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Abstract. The classification is obtained by expanding the ideas pertaining to the so-called last possible place of unitarity as well as a certain diagrammatic presentation of $\triangle_n^+$. In this approach it is a key step to establish that in the singular case, certain maximal submodules are generated by a single highest weight vector. Using a duality between highest weight modules and holomorphically induced modules previously established, these ideas can be further extended to also give a direct classification of the annihilators. In particular, at no place do we use induction from lower rank.

1. Introduction

That spaces of holomorphic sections over a hermitian symmetric space $\mathcal{D}$ of the non-compact type could carry unitary representations of the group G of holomorphic transformations of $\mathcal{D}$ (or a suitable covering group $\tilde{G}$ thereof) was first discovered and utilized by Harish-Chandra in his work on the holomorphic discrete series [11].

The idea of the analytic continuation of the holomorphic discrete series was investigated in [22] and reached its first peak in two independent studies in the mid 1970s of the case of holomorphic *functions* on $\mathcal{D}$. We refer to this case as the *scalar* case. It seems to be generally agreed upon that N. Wallach [28] with his algebraic approach was a little earlier than the analytic approach of H. Rossi and M. Vergne [25], and (maybe) for this reason, the discrete part of the set of unitary representations for this situation became known as the Wallach set. Nevertheless, the description of the most singular spaces $\mathcal{H}_i$ as Fourier-Laplace transforms – in the unbounded realization $\mathcal{D}_u$ of $\mathcal{D}$ – of distributions supported by subsets $\mathcal{O}_i$ of the boundary of a certain cone Ω used in the definition of $\mathcal{D}_u$, as furnished by the Rossi and Vergne, has for the current investigation been the most important result. Indeed, the spaces $\mathcal{O}_i$ are orbits under a certain subgroup of G and are as such immediately recognizable as open subsets of the real points of algebraic varieties. Specifically, in this realization, the space of holomorphic constant coefficient differential operators that annihilate the representation connected with the orbit $\mathcal{O}_i$ is equivalent, via the Fourier–Laplace transformation, to the *prime* ideal defined by $\mathcal{O}_i$.

To make our presentation as self-contained as possible, we also present our own approach to these representations. For use below, define $W((0, \lambda_2^{sc})')$ to be the most

singular non-trivial unitary representation in the scalar case. This representation is annihilated by a prime ideal generated by some second order differential operators.

Let us make some more remarks about the history of the subject: For reasons of physics, the conformal group $SU(2,2)$ early received special attention ([13], see also [23] and [26]) but after that, a case-by-case hunt for the full set of unitary holomorphic representations (or, equivalently, unitarizable highest weight modules) took place for some years: $SU(p,q)$: [15], $Mp(N,\mathbb{R})$: [16] (whereby the first proof of the Kashiwara-Vergne Conjecture ([21]) was completed), and $O(n,2)$: [10] (see also [24]). Finally, the classification was completed in [17] and, independently, in [8].

In the last few years there has been a renewed interest in certain geometrical aspects of the representation theory connected with these spaces, see e.g. [5], [6], [9], [20], and this has then provoked the present article which is the completion of a project undertaken in connection with the classification problem but which was never completed because one could get by without it. An example of what we have in mind is the reported result of Davidson, Enright and Stanke [6] stating that in the singular unitary case, a certain maximal submodule is generated by a single highest weight vector. This is a result which we knew was very important and which we, at least implicitly, established for the exceptional cases *en route* to our classification ([17]). The reason is, provided of course it is proved without using the classification, that it leads almost directly to the classification.

So-far, the approaches have either been case-by-case, or have used induction on rank, or both. The approach we present here is, in our opinion, very different. With it, almost all arguments are reduced to simple combinatorics on certain two-dimensional diagrams. Of course, it must be mentioned that to obtain this reduction, we have to rely on a celebrated theorem of Bernstein, Gelfand and Gelfand [1]. At this time it should also be said that even though our methods are algebraic or combinatoric, the theorems we are proving either directly belong to harmonic analysis or have immediate translations into some that do.

The method which is presented here begins by attacking the so-called 'last possible place of unitarity'. In fact, the method is successful exactly because there *is* unitarity at that point. Before going into more technical details, it may be worthwile to quote the theorem which guided us into the philosophy of that method, and which still, not the least on the intuitive level, is very important.

THEOREM 1.1. ([15]) *Let $W(\tau)$ be the unique irreducible submodule of a holomorphically induced representation $\mathcal{P}(V_\tau)$ corresponding to the irreducible unitary representation τ of a maximal compact subgroup. Suppose $W(\tau)$ is unitary and that the representation (necessarily unitary) $W(\tau + (0, \lambda_2^{sc})')$ obtained from $W(\tau) \otimes W((0, \lambda_2^{sc})')$ by restricting the tensor product to the diagonal in $\mathcal{D} \times \mathcal{D}$ is annihilated by a dth order differential operator. Then $W(\tau)$ is annihilated by a $(d-1)$th order differential operator.*

Since the spaces cannot be annihilated by 0th order operators, this explains why a unitary representation which is annihilated by a 1st order differential operator must correspond, in an appropriate sense, to the 'last possible place of unitarity'.

Interpreted correctly the whole question of unitarity at the singular points, also in the general 'vector-valued' case, is equivalent to the study of the differential operators that annihilate the representation. Indeed, by duality, the radical of the canonical hermitian form (occasionally called the 'missing K-types', cf. [15]) can be seen to determine the space of constant coefficient differential operators that annihilate the representation spaces. As a consequence, once there is a differential operator that annihilates the space, there is also a *covariant* differential operator (cf. [18] and below).

The main results established here are the following:

- An intrinsic classification of the set of unitarizable highest weight modules.
- An explicit description of the annihilator $\text{Ann}_{\mathfrak{p}^-}(\tau)$ of $W(\tau)$ in $\mathcal{U}(\mathfrak{p}^-)$.
- In the singular case, the *polar* $W(\tau)^0$ (the maximal submodule of the dual module to $\mathcal{P}(V_\tau)$) is generated by a single highest weight vector. We prove this explicitly in the case where γ_0 is on a wall (see below).

More generally denote, for $f \in W(\tau)$, the annihilator of f in $\mathcal{U}(\mathfrak{p}^-)$ by $\text{Ann}_{\mathfrak{p}^-}(f, \tau)$. As noted by Davidson, Enright and Stanke [6] and Joseph [20], not only are the spaces $\text{Ann}_{\mathfrak{p}^-}(\tau)$ prime ideals, but for all $f \in W(\tau)$, $\text{Ann}_{\mathfrak{p}^-}(f, \tau) = \text{Ann}_{\mathfrak{p}^-}(\tau)$. These remarks can also easily be obtained from the explicit description of the annihilators given in Theorem 7.3. In fact, the result of Rossi and Vergne can be used in connection with the following proposition from Section 2 and the classical Payley–Wiener Theorem by going to the unbounded realization and there use that an appropriate group of translations, after complexification, is conjugate to $\mathfrak{p}^-$.

PROPOSITION 2.9. *Let g be an element of a connected complex Lie group $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$ and set $\mathfrak{p}^g = \text{Ad}(g)(\mathfrak{p}^-)$. Then,*

$$\text{Ann}_{\mathfrak{p}^g}(\tau) = \text{Ad}(g)(\text{Ann}_{\mathfrak{p}^-}(\tau)). \tag{28}$$

However, we shall not pursue this issue any further since our interest here is to give the mentioned intrinsic proofs of the above results, and since, as demonstrated elegantly by Joseph [20], the general results can be obtained directly from these.

Instead we finish this introduction with a more detailed account of the content:

Section 2 contains the basic definitions of the modules, unitarity, annihilators, etc., as well as some important results relating them. Section 3 describes our approach to the result of Bernstein, Gelfand and Gelfand and in Section 4, spaces of vector-valued polynomials are investigated. Also there, the important *last possible point of unitarity* is defined, as is the corresponding positive non-compact root γ_0. In Section 5 we give the definitions and results leading to the presentation of

the set of positive non-compact roots by 2-dimensional diagrams. For the sake of completeness, proofs are included, but the reader is referred to [17], where these results were first proved, for a list of all the diagrams. The usefulness of this presentation is illustrated by two results that are used in the following sections. Section 6 is devoted to translating results about homomorphisms between highest-weight modules into propositions about possible configurations of non-compact roots, in particular their relation to the γ_0 mentioned above. The unitarity of so-called scalar modules is then settled. At the end of the section, special attention is given to the case where γ_0 is on a wall of the diagram. Here some finer results can be obtained by simple arguments because the possible configurations are very few and easy. Moreover, these results turn out to be sufficient for our purposes. Finally, the presentation culminates in Section 7 where the main results are stated and proved.

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2. Background

2.1. MODULES

Let $\mathfrak{g}_0$ be a simple Lie algebra over $\mathbb{R}$ and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ a Cartan decomposition of $\mathfrak{g}_0$. We assume that $\mathfrak{k}_0$ has a non-empty center η; in this case $\eta = \mathbb{R} \cdot h_0$ for an $h_0 \in \eta$ whose eigenvalues under the adjoint action on $\mathfrak{p}_0^{\mathbb{C}}$ are $\pm i$. From the next displayed equation on we adopt the convention that, unless otherwise indicated, Gothic letters denote the complexified algebras. Thus, $\mathfrak{g}$ denotes $\mathfrak{g}_0^{\mathbb{C}}$ and $\mathfrak{p}_0^{\mathbb{C}}$ plainly becomes $\mathfrak{p}$. Let

$$\mathfrak{p}^+ = \{z \in \mathfrak{p} \mid [h_0, z] = iz\}, \tag{1}$$

and

$$\mathfrak{p}^- = \{z \in \mathfrak{p} \mid [h_0, z] = -iz\}. \tag{2}$$

Let $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$ and let $\mathfrak{h}_0$ be a maximal abelian subalgebra of $\mathfrak{k}_0$. Then $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathbb{C} \cdot h_0$, $\mathfrak{h} = [\mathfrak{h} \cap \mathfrak{k}_1] \oplus \mathbb{C} \cdot h_0$, $(\mathfrak{h} \cap \mathfrak{k}_1)$ is a Cartan subalgebra of $\mathfrak{k}_1$, and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Moreover,

$$\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+. \tag{3}$$

We let ω denote (-1) times the conjugation in $\mathfrak{g}$ relative to the real form $\mathfrak{g}_0$ of $\mathfrak{g}$ and let $\sigma = -\omega$. Thus, $\mathfrak{g}_0$ is the -1 eigenspace. We extend ω to an anti-linear anti-involution of $\mathcal{U}(\mathfrak{g})$. The sets of compact and non-compact roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ are denoted $\triangle_c$ and $\triangle_n$, respectively. $\triangle = \triangle_c \cup \triangle_n$. Let Σ_c denote a fixed set of simple compact roots. We choose an ordering of $\triangle$ such that

$$\mathfrak{p}^+ = \sum_{\alpha \in \triangle_n^+} \mathfrak{g}^\alpha, \tag{4}$$

and set

$$\mathfrak{g}^+ = \sum_{\alpha \in \triangle^+} \mathfrak{g}^\alpha, \qquad \mathfrak{g}^- = \sum_{\alpha \in \triangle^-} \mathfrak{g}^\alpha, \tag{5}$$

and

$$\rho = \frac{1}{2} \sum_{\alpha \in \triangle^+} \alpha. \tag{6}$$

Throughout, β denotes the unique simple non-compact root. For $\gamma \in \triangle$ let H_γ be the unique element of $i\mathfrak{h}_0 \cap [\mathfrak{g}^\gamma, \mathfrak{g}^{-\gamma}]$ for which $\gamma(H_\gamma) = 2$. Then for all γ_1 in $\triangle$,

$$\langle \gamma_1, \gamma \rangle = \frac{2(\gamma_1, \gamma)}{(\gamma, \gamma)} = \gamma_1(H_\gamma), \tag{7}$$

where $(\cdot, \cdot)$ is the bilinear form on $(\mathfrak{h})^*$ obtained from the Killing form of $\mathfrak{g}$. The reflexion corresponding to $\gamma \in \triangle$ is denoted by σ_γ;

$$\sigma_\gamma(\gamma_1) = \gamma_1 - \langle \gamma_1, \gamma \rangle \gamma. \tag{8}$$

For $\alpha \in \triangle_n^+$ choose $z_\alpha \in \mathfrak{g}^\alpha$ such that

$$[z_\alpha, z_\alpha^\sigma] = H_\alpha, \tag{9}$$

and let $z_{-\alpha} = z_\alpha^\sigma$. Essentially following the notation of [25] we let $\gamma_\mathfrak{r}$ denote the highest root. Then $\gamma_\mathfrak{r} \in \triangle_n^+$, and $H_{\gamma_\mathfrak{r}} \notin [\mathfrak{h} \cap \mathfrak{k}_1]$.

If Λ_0 is a dominant integral weight of $\mathfrak{k}_1$ and if $\lambda \in \mathbb{R}$ we denote by $\Lambda = (\Lambda_0, \lambda)$ the linear functional on $\mathfrak{h}^\mathbb{C}$ given by

$$\Lambda\big|_{(\mathfrak{h} \cap \mathfrak{k}_1)}\mathbb{C} = \Lambda_0, \qquad \Lambda(H_{\gamma_\mathfrak{r}}) = \lambda. \tag{10}$$

Such a Λ determines an irreducible finite-dimensional $\mathcal{U}(\mathfrak{k})$-module which we, for convenience, denote by V_τ, where $\tau = \tau_\Lambda$ is the corresponding representation of the connected, simply connected Lie group $\tilde{K}$ with Lie algebra $\mathfrak{k}$. Further, let

$$M(V_\tau) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k} \oplus \mathfrak{p}^+)} V_\tau \tag{11}$$

denote the generalized Verma module of highest weight Λ, and let M_Λ denote the Verma module of which $M(V_\tau)$ is a quotient.

In what follows we choose to represent our Hermitian symmetric space $\mathcal{D}$ as a bounded domain in $\mathfrak{p}^-$. Consider an (irreducible) finite-dimensional $\mathcal{U}(\mathfrak{k})$-module V_τ. Through the process of holomorphic induction the space $\mathcal{P}(V_\tau)$ of V_τ-valued polynomials on $\mathfrak{p}^-$ becomes a $\mathcal{U}(\mathfrak{g})$-module consisting of $\mathfrak{k}$- (or $\tilde{K}$-)

finite vectors. We maintain the notation $\mathcal{P}(V_\tau)$ for this module and let dU_τ denote the corresponding representation of $\mathfrak{g}$. Explicitly, let

$$(\delta(z_0)f)(z) = \left.\frac{d}{dt}\right|_{t=0} f(z + tz_0) \tag{12}$$

for $z_0, z \in \mathfrak{p}^-$, and $f \in C^\infty(\mathfrak{p}^-)$. Then, for $p \in \mathcal{P}(V_\tau)$ we have [18]:

$$(dU_\tau(x)p)(z) = -(\delta(x)p)(z) \quad \text{for } x \in \mathfrak{p}^-, \tag{13}$$
$$(dU_\tau(x)p)(z) = d\tau(x)p(z) - (\delta([x,z])p)(z) \text{ for } x \in \mathfrak{k}, \tag{14}$$

and

$$(dU_\tau(x)p)(z) = d\tau([x,z])p(z) - \tfrac{1}{2}(\delta([[x,z],z])p)(z) \text{ for } x \in \mathfrak{p}^+. \tag{15}$$

It follows from these formulas (especially the first) that the space

$$W(\tau) = \mathrm{Span}\{dU_\tau(u) \cdot v \mid v \in V_\tau, u \in \mathcal{U}(\mathfrak{g})\}, \tag{16}$$

is contained in any invariant subspace. In particular, $W(\tau)$ is irreducible.

Remark 2.1. Notice also that $W(\tau)$ is a highest weight module in its own right. However, the roles of $\mathfrak{p}^+$ and $\mathfrak{p}^-$ have been interchanged. But, this is just equivalent to replacing h_0 with $-h_0$, or, to '*forget*' the definitions of $\mathfrak{p}^\pm$. This is both a very useful and sometimes confusing phenomenon. In the determination of covariant differential operators (defined below) it is quite powerful. See also [3], [4].

Let V_τ and V_{τ_1} be finite-dimensional (irreducible) $\mathcal{U}(\mathfrak{k})$-modules, and let D be a constant coefficient holomorphic differential operator on $\mathfrak{p}^-$ with values in $\mathrm{Hom}(V_\tau, V_{\tau_1})$.

DEFINITION 2.2. $D\colon \mathcal{P}(V_\tau) \to \mathcal{P}(V_{\tau_1})$ is covariant iff

$$\forall x \in \mathfrak{g}; \quad D\, dU_\tau(x) = dU_{\tau_1}(x)D. \tag{17}$$

Let $\tilde{G}$ denote the connected, simply connected Lie group with Lie algebra $\mathfrak{g}$. We remark here that dU_τ is always the differential of a representation U_τ of $\tilde{G}$ on the space of holomorphic V_τ-valued functions on $\mathcal{D}$. By holomorphy and analyticity, Definition 2.2 is then equivalent to demanding that D should intertwine U_τ and U_{τ_1}.

