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An improvement of the quantitative Subspace theorem

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1. Introduction

Let n be an integer and $l_1, \ldots, l_n$ linearly independent linear forms in n variables with (real or complex) algebraic coefficients. For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ put

$$|\mathbf{x}| := \sqrt{x_1^2 + \cdots + x_n^2}.$$

In 1972, W. M. Schmidt [17] proved his famous *Subspace theorem*: for every $\delta > 0$, there are finitely many proper linear subspaces $T_1, \ldots, T_t$ of $\mathbb{Q}^n$ such that the set of solutions of the inequality

$$|l_1(\mathbf{x}) \cdots l_n(\mathbf{x})| < |\mathbf{x}|^{-\delta} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n$$

is contained in $T_1 \cup \cdots \cup T_t$.

In 1989, Schmidt managed to prove the following quantitative version of his Subspace theorem. Suppose that each of the above linear forms l_i has height $H(l_i) \leqslant H$ defined below and that the field generated by the coefficients of $l_1, \ldots, l_n$ has degree D_0 over $\mathbb{Q}$. Further, let $0 < \delta < 1$. Denote by $\det(l_1, \ldots, l_n)$ the coefficient determinant of $l_1, \ldots, l_n$. Then there are proper linear subspaces $T_1, \ldots, T_t$ of $\mathbb{Q}^n$ with

$$t \leqslant (2D_0)^{2^{26n}\delta^{-2}}$$

such that the set of solutions of

$$|l_1(\mathbf{x}) \cdots l_n(\mathbf{x})| < |\det(l_1, \ldots, l_n)| \cdot |\mathbf{x}|^{-\delta} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n \tag{1.1}$$

is contained in

$$\{\mathbf{x} \in \mathbb{Z}^n : |\mathbf{x}| < \max\left((n!)^{8/\delta}, H\right)\} \cup T_1 \cup \cdots \cup T_t.$$

In 1977, Schlickewei extended Schmidt's Subspace theorem of 1972 to the p-adic case and to number fields. In 1990 [15] he generalised Schmidt's quantitative

Subspace theorem to the p-adic case over $\mathbb{Q}$ and later, in 1992 [16] to number fields. Below we state this result of Schlickewei over number fields and to this end we introduce suitably normalised absolute values and heights.

Let K be an algebraic number field. Denote its ring of integers by O_K and its collection of places (equivalence classes of absolute values) by M_K. For $v \in M_K$, $x \in K$, we define the absolute value $|x|_v$ by

(i) $|x|_v = |\sigma(x)|^{1/[K:\mathbb{Q}]}$ if v corresponds to the embedding $\sigma : K \hookrightarrow \mathbb{R}$;
(ii) $|x|_v = |\sigma(x)|^{2/[K:\mathbb{Q}]} = |\bar{\sigma}(x)|^{2/[K:\mathbb{Q}]}$ if v corresponds to the pair of conjugate complex embeddings $\sigma, \bar{\sigma} : K \hookrightarrow \mathbb{C}$;
(iii) $|x|_v = (N\mathfrak{p})^{-\text{ord}_\mathfrak{p}(x)/[K:\mathbb{Q}]}$ if v corresponds to the prime ideal $\mathfrak{p}$ of O_K.

Here $N\mathfrak{p} = \#(O_K/\mathfrak{p})$ is the norm of $\mathfrak{p}$ and $\text{ord}_\mathfrak{p}(x)$ the exponent of $\mathfrak{p}$ in the prime ideal decomposition of (x), with $\text{ord}_\mathfrak{p}(0) := \infty$. In case (i) or (ii) we call v real infinite or complex infinite, respectively and write $v|\infty$; in case (iii) we call v finite and write $v \nmid \infty$. These absolute values satisfy the *Product formula*

$$\prod_v |x|_v = 1 \quad \text{for } x \in K^*$$

(product taken over all $v \in M_K$) and the *Extension formulas*

$$\prod_{w|v} |x|_w = |N_{L/K}(x)|_v^{1/[L:K]} \quad \text{for } x \in L, v \in M_K;$$

$$\prod_{w|v} |x|_w = |x|_v \quad \text{for } x \in K, v \in M_K,$$

where L is any finite extension of K and the product is taken over all places w on L lying above v.

The *height* of $\mathbf{x} = (x_1, \ldots, x_n) \in K^n$ with $\mathbf{x} \neq \mathbf{0}$ is defined as follows: for $v \in M_K$ put

$$|\mathbf{x}|_v = \left(\sum_{i=1}^n |x_i|_v^{2[K:\mathbb{Q}]}\right)^{1/2[K:\mathbb{Q}]} \quad \text{if } v \text{ is real infinite,}$$

$$|\mathbf{x}|_v = \left(\sum_{i=1}^n |x_i|_v^{[K:\mathbb{Q}]}\right)^{1/[K:\mathbb{Q}]} \quad \text{if } v \text{ is complex infinite,}$$

$$|\mathbf{x}|_v = \max(|x_1|_v, \ldots, |x_n|_v) \quad \text{if } v \text{ is finite}$$

