the function $t - F_i$ is in the ideal generated by $(f_1, \dots, f_k)$ in $\mathcal{N}(V_i \times R)$, there exist C^1 semialgebraic functions h_j on M_1 such that $h = \sum_{j=1}^k h_j f_j$. Let h_j^* be a Nash approximation of h_j for $1 \leqslant j \leqslant k$. Then the function $h^* = \sum_{i=1}^k h_j^* f_j$ is a Nash approximation of h. Choose the approximation so strong that h^* is regular with respect to t. Then the zero set $(h^*)^{-1}(0)$ is the graph of a Nash function F on M, and h^* is equal to the product of $t - F$ by an invertible Nash function on M_1. Since for any $(x_0, t_0) \in X_1 \cap M_1$ the germ of h^* is divisible by $t - \Phi_{x_0}$ modulo $\mathcal{I}_{x_0}\mathcal{N}_{M_1,(x_0,t_0)}$, we know that the germs F_{x_0} and Φ_{x_0} are equal modulo $\mathcal{I}_{x_0}$. Hence F is the Nash function we are looking for.

The last assertion of the proposition comes from the fact that $\mathcal{I}$ is generated by the global Nash functions $f_i \circ (\mathrm{Id}_M, F)$ for $i = 1, \dots, k$. □

At this point, one could be tempted to conclude that 'global equations' implies 'extension'. But Proposition 2.2 only says that $\mathbf{Glob}^r(M_1)$ implies $\mathbf{Ext}^r(M)$, and we want to have the implication between these two properties for the *same* manifold. So there is still work to do.

COROLLARY 2.3 *Let $\mathcal{I}$ be a finite sheaf of radical ideals of $\mathcal{N}$, and let φ be a global section of the quotient sheaf $\mathcal{N}/\mathcal{I}$. Then there exists a* finite *covering of M by semialgebraic open sets U_i, and for each i a Nash function $F_i \in \mathcal{N}(U_i)$, such that for each x in U_i, the class of $F_{i,x}$ modulo $\mathcal{I}_x$ is φ_x.*

In other words, any global section of $\mathcal{N}/\mathcal{I}$ lifts to sections of $\mathcal{N}$ over a *finite* semialgebraic covering of M.

Proof. We know from Lemma 2.1 that the sheaf $\mathcal{I}(\varphi)$ is finite. We can cover M_1 with finitely many semialgebraic open sets W_i such that $\mathcal{I}(\varphi)|_{W_i}$ is generated by Nash functions on W_i. We choose semialgebraic open subsets U_i of M covering M such that W_i is a neighborhood of $X_1 \cap (U_i \times \mathbf{R})$. Then $\mathcal{I}(\varphi)|_{(U_i \times \mathbf{R}) \cap W_i}$ is generated by Nash functions on $(U_i \times \mathbf{R}) \cap W_i$. We apply Proposition 2.2 to each U_i. □

For the following three Lemmas, we have a finite sheaf $\mathcal{I}$ of radical ideals of $\mathcal{N}$, with which we perform the construction of X_1.

LEMMA 2.4 *Let $q: M_1 \to M$ be a Nash map very close to p in the C^1 topology, and $q^{\mathbf{C}}$ a complexification of q. Then for any sufficiently small invariant semialgebraic* *neighborhood W of M_1 in $M_1^{\mathbf{C}}$ and semialgebraic realization X_1^* of X_1 in W, the germ of $q^{\mathbf{C}}(X_1^*)$ at M is a semialgebraic invariant M-germ.*

Proof. We know from Corollary 2.3 that there is an invariant semialgebraic neighborhood U of M in $M^{\mathbf{C}}$, a semialgebraic C^1 function Ψ from U to $\mathbf{C}$, equivariant under conjugation, and a semialgebraic realization X^* of X in U, such that the graph of the restriction $\Psi|_{X^*}$ is a semialgebraic realization X_1^* of X_1. Indeed, consider the extensions $F_i^{\mathbf{C}}$ of the Nash functions of Corollary 2.3 to some semialgebraic invariant neighborhoods $V_i^{\mathbf{C}}$ of V_i in $M^{\mathbf{C}}$, and glue them together using some C^1 semialgebraic equivariant partition of unity subordinate to the $V_i^{\mathbf{C}}$'s. We consider the graph map $\gamma = (\mathrm{Id}_U, \Psi) : U \to U \times \mathbf{C}$.

We can choose q so close to p that (see [Sh2]):

- $q \circ \gamma|_M$ is a semialgebraic C^1 diffeomorphism from M to itself,
- at each point x of M, the differential of $q^{\mathbf{C}} \circ \gamma$ is an $\mathbf{R}$-linear automorphism from the tangent space $T_x(M^{\mathbf{C}})$ to the tangent space $T_{q(x,\Psi(x))}(M^{\mathbf{C}})$,
- the Nash map $\pi : M_1 \to M_1$ defined by $\pi(x,t) = (q(x,t),t)$ for $(x,t) \in M_1$ is a diffeomorphism.

Then, possibly shrinking U, we can suppose that $q^{\mathbf{C}} \circ \gamma$ is a semialgebraic C^1 equivariant diffeomorphism from U to another invariant semialgebraic neighborhood U' of M in $M^{\mathbf{C}}$. Moreover we can suppose that we have W and W' invariant semialgebraic neighborhoods of M_1 in $M_1^{\mathbf{C}}$ such that π complexifies to a diffeomorphism $\pi^{\mathbf{C}} : W \to W'$, $p^{\mathbf{C}}(W) = U$, and W contains the graph Γ of $\Psi : U \to \mathbf{C}$. Then $\pi^{\mathbf{C}}(\Gamma)$ is the graph of a C^1 equivariant semialgebraic function from U' to $\mathbf{C}$. Recall that $X_1^* = \Gamma \cap (p^{\mathbf{C}})^{-1}(X^*)$, and set $Y_1^* = \pi^{\mathbf{C}}(X_1^*)$. Then Y_1^* is a closed subset of $\pi^{\mathbf{C}}(\Gamma)$, hence the restriction of $p^{\mathbf{C}}$ from Y_1^* to U' is proper. On the other hand, Y_1^* is a complex analytic subset of W', hence $p^{\mathbf{C}}(Y_1^*) = q^{\mathbf{C}}(X_1^*)$ is a complex analytic subset of U'. □

LEMMA 2.5 *We can find a finite number of Nash maps $q_1, \ldots, q_k$ from M_1 to M, arbitrarily close to p in the C^1 topology, such that for any small invariant semialgebraic neighborhood W of M_1 in $M_1^{\mathbf{C}}$ and realization X_1^* of X_1 in W, the germ at M_1 of $\bigcap_i (q_i^{\mathbf{C}})^{-1}(q_i^{\mathbf{C}}(X_1^*))$ is equal to X_1.*

Proof. We know, according to Lemma 2.4, that the germ at M_1 of $\bigcap_i (q_i^{\mathbf{C}})^{-1} \times (q_i^{\mathbf{C}}(X_1^*))$ is a semialgebraic invariant M_1-germ. So, by an easy Noetherian induction, it is sufficient to see that if Y is a semialgebraic irreducible invariant M_1-germ which is not contained in X_1, then we can find a Nash map $q : M_1 \to M$ arbitrarily close to p in the C^1 topology such that for any small invariant semialgebraic neighborhood W of M_1 in $M_1^{\mathbf{C}}$ and realization X_1^* of X_1 in W, Y is not contained in the germ of $(q^{\mathbf{C}})^{-1}(q^{\mathbf{C}}(X_1^*))$. Let Y^* be a semialgebraic realization of Y and choose a real analytic curve $\gamma : [-1,1] \to M_1^{\mathbf{C}}$ with $\gamma((0,1]) \subset Y^* \backslash X_1^*$ and $\gamma(0) \in M_1$. It is sufficient to choose q such that, for small W where q is defined

$$q^{\mathbf{C}}(\gamma((0,1]) \cap W) \not\subset q^{\mathbf{C}}(X_1^*).$$

This can be done since there exists a ν such that for any q whose ν-jet at $\gamma(0)$ belongs to some dense subset, the condition is satisfied. □