Along with $\mathcal{P}(V_\tau)$ we consider the space $\mathcal{E}(V_{\tau'})$ of holomorphic constant coefficient differential operators on $\mathfrak{p}^-$ with values in the contragredient module $V_\tau' = V_{\tau'}$ to V_τ. For $p \in \mathcal{P}(V_\tau)$ and $q \in \mathcal{E}(V_{\tau'})$ let

$$(q, p) = \left(q\left(\frac{\partial}{\partial z}\right), p(\cdot) \right)(0). \tag{18}$$

This bilinear pairing clearly places $\mathcal{P}(V_\tau)$ and $\mathcal{E}(V_{\tau'})$ in duality and, as a result, $\mathcal{E}(V_{\tau'})$ becomes a $\mathcal{U}(\mathfrak{g})$-module. Specifically,

$$\left((\mathrm{d}U'_\tau(x)q) \left(\frac{\partial}{\partial z} \right), p(\cdot) \right) = \left(q \left(\frac{\partial}{\partial z} \right), (\mathrm{d}U_\tau(-x)p)(\cdot) \right)(0), \qquad (19)$$

for $x \in \mathfrak{g}$. The following result was stated in [18]. The proof is straightforward (cf. the appendix to [12]).

PROPOSITION 2.3. *As* $\mathcal{U}(\mathfrak{g})$*-modules,* $\mathcal{P}(V_\tau)' = \mathcal{E}(V_{\tau'}) = M(V_{\tau'})$.

From this it follows [18]:

PROPOSITION 2.4. *A homomorphism* $\varphi\colon M(V_{\tau_1}) \to M(V_\tau)$ *gives rise, by duality, to a covariant differential operator* $D_\varphi\colon \mathcal{P}(V_{\tau'}) \to \mathcal{P}(V_{\tau'_1})$, *and conversely.*
    Proof. By Proposition 2.3, we may view φ as a homomorphism from $\mathcal{E}(V_{\tau_1})$ to $\mathcal{E}(V_\tau)$. $V_{\tau_1} \subset \mathcal{E}(V_{\tau_1})$ and thus there exists an element T_φ in $\mathcal{E}(\mathrm{Hom}(V_{\tau_1}, V_\tau))$ such that $\varphi(v) = T_\varphi(v)$ for $v \in V_{\tau_1}$. Since φ is a module map it then follows that

$$\forall q \in \mathcal{E}(V_{\tau_1}), \; \varphi(q) = T_\varphi(q) \quad \text{(pointwise)}. \qquad (20)$$

D_φ is then the transpose of T_φ. The converse is equally obvious. $\square$

2.2. UNITARITY

Corresponding to the decomposition (3) we have

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})\mathfrak{p}^+ \oplus \mathcal{U}(\mathfrak{k}) \oplus \mathfrak{p}^-\mathcal{U}(\mathfrak{g}). \qquad (21)$$

Let $P_\mathfrak{k}$ denote the projection of $\mathcal{U}(\mathfrak{g})$ onto $\mathcal{U}(\mathfrak{k})$ according to this and let $(\cdot, \cdot)_{V_\tau}$ denote the inner product on V_τ for which τ is unitary. Then, as is well known, the prescription

$$\forall u_1, u_2 \in \mathcal{U}(\mathfrak{g}); \forall v_1, v_2 \in V_\tau:$$
$$B_\tau(u_1 \otimes v_1, u_2 \otimes v_2) \;=\; (v_1, \mathrm{d}\tau(P_\mathfrak{k}(\omega(u_1)u_2))v_2)_{V_\tau} \qquad (22)$$

defines, by sesqui-linearity, a hermitian form B_τ on $M(V_\tau)$. It is $\mathfrak{g}$-contravariant in the following sense:

$$\forall m_1, m_2 \in M(V_\tau); \forall x \in \mathfrak{g}: \quad B_\tau(x \cdot m_1, m_2) = B_\tau(m_1, \omega(x) \cdot m_2). \quad (23)$$

Moreover, up to multiplication by a non-zero, real constant, it is unique in this respect. We let N_τ (or N_Λ) denote the radical of B_τ;

$$N_\tau = \{m_0 \in M(V_\tau) \mid \forall m \in M(V_\tau)\colon B_\tau(m_0, m) = 0\}. \qquad (24)$$

It is clear that any $\mathfrak{k}$-type $V \subseteq M(V_\tau)$, $V \neq V_\tau$, which is annihilated by $\mathfrak{p}^+$ is contained in N_τ. Thus, all images under non-surjective homomorphisms of highest weight modules into $M(V_\tau)$ are contained in N_τ. From this it follows easily that any proper invariant subspace is contained in N_τ.

Unitarity is in this connection equivalent to the induced form on $M(V_\tau)/N_\tau$ being positive definite. In many cases, one is more interested in $W(\tau)$. Since the duality (18) is defined on $\mathfrak{k}$-types, it is possible, though the pairing is not topological, to transport a (pre-) Hilbert space structure back and forth. For this reason, we shall not distinguish between the spaces as far as unitarity is concerned. In fact, we may summarize our position by the following

DEFINITION 2.5. We say that dU_τ is unitarizable on $W(\tau)$, or simply, unitary, if the hermitian form induced by $B_{\tau'}$ on $M(V_{\tau'})/N_{\tau'}$ is positive definite.

Observe that the vector space decomposition of $\mathcal{P}(V_\tau)$ into $\mathfrak{k}$-types is independent of the λ in $\tau = (\Lambda_0, \lambda)$. We shall occasionally use the following terminology:

DEFINITION 2.6. If a $\mathfrak{k}$-type $q \in \mathcal{P}(V_\tau)$ belongs to N_τ for a $\tau = (\Lambda_0, \lambda)$ we say that q *vanishes* at λ.

2.3. ANNIHILATORS

Using the action (13) as well as the duality (18) there are some natural annihilators one can associate to $W(\tau)$:

DEFINITION 2.7.

$$W^0(\tau') = \{q \in \mathcal{E}(V_{\tau'}) \mid \forall p \in W(\tau): (q, p) = 0\}. \tag{25}$$

$$\mathrm{Ann}_{\mathfrak{p}^-}(\tau) = \{q \in \mathcal{U}(\mathfrak{p}^-) = \mathcal{P} \mid \forall p \in W(\tau): dU_\tau(q)(p) \equiv 0\}. \tag{26}$$

Since $W(\tau)$ is translation invariant,

$$q \in \mathrm{Ann}_{\mathfrak{p}^-}(\tau) \Leftrightarrow \forall p \in W(\tau): \left(q\left(\frac{\partial}{\partial z}\right)p\right)(z) = 0$$

for a fixed z, say, $z = 0$. Thus, it follows that

$$\mathrm{Ann}_{\mathfrak{p}^-}(\tau) = \{q \in \mathcal{P} \mid \forall v' \in V_{\tau'}: q \cdot v' \in W^0(\tau)\}. \tag{27}$$

This fact immediately implies

LEMMA 2.8. $\mathrm{Ann}_{\mathfrak{p}^-}(\tau)$ *is invariant under the adjoint action of $\mathfrak{k}$ on $\mathfrak{p}^-$.*

In certain situations it may be of interest to consider $\mathrm{Ann}_{\mathfrak{p}^+}(\tau)$ instead of $\mathrm{Ann}_{\mathfrak{p}^-}(\tau)$, in fact, it is interesting to define $\mathrm{Ann}_{\mathfrak{p}}(\tau)$ for any abelian subalgebra $\mathfrak{p}$

of $\mathfrak{g}$. In this connection we have the following somewhat surprising result:

PROPOSITION 2.9. *Let g be an element of a connected complex Lie group $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$ so that $\mathfrak{p}^g = \mathrm{Ad}(g)(\mathfrak{p}^-)$. Then,*

$$\mathrm{Ann}_{\mathfrak{p}^g}(\tau) = \mathrm{Ad}(g)(\mathrm{Ann}_{\mathfrak{p}^-}(\tau)). \tag{28}$$

Proof. It clearly suffices to consider elements of the form $g = e^{t \cdot X}$ for $X \in \mathfrak{g}$ and t small. But, everything in sight being holomorphic, $e^{t \cdot X} p$, for $p \in W(\tau)$, is given by a convergent power series, which converges uniformly on compact subsets of $\mathbb{R} \times \mathfrak{p}^-$. Since each summand is in $W(\tau)$, if $dU_\tau(q)W(\tau) = 0$, then also $dU_\tau(q) e^{t \cdot X} W(\tau) = 0$. Finally, by switching the roles of $\mathfrak{p}^g$ and $\mathfrak{p}^-$ and by replacing g by g^{-1}, the other inclusion follows. □

Remark 2.10. It is well known that e.g. $\mathfrak{p}^+$ satisfies the requirements for Proposition 2.9. Thus, any algebraic property of $\mathrm{Ann}_{\mathfrak{p}^-}$ is inherited by $\mathrm{Ann}_{\mathfrak{p}^+}$ (and vice versa). One such property is, clearly, that they are ideals.

3. BGG

The most fundamental result used in this article is the well-known theorem of Bernstein, Gelfand and Gelfand [1]. It emphasizes the importance of the Weyl group and the Bruhat order, but, as we shall see below, one particular feature of the present framework is that only certain subsets of the Weyl group are relevant. One such subset, specialized to our situation is

$$W^c = \{w \in W \mid w^{-1}\triangle_c^+ \subset \triangle^+\}. \tag{29}$$

The general analogues of this have been studied by Deodhar [7] and Boe [2].

We present here our approach [17]. Instead of W^c we focus directly on all Weyl group elements that can be written as products of reflections by non-compact roots. The following proposition is essentially Proposition 3.6 of [17].

DEFINITION 3.1. Let χ, $\psi \in \mathfrak{h}^*$. A sequence of roots $\alpha_1, \ldots, \alpha_k \in \triangle^+$ is said to satisfy condition (A) for the pair (χ, ψ) if

- $\chi = \sigma_{\alpha_k} \cdots \sigma_{\alpha_1} \psi$.
- Put $\chi_0 = \psi$ and $\chi_i = \sigma_{\alpha_i} \cdots \sigma_{\alpha_1} \psi$. Then $\chi_{i-1} - \chi_i = n_i \alpha_i$, where $n_i = \langle \chi_{i-1}, \alpha_i \rangle \in \mathbb{N}$ for all $i = 1, \ldots, k$.

We will occasionally refer to such a sequence as a BGG-chain.

PROPOSITION 3.2. *Let $\tau_i = \tau_{\Lambda_i}$ for $i = 1, 2$. Let $\varphi \neq 0$ be a homomorphism from $M(V_{\tau_1})$ to $M(V_{\tau_2})$. Then there exists a sequence $\alpha_1, \ldots, \alpha_k$ of elements of $\triangle_n^+$ which satisfies condition (A) for the pair $(\Lambda_1 + \rho, \Lambda_2 + \rho)$.*

In fact, the following stronger result holds (with a slight adaptation to our needs). Since we want to keep track of what we use, and where, we list it separately:

PROPOSITION 3.3. (Strong BGG) *Let $\tau_i = \tau_{\Lambda_i}$ for $i = 1, 2$. For the conclusion of Proposition 3.2 to hold, it is sufficient that $M(V_{\tau_1})$ occurs in the composition series of $M(V_{\tau_2})$.*

Observe that in Proposition 3.2 (and Proposition 3.3),

$$\Lambda_1 = \Lambda_2 - \left(\sum_{i=1}^{k} n_i \alpha_i \right). \tag{30}$$

DEFINITION 3.4. The decomposition

$$-p = \Lambda_1 - \Lambda_2 = - \left(\sum_{i=1}^{k} n_i \alpha_i \right) \tag{31}$$

corresponding to the above will be referred to as the canonical presentation of $\Lambda_1 - \Lambda_2$. We will also refer to the 'p' occuring in (31) as the p occuring in the canonical presentation.

Of course, there may be several canonical presentations, corresponding to different sequences of roots satisfying condition (A).

The following result, which is of interest in its own right, will be used in Section 7:

PROPOSITION 3.5. *Let $\alpha_1, \ldots, \alpha_k \in \triangle_n^+$ be a sequence of non-compact roots, satisfying condition (A) for a pair $(\chi, \psi) = (\Lambda_1 + \rho, \Lambda + \rho) \in \mathfrak{h}^*$. Put $\chi_0 = \psi$ and $\chi_i = \sigma_{\alpha_i} \cdots \sigma_{\alpha_1} \psi$ for $i = 1, \ldots, k$. Then $\chi_i = (\Lambda + \rho) - p_i$ for a $p_i = n_1 \alpha_1 + \cdots + n_i \alpha_i$, with $n_1, \ldots, n_i \in \mathbb{N}$, and*

$$\langle \Lambda + \rho, p_i \rangle = 1. \tag{32}$$

Proof. This follows by induction: For $i = 1, p_1 = n_1 \alpha_1$, where $n_1 = \langle \Lambda + \rho, \alpha_1 \rangle$. Thus, $\langle \Lambda + \rho, p_1 \rangle = \frac{2(\Lambda + \rho, n_1 \alpha_1)}{n_1^2(\alpha_1, \alpha_1)} = 1$. Let us then assume that $\chi_{i-1} = \Lambda + \rho - p_{i-1}$ satisfies the condition and consider $\chi_i = \Lambda + \rho - p_{i-1} - n_i \alpha_i$ with $n_i = \frac{2(\Lambda + \rho - p_{i-1}, \alpha_i)}{(\alpha_i, \alpha_i)}$. Then

$$\begin{aligned}
2(\Lambda + \rho, p_{i-1} + n_i \alpha_i) &= (p_{i-1}, p_{i-1}) + 2n_i(\Lambda + \rho, \alpha_i) \tag{33} \\
&= (p_{i-1}, p_{i-1}) + 2n_i(\Lambda + \rho - p_{i-1} + p_{i-1}, \alpha_i) \\
&= (p_{i-1}, p_{i-1}) + n_i^2(\alpha_i, \alpha_i) + 2n_i(p_{i-1}, \alpha_i) \\
&= (p_i, p_i).
\end{aligned}$$

$\square$

Remark 3.6. Observe that (32) is true in particular for the $p_i = p$ occuring in the canonical presentation of $\Lambda_1 - \Lambda$.

4. $\mathcal{U}(\mathfrak{p}^-) \otimes V_\tau$

4.1. THE BASIC OBSERVATIONS

The key fact in what follows is the following well known lemma which is a direct reformulation of the facts that $\mathfrak{p}^\pm$ are abelian and that $[\mathfrak{p}^+, \mathfrak{p}^-] \subseteq \mathfrak{k}$:

LEMMA 4.1. *Let $\gamma \in \triangle$. The coefficient of β in γ is $-1, 0$, or 1.*

In particular, if $\gamma \in \triangle_n^+$ the coefficient is 1. Hence the following is immediately seen to hold:

COROLLARY 4.2. *Let $\gamma_1, \gamma_2 \in \triangle_n^+$. Then $\langle \gamma_1, \gamma_2 \rangle \in \{0, 1, 2\}$.*

COROLLARY 4.3. *Let $\alpha \in \triangle_n$ and $\mu \in \triangle_c$. Then*

$$\langle \mu, \alpha \rangle \in \{-1, 0, 1\} \quad \text{and} \quad \langle \alpha, \mu \rangle \in \{-2, -1, 0, 1, 2\}. \tag{34}$$

Proof. $\sigma_\alpha(\mu) = \mu - \langle \mu, \alpha \rangle \cdot \alpha$, hence the first assertion is clear. Let $\langle \alpha, \mu \rangle = n$ and observe that $\alpha - n\mu \in \triangle$. Since $\langle \alpha - n\mu, \alpha \rangle = 2 - |n|$, it follows that $\sigma_\alpha(\alpha - n\mu) = (|n| - 1)\alpha - n\mu$. $\qquad \square$

COROLLARY 4.4. *In the notation of Proposition 3.2 let $\varphi(1) = p \in M(V_{\tau_2})$. Then p is a homogeneous polynomial of degree $d = \sum_{i=1}^k n_i$.*
 Proof. It is clear, by the above observation, that $p \in \mathcal{U}(\mathfrak{p}^-) \cdot V_{\tau_2}$ has weight $\Lambda_2 - (\sum_{i=1}^k n_i \gamma_i)$. Since β occurs exactly once in each γ_i the claim follows. $\quad \square$

Remark 4.5. Should several sequences from $\triangle_n^+$ satisfy condition (A) for the same pair $(\Lambda_1 + \rho, \Lambda_2 + \rho)$, this degree, of course, is the same.

Let v_Λ denote a fixed non-zero element in the subspace of $V_\tau = V_\Lambda$ corresponding to the highest weight. Equivalently, $v_\Lambda \neq 0$ and $(\mathfrak{k})^+ v_\Lambda = 0$. We recall the following which is a well-known result from the theory of tensor products between finite-dimensional representations:

LEMMA 4.6. *Any non-zero $\mathcal{U}(\mathfrak{k})$ highest weight vector in $\mathcal{U}(\mathfrak{p}^-) \otimes V_\Lambda$ has a non-zero coefficient in $\mathcal{U}(\mathfrak{p}^-) \otimes v_\Lambda$.*

We also recall the following very useful observations:

LEMMA 4.7. *Let $\alpha \in \triangle_n^+$. Then $\Lambda_0 - \alpha$ is a highest weight for the $\mathcal{U}(\mathfrak{k}_1)$-module $\mathfrak{p}^- \otimes V_{\Lambda_0}$ if and only if the following two conditions are satisfied:*

- $\Lambda_0 - \alpha$ is $\mathfrak{k}_1$ *dominant*.
- If $\alpha = \alpha_1 + \mu$ *with* $\mu \in \Sigma_c$ *and* $\alpha_1 \in \triangle_n^+$, *then* $\Lambda_0(H_\mu) > 0$.