LEMMA 2.6 *Suppose that* $\mathbf{Glob}^r(M)$ *holds. Then the* M_1*-germ* X_1 *is a Nash* M_1*-germ.*

Proof. Consider a finite number of Nash maps $q_1, \dots, q_k \colon M_1 \to M$ sufficiently close to p. According to Lemma 2.4, we find an invariant semialgebraic neighborhood W of M_1 in $M_1^{\mathbf{C}}$ and a realization X_1^* of X_1 in W so that the germ of each $q_i^{\mathbf{C}}(X_1^*)$ is a semialgebraic invariant M-germ, hence of the form $\mathfrak{X}(\mathcal{I})$ for some finite sheaf of radical ideals $\mathcal{I}$ of $\mathcal{N}$ by Theorem 1.10. So if $\mathbf{Glob}^r(M)$ holds, the germ of $q_i^{\mathbf{C}}(X_1^*)$ at M, and consequently the germ Y_i of $(q_i^{\mathbf{C}})^{-1}(q_i^{\mathbf{C}}(X_1^*))$ at M_1 is a Nash germ. Since, according to Lemma 2.5, we can choose $q_1, \dots, q_k$ so that $Y_1 \cap \dots \cap Y_k = X_1$, we are done. □

PROPOSITION 2.7 *If* $\mathbf{Glob}^r(M)$ *holds, then* $\mathbf{Ext}^r(M)$ *holds.*

Proof. Let $\mathcal{I}$ be a finite sheaf of radical ideals of $\mathcal{N}$, and let φ be a global section of the quotient sheaf $\mathcal{N}/\mathcal{I}$. The Lemma 2.6 tells us that the sheaf of ideals $\mathcal{I}(\varphi) = \mathfrak{I}(X_1)$ is generated by a finite number of global Nash functions $(f_1, \dots, f_k)$ on M_1. Then we apply Proposition 2.2.

PROPOSITION 2.8 *If* $\mathbf{Ext}^r(M)$ *holds, then* $\mathbf{Sep}(M)$ *holds.*

Proof. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{N}(M)$. By way of contradiction, assume that $\mathfrak{p}\mathcal{O}(M)$ is not prime. Then $\mathfrak{X}(\mathfrak{p})$ is not an irreducible invariant M-germ, and hence by Lemma 1.9 there are finite sheaves of radical ideals of $\mathcal{N}_M$, $\mathcal{I}_1$ and $\mathcal{I}_2$, such that $\mathfrak{p}\mathcal{N} = \mathcal{I}_1 \cap \mathcal{I}_2$ and

$$\dim \mathfrak{X}(\mathcal{I}_1) = \dim \mathfrak{X}(\mathcal{I}_2) > \dim \mathfrak{X}(\sqrt{\mathcal{I}_1 + \mathcal{I}_2}).$$

Let J be the ideal of Nash functions on M vanishing on $\mathfrak{X}(\sqrt{\mathcal{I}_1 + \mathcal{I}_2})$. By 1.4 we have $\dim \mathfrak{X}(J) < \dim \mathfrak{X}(\mathcal{I}_1)$. Hence there exists a Nash function f on M whose complexification has a germ at M which vanishes on $\mathfrak{X}(\sqrt{\mathcal{I}_1 + \mathcal{I}_2})$ but not on $\mathfrak{X}(\mathcal{I}_1)$. Let $\overline{f}$ denote the image of f in $H^0(M, (\mathcal{I}_1 + \mathcal{I}_2)/\mathcal{I}_1) = H^0(M, \mathcal{I}_2/\mathfrak{p}\mathcal{N})$. Regard $\overline{f}$ as an element of $H^0(M, \mathcal{N}/\mathfrak{p}\mathcal{N})$. Then, by $\mathbf{Ext}^r(M)$, there exists a Nash function F on M whose image in $H^0(M, \mathcal{N}/\mathfrak{p}\mathcal{N})$ is $\overline{f}$. Clearly, the germ at M of a complexification of F vanishes on $\mathfrak{X}(\mathcal{I}_2)$ and does not vanish on $\mathfrak{X}(\mathcal{I}_1)$. In the same way, we obtain a Nash function G on M whose complexification has a germ at M which vanishes on $\mathfrak{X}(\mathcal{I}_1)$ and does not vanish on $\mathfrak{X}(\mathcal{I}_2)$. Then neither F nor G belong to $\mathfrak{p}$, while FG does, which contradicts the primeness of $\mathfrak{p}$. □

PROPOSITION 2.9 *If* $\mathbf{Sep}(M)$ *holds, then* $\mathbf{Glob}^r(M)$ *holds.*

Proof. Let $\mathfrak{X}(\mathcal{I}) = X_1 \cup \dots \cup X_p$ be the decomposition into irreducible invariant components. Set $\mathcal{I}_i = \mathfrak{I}(X_i)$ for $i = 1, \dots, p$. We know that $\mathcal{I}_i$ is a finite sheaf of radical ideals of $\mathcal{N}$. Suppose that each $\mathcal{I}_i$ is generated by the ideal

$I_i = H^0(M, \mathcal{I}_i)$ of its global sections, and let $I = \sqrt{I_1 \cdots I_p}$. Since $\mathfrak{X}(I\mathcal{N}) = \mathfrak{X}(\mathcal{I})$, we have $I\mathcal{N} = \mathcal{I}$ and hence $\mathcal{I}$ is generated by its global sections. So it is sufficient to consider the case where $\mathfrak{X}(\mathcal{I})$ is an irreducible invariant M-germ. Then we know that there is a prime ideal $\mathfrak{p}$ of $\mathcal{N}(M)$ such that $\mathfrak{X}(\mathcal{I})$ is an irreducible invariant component of $\mathfrak{X}(\mathfrak{p})$. Since, by **Sep**$(M)$, $\mathfrak{X}(\mathfrak{p})$ is irreducible as invariant M-germ, we have that $\mathfrak{X}(\mathcal{I}) = \mathfrak{X}(\mathfrak{p})$, and hence $\mathcal{I} = \mathfrak{p}\mathcal{N}$ is generated by its global sections. □

Up to now, we have proved the equivalence of **Sep**(M), **Glob**$^r(M)$ and **Ext**$^r(M)$. The properties **Glob**(M) and **Ext**(M) are obviously stronger than **Glob**$^r(M)$ and **Ext**$^r(M)$ respectively. So, to complete the proof of Theorem 0.1, we just have to see:

PROPOSITION 2.10 *If the conjunction of* **Glob**$^r(M)$ *and* **Ext**$^r(M)$ *holds, then* **Glob**(M) *and* **Ext**(M) *hold.*

The argument, which can be considered as standard, is detailed in [CoRzSh] in the compact case, and can be used word for word (with ‘finite’ instead of ‘coherent’) here. So, we do not repeat it. Actually, the following result is proved as an intermediate step:

> *Let $\mathcal{I}$ be a finite sheaf of ideals of $\mathcal{N}_M$. Suppose that global equations and extension hold for all finite sheaves of ideals $\mathcal{I}'$ of $\mathcal{N}_M$ such that $\sqrt{\mathcal{I}} \subset \mathcal{I}'$. Then both hold also for $\mathcal{I}$.*

Using this, we get a more precise version of Proposition 2.10.