Proof. [17].   □

LEMMA 4.8. *Let* $\alpha \in \triangle_n^+$ *and assume* $\alpha - \mu_j \in \triangle_n^+$ *for* $\mu_j \in \Sigma_c$ *for* $j = 1, \ldots, i$. *Then* $\Lambda_0 - \alpha$ *is a highest weight for the* $\mathcal{U}(\mathfrak{k}_1)$ *module* $\mathfrak{p}^- \otimes V_\tau$ *if and only if*

$$\forall j = 1, \ldots, i : \langle \Lambda_0, \mu_j \rangle \geqslant \max\{1, \langle \alpha, \mu_j \rangle\}. \tag{35}$$

Proof. [17]. (Observe that a minor misprint has been corrected.)   □

Remark 4.9. We shall later see that the 'i' above is at most 2.

4.2. POLYNOMIALS

We begin by quoting a key fact from the existing literature:

Let $\gamma_1 = \beta, \gamma_2, \ldots \gamma_r$, be a maximal set of orthogonal roots in $\triangle_n^+$, constructed so that γ_i is the element in $\triangle_n^+ \cap \{\gamma_1 \ldots, \gamma_{i-1}\}^\perp$ with the smallest height; $i = 2, \ldots, r$. (The real rank of $\mathfrak{g}_0$ is then equal to r). Let $\delta_i = \gamma_1 + \cdots + \gamma_i, i = 1, \ldots, r$.

PROPOSITION 4.10. ([27]) *The set of highest weights of the irreducible submodules of the* $\mathfrak{k}$-*module* $\mathcal{U}(\mathfrak{p}^-)$ *are*

$$\{-i_1\delta_1 - \cdots - i_r\delta_r \mid (i_1, \ldots, i_r) \in (\mathbb{Z}_+)^r\}. \tag{36}$$

There are no multiplicities.

The following observation will be interesting in Sections 6 and 7:

PROPOSITION 4.11. *For* $i = 1, \ldots, \cdot\cdot$, *let* m_i *denote the* $\mathfrak{k}$-*type in* $\mathcal{U}(\mathfrak{p}^-)$ *of highest weight* $-\delta_i$. *Then, for* $i = 1, \ldots, r-1$, $m_{i+1} \in m_1 \otimes m_i$, *whereas* $m_{i+j} \notin m_1 \otimes m_i$ *for* $j > 1$.

Proof. Of course, $m_1 = \mathfrak{p}^-$ so, since m_{i+1} has degree $i + 1$, by definition of $\mathcal{U}(\mathfrak{p}^-)$ there has to be some $\mathfrak{k}$-type m of degree i such that $m_{i+1} \in m \otimes m_1$. Let the weight of m be $\omega = -r_1\delta_1 + \cdots - r_i\delta_i$. Since the highest weights in $m \otimes m_1$ are of the form $\omega + \gamma$, for certain weights γ of m_1, that is, $\gamma \in \triangle_n^-$, it thus follows that

$$\gamma = (r_1 + \cdots + r_i - 1)\gamma_1 + (r_2 + \cdots + r_i - 1)\gamma_2$$
$$+ \cdots + (r_i - 1)\gamma_i - \gamma_{i+1}, \tag{37}$$

hence $r_1 = \cdots = r_{i-1} = 0$, $r_i = 1$, and $\gamma = -\gamma_{i+1}$.

The second half of the statement follows by essentially the same argument. □

4.3. THE LAST POSSIBLE PLACE OF UNITARITY

We briefly recall the following from [17]:

As a $\mathcal{U}(\mathfrak{k})$-module, $M(V_\tau) = \mathcal{U}(\mathfrak{p}^-) \otimes V_\tau$. The restriction of B_Λ to each $\mathfrak{k}$-irreducible subspace is then either positive definite, negative definite, or identically zero. For any irreducible $\mathfrak{k}$-representation there is a fixed degree so that any subspace of $M(V_\tau)$ that transforms according to this type has got this degree. It is furthermore clear, because of the center of $\mathfrak{k}$, that types of different degree are perpendicular with respect to B_τ. Likewise, it is clear that two different $\mathfrak{k}$-types are perpendicular.

We will throughout this article be interested in the one-parameter family of representations $\Lambda = \Lambda(\lambda) = (\Lambda_0, \lambda)$ obtained by holding Λ_0 fixed and letting λ vary throughout $\mathbb{R}$.

As a $\mathfrak{k}_1$-representation, $V_{\Lambda_0} = V_\Lambda$ for all λ. In fact, let V be a subspace of $M(V_\tau)$ which is invariant and irreducible under $\mathfrak{k}$ and let q be a fixed highest weight vector. Assume q has degree d. Then q is a highest weight vector for all λ. Consider the function

$$b_q(\lambda) = B_\Lambda(q, q). \tag{38}$$

Then we have (cf. [17])

LEMMA 4.12. $b_q(\lambda) = (-1)^d C_q \lambda^d + $ *lower order terms in λ and $C_q > 0$.*

The zeros of b_q are the only places where the restriction of B_Λ to the irreducible subspace in question can change signature.

For a general $\mathfrak{k}$-type in $M(V_\tau)$ it is clear, since it is finite-dimensional, that for λ sufficiently negative the hermitian form will be positive definite. Likewise, for λ sufficiently positive, the type will be either posive definite or negative definite. Furthermore, it is obvious that the signature of the hermitian form on any $\mathfrak{k}$-type, varying with λ, can only change at specific values of λ at which the form becomes degenerate. Due to the observation that different types are perpendicular, it then follows that any sign change on a given $\mathfrak{k}$-type implies that the hermitian form on all of $M(V_\tau)$ is degenerate. Thus, there is a non-trivial invariant subspace, and hence, there is at least one non-trivial homomorphism $\varphi: M(V_{\tau_1}) \to M(V_\tau)$ for some τ_1. This observation will be crucial later on. For the moment we will focus on $\mathfrak{p}^- \otimes V_\tau$:

If q is a first order polynomial and if $b_q(\lambda_q) = 0$ it is clear that $W_{(\Lambda_0, \lambda)'}$ cannot be unitary for $\lambda < \lambda_q$. The smallest λ_q determined by a highest weight vector of degree 1; λ_0, was named 'the last possible place of unitarity' and was explicitly determined, for an arbitrary Λ_0 in [16]:

PROPOSITION 4.13. *Let $\Lambda_0 - \alpha_1, \ldots, \Lambda_0 - \alpha_s$ be the set of highest weights of*

the $\mathcal{U}(\mathfrak{k}_1)$-*module* $\mathfrak{p}^- \otimes V_{\Lambda_0}$ *for* $\alpha_1, \ldots, \alpha_s \in \triangle_n^+$. *Let, for* $i = 1, \ldots, s$, λ_i *be determined by the equation*

$$((\Lambda_0, \lambda_i) + \rho)(H_{\alpha_i}) = 1. \tag{39}$$

Then the last possible place of unitarity, λ_0 *is given by*

$$\lambda_0 = min\{\lambda_1, \ldots, \lambda_s\}. \tag{40}$$

DEFINITION 4.14. The unique element of $\triangle_n^+$ for which this minimum is attained is denoted by $\gamma_0 = \gamma_0(\Lambda_0)$.

Remark 4.15. That γ_0 indeed **is** unique follows from the discussions in the following 2 sections.

5. Diagrams

We present here the lemmas leading to the presentation of $\triangle_n^+$ as a two-dimensional diagram. A substantial part of this can be found in [17], to which article we also refer for the actual diagrams. We include it here since we need to use portions of the proofs later on. Also, we have tried to make the construction more pedagogical.

LEMMA 5.1. *Let* $\alpha \in \triangle_n^+$, *let* $\mu_1, \ldots, \mu_i$ *be distinct elements of* Σ_c, *and assume that* $\alpha + \mu_j \in \triangle_n^+$ *for all* $j = 1, \ldots, i$. *Then* $i \leqslant 2$. *If* $i = 2$ *then* $\alpha + \mu_1 + \mu_2 \in \triangle_n^+$.
   Proof. (i) Assume $i \geqslant 2$ and $\forall j$: $(\alpha, \mu_j) = 0$.
   Since in this case $(\alpha + \mu_j, \alpha + \mu_j) = (\alpha, \alpha) + (\mu_j, \mu_j)$, it follows that $(\alpha + \mu_j, \alpha + \mu_j) = 2(\alpha, \alpha) = 2(\mu_j, \mu_j)$, and α and μ_j are both short. Consider two distinct elements μ_{j_1} and μ_{j_2} from $\{\mu_1, \ldots, \mu_i\}$, and let $\langle \alpha + \mu_{j_1}, \alpha + \mu_{j_2} \rangle = n$. By Corollary 4.2, $n = 0, 1$, or 2, but $n = 1$ is excluded since $\mu_{j_1} - \mu_{j_2}$ is not a root. Because $\alpha + \mu_{j_2}$ is long, it follows that $\langle \mu_{j_1}, \mu_{j_2} \rangle = \pm 2$ and thus, since the roots are simple, $\langle \mu_{j_1}, \mu_{j_2} \rangle = -2$. This, however, is not possible since by symmetry, $\langle \mu_{j_2}, \mu_{j_1} \rangle = -2$, and μ_{j_1} is not proportional to μ_{j_2}. Thus we conclude that there can be at most one μ_j for which $(\alpha, \mu_j) = 0$. If this happens then α and μ_j are short and $\alpha + \mu_j$ is long.
   (ii) Assume $i \geqslant 3$, $(\alpha, \mu_{j_1}) = 0$, $(\alpha, \mu_{j_2}) \neq 0$, and $(\alpha, \mu_{j_3}) \neq 0$.
   Set $n_2 = \langle \alpha, \mu_{j_2} \rangle$ and $n_3 = \langle \alpha, \mu_{j_3} \rangle$. By Corollary 4.3 it follows easily that root strings containing α are at most of length 3. As in i), α is short and hence $n_2 = n_3 = -1$. Again by Corollary 4.3, $\langle \mu_{j_2}, \alpha \rangle = \langle \mu_{j_3}, \alpha \rangle = -1$, and, in particular, μ_{j_2} and μ_{j_3} are short. Clearly, $\alpha + \mu_{j_2}$ and $\alpha + \mu_{j_3}$ are short. It follows that $\langle \alpha + \mu_{j_1}, \alpha + \mu_{j_2} \rangle = 1 + \langle \mu_{j_1}, \mu_{j_2} \rangle$ and, as in i), this implies that $\langle \mu_{j_1}, \mu_{j_2} \rangle = -1$. Likewise, $\langle \mu_{j_1}, \mu_{j_3} \rangle = -1$. Consider now $\langle \alpha + \mu_{j_2}, \alpha + \mu_{j_3} \rangle$. By the previous observations, this equals $\langle \mu_{j_2}, \mu_{j_3} \rangle$, but, on the other hand, it must clearly equal zero. Hence, μ_{j_2} and μ_{j_3} are perpendicular, and hence $\alpha + \mu_{j_2} + \mu_{j_3} \in \triangle_n^+$. But, $\langle \alpha + \mu_{j_2} + \mu_{j_3}, \alpha + \mu_{j_1} \rangle = -1$, and this is a contradiction. So, if a μ_{j_1} is present and satisfies $\langle \alpha, \mu_{j_1} \rangle = 0$, at most one other simple compact root, say, μ_{j_2} can be

present. In this case, α, μ_{j_1}, and μ_{j_2} are short, $\alpha + \mu_{j_1}$ is long and $\alpha + \mu_{j_1} + \mu_{j_2}$ is also a root. Observe that $\alpha - \mu_{j_1}$ also is a long non-compact root.

(iii) Assume $i \geqslant 2$ and $\langle \alpha, \mu_1 \rangle = -2$.

It follows from ii) that $\langle \alpha, \mu_2 \rangle = -n < 0$, that α is long, and that μ_1 is short. But then $\langle \alpha + n\mu_2, \alpha + 2\mu_1 \rangle < 0$ which is impossible. Thus, if $\langle \alpha, \mu_1 \rangle = -2$ then $i = 1$.

(iv) Assume $\langle \alpha, \mu_1 \rangle = \langle \alpha, \mu_2 \rangle = \langle \alpha, \mu_3 \rangle = -1$.

Then α, μ_1, μ_2 and μ_3 are of equal length. By computing the various $\langle \alpha + \mu_i, \alpha + \mu_j \rangle$ terms, it follows that the compact roots must be pairwise orthogonal. But then, say, $\alpha + \mu_1 + \mu_2 \in \triangle_n^+$ and the only possible value of $\langle \alpha + \mu_1 + \mu_2, \alpha + \mu_3 \rangle$ is -1, which is impossible. Thus, there can be at most two simple compact roots, say, μ_1 and μ_2 such that $\langle \alpha, \mu_1 \rangle = \langle \alpha, \mu_2 \rangle = -1$. In this case, $\langle \mu_1, \mu_2 \rangle = 0$ and $\alpha + \mu_1 + \mu_2 \in \triangle_n^+$. $\square$

Naturally, the following result also holds:

COROLLARY 5.2. *Let* $\alpha \in \triangle_n^+$, *let* $\mu_1, \dots, \mu_i$ *be distinct elements of* Σ_c, *and assume that* $\alpha - \mu_j \in \triangle_n^+$ *for all* $j = 1, \dots, i$. *Then* $i \leqslant 2$. *If* $i = 2$ *then* $\alpha - \mu_1 - \mu_2 \in \triangle_n^+$.

With these results at our disposal, we can easily construct the diagram of $\triangle_n^+$. We start with β. If there is only a single compact root μ_1 for which $\beta + \mu_1 \in \triangle_n^+$ then we just draw an arrow with the label μ_1, and (occasionally) a β at its beginning. The arrowhead then represents the non-compact root $\beta + \mu_1$. We proceed like this until we get to a situation in which two simple compact roots can be added to the non-compact root we have reached. From then on we proceed as in the case where already for β do two distinct simple compact roots μ_1 and μ_2 exist such that $\beta + \mu_i$, $i = 1, 2$, are in $\triangle_n^+$. Then, using Lemma 5.1, we draw two perpendicular roots, usually in the NE and SW directions, representing these and then complete the square (or 'diamond' as we shall sometimes call it). After this we proceed from the three new arrowheads as in the previous cases.

We shall use the following terminology:

DEFINITION 5.3. The diagram such obtained is said to be of hermitian type.

For later use we now state the following, which also amplifies the soundness of our construction.

LEMMA 5.4. *The configurations in Figure 1 are impossible. In fact, for no* $\alpha \in \triangle_n^+$ *do there exist three different roots,* μ_0, μ_1 *and* $\mu_2 \in \Sigma_c$ *such that* $\alpha - \mu_1$, $\alpha - \mu_2$, $\alpha - \mu_1 + \mu_0$, *and* $\alpha - \mu_2 + \mu_0$ *all are elements of* $\triangle_n^+$.

Proof. (This is essentially Lemma 4.2 and Figure 1 in [17].) By e.g. reversing to an ordering of $\mathfrak{k}$ by $-\Sigma_c$ it follows from the proof of Lemma 5.1, part (iv),
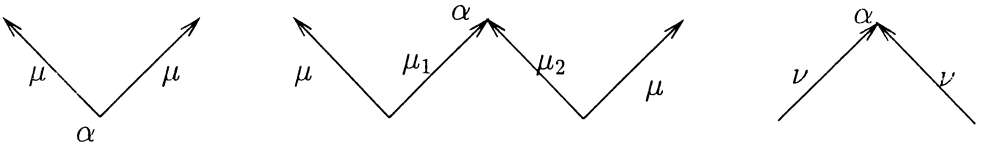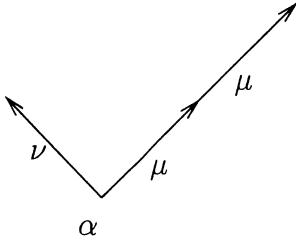
Figure 1.

Figure 2.

that $\langle \alpha, \mu_1 \rangle = \langle \alpha, \mu_2 \rangle = 1$ and that $\tilde{\alpha} = \alpha - \mu_1 - \mu_2$ is a positive non-compact root. Observe that case ii) is excluded since we assume the compact roots to be distinct. Moreover, $\tilde{\alpha} + \mu_1$ is perpendicular to $\tilde{\alpha} + \mu_2$. The only possible value of $\langle \tilde{\alpha} + \mu_1 + \mu_0, \tilde{\alpha} + \mu_2 \rangle$ is zero, so μ_0 must be perpendicular to $\tilde{\alpha} + \mu_2$. Likewise, μ_0 is perpendicular to $\tilde{\alpha} + \mu_1$. But this implies that $\langle \tilde{\alpha} + \mu_1 + \mu_0, \tilde{\alpha} + \mu_2 + \mu_0 \rangle > 0$, in fact, since μ_0 is perpendicular to $\tilde{\alpha}, \mu_1$ and μ_2, it follows that the latter quantity involves long roots and hence is equal to 1. That, of course, is impossible.   □

The natural partial ordering in $\triangle_n^+$ given by $\alpha_1 \geqslant \alpha_2 \Leftrightarrow \alpha_1 - \alpha_2 \in R_c^+$, where R_c^+ denotes the positive root lattice corresponding to $\triangle_c^+$, has an obvious interpretation in terms of paths in the direction of the arrows and the following is a useful observation:

LEMMA 5.5.

$$\forall \alpha \in \triangle_n^+: \quad \beta \leqslant \alpha \leqslant \gamma_r. \tag{41}$$

Proof. This is just a reformulation of the fact that $\mathfrak{p}^+$ is an irreducible $\mathfrak{k}$-module together with Corollary 5.2.   □

For future use we list a few lemmas about the finer structure of these diagrams.