PROPOSITION 2.11 *If both global equations and extension hold for all finite sheaves of radical ideals of $\mathcal{N}_M$ whose stalks are everywhere of height $\geqslant r$, then they hold for all finite sheaves of ideals of $\mathcal{N}_M$ whose stalks are everywhere of height $\geqslant r$.*

Proof. If it is not the case, consider the non empty family of finite sheaves $\mathcal{J}$ of radical ideals of $\mathcal{N}_M$ whose stalks are everywhere of height $\geqslant r$ such that there exists a finite sheaf of ideals $\mathcal{I}$ with $\mathcal{J} = \sqrt{\mathcal{I}}$, for which global equations or extension do not hold. All those $\mathcal{J}$'s are generated by their global sections, hence by noetherianity we have a maximal element in this family, which we will denote by $\mathcal{J}$. If $\mathcal{I}'$ is a finite sheaf of ideals containing $\mathcal{J}$, either $\mathcal{I}' = \mathcal{J}$ or $\sqrt{\mathcal{I}'}$ strictly contains $\mathcal{J}$. In both cases, global equations and extension hold for $\mathcal{I}'$. Hence, by the result quoted above, both hold also for any finite sheaf of ideals $\mathcal{I}$ with $\sqrt{\mathcal{I}} = \mathcal{J}$. This contradicts the choice of $\mathcal{J}$. □

It is possible to generalize **Glob**(M) and **Ext**(M) to sheaves of $\mathcal{N}$-modules. We have first to discuss what are the appropriate sheaves of $\mathcal{N}$-modules. We have already seen, concerning sheaves of ideals, that the coherent ones are not good enough. We had to consider finite sheaves of ideals. We can of course define in a similar way *finite sheaves of $\mathcal{N}$-modules*. Let $\mathcal{F}$ be a sheaf of $\mathcal{N}$-modules. We will say that $\mathcal{F}$ is finite when there is a *finite* covering of M by semialgebraic open subsets U_i, such that each restriction $\mathcal{F}|_{U_i}$ is generated (as $\mathcal{N}|_{U_i}$-module) by a finite number of sections of $\mathcal{F}$ over U_i. Thus, a finite sheaf of $\mathcal{N}$-modules is coherent, but when M is not compact there are coherent sheaves of $\mathcal{N}$-modules which are not finite. Still, these finite sheaves of $\mathcal{N}$-modules are not the good choice. Hubbard ([Hd] or [BoCoRo, 12.7.9]) has given the example of a Nash line bundle over **R**, which is trivial over the two intervals $]-\infty, 1[$ and $]-1, +\infty[$, and does not have any global nonzero Nash section. Hence, the sheaf of Nash sections of this bundle is finite, but it is surely not generated by its global sections.

Thus, we have to define a special subcategory, as was already done in the compact case in [CoRzSh]. This is related to the notion of 'A-coherent' sheaves, and also to that of 'strongly algebraic' vector bundles, discussed in the context of regular functions [To;BeTo].

A sheaf of $\mathcal{N}$-modules $\mathcal{F}$ is called *strongly coherent* if there is an exact sequence

$$\mathcal{N}^q \to \mathcal{N}^p \to \mathcal{F} \to 0.$$

Of course, a strongly coherent sheaf is finite, and hence coherent.

Now, let F be a finitely generated $\mathcal{N}(M)$-module. Since $\mathcal{N}(M)$ is noetherian, there is an exact sequence

$$\mathcal{N}(M)^q \to \mathcal{N}(M)^p \to F \to 0.$$

Let $\mathcal{N} \otimes_{\mathcal{N}(M)} F$ be the sheaf of $\mathcal{N}$-modules generated by F. We get the exact sequence of sheaves

$$\mathcal{N}^q \to \mathcal{N}^p \to \mathcal{N} \otimes_{\mathcal{N}(M)} F \to 0.$$

Hence, the assignment: $F \mapsto \mathcal{N} \otimes_{\mathcal{N}(M)} F$ defines a functor from the category of finitely generated $\mathcal{N}(M)$-modules to the category of strongly coherent sheaves of $\mathcal{N}$-modules.

The idea that the category of strongly coherent sheaves of $\mathcal{N}$-modules is the good one for sheaf theory in the Nash case is clearly supported by the following result.

PROPOSITION 2.12 *Suppose that the equivalent properties of Theorem* 0.1 *hold. Then:*

(a) *A finite subsheaf of a strongly coherent sheaf of $\mathcal{N}$-modules is strongly coherent.*
(b) *Strongly coherent sheaves of $\mathcal{N}$-modules form an abelian subcategory of the category of sheaves of $\mathcal{N}$-modules.*

(c) *The global sections functor $H^0(M,-)$ induces an equivalence of abelian categories from strongly coherent sheaves of $\mathcal{N}$-modules to finitely generated $\mathcal{N}(M)$-modules, with inverse functor $\mathcal{N} \otimes_{\mathcal{N}(M)} -$.*

The derivation of these results is made exactly as for the compact case in [CoRzSh] (see also [Sh2, I.6.15]). Again, we will not repeat the arguments here.

3. Equivalence of factorization and the height 1 cases of the other problems

The proof that $\mathbf{Sep}_1(M)$, $\mathbf{Glob}_1^r(M)$ and $\mathbf{Ext}_1^r(M)$ are equivalent is the same as the proof we have done for the equivalence without the index 1. Remark that a finite sheaf of radical ideals $\mathcal{I}$ is locally principal if and only if $\mathfrak{X}(\mathcal{I})$ has complex codimension 1 in $M^{\mathbf{C}}$ at each point of $\mathfrak{X}(\mathcal{I}) \cap M$.

PROPOSITION 3.1 *If* $\mathbf{Sep}_1(M)$ *holds, then* $\mathbf{Fact}(M)$ *holds.*

Proof. For the proof, we can assume that M is connected. Let f be a Nash function on M and $f = f_1 f_2$ an analytic factorization. Since $\mathcal{N}(M)$ is a noetherian normal domain, we have $(f) = \mathfrak{p}_1^{\alpha_1} \cdots \mathfrak{p}_k^{\alpha_k}$ where the $\mathfrak{p}_i$'s are height one prime ideals of $\mathcal{N}(M)$, and the α_i's positive integers. We know by the separation property that the extensions $\mathfrak{p}_i\mathcal{O}(M)$ are prime. Let $x \in M$ be any point, and $\mathfrak{m}_x \subset \mathcal{O}(M)$ its maximal ideal; also, set $A_x = \mathcal{O}(M)_{\mathfrak{m}_x}$. Then, the ideal $(f)A_x$ is equal to $\mathfrak{p}_1^{\alpha_1} \cdots \mathfrak{p}_k^{\alpha_k} A_x$, and, since A_x is factorial, $(f_1)A_x$ is equal to a product $\mathfrak{p}_1^{\beta_1} \cdots \mathfrak{p}_k^{\beta_k} A_x$, where $0 \leqslant \beta_i \leqslant \alpha_i$. Fix $i = 1, \ldots, k$. Since $\mathfrak{p}_i\mathcal{O}(M)$ is finitely generated, its zero set Z_i in M is not empty. Thus, the localization $B_i = \mathcal{O}(M)_{\mathfrak{p}_i}$ coincides with $(A_x)_{\mathfrak{p}_i}$ for every $x \in Z_i$. This shows on the one hand that B_i is a discrete valuation ring whose valuation we denote by v_i, and on the other hand that the exponent β_i of the factorization of $f_1 A_x$ coincides with $v_i(f_1)$, hence it is the same for all $x \in Z_i$. Furthermore, if $x \notin Z_i$, the exponent does not matter because $\mathfrak{p}_i$ disappears in A_x. Then a standard application of Cartan's Theorem B yields $(f_1)\mathcal{O}(M) = \mathfrak{p}_1^{\beta_1} \cdots \mathfrak{p}_k^{\beta_k}\mathcal{O}(M)$. Choose a finite system of generators of $\mathfrak{p}_i$ in $\mathcal{N}(M)$, and let F_i be the sum of the squares of these generators. Then $\mathfrak{p}_i^2 = (F_i)$, and hence there is a strictly positive analytic function G_1 on M such that $f_1^2 = G_1 F_1^{\beta_1} \cdots F_k^{\beta_k}$. Then $g_1 = \sqrt{G_1}$ is a strictly positive analytic function on M, and $f_1/g_1 = \varphi_1$ is a Nash function since it is analytic, and its square is a Nash function. □

PROPOSITION 3.2 *If* $\mathbf{Fact}(M)$ *holds then* $\mathbf{Glob}_1^r(M)$ *holds.*

Proof. Let $\mathcal{I}$ be a finite sheaf of radical ideals of $\mathcal{N}$ such that all the ideals $\mathcal{I}_x$, $x \in M$, are principal. We need to prove that $\mathfrak{X}(\mathcal{I})$ is a $\mathbf{C}$-Nash M-germ. By Lemmas 1.4 and 1.9 there exists a sheaf of $\mathcal{N}$-ideals $\mathcal{J}$ with the same properties such that $\mathcal{I} \cap \mathcal{J} = \mathcal{I}\mathcal{J}$ is generated by global Nash functions $f_1, \ldots, f_k$ and the dimension of $\mathfrak{X}(\sqrt{\mathcal{I} + \mathcal{J}})$ is smaller than $\dim M - 1$. Let f denote the sum of squares $f_1^2 + \cdots + f_k^2$. Then f is a generator of $(\mathcal{I} \cap \mathcal{J})^2 = \mathcal{I}^2 \cap \mathcal{J}^2 = \mathcal{I}^2\mathcal{J}^2$. Let Ω be a Stein open neighborhood of M in $M^{\mathbf{C}}$, and $\mathcal{I}^{\mathbf{C}}$ an extension of $\mathcal{I}$ to

Ω such that $\mathcal{I}^{\mathbf{C}}$ keeps the properties of $\mathcal{I}$. Then by Cartan's Theorem A we have complex analytic functions $g_1, g_2, \dots$ on Ω which generate $\mathcal{I}^{\mathbf{C}}$ and are real-valued on M. Let $c_1, c_2, \dots$ be sufficiently small positive real numbers, and let g denote the restriction to M of $\sum_{i=1}^{\infty} c_i g_i^2$. Then g is an analytic function on M and it is a generator of $\mathcal{I}^2$. In the same way we obtain an analytic function h on M which is a generator of $\mathcal{J}^2$. Thus we have an analytic factorization $f = ghf'$ for some positive analytic function f'. Since we assume **Fact**(M), we can replace g by a Nash function, whose complexification has a zero set germ at M that coincides with $\mathfrak{X}(\mathcal{I})$. Hence $\mathfrak{X}(\mathcal{I})$ is a **C**-Nash M-germ. □

The reduction of the global equations and extension problems from the general case to the radical one cannot be done for locally principal finite sheaves as for finite sheaves (or coherent sheaves in the compact case). Proposition 2.11 does not help here. So we have to produce a specific argument.

LEMMA 3.3 *Let $\mathcal{I}$ be a finite sheaf of $\mathcal{N}$-ideals such that all the ideals $\mathcal{I}_x$, $x \in M$, are principal. Then there is a covering of M by finitely many semialgebraic open subsets U_i of M such that for each i, the restriction of the sheaf of ideals $\mathcal{I}|_{U_I}$ is generated by a principal ideal of $\mathcal{N}(U_i)$.*

Proof. We can find a covering of M by finitely many semialgebraic open subsets U_i such that the restriction $\mathcal{I}|_{U_i}$ is generated by an ideal of $\mathcal{N}(U_i)$. Moreover, by refining the covering, we can suppose that each U_i is contractible. This implies that all rings $\mathcal{N}(U_i)$ are factorial, see [BoCoRo, 12.7.17]. Hence we can suppose that $\mathcal{I}$ is generated by an ideal of $\mathcal{N}(M)$, say $I = (f_1, \dots, f_k)$, and that $\mathcal{N}(M)$ is factorial.

For any $x \in M$, the ideal $\mathcal{I}_x = I\mathcal{N}_x$ is principal, hence by descent, so is the localization $I_{\mathfrak{m}_x}$ at the maximal ideal $\mathfrak{m}_x$ of x in $\mathcal{N}(M)$. Thus, the localization of I at any maximal ideal of $\mathcal{N}(M)$ is principal, and since $\mathcal{N}(M)$ is factorial we find that I is principal. □

PROPOSITION 3.4 *Let $\mathcal{I}$ be a finite sheaf of $\mathcal{N}$-ideals such that all the ideals $\mathcal{I}_x$, $x \in M$, are principal. Then $\mathcal{I}$ can be written as a finite product $\mathcal{I} = \prod_{i=1}^{k} \mathcal{I}_i^{\alpha(i)}$ where $\alpha(i)$ is a positive integer, and each $\mathcal{I}_i$ is a finite sheaf of radical ideals such that $\mathcal{I}_{i,x}$ is principal for all $x \in M$ and $\mathfrak{X}(\mathcal{I}_i)$ is an irreducible invariant M-germ.*

Proof. Let X_i, $1 \leqslant i \leqslant k$ be the irreducible invariant components of $\mathfrak{X}(\mathcal{I})$, and set $\mathcal{I}_i = \mathfrak{I}(X_i)$. Let x be a point of M. Then all the invariant irreducible components of the germs $X_{i,x}$ at x have complex codimension 1 and hence all the $\mathcal{I}_{i,x}$'s are principal, since prime ideals of height 1 are principal in $\mathcal{N}_x$.

We can choose a covering of M by finitely many semialgebraic open subsets U_j such that:

- for each j, the restrictions to U_j of the sheaves $\mathcal{I}$ and $\mathcal{I}_i$, $1 \leqslant i \leqslant k$, are respectively generated by the elements f_j and $f_{j,i}$, $1 \leqslant i \leqslant k$, of $\mathcal{N}(U_j)$,
- all U_j and all non empty intersections $U_j \cap U_{j'}$ are contractible.

Indeed, the first condition may be realized by Lemma 3.3. Then, we get the second by refining the covering using a semialgebraic triangulation compatible with the first covering. Again, this condition implies that all the $\mathcal{N}(U_j)$'s and $\mathcal{N}(U_j \cap U_{j'})$'s are factorial domains.

Now write a factorization in $\mathcal{N}(U_j)$

$$f_j = \prod_{\lambda=1}^{l_j} u_j p_{j,\lambda}^{\beta(j,\lambda)},$$

where u_j is invertible, $p_{j,\lambda}$ are non associated irreducible elements, and $\beta(j,\lambda)$ are positive integers. The radical ideal of $f_j\mathcal{N}(U_j)$ is generated by $\prod_{\lambda=1}^{l_j} p_{j,\lambda}$ and also by $\prod_{i=1}^{k} f_{j,i}$. So each $p_{j,\lambda}$ is an irreducible factor of exactly one $f_{j,i}$, and all the irreducible factors of the $f_{j,i}$'s are obtained in this way. We claim that:

there is a positive integer $\alpha(i)$ such that, for any j, if $p_{j,\lambda}$ is any irreducible factor of $f_{j,i}$, then $\beta(j,\lambda) = \alpha(i)$.