LEMMA 5.6. *The configuration in Figure 2 is impossible.*
   Proof. It follows easily (cf. the proof of Lemma 4.1 in [17]) that in such a configuration, α must be long. Since $\alpha + \nu$ is a root, we must have $2\frac{(\alpha,\nu)}{(\alpha,\alpha)} = -1$ and furthermore, $2\frac{(\alpha,\nu)}{(\nu,\nu)} = -1$ since otherwise $\gamma = \alpha + \mu + \nu$ would be a short root for which $\gamma \pm \mu$ as well as $\gamma \pm \nu$ are roots and this is excluded by the proof of
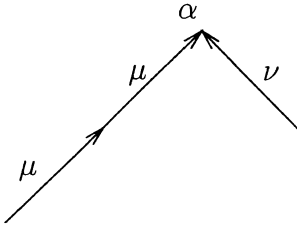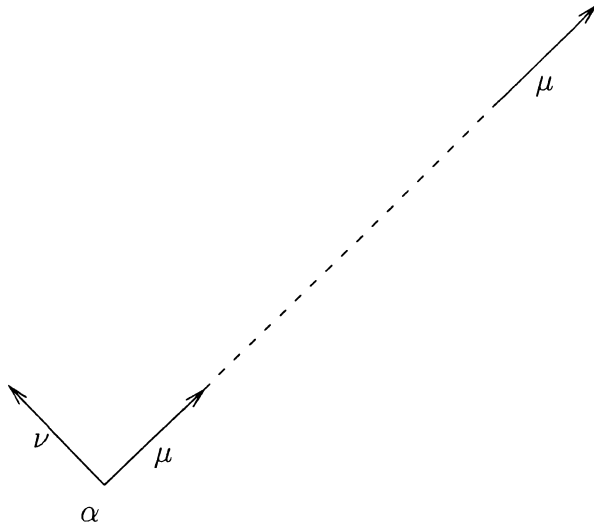
Figure 3.

Figure 4.

the above mentioned Lemma 4.1. Thus ν is long. Now observe that since $2\mu - \nu$ is not a root, $\langle \alpha + 2\mu, \alpha + \nu \rangle$ must equal 0. Thus,

$$\begin{aligned}
0 &= \langle \alpha + 2\mu, \alpha + \nu \rangle \\
&= \langle \alpha + 2\mu, \nu \rangle \\
&= -1 + 2\langle \mu, \nu \rangle,
\end{aligned} \tag{42}$$

an the last equation is absurd since it expresses 0 as a sum of an even number with an odd. □

In a similar way one can establish

LEMMA 5.7. *The configuration in Figure 3 is impossible.*

COROLLARY 5.8. *The configurations in Figure 4 are impossible.*
    Proof. Let us first consider the case in which the dots in Figure 4 just correspond to a single simple compact root, μ_2. It is clear that $\mu_2 \neq \nu$. Moreover, $\langle \alpha, \mu \rangle = -1$

since it follows from the diagram that $\langle \alpha + \mu + \mu_2, \mu \rangle = -1$ and thus $\langle \alpha, \mu \rangle = 0$ would imply that $\langle \mu_2, \mu \rangle = -3$ which is impossible. As a result, $\langle \alpha + \mu, \mu_2 \rangle = -1 = \langle \mu, \mu_2 \rangle$. But by the proof of Lemma 5.1, all roots involved are short, and it follows that $\langle \alpha + \mu + \mu_2, \mu \rangle = 0$ which is absurd.

In the general case it is easy to see that the root μ_2 pointing away from $\alpha + \mu$ also must be the one that points into the μ to the right. Hence we get a similar picture with the μ_2's, except that the chain of simple roots represented by the dots has been shortened by 2. $\square$

Many things can be proved directly by appealing to the diagrams and even though it does tend to have a flavor of 'case-by-case' it is most of the time clear that a general phenomenon is at work. Sometimes the effort needed to unravel this principle rigorously seems too big, and at other times it seems to be worth it. We will try to balance between the two viewpoints, and finish this section by two propositions that will be used in our treatment of annihilators (but not unitarity). They can both, in an obvious and easy way, be established by looking at the various diagrams:

PROPOSITION 5.9. *Consider q_ω for $\omega = -\delta_j$. For $j \geqslant 2$ there is a maximal non-trivial reductive subalgebra $\mathfrak{k}_j$ of $\mathfrak{k}$ that fixes $q_{-\delta_j}$. In fact, $\mathfrak{k}_j$ is the compact subalgebra of a hermitian symmetric space compatible with the given, and with the same 'β'.*

PROPOSITION 5.10. *The polynomial $q_{-\delta_j}$ has an expression*

$$q_{-\delta_j} = \sum_{I=\{i_1,\dots,i_j\}} a_I \cdot z_{-\gamma_{i_1}} \cdot \dots \cdot z_{-\gamma_{i_j}} \tag{43}$$

with complex coefficients a_I and elements $z_{-\gamma_{i_1}}, \dots, z_{-\gamma_{i_j}} \in \mathfrak{p}^-$, where, furthermore, there is an index I such that $ht(\gamma_{i_1}) = \dots = ht(\gamma_{i_j}) = \frac{ht(\delta_j)}{j}$ and $a_I \neq 0$. Thus, if we plot these points $\gamma_{i_1}, \dots, \gamma_{i_j}$ in our diagram of $\triangle_n^+$, they fall on a horizontal line.

6. γ_0-lemmas

In this section we shall establish some important lemmas which narrow down the possible configurations in the diagrammatic presentation of $\triangle_n^+$ of certain subsets obtained in connection with a homomorphism between generalized Verma modules. Specifically, the location relative to γ_0 is essential.

LEMMA 6.1. *If, in Figure 5, $\langle \Lambda, \nu_2 \rangle \neq 0$, then $\langle \nu_2, \gamma_0 \rangle = 0$ and $\langle \Lambda, \nu_2 \rangle = 1$. Hence, in particular, there are two root lengths and $\nu_2 = \nu_1$ or $\nu_2 = \mu_1$.*
    Proof. Let $n_2 = \langle \Lambda, \nu_2 \rangle$. By the fact that $\Lambda - \gamma_0$ is a highest weight and from the fact that μ_1 and ν_2 correspond to simple roots, it follows that $\langle \Lambda - (\gamma_0 + \nu_2), \mu_1 \rangle \geqslant 0$.
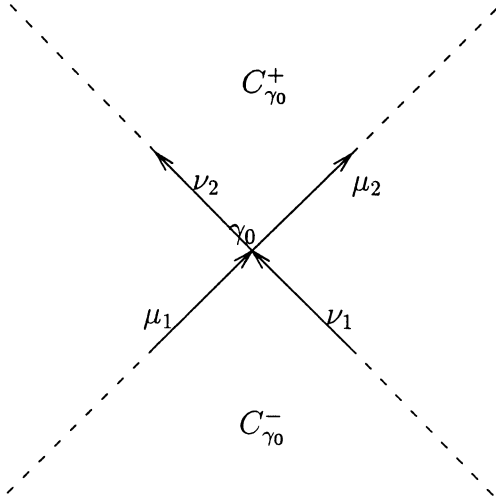
Figure 5.

Furthermore, by the definition of γ_0 and Lemma 4.7, $\Lambda - (\gamma_0 + \nu_2)$ cannot be a highest weight, hence

$$\langle \Lambda - (\gamma_0 + \nu_2), \nu_2 \rangle < 0. \tag{44}$$

From Corollary 4.3 it follows that $\langle \gamma, \nu_2 \rangle \leqslant 2$ for any non-compact root γ. Thus, we must have $n_2 = 1$ and $\langle \gamma_0 + \nu_2, \nu_2 \rangle = 2$. In particular, $\gamma_0 - \nu_2$ is also a root. $\square$

Remark 6.2. Observe that we only used Figure 5 in the generic sense. Some of the roots need not be present.

DEFINITION 6.3.

$$C_{\gamma_0}^+ = \{\alpha \in \triangle_n^+ \mid \alpha \geqslant \gamma_0\}, \tag{45}$$

$$C_{\gamma_0}^- = \{\alpha \in \triangle_n^+ \mid \alpha \leqslant \gamma_0\}, \qquad \text{and} \tag{46}$$

$$C_{\gamma_0}^+(i) = \{\alpha \in C_{\gamma_0}^+ \mid \text{height}(\alpha - \gamma_0) \geqslant i\}. \tag{47}$$

LEMMA 6.4. *Assume that* $\langle \Lambda + \rho, \gamma_0 \rangle = 1$ *and let* $\gamma \in C_{\gamma_0}^+$, $\gamma \neq \gamma_0$. *Then* $\langle \Lambda + \rho, \gamma \rangle > 1$. *Equivalently, if* $\langle (\Lambda_0, \lambda) + \rho, \gamma \rangle = 1$, *then* $\lambda < \lambda_0$.
Proof. Since $\langle (0, \lambda), \gamma \rangle = \lambda$ or 2λ (depending on whether γ is long or short), the equivalence between the two statements is clear. Thus, it suffices to prove the first.
Let $\gamma = \gamma_0 + \mu_1 + \cdots + \mu_i$, where $\mu_1, \ldots, \mu_i$ are simple compact roots. It follows that

$$\langle \Lambda + \rho, \gamma \rangle = \frac{2(\Lambda + \rho, \gamma)}{(\gamma, \gamma)} \tag{48}$$
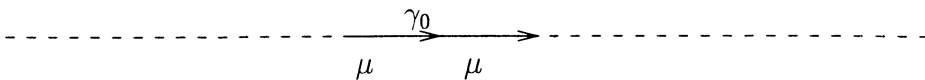
Figure 6.

$$= \frac{2(\Lambda + \rho, \gamma_0)}{F_0(\gamma_0, \gamma_0)} + \sum_{j=1}^{i} \frac{2(\Lambda + \rho, \mu_j)}{F_j(\mu_j, \mu_j)},$$

where the constants $F_0, F_1, \ldots, F_i$ take values in $\{2, 1, \frac{1}{2}\}$ depending on whether the length of the relevant root is shorter, equal, or longer than that of γ. Since Λ is $\mathfrak{k}_1$-dominant all cases are immediately clear except the one where γ is long, γ_0 is short, and $\gamma = \gamma_0 + \mu_1$ with μ_1 (necessarily) short. But even in this case we only get a problem if $\Lambda_0(H_{\mu_1}) = 0$. However, due to the assumptions on the lengths it follows that in the latter case there is a chain of non-compact roots $\gamma_1 = \gamma_0 - \mu_1, \gamma_0, \gamma_0 + \mu_1$, and since $\gamma_0 = \gamma_1 + \mu_1$, this is, by Lemma 4.7, a contradiction.   □

LEMMA 6.5. *With the notation of Figure 5, it is not possible to have μ_1 occur as a summand in $\gamma_1 - \gamma_0$ (expanded on simple roots) for some $\gamma_1 \in C_{\gamma_0}^+$, and, at the same time, ν_1 occur as a summand in $\gamma_2 - \gamma_0$ for some $\gamma_2 \in C_{\gamma_0}^+$. If, say, μ_1 takes part as such a summand, then there are two root lengths and either the case of Figure 6 or of Figure 7 occurs.*

Proof. Suppose μ_1 occurs. Let $\gamma \in C_{\gamma_0}^+$ be of minimal height such that μ_1 is pointing into γ. If there exists another root μ pointing into γ, then $\gamma - \mu \notin C_{\gamma_0}^+$ and γ is thus on one of the walls of $C_{\gamma_0}^+$ passing through γ_0. Thus, $\mu = \nu_1$, which is impossible by Corollary 5.8. Hence μ_1 is the only root which points into γ. Using the fact that Λ is not a highest weight of $\mathfrak{p}^- \otimes V_\Lambda$ it follows that $\langle \gamma, \mu_1 \rangle = 2$ and thereby $\gamma - 2\mu_1 \in \triangle_n^+$. Thus, $\gamma, \gamma - \mu_1$, and γ_0 are on the same wall of $C_{\gamma_0}^+$; the one into which μ_1 is pointing.

Suppose now that also ν_1 occurs. Then, by the previous argument, there exists a $\gamma' \in C_{\gamma_0}^+$ such that $\gamma', \gamma' - \nu_1$, and γ_0 are on the wall of $C_{\gamma_0}^+$ into which ν_1 is pointing. But from the established properties of the diagram it then follows that $\tilde{\gamma} = \gamma + \gamma' - \gamma_0$ is a root in $C_{\gamma_0}^+$ into which both μ_1 and ν_1 is pointing. This is impossible by the minimality of γ_0.   □

Remark 6.6. In Figure 7 the subalgebras u_1 and u_2 corresponding to $\{\nu_2, \cdots, \nu_i\}$ and $\{\mu_2, \cdots, \mu_j\}$, respectively, clearly commute. This case corresponds to C_n.

For a given Λ it is very easy to describe $C_{\gamma_0}^+$, and one may adobt the informal attitude that Λ is almost always completely trivial on the roots corresponding to the arrows inside $C_{\gamma_0}^+$. More precisely, we have:

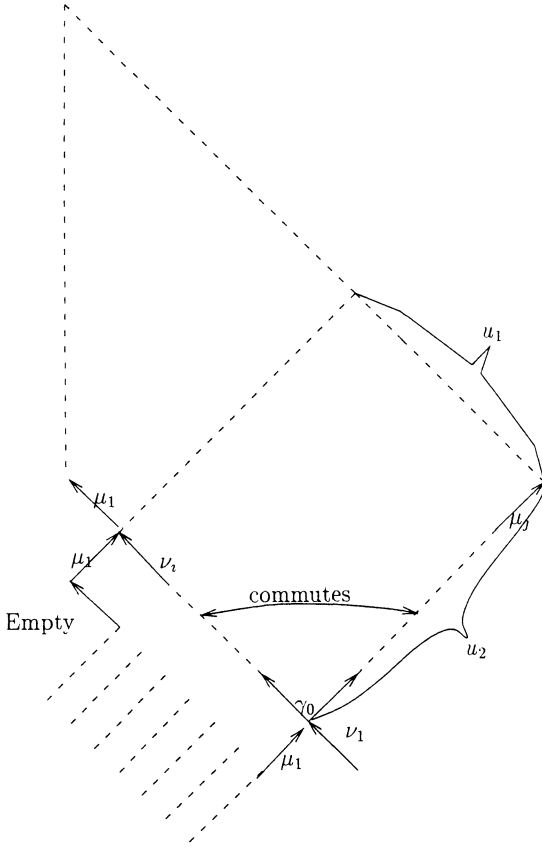LEMMA 6.7. *If μ occur as a summand in $\gamma_1 - \gamma_0$ (expanded on simple roots) for*

Figure 7.

some $\gamma_1 \in C_{\gamma_0}^+$ and if $\Lambda(H_\mu) \neq 0$, then, in the notation of Figure 5, $\mu = \mu_1$ or $\mu = \nu_1$. In particular, there are two root lengths.

Proof. The last assertion is obvious in view of Lemma 6.5. Suppose then that μ satisfies the assumptions above. By passing to the boundary of $C_{\gamma_0}^+$ it is clear that we can find a $\tilde{\gamma} \in C_{\gamma_0}^+$, $\tilde{\gamma} \neq \gamma_0$, so that either μ is the only simple compact root pointing into $\tilde{\gamma}$, or the other simple root pointing into $\tilde{\gamma}$ is μ_1 or ν_1. As in the proof of the previous lemma it follows that $\langle \tilde{\gamma}, \mu \rangle = 2$ and hence there are two root lengths. But then $\tilde{\gamma} - \mu$ is also a positive non-compact root having μ pointing into it and either there are no other roots pointing into $\tilde{\gamma} - \mu$, which then must equal γ_0, or some other root ν_s does point into it. In the latter case we then repeat the argument with $\tilde{\gamma}$ replaced by $\tilde{\gamma} - \nu_s$, as in the proof of Lemma 6.5 (cf. Figure 7). $\square$

LEMMA 6.8. $C_{\gamma_0}^+$ *is a union of at most two diagrams of hermitian type. If there are two they are connected by the arrows corresponding to a single* $\mu \in \Sigma_c$.

Proof. Let u_{top} be the subalgebra of $\mathfrak{k}$ generated by the simple roots μ occuring in $C_{\gamma_0}^+$ and satisfying (in the language of Figure 5) $\mu \neq \mu_1, \nu_1$. Let

$u_{\text{top}} = u_1 \times u_2 \times \cdots \times u_r$ be a decomposition into simple factors. Then $r \leqslant 2$ since otherwise, the u_i's have to 'interact' due to the 2-dimensional nature of the diagrams. If neither μ_1 nor ν_1 occurs later on, then $C_{\gamma_0}^+$ is invariant, and since γ_0 is connected to γ_r the claim follows easily. The other case follows by arguments similar to those used around Figure 6 and Figure 7 in Lemma 6.5. $\qquad \square$

Observe that γ_0 will be the unique simple non-compact root of one of these diagrams, i.e. the 'β' for that structure.

DEFINITION 6.9. The hermitian symmetric spaces corresponding to the above are denoted by $\mathcal{D}^{\downarrow}(\gamma_0)$ and $\mathcal{D}^{\uparrow}(\gamma_0)$, and are called the hermitian symmetric spaces defined by γ_0. We will always assume that the one indexed by the $\downarrow$ is the one whose diagram contains γ_0. If the one indexed by the $\uparrow$ is not present, we drop the arrows and simply talk about the hermitian symmetric space defined by γ_0.

The interrelation between polynomials on these hermitian symmetric subspaces and on the full hermitian symmetric space will be of interest later on:

LEMMA 6.10. *Suppose that $\mathfrak{g}_1 \subseteq \mathfrak{g}$ is a simple subalgebra of $\mathfrak{g}$ which, in its own right, corresponds to a hermitian symmetric space and assume that this structure is compatible with that of $\mathfrak{g}$, that is, if*

$$\mathfrak{g}_1 = \mathfrak{p}_1^- \oplus \mathfrak{k}_1 \oplus \mathfrak{p}_1^+, \tag{49}$$

then $\mathfrak{p}_1^{\pm} \subseteq \mathfrak{p}^{\pm}$ and $\mathfrak{k}_1 \subseteq \mathfrak{k}$. More precisely, assume that the hermitian symmetric space corresponding to $\mathfrak{g}_1$ is either $\mathcal{D}^{\downarrow}(\gamma_0)$ or, where applicable, $\mathcal{D}^{\uparrow}(\gamma_0)$. Let $\delta_1^{(1)}, \ldots, \delta_{r_1}^{(1)}$ be the quantities corresponding to the δ_i's, $i = 1, \ldots, r$ for $\mathfrak{g}$, and let $m_i^{(1)}$ be the $\mathfrak{k}_1$-type corresponding to the weight $-\delta_i^{(1)}$, $i = 1, \ldots, r_1$. Then $\forall i = 1, \ldots, r_1$, $m_i^{(1)} \subseteq m_i$, and m_i is the only $\mathfrak{k}$-type in $\mathcal{U}(\mathfrak{p}^-)$ for which $m_i^{(1)}$ can be obtained by restricting the variables to $\mathfrak{p}_1^-$.

Proof. Obviously, $m_1^{(1)} \subseteq m_1$ always. Consider the case in which there is only one hermitian symmetric space defined by $C_{\gamma_0}^+$. Then the root subspace corresponding to $-\gamma_r$ is contained in $\mathfrak{p}_1^-$, and therefore, in this case, we can use a conjugation by the Weyl group element which preserves $\triangle_n^+$ and exchanges β and γ_r to get to a situation in which the 'β' of the 'small' hermitian space is the same as the 'β' of the big. But then the conclusion is obvious. The remaining cases are those that correspond to Figures 6 and 7. The one corresponding to Figure 6 is very trivial, and the one corresponding to Figure 7 is almost as simple. For the latter, observe that either u_1 or u_2 (cf. above) is the compact subalgebra corresponding to the space $\mathcal{D}^{\uparrow}(\gamma_0)$ which was taken care of by the previous argument. But then the case of $\mathcal{D}^{\downarrow}(\gamma_0)$, which involves both u_1 and u_2, follows directly, being one of the previous (A_n) cases. $\qquad \square$

Remark 6.11. By pointing at the various diagrams for the hermitian symmetric spaces, one can easily see that the lemma above remains valid without the restriction imposed in the sentence 'More precisely ... '.

Returning to the setup in subsection 2.1, a related result is the following:

LEMMA 6.12. *Let γ_0 be defined as previously, let $\mathfrak{p}_\downarrow^-$ be the '$\mathfrak{p}^-$' of $\mathcal{D}^\downarrow(\gamma_0)$, and let $\mathfrak{p}_\uparrow^-$ be that of the other (when it exists). Construct the standard set of strongly orthogonal non-compact roots $\gamma_0 = \tilde{\gamma}_1, \ldots, \tilde{\gamma}_t$ for $\mathfrak{p}_\downarrow^-$, and let $\tilde{\tilde{\gamma}}_1, \ldots, \tilde{\tilde{\gamma}}_s$ be the analogues for $\mathfrak{p}_\uparrow^-$. Then there exists a unique $\mathfrak{k}$-type in $\mathcal{U}(\mathfrak{p}^-) \otimes V_\Lambda$ of highest weight $\Lambda - (\tilde{\gamma}_1 + \cdots + \tilde{\gamma}_l) = \Lambda - \xi(\gamma_0, l)$ for any $l \leqslant t$. Moreover, the coefficient of the corresponding highest weight vector with respect to the highest weight vector in V_Λ is exactly the highest weight vector in $\mathcal{U}(\mathfrak{p}_\downarrow^-)$ whose weight is $-(\tilde{\gamma}_1 + \cdots + \tilde{\gamma}_l)$. Furthermore, consider those values of $\hat{s} \leqslant s$ for which the set $\tilde{\tilde{\gamma}}_1, \ldots, \tilde{\tilde{\gamma}}_{\hat{s}}$ can be complemented with a set of orthogonal roots $\tilde{\gamma}_{\hat{s},1}, \ldots, \tilde{\gamma}_{\hat{s},t}$ corresponding to $\mathfrak{p}_\downarrow^-$ so that $\Lambda_{\hat{s}} = \Lambda - \tilde{\gamma}_{\hat{s},1} - \cdots - \tilde{\gamma}_{\hat{s},t} - \tilde{\tilde{\gamma}}_1 - \cdots - \tilde{\tilde{\gamma}}_{\hat{s}} = \Lambda - \xi(\gamma_0, \hat{s} + t)$ is $\mathfrak{k}$-dominant. Then there exists a unique $\mathfrak{k}$-type in $\mathcal{U}(\mathfrak{p}^-) \otimes V_\Lambda$ of highest weight $\Lambda_{\hat{s}}$.*

Proof. Let $\{v_\Lambda, v_1, \ldots, v_d\}$ be a basis of V_Λ such that v_Λ is the highest weight vector and such that $\forall i = 1, \ldots, d$: $v_i \in \mathfrak{k}^- \cdot \mathcal{U}(\mathfrak{k}^-) \cdot v_\Lambda$. Consider the highest weight vector p_{γ_0} corresponding to the $\mathfrak{k}$-type $V_{\Lambda-\gamma_0}$ in $\mathfrak{p}^- \otimes V_\Lambda$ of highest weight $\Lambda - \gamma_0$. Then

$$p_{\gamma_0} = p_0 v_\Lambda + p_1 v_1 + \cdots, \tag{50}$$

and $p_0 \equiv z_{-\gamma_0} \neq 0$. We can then use $V_{\Lambda-\gamma_0}$ as our 'V_Λ' and determine the type corresponding to the last possible place of unitarity in $\mathfrak{p}^- \otimes V_{\Lambda-\gamma_0}$. The weight of the latter is, naturally, $\Lambda - \tilde{\gamma}_1 - \tilde{\gamma}_2$ since the simple root(s) pointing out from γ_0 have weight 1 in $\Lambda - \gamma_0$. By keeping track of the v_Λ-coefficient and the known behaviour of the tensor products of polynomials it follows that in the case where there is only one hermitian symmetric space, this type will be non-zero. The rest follows analogously. When there are two hermitian symmetric spaces one splits the argument into two (identical) steps. Specifically, it is only when we are forced into the region corresponding to the 'top' hermitian space that the situation is new. But we can begin, using Lemma 6.10, by considering polynomials in the top space. These are then not highest weight with respect to μ_1, but this can be repaired by throwing in a suitable polynomial from $\mathcal{D}^\downarrow(\gamma_0)$. The only limitation is that it must be a highest weight representation both for u_1 and u_2. (See Figure 15). $\square$

DEFINITION 6.13. For given l, respectively $\hat{s} + t$, the representation in Lemma 6.12 is called the leading l-minor, respectively $\hat{s} + t$-minor, representation corresponding to Λ_0. In the sequel we always write the highest weights as $\Lambda - \xi(\gamma_0, l)$ and $\Lambda - \xi(\gamma_0, \hat{s} + t)$, respectively, thus defining the expressions $\xi(\gamma_0, d)$ in certain degrees d. We let t_0 denote the biggest degree d in which there is a leading d-minor

representation.

Remark 6.14. For each i, $V(\Lambda - \xi(\gamma_0, i))$ has got multiplicity 1 in $\mathcal{U}(\mathfrak{p}^-)V(\Lambda)$ and $V(\Lambda - \xi(\gamma_0, i + 1)) \subset \mathcal{U}(\mathfrak{p}^-)V(\Lambda - \xi(\gamma_0, i))$. In fact, $V(\Lambda - \xi(\gamma_0, i))$ is the only irreducible $\mathfrak{k}$-type in the space of ith order elements in $\mathcal{U}(\mathfrak{p}^-) \otimes V(\Lambda)$ from which $V(\Lambda - \xi(\gamma_0, i + 1))$ can be reached by applying $\mathfrak{p}^-$.

In the special case where $\Lambda_0 = 0$, $t_0 = r$ (Section 4.2.)

Remark 6.15. The results and definitions above concerning $\mathcal{U}(\mathfrak{p}^-) \otimes V(\Lambda)$ and its multiplicities, as well as several to come, are independent of λ in $\Lambda = (\Lambda_0, \lambda)$. The reason is that $\mathcal{U}(\mathfrak{p}^-) \otimes V(\Lambda)$ as $\mathfrak{k}_1$-module is independent of λ since, given Λ_0, Λ is completely determined by Λ_0 together with its value on the center of $\mathfrak{k}$.

There are many more reasons that $C_{\gamma_0}^+$ is important. One is the following: Let $V_{\Lambda - \gamma_0}$ be the $\mathfrak{k}$-irreducible subspace of $\mathfrak{p}^- \otimes V_\Lambda$ of highest weight $\Lambda - \gamma_0$.

LEMMA 6.16. *Let* $z_{-\gamma}v_\Lambda + \sum_{j=1}^d z_j v_j \in (V_{\Lambda - \gamma_0})^{\Lambda - \gamma}$ *be a weight vector of weight* $\Lambda - \gamma$, *with* $v_\Lambda, v_1, \ldots, v_d$, *as previously, an orthogonal basis for* V_Λ. *Then* $z_{-\gamma} \neq 0 \Rightarrow \gamma \in C_{\gamma_0}^+$.

Proof. It is clear that we may assume that $\forall i = 1, \ldots, d$: $v_i \in \mathcal{U}(\mathfrak{k}^-) \cdot \mathfrak{k}^- \cdot v_\Lambda$. Furthermore, it is clear that the highest weight vector $p_{\Lambda - \gamma_0}$ in $V_{\Lambda - \gamma_0}$ can be chosen to be of the form

$$p_{\Lambda - \gamma_0} = z_{\gamma_0}v_\Lambda + \sum_{j=1}^d z_j v_j \tag{51}$$

with a non-zero $z_{\gamma_0} \in (\mathfrak{p}^-)^{-\gamma_0}$ as in (9) and $z_1, \ldots, z_d \in \mathfrak{p}^-$. Since $V_{\Lambda - \gamma_0} = \mathcal{U}(\mathfrak{k}^-) \cdot p_{\Lambda - \gamma_0}$, the conclusion is evident. $\square$

We will now investigate, under some special extra assumptions, homomorphisms between generalized Verma modules. Thus, in the following also Λ^1 is $\mathfrak{k}$-finite, and this is essential.

PROPOSITION 6.17. *Suppose that* $\langle \Lambda + \rho, \gamma_0 \rangle \leqslant 1$ *and that there is a non-trivial homomorphism* $\varphi \colon M(V_{\Lambda^1}) \to M(V_\Lambda)$. *Then, in the canonical presentation of* $\Lambda^1 - \Lambda$, *something from* $C_{\gamma_0}^+$ *has to occur.*

Proof. Let $\chi_0 = \Lambda + \rho$. Points occuring in the BGG-chain have got to have $\langle \chi_0, \alpha \rangle \geqslant 1$. This follows because $\langle \chi_{i-1}, \alpha_i \rangle \geqslant 1$, $\chi_{i-1} = \chi_0 - \sum_{j=1}^{i-1} n_j \alpha_j$ and, by Corollary 4.2, $\forall i, j$: $\langle \alpha_i, \alpha_j \rangle \geqslant 0$.

Suppose on the contrary that none of the roots are in $C_{\gamma_0}^+$. The γ's in the BGG-chain then fall in two disjoint sets, L and R (see Figure 8). Let us look at the points in L. Draw a SW–NE line through γ_0 and define this as the level 0 base line. We may then divide L into disjoint subsets $L_i = \{\gamma \in L \mid \gamma$ is on a SW–NE line at level i
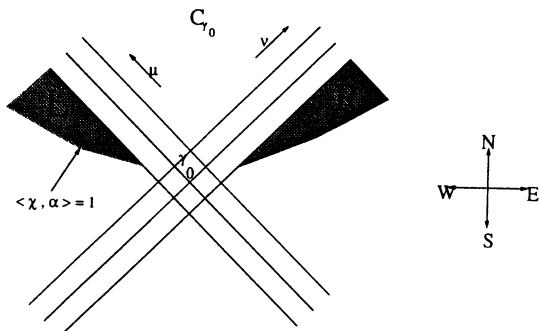
Figure 8.

above the baseline}. Here, the level is the number of simple roots one has to add to γ_0 to get to the first intersection between L_i and $C_{\gamma_0}^+$. Now observe that $L_1 = \emptyset$. This follows by looking at the simple compact root μ_1 in Figure 5 pointing into γ_0 in the SW–NE direction. The only possible element of L_1 is $\gamma_0 - \mu_1 + \nu_2$, and for this to satisfy the requirement we must be in a situation as in Lemma 6.1. Furthermore, by Lemma 5.4 and Lemma 5.6, either $\mu_1 = \nu_2$ or ν_2 is not present at all. In both cases, $L_1 = \emptyset$. Now let i_0 be the smallest positive integer such that $L_{i_0} \neq \emptyset$. Let $\gamma \in L_{i_0}$ and let ν_{i_0+1} be the simple compact root pointing into γ in the SE-NW direction. By appealing to Lemma 6.7 we may assume that $\langle \Lambda, \nu_{i_0+1} \rangle = 0$. Already by assumption, $\langle \gamma, \nu_{i_0+1} \rangle > 0$. Hence, in $\Lambda^1 = \Lambda - n_\gamma \gamma - \sum n_i \gamma_i$, which is ℓ-dominant, there has got to be some γ_i with negative inner product with ν_{i_0+1}. But these cannot come from the set L by Lemma 5.4, Lemma 5.6, Corollary 5.8 and/or the definition of i_0. Again by Lemma 5.4, Lemma 5.6, they cannot come from the right hand side, R, either, since ν_{i_0+1} then would occur with multiplicity $\geqslant 2$ in one of the configurations which have been ruled out by Lemma 5.4. $\square$

PROPOSITION 6.18. *If, under the same assumptions as in Proposition 6.17,* $\Lambda^1 = \Lambda - \sum_1^s n_i \alpha_i$ *with* $n_i \in \mathbb{N}$ *and* $\forall i = 1, \ldots, s$: $\alpha_i \in \triangle_n^+$, *then for at least one* i, $\alpha_i \in C_{\gamma_0}^+$.

Proof. Let us first assume that we are in a situation as in Figure 5, with both μ_1 and ν_1 present. Due to Lemma 5.4, Lemma 5.6, and Corollary 5.8 it is clear that we may assume that μ_1 and ν_1 do not both occur further down towards β, i.e. when $\gamma_0 - \beta$ is written as a sum of simple compact roots then the two mentioned ones do not both have multiplicity greater than 1. Assume that μ_1 has multiplicity 1. In the canonical decomposition, let M denote the number of elements, counted with multiplicitity, that belong to $C_{\gamma_0}^+$.

Now suppose that we were to take the canonical expression and rewrite it in such a way that nothing from $C_{\gamma_0}^+$ occurs. Clearly, then, the M abovementioned elements must be relegated to the shaded area in Figure 9. Moreover, observe that the degree stays fixed (see Remark 4.5). However these new points do not help us
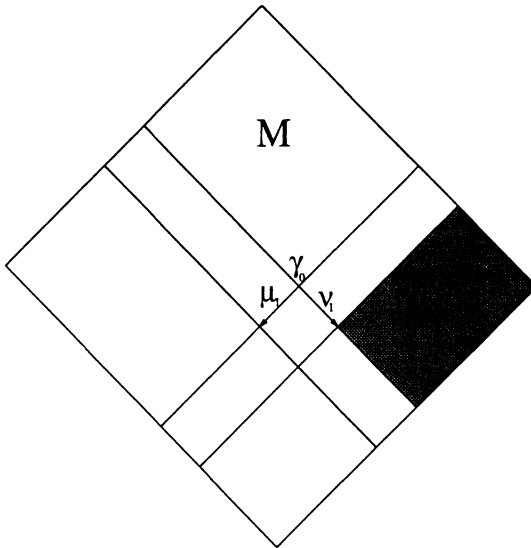
Figure 9.

to get rid of the M ν_1-summands stemming from $C_{\gamma_0}^+$. On the other hand, due to degree considerations, we have used our quota, so this problem cannot be remedied.