The Proposition follows from this claim. So let us prove the claim. By Lemma 1.3, it is sufficient to prove that, when the restrictions of $p_{j,\lambda}$ and $p_{j',\lambda'}$ to $U_j \cap U_{j'}$ have a common irreducible factor q (and hence are irreducible factors of $f_{j,i}$ and $f_{j',i}$ with the same i), then $\beta(j,\lambda) = \beta(j',\lambda')$. The q-adic valuations of the restrictions to $U_j \cap U_{j'}$ of $p_{j,\lambda}$ and $p_{j',\lambda'}$ are both 1, so the q-adic valuations of the restrictions of f_j and $f_{j'}$ are respectively $\beta(j,\lambda)$ and $\beta(j',\lambda')$. These two numbers must be the same since the restrictions of f_j and $f_{j'}$ differ only by an invertible factor. □

PROPOSITION 3.5 *If* $\mathbf{Glob}_1^r(M)$ *holds, then* $\mathbf{Glob}_1(M)$ *holds.*

Proof. Let $\mathcal{I}$ be a finite sheaf of ideals of $\mathcal{N}$ such that for any $x \in M$, $\mathcal{I}_x$ is principal. We know from Lemma 3.4 that we can write $\mathcal{I} = \prod_{i=1}^{k} \mathcal{I}_i^{\alpha(i)}$ where each $\mathcal{I}_i$ is an irreducible finite sheaf of radical ideals such that $\mathcal{I}_{i,x}$ is principal for all $x \in M$ and $\alpha(i)$ is a positive integer. By the hypothesis $\mathbf{Glob}_1^r(M)$, each $\mathcal{I}_i$ is generated by an ideal I_i of $\mathcal{N}(M)$. Then $\mathcal{I}$ is generated by $\prod_{i=1}^{k} I_i^{\alpha(i)}$. □

PROPOSITION 3.6 *If* $\mathbf{Ext}_1^r(M)$ *holds, then* $\mathbf{Ext}_1(M)$ *holds.*

Proof. Let $\mathcal{I}$ be a finite sheaf of ideals of $\mathcal{N}$ such that for any $x \in M$, $\mathcal{I}_x$ is principal. Using Lemma 3.4 and grouping together all the factors with the same exponents, we can write $\mathcal{I} = \prod_{i=1}^{k} \mathcal{I}_i^{\alpha(i)}$ where each $\mathcal{I}_i$ is a finite sheaf of radical ideals such that $\mathcal{I}_{i,x}$ is principal for all $x \in M$ and $\alpha(1) > \alpha(2) > \cdots > \alpha(k) > 0$. We prove the Proposition by induction on $\alpha(1)$. The key idea of the proof is as in [Sh1] to use the fundamental class of a Nash set and Thom's realization Theorem [Th]. The case $\alpha(1) = 1$ is just the hypothesis $\mathbf{Ext}_1^r(M)$. Hence assume $\alpha(1) > 1$, and that the extension property holds for locally principal finite sheaves of ideals with smaller $\alpha(1)$. Let $\varphi \in H^0(M, \mathcal{N}/\mathcal{I})$. We first reduce the problem to the following easier case

$$\mathcal{I} = \mathcal{I}_1^2. \tag{$*$}$$

Let f_1 denote the sum of squares of a finite system of generators of $H^0(M, \mathcal{I}_1)$ (recall that $\mathbf{Glob}_1^r(M)$ holds by hypothesis). Then f_1 is a generator of $\mathcal{I}_1^2$, which induces an isomorphism from $\mathcal{N}$ to $\mathcal{I}_1^2$ defined by

$$\mathcal{N}_x \ni \rho \mapsto \rho f_{1,x} \in \mathcal{I}_{1,x}^2, \quad x \in M.$$

This isomorphism induces a commutative diagram

$$\begin{array}{ccc} H^0(M, \mathcal{N}) & \xrightarrow{\cong} & H^0(M, \mathcal{I}_1^2) \\ \downarrow & & \downarrow \\ H^0\left(M, \mathcal{N}/\mathcal{I}_1^{\alpha(1)-2} \prod_{i=2}^{k} \mathcal{I}_i^{\alpha(i)}\right) & \xrightarrow{\cong} & H^0(M, \mathcal{I}_1^2/\mathcal{I}). \end{array}$$

(Here the vertical maps are the natural ones.) If φ belongs to $H^0(M, \mathcal{I}_1^2/\mathcal{I})$, then, by the induction hypothesis, φ is the image of some element of $H^0(M, \mathcal{I}_1^2)$. Therefore, it suffices to reduce the problem to the case when $\varphi \in H^0(M, \mathcal{I}_1^2/\mathcal{I})$. Let φ_1 be the image of φ under the natural homomorphism

$$H^0(M, \mathcal{N}/\mathcal{I}) \to H^0(M, \mathcal{N}/\mathcal{I}_1^2).$$

Assume $\mathbf{Ext}_1(M)$ holds in the case $(*)$. Then we have a global Nash function $\Phi_1 \in H^0(M, \mathcal{N})$ whose image in $H^0(M, \mathcal{N}/\mathcal{I}_1^2)$ is φ_1. Replace φ by

$$\varphi - (\text{the image of } \Phi_1 \text{ in } H^0(M, \mathcal{N}/\mathcal{I})).$$

Then we can assume $\varphi \in H^0(M, \mathcal{I}_1^2/\mathcal{I})$, which implies the reduction to $(*)$.

Next apply the same arguments as above to the natural homomorphism

$$H^0(M, \mathcal{N}/\mathcal{I}_1^2) \to H^0(M, \mathcal{N}/\mathcal{I}_1),$$

and obtain a second reduction: to the case when

$$\varphi \in H^0(M, \mathcal{I}_1/\mathcal{I}_1^2). \tag{$**$}$$

To progress further, we need to clarify the structures of M and $\mathcal{I}^{-1}(0)$. By [Sh2, VI.2.1], we can assume that M is the interior of a compact Nash manifold N possibly with boundary. Let $\dim M = n$. For a Nash set Z in M of dimension $< n$, let $[Z]$ denote the fundamental class of Z in $H_{n-1}(Z \cup \partial N, \partial N; \mathbf{Z}_2)$. (If the dimension of Z is smaller than $n-1$, then $[Z] = 0$.) Let $[Z]_*$ denote the image of $[Z]$ in $H_{n-1}(N, \partial N; \mathbf{Z}_2)$. We recall the following fact. Let $Z(n-1) \subset Z$ be the set of the points z such that the germ Z_z has dimension $n-1$, and let Z_1 be a connected component of $Z(n-1)$. Then Z_1 also has a fundamental class in $H_{n-1}(Z \cup \partial N, \partial N; \mathbf{Z}_2)$, and if $Z_1 = Z(n-1)$ then the class equals $[Z]$.