It remains to consider the case in which, say, the μ_1 is not present in Figure 5. Let n_1 denote the multiplicity of ν_1 in $\gamma_0 - \beta$. It is then clear that n_1 (or $n_1 + 1$) also is the multiplicity of ν_1 in $\alpha - \beta$ for any $\alpha \in C_{\gamma_0}^+$. (Usually, $n_1 = 1$, but in a few cases, for sp$(n, \mathbb{R})$, it may equal 2.) But this means that if we are to replace the α's from $C_{\gamma_0}^+$ with some other elements of $\triangle_n^+$ there will be a deficit in the ν_1 account, of magnitude at least M (M being the same as above). Hence this is also impossible. $\square$

Let us now take a general look at the conditions $\langle \Lambda + \rho, \gamma_0 \rangle \leqslant 1$ and $\langle \Lambda + \rho, \gamma \rangle \geqslant 1$ for any root γ occuring in a BGG-chain (cf. the proof of Proposition 6.17).

Again, set $\chi_0 = \Lambda + \rho$ and let $x = (\chi_0, \gamma_0)$. As mentioned previously, if γ takes part in a BGG-chain then necessarily $(\chi_0, \gamma) > 0$. Thus, γ must satisfy

$$x + (\Lambda + \rho, \gamma - \gamma_0) > 0. \tag{52}$$

Let us assume that we are not in a situation as in Figure 6 or Figure 7. Suppose further that $\gamma, \gamma - \nu$, and $\gamma - \nu + \mu \in C_{\gamma_0}^+$ for some simple compact roots ν, μ. It follows from the proof of Lemma 5.1 that in this situation, ν and μ have the same square length (equal to either (γ, γ) or $\frac{1}{2}(\gamma, \gamma)$). Thus, since $(\Lambda, \nu) = (\Lambda, \mu) = 0$ (cf. Lemma 6.7),

$$\langle \Lambda + \rho, \gamma - \nu + \mu \rangle = F \langle \Lambda + \rho, \gamma \rangle, \tag{53}$$

for some constant F (which further may be seen to be either 1 or 2).

As a consequence, the set of $\gamma \in C_{\gamma_0}^+$ for which $\langle \chi_0, \gamma \rangle > 0$ is bounded by a line in the W–E direction given by the set of points $\gamma \in C_{\gamma_0}^+$ for which $\gamma - \gamma_0$ has a fixed

height $h = h(x)$. Furthermore, it is clear that in Proposition 6.17 and Proposition 6.18 we can improve the statement there by replacing $C_{\gamma_0}^+$ by $C_{\gamma_0}^+(h(x))$.

In the case described by, in particular, Figure 7, the situation is a little more complicated, but essentially the same.

We collect some of these remarks into the following:

COROLLARY 6.19. *Suppose that the hermitian symmetric space defined by γ_0 has only one root length. Assume that $(\gamma_0, \gamma_0) = 2$, let $x = (\chi_0, \gamma_0)$ be as above, and assume $x \in \{1, 0, -1, \ldots, -n, \ldots\}$. Then the elements of $C_{\gamma_0}^+$ referred to in Proposition 6.17 and Proposition 6.18 belong to $C_{\gamma_0}^+(1 - x)$.*

Proof. It is clear that Λ is trivial on the compact roots corresponding to the hermitian symmetric space defined by γ_0. Furthermore, let γ and $\gamma + \nu$ belong to $C_{\gamma_0}^+$ for some simple compact root ν. Then, according to the assumptions,

$$\langle \Lambda + \rho, \gamma + \nu \rangle = \langle \Lambda + \rho, \gamma \rangle + 1. \tag{54}$$

Since furthermore $x = \langle \Lambda + \rho, \gamma_0 \rangle$, the conclusion follows. $\square$

Remark 6.20. Of course, analogous formulas may be given for the remaining cases.

The implications of these rather simple observations will be made much more explicit below and in the next section but for now let us assume that the diagram representing $\triangle_n^+$ is *contained* in a diagram as described in Figure 10, where we further assume that there is only one root length and that all the simple compact roots $\mu_1, \ldots, \mu_s, \nu_1, \ldots, \nu_r$ are distinct. Let us also assume that $r \geqslant s$.

We can now see that as soon as the value of $\langle \Lambda + \rho, \gamma_0 \rangle$ is so small (this is a condition on the 'λ' of Λ) that any $\gamma \in \triangle_n^+$ satisfying that $\langle \Lambda + \rho, \gamma \rangle > 0$ is forced to be in the shaded area, then there can be no homomorphism of a generalized Verma module $M(V_{\Lambda^1})$ into $M(V_\Lambda)$. The reason is that the shaded area cannot define a Λ^1 which is $\mathfrak{k}_1$-dominant:

Suppose, namely, that $\Lambda^1 = \Lambda - \sum_{j=1}^i \gamma_j$ with the γ_j's coming from the shaded area. Assume that they in fact all are above the line indicated at ν_{i_0} but that they are not all above the line at ν_{i_0+1}. Then, since our assumptions imply that $\langle \Lambda, \nu_{i_0} \rangle = 0$, it is clear that $\langle \Lambda^1, \nu_{i_0} \rangle < 0$. This, of course, is impossible.

We can now settle the unitarity in an important special case:

THEOREM 6.21. *Let $W(\lambda) = W((0, \lambda)')$ be a scalar module of weight λ. Assume that the real rank r of $\mathfrak{g}$ is greater than one. Then there is a unique negative $\lambda = \lambda_2^{sc}$ at which the $\mathfrak{k}$-type m_2 in $\mathcal{U}(\mathfrak{p}^-)$ of weight $-\gamma_1 - \gamma_2$ is in the radical of the hermitian form. At this point there is unitarity for $W(\lambda_2^{sc})$. The values are listed below. More generally, there is a negative λ_i at which the $\mathfrak{k}$-type m_i in $\mathcal{U}(\mathfrak{p}^-)$ of weight $-\gamma_1 - \cdots - \gamma_i$ is in the radical of the hermitian form but $m_1, \ldots, m_{i-1}$ are not. At this point, there is unitarity, in fact, $\lambda_i = (i - 1)\lambda_2^{sc}$ and $W(\lambda_i)$ is*
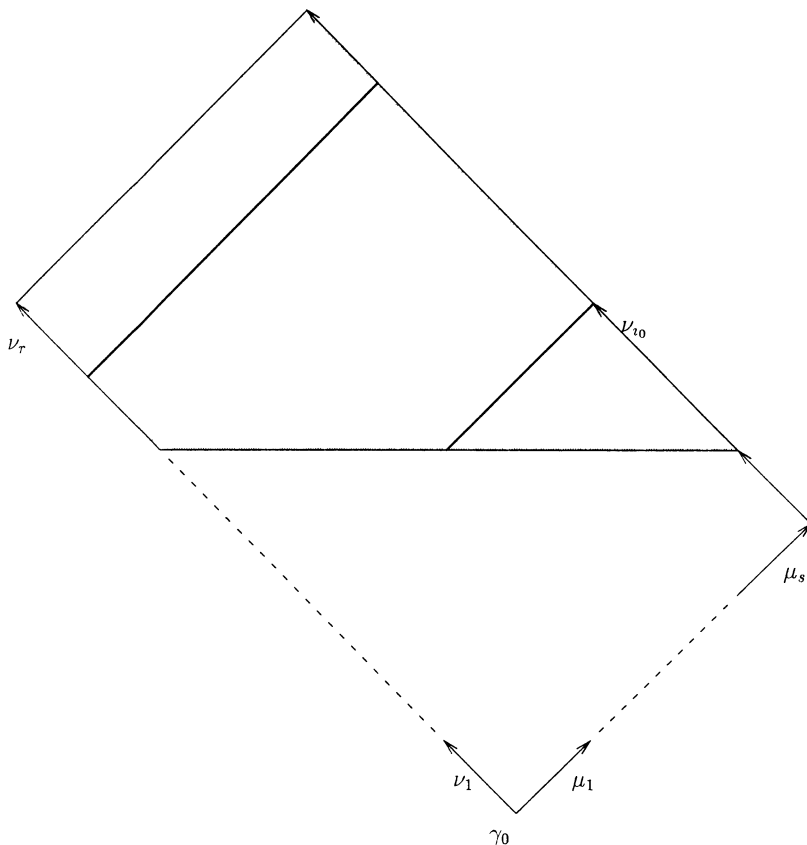
Figure 10.

contained in the $(i - 1)$ *fold tensor product of* $W(\lambda_2^{sc})$ *with itself. If* $\lambda \geqslant (\lambda_r)$ *and* $\lambda \notin \{0, \lambda_2^{sc}, \ldots, \lambda_r\}$, $W(\lambda)$ *is not unitary, but for* $\lambda < \lambda_r$, $W(\lambda)$ *is unitary.*

Proof (Sketch). By locating either a su(2, 2) (a 'diamond' $\diamondsuit$) or a sp(2, $\mathbb{R}$) (a 'hook') inside the diagram, which is possible by the assumption on the real rank (notice the sp(2, $\mathbb{R}$) inside so(2n + 1, 2)) it follows from the well established theory of these two algebras (or a very simple direct calculation) that some second order polynomial is perpendicular to itself under the hermitian form at a strictly negative value of λ. Thus, since the polynomials whose weights are of the form $-n \cdot \gamma_1$ have their zeros for positive λ's (a trivial fact), the type m_2 must vanish at a negative λ. But recalling how the algebra $\mathcal{U}(\mathfrak{p}^-)$ is built up, it follows easily that at that λ, the only polynomials that can stay outside the radical of the hermitian form are those of the above mentioned weights $-n \cdot \gamma_1$. These, since their vanishing takes place at positive λ's, and because of the way the inner product depends on λ, define a subspace on which the hermitian form is positive definite. To finish this case we need only determine said $\lambda = \lambda_2^{sc}$. A major point here is of course that there must be a homomorphism corresponding to this λ, a homomorphism defined
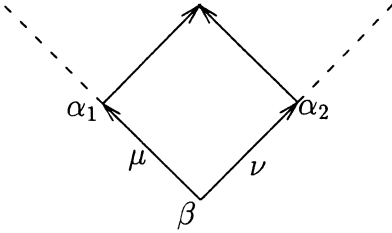
Figure 11. The BGG-chain $\{\alpha_1, \alpha_2\}$ that defines the homomorphism at λ_2^{sc} in the case of $su(p,q)(p,q \geqslant 2)$. Naturally, they both have multiplicity 1.

by either two different points α_1, α_2 in $\triangle_n^+$ or a single α with multiplicity 2. It is then clear that the diamond (resp. hook) closest to β will be the one that defines the homomorphism corresponding to the vanishing of m_2 at λ_2^{sc}. If we let α_1 and α_2 denote the western and eastern corner of the diamond, respectively, then clearly (cf. Figure 11)

$$\langle (0, \lambda_2^{sc}), \alpha_1 \rangle = \langle (0, \lambda_2^{sc}), \alpha_2 \rangle = 1. \tag{55}$$

The last remark for this case is that of course there can be no unitarity in-between since m_2 also vanishes at 0 and has a positive inner product with itself for λ sufficiently negative.

Let us now look at $W(\lambda_2^{sc}) \otimes W(\lambda_2^{sc})$. We decompose this by viewing it as a space of polynomials on the product space of the hermitian symmetric space with itself and then restricting to the diagonal [14]. In particular, $W(\lambda_2^{sc} + \lambda_2^{sc})$ is the quotient of the whole space by those polynomials that vanish on the diagonal. Now, clearly, as a space of polynomials on the hermitian symmetric space, $W(\lambda_2^{sc} + \lambda_2^{sc}) = W(\lambda_2^{sc}) \cdot W(\lambda_2^{sc})$. Hence, since $m_2 \notin W(\lambda_2^{sc})$ whereas $m_1 \in W(\lambda_2^{sc})$, by Proposition 4.11 $m_2 \in W(2\lambda_2^{sc})$ and $m_3 \notin W(2\lambda_2^{sc})$. In particular, m_3 vanishes at $\lambda_3 \stackrel{\text{Def.}}{=} 2\lambda_2^{sc}$.

The unitarity of the higher order ones follow similarly by tensoring $W(\lambda_2^{sc})$ with itself. By Proposition 4.11, the $\mathfrak{k}$-type m_i that vanishes at λ_i is in the ideal generated by m_{i-1} and hence also vanishes at λ_{i-1}. Thus, m_i vanishes at $0, \lambda_2, \ldots, \lambda_i$, which exhaust the list of possible zeros for the restriction of the hermitian form to m_i. Since the λ dependence of the form is given by Lemma 4.12 it follows that it is negative definite on m_i for $\lambda \in]\lambda_{i-1}, \lambda_i[$.

That there is unitarity for $\lambda < \lambda_r$ can be seen as follows: First of all we know, as remarked following Lemma 4.12, that for a given $\mathfrak{k}$-type we have positivity for λ sufficiently negative. Suppose a $\mathfrak{k}$-type $-p$ changes sign at a $\tilde{\lambda} < \lambda_r$. It then follows easily (here we use the strong version of BGG, Proposition 3.3) that there is a homomorphism between generalized highest weight modules at this point. With no loss of generality, we may assume that this homomorphism is defined by $-p$. Thus, (Proposition 3.5)
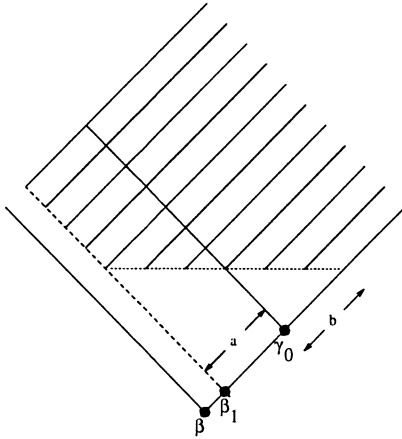
$$\langle (0, \lambda) + \rho, p \rangle = 1. \tag{56}$$

Figure 12.

Let $p = n_1\gamma_1 + \cdots + n_r\gamma_r$ and let $\rho_i = \langle \rho, \gamma_i \rangle$. Then (56) becomes

$$\lambda = \frac{n_1^2 + \cdots + n_r^2 - (n_1\rho_1 + \cdots + n_r\rho_r)}{n_1 + \cdots + n_r}. \qquad (57)$$

Under the assumption that all $n_i \neq 0$, the righthand side of (57), viewed as a function of the n_i's, is easily seen to have its minimum for $n_1 = \ldots = n_r = 1$ and this corresponds exactly to the $-p = m_r$. $\qquad\qquad \square$

Remark 6.22. A different, though related proof may be given using the observations following Remark 6.20 about the limitations on the allowed subsets of the diagrams. Finally, one can use the established results ([18]) about homomorphisms into scalar modules to give a third proof of the last part of the theorem. Actually the latter results assume the classification of unitarizable, scalar modules, but only in a mild, and for the present purposes, removable way.

Remark 6.23. The values of λ_2^{sc} can easily be computed directly from the diagrams (cf. [17]). They are

- $\mathrm{su}(p, q)$: -1.
- $\mathrm{sp}(n, \mathbb{R})$: $-\frac{1}{2}$.
- $\mathrm{so}^*(2n)$: -2.
- $\mathrm{so}(2, n)$: $-\frac{n-2}{2}$.
- e_6: -3.
- e_7: -4.

THEOREM 6.24. *Let* $1 \leqslant j \leqslant t_0$, *let* $\lambda = \lambda_{j-1} = \lambda_0 + (j-1) \cdot \lambda_2^{\text{sc}}$, *with* λ_2^{sc} *as in Theorem 6.21, let* $\tau_j = (\Lambda_0, \lambda_{j-1})$, *and let* $\check{\tau}_j = (\Lambda_0, \lambda_{j-1}) - \xi(\gamma_0, j)$. *Then there is a homomorphism*

$$M(V(\check{\tau}_j)) \to M(V(\tau_j)). \tag{58}$$

Proof. If $j = 1$, the claim follows by the definition of γ_0. Let $M_2^{\text{sc}} = M(0, \lambda_2^{\text{sc}})$ and consider the $\mathfrak{g}$-module

$$M(\tau_1) \otimes M_2^{\text{sc}}. \tag{59}$$

Inside $M(\tau_1)$ we have the submodule $\tilde{M}(\tau_1) = \mathcal{U}(\mathfrak{p}^-)V(\Lambda - \gamma_0)$ and inside M_2^{sc} the submodule $\tilde{M}_2^{\text{sc}} = \mathcal{U}(\mathfrak{p}^-)m_2$. The highest weight modules $S_1 = M(\tau_1)/\tilde{M}(\tau_1)$ and $S_2 = M_2^{\text{sc}}/\tilde{M}_2^{\text{sc}}$ may then, as $\mathfrak{k}$-modules, be viewed as certain subspaces $\mathcal{P}_1$ and $\mathcal{P}_2$ of $\mathcal{U}(\mathfrak{p}^-) \otimes V(\Lambda)$ and $\mathcal{U}(\mathfrak{p}^-)$, respectively. The $\mathfrak{k}$-type of highest weight $\Lambda - \gamma_0$ is missing from S_1 and the set of the highest weights inside S_2 is $\{-i\delta_1 \mid i = 0, 1, \ldots\}$. Inside the tensor product $S_1 \otimes S_2$, the vector $\tilde{v} = v_{\tau_1} \otimes 1_2^{\text{sc}}$, where v_{τ_1} and 1_2^{sc} are highest weight vectors in S_1 and S_2, respectively, defines a highest weight representation of $\mathfrak{g}$. This representation will clearly be a quotient of $M(\tau_2)$. Furthermore, the subrepresentation generated by $\tilde{v}$ will have a set of $\mathfrak{k}$-types which is contained in $\mathcal{P}_1 \otimes \mathcal{P}_2$, indeed contained in $\mathcal{P}_1(\mathcal{P}_2)$. But, cf. Remark 6.14, the $\mathfrak{k}$-type of highest weight $\Lambda - \xi(\gamma_0, 2)$ does not appear in this space. It then follows that there is a proper $\mathfrak{g}$-invariant subspace of $M(\tau_2)$ and this subspace contains the unique $\mathfrak{k}$-type $V(\check{\tau}_2)$ of the mentioned highest weight. Finally it is easy to see that $\mathfrak{p}^+(V(\check{\tau}_2)) = 0$. This last claim follows by observing that if $\mathfrak{p}^+(V(\check{\tau}_2)) \neq 0$ it would have to be a space of highest weight $\Lambda - \gamma_0$ and this would give rise to a homomorphism $M(V(\Lambda - \gamma_0)) \to M(V(\Lambda))$ at a value of λ equal to λ_1, and this is of course impossible. (See also Remark 6.25 below).