Assume $[\mathcal{I}_1^{-1}(0)]_* = 0$. Then we prove that $H^0(M, \mathcal{I}_1)$ is principal as follows. We have the homology exact sequence

$$\begin{aligned} 0 &\to H_n(N, \partial N; \mathbf{Z}_2) \to H_n(N, \mathcal{I}_1^{-1}(0) \cup \partial N; \mathbf{Z}_2) \\ &\to H_{n-1}(\mathcal{I}_1^{-1}(0) \cup \partial N, \partial N; \mathbf{Z}_2). \end{aligned}$$

Let N_1 be a union of some connected components of $M \backslash \mathcal{I}_1^{-1}(0)$ such that the fundamental class in $H_n(N, \mathcal{I}_1^{-1}(0) \cup \partial N; \mathbf{Z}_2)$ of the closure $\overline{N_1}$ is carried to $[\mathcal{I}_1^{-1}(0)]$. Then $\overline{N_1}$ contains $(\mathcal{I}_1^{-1}(0))(n-1)$ by the definition of the above sequence, and for each $x \in M$ there exists a function germ $\rho \in \mathcal{I}_{1x}$ such that $\{\rho \geqslant 0\} = \overline{N_1} \cup \mathcal{I}_1^{-1}(0)$ as germs at x. Let $\gamma_1 : M \backslash \mathcal{I}^{-1}(0) \to \{1, -1\}$ denote the map defined by

$$\gamma_1(x) = \begin{cases} 1 & \text{for } x \in N_1, \\ -1 & \text{for } x \in (M \backslash \mathcal{I}^{-1}(0)) \backslash N_1. \end{cases}$$

Let f_1 be as above a generator of $\mathcal{I} = \mathcal{I}_1^2$. Then we can define a generator g_1 of $H^0(M, \mathcal{I}_1)$ as follows

$$g_1^2 = f_1 \quad \text{on } \mathcal{I}^{-1}(0) \quad \text{and } g_1 = |\sqrt{f_1}|\gamma_1 \quad \text{on } M \setminus \mathcal{I}^{-1}(0).$$

Hence $H^0(M, \mathcal{I}_1)$ is principal.

Continue to assume $[\mathcal{I}_1^{-1}(0)]_* = 0$. As above, by multiplying by a generator of $H^0(M, \mathcal{I}_1)$, we obtain a commutative diagram

$$\begin{array}{ccc} H^0(M, \mathcal{N}) & \xrightarrow{\cong} & H^0(M, \mathcal{I}_1) \\ \downarrow & & \downarrow \\ H^0(M, \mathcal{N}/\mathcal{I}_1) & \xrightarrow{\cong} & H^0(M, \mathcal{I}_1/\mathcal{I}_1^2), \end{array}$$

hence reducing the case $(**)$ to the case when $\varphi \in H^0(M, \mathcal{N}/\mathcal{I}_1)$, and, consequently, proving $\mathbf{Ext}_1(M)$.

Let us consider the general case of $[\mathcal{I}_1^{-1}(0)]_*$. By [Th] there exists a compact C^∞ submanifold L_1 of N of dimension $n-1$ such that if $\partial L_1 \neq \emptyset$ then ∂L_1 is included in ∂N, L_1 is transversal to ∂N in N, $[L_1 \backslash \partial L_1]_* = [\mathcal{I}_1^{-1}(0)]_*$, and the dimension of $L_1 \cap \mathcal{I}_1^{-1}(0)$ is smaller than $n-1$. (Here the last condition is not stated in [Th], but it is satisfied once the other conditions hold and we modify L_1.) By the Stone–Weierstrass approximation Theorem we can assume, moreover, that L_1 is of class Nash. Let $\mathcal{J}_1$ denote the sheaf of germs of Nash functions vanishing on $L_1 \backslash \partial L_1$. Set $\mathcal{K}_1 = \mathcal{I}_1 \mathcal{J}_1$. Then $\mathcal{K}_1$ is a finite sheaf of radical ideals of $\mathcal{N}$, and we have $[\mathcal{K}_1^{-1}(0)]_* = 0$. As above, we have a commutative diagram

$$\begin{array}{ccc} H^0(M, \mathcal{N}) & \xrightarrow{\cong} & H^0(M, \mathcal{J}_1^2) \\ \downarrow & & \downarrow \\ H^0(M, \mathcal{N}/\mathcal{I}_1^2) & \xrightarrow{\cong} & H^0(M, \mathcal{J}_1^2/\mathcal{K}_1^2). \end{array}$$

This reduces the problem to the case where $\varphi \in H^0(M, \mathcal{J}_1^2/\mathcal{K}_1^2)$, and by the proof above in the case $[\mathcal{I}_1^{-1}(0)]_* = 0$, φ is the image of some function of $H^0(M, \mathcal{J}_1^2)$. This completes the proof of $\mathbf{Ext}_1(M)$. □

4. Positive answers in particular cases

We first consider the case where the real zero set is zero dimensional.

PROPOSITION 4.1 *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{N}(M)$. If $\mathfrak{p}^{-1}(0)$ has dimension zero, then separation holds for $\mathfrak{p}$.*

Proof. Then $\mathfrak{p}^{-1}(0)$ consists of one single point a, and it suffices to see that $\mathfrak{p}\mathcal{O}_a$ is a prime ideal. By Artin's approximation Theorem this is equivalent to see that $\mathfrak{p}\mathcal{N}_a$ is prime. Suppose $fg \in \mathfrak{p}\mathcal{N}_a$, where $f, g \in \mathcal{N}_a$. Let $h_1, \dots, h_s$ be generators of $\mathfrak{p}$ and set

$$f' = f^2 + \sum_i h_i^2, \qquad g' = g^2 + \sum_i h_i^2.$$

Since $\mathfrak{p}\mathcal{N}_a$ is a radical ideal, it is enough to show that either f' or g' is in $\mathfrak{p}\mathcal{N}_a$. In other words, we may assume that a is an isolated zero of both f and g. Then, by [Sh3], there are global Nash functions F, G such that $F_a = uf$, $G_a = vg$, where u, v are units of $\mathcal{N}_a$, and $F^{-1}(0) = G^{-1}(0) = \{a\}$. It follows that $FG \in$

$\mathfrak{p}$, and since $\mathfrak{p}$ is prime, either $F \in \mathfrak{p}$ or $G \in \mathfrak{p}$. Hence, either $f \in \mathfrak{p}\mathcal{N}_a$ or $g \in \mathfrak{p}\mathcal{N}_a$. □

PROPOSITION 4.2 *If $\mathcal{I}$ is a finite sheaf of ideals of $\mathcal{N}$ such that $\mathcal{I}^{-1}(0)$ has dimension* 0, *then global equations and extension hold for $\mathcal{I}$.*

Proof. By hypothesis $\mathcal{I}^{-1}(0)$ consists of finitely many points. Let a be one of them. We will find an ideal $J(a)$ of $\mathcal{N}(M)$ which generates $\mathcal{I}_a$ and whose only zero in M is a; then the product of $J(a)$'s will generate $\mathcal{I}$, and so we have global equations. Pick finitely many Nash functions g_i, $1 \leqslant i \leqslant s$, defined on an open semialgebraic neighborhood of a, whose germs at a generate $\mathcal{I}_a$. Set $h_i = g_i + c\sum_{j=1}^{s} g_j^2$, $1 \leqslant i \leqslant s$. We can find a sufficiently large $c > 0$, so that the zero set of h_i in some open neighborhood of a is compact. We can find a Nash function ϕ_i defined and positive on a neighborhood of a, such that the germ of $\phi_i h_i$ at a coincides with the germ of a global Nash function F_i ([Sh3]). Analogously, after multiplication by a positive function, we extend $\sum_{j=1}^{s} g_j^2$ to a global Nash function F_0 whose only zero is a (see again [Sh3]). Then $J(a) = (F_0, \dots, F_s)$ is what we want.

Now let φ be a global section of $\mathcal{N}/\mathcal{I}$. Then the sheaf $\mathcal{I}(\varphi)$ of ideals of $\mathcal{N}_{M\times\mathbf{R}}$ is obviously finite and $\mathcal{I}(\varphi)^{-1}(0)$ is a finite number of points. Hence $\mathcal{I}(\varphi)$ is generated by its global sections, and Proposition 2.2 allows to lift φ to a global Nash function on M. So extension holds for $\mathcal{I}$. □

Next, we turn to the case of real ideals whose singularities are isolated in the real zero set. A point a of a Nash set N is called *Nash nonsingular* if a is a smooth point of N of the maximal dimension and if for some Nash functions f_i vanishing on N, grad f_i span the normal vector space.

PROPOSITION 4.3 *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{N}(M)$. If $\mathfrak{p}$ is real and $\mathfrak{p}^{-1}(0)$ has only isolated Nash singularities, then separation holds for $\mathfrak{p}$.*

Proof. We need to prove that $\mathfrak{X}(\mathfrak{p})$ is irreducible as an invariant M-germ. Assume $\mathfrak{X}(\mathfrak{p})$ were not so. There would exist two finite sheaves of radical $\mathcal{N}_M$-ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $\mathcal{I}_1 \cap \mathcal{I}_2 = \mathfrak{p}\mathcal{N}$ and $\mathcal{I}_1^{-1}(0) \cap \mathcal{I}_2^{-1}(0)$ is a finite number of points. This comes from Lemma 1.9. Hence it suffices to prove the following statement.

> *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be finite sheaves of radical $\mathcal{N}_M$-ideals such that all irreducible invariant components of $\mathfrak{X}(\mathcal{I}_1)$ and $\mathfrak{X}(\mathcal{I}_2)$ have the same dimension,* $\dim \mathfrak{X}(\mathcal{I}_1) \cap \mathfrak{X}(\mathcal{I}_2) < \dim \mathfrak{X}(\mathcal{I}_1)$, *$\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2$ is generated by global Nash functions, and $\mathcal{I}_1|_U$ and $\mathcal{I}_2|_U$ are generated by Nash functions on U, for an open semialgebraic neighborhood U of $\mathcal{I}_1^{-1}(0) \cap \mathcal{I}_2^{-1}(0)$ in M. Then $\mathcal{I}_1$ and $\mathcal{I}_2$ are generated by global Nash functions.*

First we show that for a small open semialgebraic neighborhood U_2 of $\mathcal{I}_2^{-1}(0)$ in M, $\mathcal{I}_1|_{U\cup U_2}$ is generated by Nash functions on $U \cup U_2$. Let $f_1, \dots, f_k \in \mathcal{N}(U)$ be generators of $H^0(U, \mathcal{I}_1)$. Replace f_i by $f_i + f' \sum_{j=1}^{k} f_j^2$, $1 \leqslant i \leqslant k$, for a sufficiently large positive Nash function f' on U, and add $\sum_{j=1}^{k} f_j^2$ to the generators. Then we can assume that each f_i is positive on

$$U \cap \mathcal{I}_2^{-1}(0)\backslash(\text{a neighborhood of } \mathcal{I}_1^{-1}(0) \text{ in } M).$$

Hence by [Sh3], multiplying f_i by a positive Nash function on U, we can extend f_i to $U \cup U_2$ for a small U_2 so that the extension is positive on $U_2\backslash U$. Hence $\mathcal{I}_1|_{U\cup U_2}$ is generated by these extensions.

Secondly we want to show that the restriction $\mathcal{I}_2|_{U\cup U_2}$ is generated by Nash functions on $U \cup U_2$. We know that $\mathcal{I}|_{U\cup U_2}$ and $\mathcal{I}_1|_{U\cup U_2}$ are generated respectively by radical ideals I and I_1 of $\mathcal{N}(U \cup U_2)$. We have

$$\mathfrak{X}(I) = \mathfrak{X}(I_1) \cup \mathfrak{X}(\mathcal{I}_2|_{U\cup U_2}),$$

$$\dim \mathfrak{X}(I_1) \cap \mathfrak{X}(\mathcal{I}_2|_{U\cup U_2}) < \dim \mathfrak{X}(I_1),$$

and all the irreducible invariant components of $\mathfrak{X}(I_1)$ and $\mathfrak{X}(\mathcal{I}_2|_{U\cup U_2})$ have the same dimension. From this it follows that

- any irreducible invariant component of $\mathfrak{X}(I_1)$ or of $\mathfrak{X}(\mathcal{I}_2|_{U\cup U_2})$ is an irreducible invariant component of $\mathfrak{X}(I)$, and hence an irreducible invariant component of $\mathfrak{X}(\mathfrak{q})$ for some minimal prime divisor $\mathfrak{q}$ of I,
- $\mathfrak{X}(I_1)$ and $\mathfrak{X}(\mathcal{I}_2|_{U\cup U_2})$ have no common irreducible invariant component.

Hence I_1 must be the intersection of some of the minimal prime divisors of I, and therefore $\mathcal{I}_2|_{U\cup U_2}$ is generated by the intersection of the other minimal prime divisors. Finally apply the arguments in the above first step to $\mathcal{I}_2|_{U\cup U_2}$. Then we can choose generators of $\mathcal{I}_2|_{U\cup U_2}$ so that they are extensible to M and the extensions are generators of $\mathcal{I}_2$. In the same way we see that $\mathcal{I}_1$ is generated by global Nash functions. □

PROPOSITION 4.4 *Let $\mathcal{I}$ be a finite sheaf of ideals of $\mathcal{N}$. Global equations holds for $\mathcal{I}$ if each stalk $\mathcal{I}_x$ is real, and the intersection of M with the Zariski closure of $\mathcal{I}^{-1}(0)$ in $\mathbf{R}^m$ has only isolated Nash singularities.*

Proof. Set $X = \mathfrak{X}(\mathcal{I})$. It suffices to prove that X is Nash. Without loss of generality we can assume X irreducible as an invariant M-germ, and $M = \mathbf{R}^m$. Let X' denote the smallest Nash $\mathbf{R}^m$-germ containing X, and let X'' denote the germ at $\mathbf{R}^m$ of the Zariski closure of X in $\mathbf{C}^m$. Then X' is a union of some components of the decomposition of X'' into irreducible invariant M-germs. Hence by hypothesis, if we set $Y = X' \cap \mathbf{R}^m$, then X' is the germ at $\mathbf{R}^m$ of a complexification of Y, Y is irreducible as a Nash set, and Y has only isolated Nash singularities. Let $\mathfrak{p}$ be the ideal of global Nash functions vanishing on X'. Then $\mathfrak{p}^{-1}(0) = Y$, $\mathfrak{X}(\mathfrak{p}) = X'$, and $\mathfrak{p}$ satisfies the condition of 4.3. Hence X' is irreducible as an invariant M-germ. It follows that $X' = X$, i.e. X is Nash. □

The next case is when the complex zero set germ has dimension 1.

PROPOSITION 4.5 *If $\mathfrak{p}$ is a prime ideal of $\mathcal{N}(M)$ of coheight $\leqslant 1$, then separation holds for $\mathfrak{p}$. If a finite sheaf of ideals $\mathcal{I}$ of $\mathcal{N}$ is everywhere of coheight $\leqslant 1$, then global equations and extension hold for $\mathcal{I}$.*

Proof. For separation there are two cases: the dimension of $\mathfrak{p}^{-1}(0)$ is 0 or 1. If the dimension is 0, then we apply Proposition 4.1; if the dimension is 1, then $\mathfrak{p}$ satisfies the condition in Proposition 4.3.