The cases $j = 3, \ldots$ follow analogously. □

Remark 6.25. It is very easy to describe the roots γ taking part in the BGG-chains corresponding to Theorem 6.24. They lie on a single horizontal line inside $C_{\gamma_0}^+$ in an appropriate height above γ_0. In the cases $\text{su}(p, q)$, $\text{so}(2n - 2, 2)$, $\text{so}^*(2n)$, e_6, and e_7 the value of $\langle \Lambda + \rho, \alpha \rangle$ is constant and equal to 1. For $\text{so}(2n - 3, 2)$ and $\text{sp}(n, \mathbb{R})$ it is 1 or 2 depending on the length of γ_0, the length of γ, and the location of γ relative to $\mathcal{D}^{\downarrow}(\gamma_0)$.

The following theorems are crucial in several places in the next chapter.

Many of the general results above can be strengthened by using more explicitly the representation theory of the algebras $\mathfrak{k}$. Indeed, the cases corresponding to $\text{so}(n, 2)$, e_6, and e_7 are almost completely trivial, but in the remaining cases one is left with some questions about the representation theory of $u(n)$. These are, in general, somewhat complicated. Another way of strengthening the above results
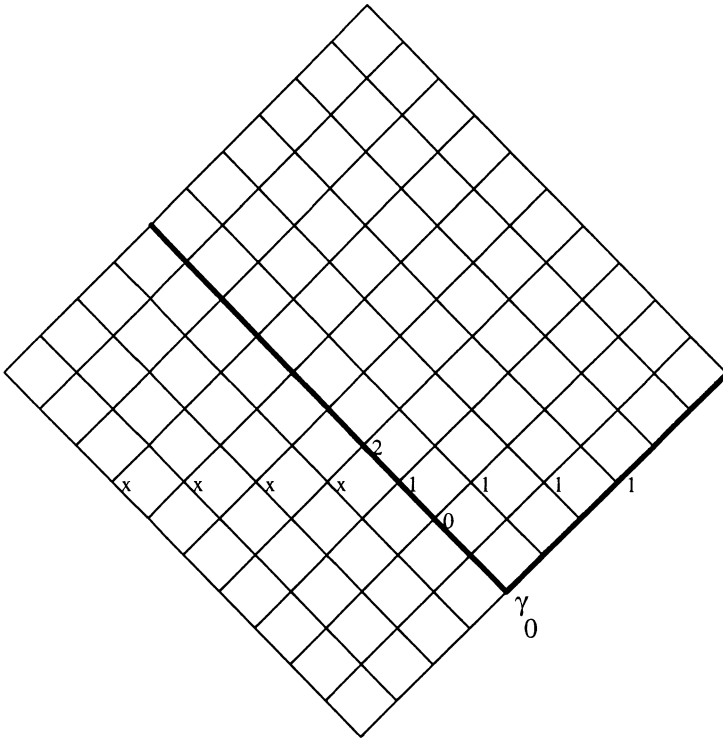
Figure 13. Typical situation in the proof of Proposition 6.26. Here, γ_0 is short. Inside $C^+_{\gamma_0}$, $\langle \Lambda + \rho, \alpha \rangle \geqslant 2$ above the string of 1's and is non-positive below. Outside, at each of the x's the value is less than or equal to 0 (the values may differ). Any occurrence of a number $\geqslant 2$ is incompatible with the weight coming from a m_i.

is by restricting to the cases where γ_0 is on a 'wall' of the diagram (i.e. the total number of simple compact roots that can be added/subtracted to/from γ_0 to give non-compact roots is at most 3). Here, the 'nature' of the configurations is very simple and so is the representation theory (again it is essentially only $u(n)$ arguments, but here very easy).

We choose then to consider γ_0 on a wall since it is possible to give the following general proofs and since it in the end turns out to be sufficient.

PROPOSITION 6.26. *Suppose that γ_0 is on a wall. In the notation of Definition 6.13, let $\Lambda - \xi(\gamma_0, j) - \breve{\omega}$ be a highest weight in $\mathcal{U}(\mathfrak{p}^-) \otimes V(\Lambda - \xi(\gamma_0, j))$ as well as in $m_i \otimes V(\Lambda)$ for some m_i ($i \geqslant j$). Then the natural projection of $\mathcal{U}(\mathfrak{p}^-) \otimes V(\Lambda - \xi(\gamma_0, j))$ into $\mathcal{U}(\mathfrak{p}^-) \otimes V(\Lambda)$ is injective on the space(s) corresponding to this highest weight.*

Proof. Let us consider the special value λ_{j-1} for which $\Lambda - \xi(\gamma_0, j)$ is a primitive weight in the $\mathfrak{g}$-module $\mathcal{U}(\mathfrak{p}^-) \otimes V(\Lambda)$ (Theorem 6.24). We then have a homomorphism $M(V(\Lambda - \xi(\gamma_0, j))) \to M(V(\Lambda))$ and the kernel for that is exactly the kernel for the natural projection. Let us consider a $\mathfrak{k}$-type in this kernel
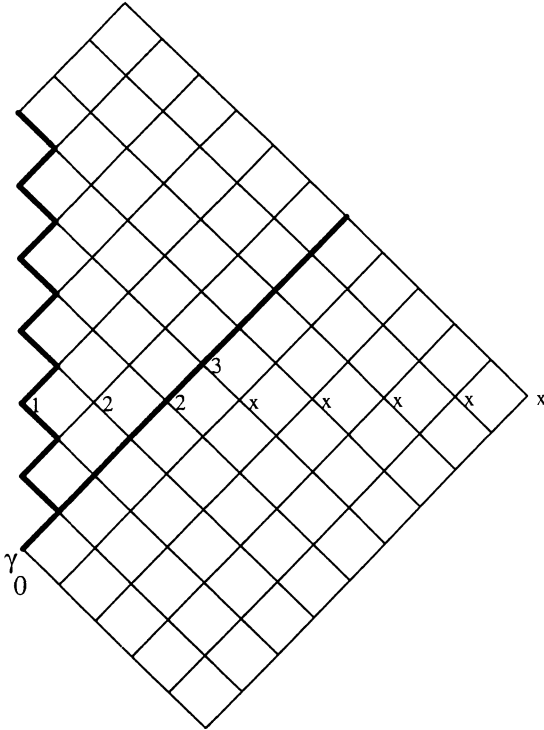
Figure 14. Situation from the proof of Proposition 6.26. Here, γ_0 is long. Inside $C_{\gamma_0}^+$, $\langle \Lambda + \rho, \alpha \rangle \geqslant 3$ above the string of 2's and is less than or equal to 1 below. Outside, at each of the x's the value is less than or equal to 0 (the values may differ). Any occurrence of a number $\geqslant 3$ is incompatible with the weight coming from a m_i. Such a 3 (or more) will have to appear if anything from outside of $C_{\gamma_0}^+$ takes part in the BGG-chain. This is the only non-trivial case, but we can also see that even though a string of 1's inside $C_{\gamma_0}^+$ is formally possible, this does not correspond to a polynomial representation. (Cf. the second condition in Lemma 4.7). It follows easily that we also here will be forced to consider values $\geqslant 3$ unless we consider the BGG-chain consisting of the 2's plus the 1 at the long root. But this is the chain that corresponds to $\Lambda - \xi(\gamma_0, j)$.

of smallest possible degree and let $\Lambda - \xi(\gamma_0, j) - \hat{\omega}_1$ be its highest weight. Such a type will define a homomorphism into $M(V(\Lambda - \xi(\gamma_0, j)))$ and further on into $M(V(\Lambda))$. Hence we get a BGG-chain which, of course, is not the one corresponding to $\Lambda - \xi(\gamma_0, j)$. The two conditions, (i) $\Lambda - \xi(\gamma_0, j) - \hat{\omega}_1$ should be dominant and coming from a BGG-chain and (ii) $\Lambda - \xi(\gamma_0, j) - \hat{\omega}_1$ should be equal to a weight of the form $\Lambda - \tilde{\xi}$ for a weight $-\tilde{\xi}$ in the $\mathfrak{k}$-type m_i, are now easily seen to be incompatible. See Figure 13 and Figure 14. The general statement follows readily from this. $\square$

PROPOSITION 6.27. *Suppose that γ_0 is on a wall. Suppose that $\Lambda - \xi(\gamma_0, j) - \check{\omega}$ is a highest weight in the tensor product*

$$m_i \otimes V(\Lambda), \tag{60}$$

where $i \geqslant j$. Then

$$V(\Lambda - \xi(\gamma_0, j) - \breve{\omega}) \subseteq \mathcal{U}(\mathfrak{p}^-)V(\Lambda - \xi(\gamma_0, j)). \tag{61}$$

Proof. This can be proved by checking multiplicities. We notice that the only non-trivial cases involve $u(n)$ (corresponding to $su(p, q)$, $sp(n, \mathbb{R})$, and $so^*(2n)$). Here we have the Littlewood–Richardson Rule (the LR Rule) [19] for computing multiplicities in a tensor product. Specifically, we must prove that the multiplicity of the representation $\tilde{\tau}$ in $m_i \otimes V(\Lambda)$ whose highest weight is $\Lambda - \xi(\gamma_0, j) - \breve{\omega}$ is the same as its multiplicity in $\mathcal{U}(\mathfrak{p}^-)V(\Lambda - \xi(\gamma_0, j))$. Here we begin by establishing that the multiplicities of $\tilde{\tau}$ in the first mentioned space is the same as in $\mathcal{U}(\mathfrak{p}^-) \otimes V(\Lambda - \xi(\gamma_0, j))$ (in fact, in $m_{i-j} \otimes V(\Lambda - \xi(\gamma_0, j))$. This is where the LR Rule comes in together with the assumption about γ_0. These namely make it easy to see that these multiplicities are the same. Indeed, in all cases except the one where γ_0 is a long root in $sp(n, \mathbb{R})$, the multiplicities are 1 in both places. To wit, in each of these cases there is a representation of $u(n)$ involved of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$ (a fundamental weight), and a such gives multiplicity 1. The case of the long root in $sp(n, \mathbb{R})$ may have bigger multiplicities, but again, using the LR Rule on a tensor product involving an m_i, it is elementary to see that the multiplicities are the same. We omit the details. (Cf. Remark 6.28 below). The proof is then completed by using Proposition 6.26.   □

Remark 6.28. It turns out that one, by using a tensor product argument, in the case of $sp(n, \mathbb{R})$ need only treat the cases $(r, 0, \ldots, 0)$. For these, the above claims are very easy.

PROPOSITION 6.29. *Suppose that γ_0 is on a wall. Consider a highest weight vector v_{high} for $\mathfrak{k}$ in $\mathcal{U}(\mathfrak{p}^-) \otimes V_\tau$. Let $\Lambda - \xi$ be its weight, and assume that ξ can be written as a sum of positive non-compact roots in such a way that the summands γ_j all satisfy*

$$\text{ht}(\gamma_j - \beta) \geqslant \text{ht}(\gamma_0 - \beta) + b, \tag{62}$$

for some non-negative integer b. Then

$$v_{\text{high}} \in \mathcal{U}(\mathfrak{p}^-)V(\Lambda - \xi(\gamma_0, b + 1)). \tag{63}$$

Proof. Suppose ξ can be written as a combination of elements from $C_{\beta_1}^+$ for some β_1 on the same wall as γ_0. Set $a = \text{ht}(\gamma_0 - \beta_1)$ and assume a is the smallest such integer. With no loss of generality we may assume that $a \geqslant 1$ since otherwise we need only consider $\mathcal{D}(\gamma_0)$ (possibly with arrows up and down) and then it is trivial. By looking at the other wall that contains β (corresponding to μ_i's on which Λ is trivial) it follows that there is a $\mathfrak{k}$-type m_s in $\mathcal{U}(\mathfrak{p}^-)$ of size at least $s = (a + b + 1)$ such that (cf. Figure 12)

$$v_{\text{high}} \in \mathcal{U}(\mathfrak{p}^-)(m_s \otimes V_\tau). \tag{64}$$
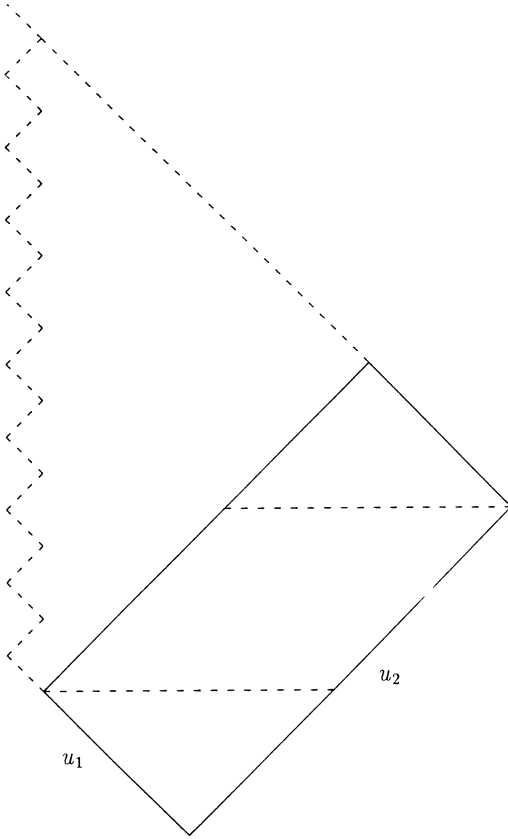
Figure 15.

More precisely, there must be a $\mathfrak{k}$-type V_1 in $m_s \otimes V(\Lambda)$ of highest weight $\Lambda - \xi(\gamma_0, b+1) - \hat{\omega}$ such that $v_{\text{high}} \in \mathcal{U}(\mathfrak{p}^-)V_1$. Now we just consider $m_s \otimes V(\Lambda)$ since the type in question must be contained in the tensor product of a such with $\mathcal{U}(\mathfrak{p}^-)$. But the claim then follows from Proposition 6.27. $\qquad\square$

7. Main Theorems

THEOREM 7.1. *Suppose that γ_0 is on a wall. Let $\tau = (\Lambda_0, \lambda)$. At $\lambda = \lambda_0$, $W(V_{\tau'})^0$ is a highest weight module generated by the first order polynomial $p_{\Lambda - \gamma_0} \in \mathfrak{p}^- \otimes V_\Lambda$ of highest weight $\Lambda - \gamma_0$. More generally, if $\lambda = \lambda_{j-1} = \lambda_0 + (j-1) \cdot \lambda_2^{\text{sc}}$, with λ_2^{sc} as in Theorem 6.21, then $N(V_\tau) = W(V_\tau')^0$ is a non-trivial highest weight module of weight $\Lambda - \xi(\gamma_0, j)$. Furthermore, for $\lambda < \lambda_{t_0-1}$, $W(V_\tau)^0 = 0$. Moreover,*

$$0 \subsetneq W(V_{(\Lambda_0, \lambda_{t_0-1})})^0 \subsetneq W(V_{(\Lambda_0, \lambda_{t_0-2})})^0 \subsetneq \cdots \subsetneq W(V_{(\Lambda_0, \lambda_0)})^0. \qquad (65)$$

$M(V_\tau)$ *is unitarizable exactly for*

$$\lambda \in] - \infty, \lambda_{t_0-1}[\cup\{\lambda_{t_0-1}, \ldots, \lambda_0\}. \tag{66}$$

Proof. Let us first turn our attention to the unitarity at $\lambda = \lambda_0$: If the hermitian form, when restricted to some $\mathfrak{k}$-type, is not positive semi-definite it is because the signature, which clearly is positive for λ sufficiently negative, has changed sign at some $\tilde{\lambda} < \lambda_0$. Assume this to be the case. It follows then easily, by looking at such a $\mathfrak{k}$-type of the smallest possible degree, that for a $\tilde{\lambda}$ for which $(\Lambda + \rho, \gamma_0) \leqslant 0$ there is a non-trivial homomorphism $\varphi \colon M(V_{\Lambda^1}) \rightarrow M(V_\Lambda)$. However, then Proposition 6.17, Proposition 6.18, and, notably, Proposition 6.29 can be applied and we conclude that if we consider the vector v_{high} which defines the homomorphism then it satisfies (63). This, on the other hand, immediately implies that v_{high} belongs to the radical of the hermitian form at $\lambda = \lambda_0$. This is a contradiction and hence all $\mathfrak{k}$-types are positive semi-definite at λ_0.

The cases involving higher order annihilators follow analogously. The main new ingredients are Corollary 6.19, the description following it, and Theorem 6.24.