Now let $\mathcal{I}$ be a finite sheaf of radical ideals of $\mathcal{N}$ everywhere of coheight $\leqslant 1$. By considering the irreducible invariant components of $\mathfrak{X}(\mathcal{I})$, we know from the first part of the Proposition that global equations holds for $\mathcal{I}$. For the same reason it holds for the sheaf $\mathcal{I}(\varphi)$ where φ is any global section of $\mathcal{N}/\mathcal{I}$. Hence, by Proposition 2.2, extension holds for $\mathcal{I}$.

Finally, using Proposition 2.11, we get that global equations and extension hold for any finite sheaf of ideals everywhere of coheight $\leqslant 1$. □

The following result is a generalization of a result in [TaTo] (see also [MoRa]). It concerns the case where the quotient is normal. It matches the result of [Qu] obtained under an assumption of normality.

PROPOSITION 4.6 *If $\mathfrak{p}$ is a prime ideal of $\mathcal{N}(M)$ such that the quotient $\mathcal{N}(M)/\mathfrak{p}$ is normal, then separation holds for $\mathfrak{p}$. If $\mathcal{I}$ is a finite sheaf of ideals of $\mathcal{N}$ such that all the quotients $\mathcal{N}_x/\mathcal{I}_x$, $x \in M$, are normal, then global equations and extension hold for $\mathcal{I}$.*

Proof. If $\mathcal{N}(M)/\mathfrak{p}$ is normal, then for any $x \in M$ the localization $\mathcal{N}(M)_{\mathfrak{m}_x}/\mathfrak{p}\mathcal{N}(M)_{\mathfrak{m}_x}$ is normal, and hence the henselization $\mathcal{N}_x/\mathfrak{p}\mathcal{N}_x$ is normal, so $\mathfrak{p}\mathcal{N}_x$ is prime. Since moreover $\mathfrak{p}^{-1}(0)$ is connected, we have that $\mathfrak{X}(\mathfrak{p})$ is an irreducible invariant M-germ, which means that separation holds for $\mathfrak{p}$.

To prove that global equations hold for $\mathcal{I}$, we use an idea of the proof of Artin–Mazur's theorem (see [BoCoRo, 8.4.4]). The fact that $\mathcal{N}_x/\mathcal{I}_x$ is normal for any x implies that $\mathcal{I}$ is a finite sheaf of radical ideals. We can suppose that $\mathcal{I}^{-1}(0)$ is connected, and then the argument above shows that $X = \mathfrak{X}(\mathcal{I})$ is an irreducible invariant M-germ. Let $\mathfrak{p}$ be the prime ideal of $\mathcal{N}(M)$ of global Nash functions vanishing on X. The normalization of $\mathcal{N}(M)/\mathfrak{p}$ is a finite $\mathcal{N}(M)/\mathfrak{p}$-module, so we can write this normalization as $\mathcal{N}(M)[y]/\mathfrak{q}$ where $y = (y_1, \dots, y_s)$ and $\mathfrak{q}$ is a prime ideal. Let X' be the M-germ $\mathfrak{X}(\mathfrak{p})$, and Y the $M \times \mathbf{R}^s$-germ $\mathfrak{X}(\mathfrak{q})$. Let $\mathfrak{r}_1, \dots, \mathfrak{r}_k$ be the minimal prime divisors of $\mathfrak{q}\mathcal{N}(M \times \mathbf{R}^s)$. Since $\mathcal{N}(M \times \mathbf{R}^s)$ is ind-étale over $\mathcal{N}(M)[y]$, the quotients $\mathcal{N}(M \times \mathbf{R}^s)/\mathfrak{r}_i$ are normal. So, by separation, the $\mathfrak{X}(\mathfrak{r}_i)$'s are the irreducible invariant components of Y.

Set $A = \mathcal{I}^{-1}(0)$, $A' = \mathfrak{p}^{-1}(0)$, $B = \mathfrak{q}^{-1}(0)$. By the assumption that the quotients $\mathcal{N}_x/\mathcal{I}_x$ are normal, the germ inclusion $f: X \to X'$ factorizes through the projection $p: Y \to X'$, which gives $g: X \to Y$. We have

$$p \circ g = f, \qquad f(A) \subset A', \qquad g(A) \subset B, \qquad p(B) \subset A'.$$

Moreover, since $f(X \backslash A) \subset X' \backslash A'$ and $f|_A$ is proper, $g(X \backslash A) \subset Y \backslash B$ and $g|_A$ is proper.

On the algebraic side, for any x in A, the local homomorphism

$$(\mathcal{N}(M)/\mathfrak{p})_{\mathfrak{m}_x} \to \mathcal{N}_x/\mathcal{I}_x,$$

induces, by normality and henselianity of $\mathcal{N}_x/\mathcal{I}_x$, a local homomorphism

$$\alpha_x : \mathcal{N}_{M\times\mathbf{R}^s,g(x)}/\mathfrak{q}\mathcal{N}_{M\times\mathbf{R}^s,g(x)} \to \mathcal{N}_x/\mathcal{I}_x.$$

This α_x is surjective, since composing it with

$$\mathcal{N}_x/\mathfrak{p}\mathcal{N}_x \to \mathcal{N}_{M\times\mathbf{R}^s,g(x)}/\mathfrak{q}\mathcal{N}_{M\times\mathbf{R}^s,g(x)}$$

gives the surjection $\mathcal{N}_x/\mathfrak{p}\mathcal{N}_x \to \mathcal{N}_x/\mathcal{I}_x$. And since both source and target of α_x are normal, and have the same dimension, α_x is an isomorphism.

Putting together these informations, we find that g is an isomorphism onto some irreducible invariant component, say $\mathfrak{X}(\mathfrak{r}_1)$, of Y. Let φ_i be the composition of the coordinate y_i with g, which can be identified with a global section of $\mathcal{N}/\mathcal{I}$: the germ $\varphi_{i,x}$ at $x \in \mathcal{I}^{-1}(0)$ is the image of the coordinate y_i under the isomorphism α_x. We can construct the sheaf $\mathcal{I}(\varphi_1,\ldots,\varphi_s)$ of ideals of $\mathcal{N}_{M\times\mathbf{R}^s}$, as it was done for $\mathcal{I}(\varphi)$ at the beginning of Section 2. We have that $\mathcal{I}(\varphi_1,\ldots,\varphi_s)$ is a finite sheaf of radical ideals by Lemma 2.1, and since

$$\mathfrak{X}(\mathcal{I}(\varphi_1,\ldots,\varphi_s)) = g(X) = \mathfrak{X}(\mathfrak{r}_1),$$

it is generated by its global sections. Then, applying s times Proposition 2.2, we conclude that $\mathcal{I}$ itself is generated by its global sections. So global equations hold for $\mathcal{I}$.

Lastly we prove extension for $\mathcal{I}$. Let $\varphi \in H^0(M,\mathcal{N}/\mathcal{I})$, and consider the sheaf $\mathcal{I}(\varphi)$ of $\mathcal{N}_{M\times\mathbf{R}}$-ideals. It is a finite sheaf of radical ideals by Lemma 2.1. For each $(x,t) \in \mathcal{I}(\varphi)^{-1}(0)$, the stalk $\mathcal{N}_{M\times\mathbf{R},(x,t)}/\mathcal{I}(\varphi)_{(x,t)}$ is isomorphic to the stalk $\mathcal{N}_{M,x}/\mathcal{I}_x$. Hence $\mathcal{N}_{M\times\mathbf{R},(x,t)}/\mathcal{I}(\varphi)_{(x,t)}$ is normal. Therefore, by the above proof, $\mathcal{I}(\varphi)$ is generated by its global sections. Then, by Proposition 2.2, φ is extensible to a global Nash function on M, which completes the proof. □

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