As in the proof of Theorem 6.24, our main tool is to form tensor products with the representation $W(\lambda_2^{\text{sc}}) = W((0, \lambda_2^{\text{sc}})')$ from Theorem 6.21. Specifically, each irreducible summand in $W((\Lambda_0, \lambda_0)') \otimes W(\lambda_2^{\text{sc}})$ is unitary. It then follows by [14] that

$$W((\Lambda_0, \lambda_0 + \lambda_2^{\text{sc}})') = \text{Res}(W((\Lambda_0, \lambda_0)') \otimes W((0, \lambda_2^{\text{sc}})')), \tag{67}$$

where Res denotes the restriction to the diagonal, is unitary.

Let us assume that $t_0 \geqslant 2$. As was hinted at in the introduction and proved in [15], it follows from this that the annihilator of $W((\Lambda_0, \lambda_0 + \lambda_2^{\text{sc}})')$ is contained in the annihilator of $W((\Lambda_0, \lambda_0)')$. This inclusion will be proper because $W((0, \lambda_2^{\text{sc}})')$ contains all first order polynomials and hence, so does $W((\Lambda_0, \lambda_0 + \lambda_2^{\text{sc}})')$. Furthermore, it is clear by Theorem 6.24 that the element of highest weight $(\Lambda_0, \lambda_0 + \lambda_2^{\text{sc}}) - \tilde{\gamma}_1 - \tilde{\gamma}_2$ is in the annihilator of $W((\Lambda_0, \lambda_0 + \lambda_2^{\text{sc}})')$. In fact, it is clear by Lemma 6.10 that we are in a situation quite analogous to the one for scalar modules. (Including the issue of the lack of unitarity in the intervals $]\lambda_i, \lambda_{i-1}[$; $i = 1, \ldots, t_0 - 1$). So, we can continue the argument to $W((\Lambda_0, \lambda_0 + 2\lambda_2^{\text{sc}})'), \ldots, W((\Lambda_0, \lambda_0 + i\lambda_2^{\text{sc}})')$ until we reach the continuous part of the range of unitarity—just as in the case of scalar modules. (The endpoint of the half line of unitarity was previously ([17]) named the 'first possible place of non-unitarity' because one can easily check that this, in the spirit of Bernstein–Gelfand–Gelfand, indeed *is* the first place, for λ coming from $-\infty$, where a homomorphism between generalized Verma modules can exist). Specifically, t_0 is the real rank of the Lie algebra corresponding to $\mathcal{D}(\gamma_0)$.

Finally there is the issue of the annihilators also here being highest weight modules. Let us consider $W((\Lambda_0, \lambda_0 + j\lambda_2^{\text{sc}})')$. We then know that the element corresponding to the weight $\Lambda - \xi(\gamma_0, j + 1)$ is in the annihilator of the module. Suppose the annihilator is not generated by this single $\mathfrak{k}$-type. Then there must be an

element of weight, say, Λ_2 in the annihilator which, modulo the subspace generated by the type corresponding to $\Lambda - \xi(\gamma_0, j+1)$, is a highest weight vector. Hence (Proposition 3.3) we get a sequence of roots $\alpha_1, \ldots, \alpha_s \in \triangle_n^+$ which satisfies condition (A) and which, therefore, fall in a certain subset of $\triangle_n^+$ restricted by the requirement of positivity in this condition. This condition can easily be translated into

$$\mathrm{ht}(\alpha_j - \beta) \geqslant \mathrm{ht}(\gamma_0 - \beta) + (j+1). \tag{68}$$

But then Proposition 6.29 gives that the type after all is in the ideal generated by the $\mathfrak{k}$-type in question.

What remains are the corresponding cases for the situations described by Figures 6 and 7. Of these the one in Figure 6 can be dismissed immediately as trivial and the one in Figure 7 is quite analogous to the previous. Indeed, for small values of 'j' in $(\lambda_0 + j\lambda_2^{\mathrm{sc}})$, there is no difference between this case and the one treated previously. It is only when we are forced into the region corresponding to the 'top' hermitian space that the situation is new. But we still have Lemma 6.12 and so everything carries over, including the explanation of the continuous range of unitarity. (Cf. Figure 15) $\square$

THEOREM 7.2. *Suppose that γ_0 be arbitrary. Let $\tau = (\Lambda_0, \lambda)$. At $\lambda = \lambda_0$, $W(V_{\tau'})^0$ is a highest weight module generated by the first order polynomial $p_{\Lambda - \gamma_0} \in \mathfrak{p}^- \otimes V_\Lambda$ of highest weight $\Lambda - \gamma_0$. More generally, if $\lambda = \lambda_{j-1} = \lambda_0 - (j-1) \cdot \lambda_2^{\mathrm{sc}}$, with λ_2^{sc} as in Theorem 6.21, then $N(V_\tau) = W(V_\tau')^0$ is a non-trivial highest weight module of weight $\Lambda - \xi(\gamma_0, j)$. Furthermore, for $\lambda < \lambda_{t_0 - 1}$, $W(V_\tau)^0 = 0$. Moreover,*

$$0 \subsetneqq W(V_{(\Lambda_0, \lambda_{t_0 - 1})})^0 \subsetneqq W(V_{(\Lambda_0, \lambda_{t_0 - 2})})^0 \subsetneqq \cdots \subsetneqq W(V_{(\Lambda_0, \lambda_0)})^0. \tag{69}$$

$M(V_\tau)$ *is unitarizable exactly for*

$$\lambda \in]-\infty, \lambda_{t_0 - 1}[\cup\{\lambda_{t_0 - 1}, \ldots, \lambda_0\}. \tag{70}$$

Proof. The only non-trivial cases are A_n and C_n. In both cases a general representation can be obtained as the tensor product of two 'wall-representations'. In case of A_n one simply projects along one side of the diagram onto the other and vice versa. In the case of C_n, in reference to Figure 7, one has to project vertically and horizontally, respectively.

One checks easily that there is unitarity at the last possible point, and then the idea of tensoring with $W(\lambda_2^{\mathrm{sc}})$ carries over with no changes.

Finally there is the question here of the annihilators being highest weight modules. We do not prove this here since it seems to involve too many specific features of the representation theory of $u(n)$ and the only point of doing this would be to show that this result is independent (or rather, precedes) unitarity. Instead we refer to [6] where the result is proved using the unitarity. $\square$

THEOREM 7.3. *Let, as in Section 4, $\delta_j = \gamma_1 + \cdots + \gamma_j$ for $j = 1, \ldots, r$. Suppose that $\mathrm{ht}(\delta_s) \geqslant s \cdot \mathrm{ht}(\gamma_0)$ for some positive integer s. Let s_0 be the smallest possible s for which the inequality is true. Then a $\mathfrak{k}$-type $V_\omega \subset \mathcal{P}$ of highest weight $\omega = -n_1\delta_1 - \cdots - n_j\delta_j$ belongs to $\mathrm{Ann}_{\mathfrak{p}-}(\Lambda_0, \lambda_0)$ if and only $n_j \geqslant 1$ for some $j \geqslant s_0$. More generally, $V_\omega \subset \mathrm{Ann}_{\mathfrak{p}-}(\Lambda_0, \lambda_0 + i\lambda_2^{\mathrm{sc}}) \Leftrightarrow n_j \geqslant 1$ for some $j \geqslant s_0 + i$, i.e. $\mathrm{Ann}_{\mathfrak{p}-}(\Lambda_0, \lambda_0 + i\lambda_2^{\mathrm{sc}})$ is generated by m_{s_0+i}.*

Proof. Let K denote the radical of the hermitian form on $M(V_\tau)$. Let us first look at $\mathrm{Ann}_{\mathfrak{p}-}(\Lambda_0, \lambda_0)$. Let V_ω be a given $\mathfrak{k}$-type in $\mathcal{P}$. Combining (27) with Theorem 7.1 we see that

$$V_\omega \subseteq \mathrm{Ann}_{\mathfrak{p}-}(\Lambda_0, \lambda_0) \Leftrightarrow V_\omega \otimes V_\tau \subset K. \tag{71}$$

Let us look at the highest weights ξ that occur in the left hand side. Among these is the $\mathfrak{k}$-type (the 'worst case') whose highest weight is $\omega + \Lambda$. Observe that the highest weight vector for this type is given by $q_\omega \cdot v_\Lambda$ where q_ω is the highest weight vector of V_ω.

Let us take a closer look at q_ω for $\omega = -\delta_j$. It follows from Proposition 5.10 that δ_j can be written as a sum of roots falling on a horizontal line in the diagram (Figure 16). This horizontal line must then intersects $C_{\gamma_0}^+$ for V_ω to be in the kernel since otherwise there will be an extremal element in the kernel that gives rise to a BGG homomorphism and this will have an expression that, in contradiction of Proposition 6.17, does not intersect $C_{\gamma_0}^+$. If it does, it is clear, by using the invariance of $q_{-\delta_j}$ under $\mathfrak{k}_j$ (Proposition 5.9), that any other expression of δ_j also will intersect $C_{\gamma_0}^+$. Thus, $n_j \geqslant 1$ for some $j \geqslant s_0$ is a necessary condition for V_ω is contained in the annihilator.

It remains to prove the converse: For this, it suffices to prove, since the annihilator is an ideal, that if $\omega_1 = -\delta_1 - \cdots - \delta_j$ with $j \geqslant s_0$, then V_{ω_1} is in the annihilator. What we can actually show is that $V_{\omega_1} \otimes V_\tau \subseteq \mathcal{U}(\mathfrak{p}^-)V(\Lambda - \gamma_0) \subseteq K$. For γ_0 on a wall this follows easily from the Proposition 6.29. What remains are then the general cases for $\mathrm{su}(p, q)$, $\mathrm{sp}(n, \mathbb{R})$, and $\mathrm{so}^*(2n)$. These can in principle be handled directly by the representation theory for $u(n)$. However, there is an easier way, namely by decomposing the general case into a tensor product of representations on the walls as we did in the case of unitarity. The result then follows from Lemma 7.4 below.

The higher order annihilators follow analogously – or by the argument in Joseph's article [20].   □

LEMMA 7.4. *Let $m_{a_i} \in \mathrm{Ann}_{\mathfrak{p}-}(\tau_i)$ for $i = 1, 2$. Assume that $a_1 + a_2 - 1 \leqslant r$. Then*

$$m_{a_1+a_2-1} \in \mathrm{Ann}_{\mathfrak{p}-}(\tau_1 \otimes \tau_2), \tag{72}$$

where $\tau_1 \otimes \tau_2$ is the tensor product as $\mathfrak{k}$-representations, and where we more precisely mean that for any $\mathfrak{k}$-type $\tilde{\tau}$ in $\tau_1 \otimes \tau_2$ (counted with multiplicity), $m_{a_1+a_2-1} \in \mathrm{Ann}_{\mathfrak{p}-}(\tilde{\tau})$.
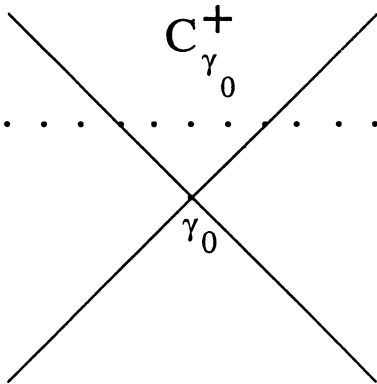
Figure 16.

Proof. $W(\tilde{\tau})$ can be obtained by restricting $W(\tau_1) \otimes W(\tau_2)$ to the diagonal. The action of $\mathfrak{p}^-$ through any dU_τ is by means of differentiation. If $p_i \in W(\tau_i), i = 1, 2$, it then follows easily from the 'determinantal' nature of the m_i's that, suppressing the representations,

$$m_{a_1+a_2-1}(p_1 \otimes p_2) \subseteq \sum_{l=1}^{a_1+a_2-1} c_l \cdot m_l(p_1) \otimes m_{a_1+a_2-1-l}(p_2), \qquad (73)$$

for certain (not interesting) constants c_l. More precisely, the action of an element of $m_{a_1+a_2-1}$ on $p_1 \otimes p_2$ can be written as a linear combination of elements of $m_l(p_1) \otimes m_{a_1+a_2-1-l}(p_2)$ for $l = 1, \ldots, a_1+a_2-1$. Since either $a_1+a_2-1-l \geqslant a_1$ or $l \geqslant a_2$, the claim follows. $\square$

Remark 7.5. From the explicit description of the ideals it follows from well-known classical results that these ideals are prime. See also the remarks in the introduction.

Remark 7.6. In the scalar case where $\Lambda_0 = 0$ it follows easily that the holomorphic functions in $W((i - 1)\lambda_2^{sc})$, in the unbounded realization of the hermitian symmetric space, are Fourier-Laplace transforms of distributions supported by a suitable real part of the variety defined by the $\mathfrak{k}$-type m_i. Also note that in the general case, if there is only a single hermitian symmetric space $\mathcal{D}(\gamma_0)$ defined by γ_0, then the representation of the algebra corresponding to this space, obtained by restricting the functions, is a *scalar* representation.

References

1. Bernstein, I. N., Gelfand, I. M. and Gelfand, S. I.: *Differential operators on the base affine space and a study of $\mathfrak{g}$-modules*, in 'Lie Groups And Their Representations', Adam Hilger, London, 1975.

2. Boe, B.: *Homomorphisms between generalized Verma modules*, Dissertation, Yale University, 1982.
3. Boe, B. and Collingwood, D.: Intertwining operators between holomorphically induced modules, *Pac. J. Math.* 124 (1986), 73–84.
4. Boe, B. and Collingwood, D.: A multiplicity one theorem for holomorphically induced representations, *Math. Z.* 192 (1986), 265–282.
5. Davidson, M. G. and Stanke, R. J.: Gradient-type differential operators and unitary highest weight representations of SU(p, q), *J. Funct. Anal.* 81 (1988), 100–125.
6. Davidson, M. G., Enright, T. J. and Stanke, R. J.: *Differential operators and highest weight representations*, Memoirs of the A.M.S. #455 (1991).
7. Deodhar, V. V.: Some characterizations of the Bruhat ordering on a Coxeter group and characterization of the relative Möbius function, *Invent. Math.* 39 (1977), 187–198.
8. Enright, T. J., Howe, R. and Wallach, N.: A classification of unitary highest weight modules, in *Representation Theory Of Reductive Groups*, P.C. Trombi (eds.) (1983), 97–143.
9. Enright, T. J. and Joseph, A.: *An intrinsic analysis of unitarizable highest weight modules*, Preprint 1990.
10. Enright, T. J. and Parthasarathy, R.: A proof of a conjecture of Kashiwara and Vergne, in Noncommutative Harmonic Analysis, Marseille-Luminy 1980, Springer Lecture Notes in Mathematics, 880 (1981), 74–90.
11. Harish-Chandra, Representations of semi-simple Lie groups IV, V, VI., *Amer. Jour. Math.* 77 (1955), 743–777; 78 (1956), 1–41; 78 (1956), 564–628.
12. Harris, M. and Jakobsen, H. P.: Covariant differential operators, in *Group Theoretical Methods In Physics, Istanbul 1982*, Springer Lecture Notes in Physics 180, 1983.
13. Jakobsen, H. P. and Vergne, M.: Wave and Dirac operators and representations of the conformal group, *J. Funct. Anal.* 24 (1977), 52–106.
14. Jakobsen, H. P. and Vergne, M.: Restrictions and expansions of holomorphic representations, *J. Funct. Anal.* 34 (1979), 29–53.
15. Jakobsen, H. P.: On singular holomorphic representations, *Invent. Math.* 62 (1980), 67–78.
16. Jakobsen, H. P.: The last possible place of unitarity for certain highest weight modules, *Math. Ann.* 256 (1981), 439–447.
17. Jakobsen, H. P.: Hermitian symmetric spaces and their unitary highest weight modules, *J. Funct. Anal.* 52 (1983), 385–412.
18. Jakobsen, H. P.: Basic covariant differential operators on hermitian symmetric spaces, *Ann. scient. Èc. Norm. Sup.* 18 (1985), 421–436.
19. James, G. D.: *'The Representation Theory of the Symmetric Groups'*, Springer Lecture Notes in Math. #682, Springer Verlag, Berlin–Heidelberg–New York 1978.
20. Joseph, A.: *Annihilators and associated varieties of unitary highest weight modules*, Preprint 1990, To appear in Ann. scient. Èc. Norm. Sup.
21. Kashiwara, M. and Vergne, M.: On the Segal-Shale-Weil representation and harmonic polynomials, *Invent. Math.* 44 (1978), 1–47.
22. Knapp, A. and Okamoto, K.: Limits of holomorphic discrete series, *J. Funct. Anal.* 9 (1972), 375–409.
23. Mack, G.: All unitary ray representations of the conformal group SU(2,2) with positive energy, *Comm. Math. Phys.* 55 (1977), 1–28.
24. Parthasarathy, K. R.: Criteria for the unitarizability of some highest weight modules, *Proc. Indian Acad. Sci.* 89 (1980), 1–24.
25. Rossi, H. and Vergne, M.: Analytic continuation of the holomorphic discrete series of a semi-simple Lie group, *Acta Math.* 136 (1976), 1–59.
26. Rühl, W.: Distributions on Minkowski space and their connection with analytic representations of the conformal group, *Comm. Math. Phys.* 27 (1972), 53–86. See also *Ibid.* 30 (1973), 287–302 and 34 (1973), 149–166.
27. Schmid, W.: Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen, *Invent. Math.* 9 (1969), 61–80.
28. Wallach, N.: Analytic continuation of the holomorphic discrete series II, *Trans. Amer. Math, Soc.* 260 (1980), 563–573.