(note that for infinite places v, $|\cdot|_v$ is a power of the Euclidean norm). Now define

$$H(\mathbf{x}) = H(x_1, \ldots, x_n) = \prod_v |\mathbf{x}|_v.$$

By the Product formula, $H(a\mathbf{x}) = H(\mathbf{x})$ for $a \in K^*$. Further, by the Extension formulas, $H(\mathbf{x})$ depends only on $\mathbf{x}$ and not on the choice of the number field K

containing the coordinates of $\mathbf{x}$, in other words, there is a unique function H from $\bar{\mathbb{Q}}^n \backslash \{\mathbf{0}\}$ to $\mathbb{R}$ such that for $\mathbf{x} \in K^n$, $H(\mathbf{x})$ is just the height defined above; here $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. For a linear form $l(\mathbf{X}) = a_1X_1 + \cdots + a_nX_n$ with algebraic coefficients we define $H(l) := H(\mathbf{a})$ where $\mathbf{a} = (a_1, \ldots, a_n)$ and if $\mathbf{a} \in K^n$ then we put $|l|_v = |\mathbf{a}|_v$ for $v \in M_K$. Further, we define the number field $K(l) := K(a_1/a_j, \ldots, a_n/a_j)$ for any j with $a_j \neq 0$; this is independent of the choice of j. Thus, $K(cl) = K(l)$ for any non-zero algebraic number c.

We are now ready to state Schlickewei's result from [16]. Let K be a *normal* extension of $\mathbb{Q}$ of degree d, S a finite set of places on K of cardinality s and for $v \in S$, $\{l_{1v}, \ldots, l_{nv}\}$ a linearly independent set of linear forms in n variables with coefficients in K and with $H(l_{iv}) \leqslant H$ for $i = 1, \ldots, n, v \in S$. Then for every δ with $0 < \delta < 1$ there are proper linear subspaces $T_1, \ldots, T_t$ of K^n with

$$t \leqslant (8sd)^{2^{34nd}s^6\delta^{-2}},$$

such that every solution $\mathbf{x} \in K^n$ of the inequality

$$\prod_{v\in S}\prod_{i=1}^{n} \frac{|l_{iv}(\mathbf{x})|_v}{|l_{iv}|_v |\mathbf{x}|_v} < H(\mathbf{x})^{-n-\delta} \tag{1.2}$$

either lies in $T_1 \cup \cdots \cup T_t$ or satisfies

$$H(\mathbf{x}) < \max\left((n!)^{9/\delta}, H^{dns/\delta}\right).$$

The restrictions that K be normal and the linear forms l_{iv} have their coefficients in K are inconvenient for applications such as estimating the numbers of solutions of norm form equations or decomposable form equations where one has to deal with inequalities of type (1.2) of which the unknown vector $\mathbf{x}$ assumes its coordinates in a finite, non-normal extension K of $\mathbb{Q}$ and the linear forms l_{iv} have their coefficients outside K.

In this paper, we improve Schlickewei's quantitative Subspace theorem over number fields. We drop the restriction that K be normal and we allow the linear forms to have coefficients outside K. Further, we derive an upper bound for the number of subspaces with a much better dependence on n and δ: our bound depends only exponentially on n and polynomially on δ^{-1} whereas Schlickewei's bound is doubly exponential in n and exponential in δ^{-1}. As a special case we obtain a significant improvement of Schmidt's quantitative Subspace theorem mentioned above.

In the statement of our main result, the following notation is used:

- K is an algebraic number field (not necessarily normal);
- S is a finite set of places on K of cardinality s containing all infinite places;
- $\{l_{1v}, \ldots, l_{nv}\} (v \in S)$ are linearly independent sets of linear forms in n variables with algebraic coefficients such that

$$H(l_{iv}) \leqslant H, \quad [K(l_{iv}):K] \leqslant D \quad \text{for } v \in S,\ i = 1, \ldots, n.$$

In the sequel we assume that the algebraic closure of K is $\bar{\mathbb{Q}}$. We choose for every place $v \in M_K$ a continuation of $|\cdot|_v$ to $\bar{\mathbb{Q}}$, and denote this also by $|\cdot|_v$; these continuations are fixed throughout the paper.

THEOREM. *Let* $0 < \delta < 1$. *Consider the inequality*

$$\prod_{v\in S}\prod_{i=1}^{n}\frac{|l_{iv}(\mathbf{x})|_v}{|\mathbf{x}|_v} < \left(\prod_{v\in S}|\det(l_{1v},\ldots,l_{nv})|_v\right)\cdot H(\mathbf{x})^{-n-\delta} \quad \text{in } \mathbf{x}\in K^n. \tag{1.3}$$

(i) *There are proper linear subspaces* $T_1, \ldots, T_{t_1}$ *of* K^n, *with*

$$t_1 \leqslant (2^{60n^2}\cdot\delta^{-7n})^s \log 4D \cdot \log\log 4D$$

such that every solution $\mathbf{x} \in K^n$ *of* (1.3) *with*

$$H(\mathbf{x}) \geqslant H$$

belongs to $T_1 \cup \cdots \cup T_{t_1}$.
(ii) *There are proper linear subspaces* $S_1, \ldots, S_{t_2}$ *of* K^n, *with*

$$t_2 \leqslant (150n^4\cdot\delta^{-1})^{ns+1}(2 + \log\log 2H)$$

such that every solution $\mathbf{x} \in K^n$ *of* (1.3) *with*

$$H(\mathbf{x}) < H$$

belongs to $S_1 \cup \cdots \cup S_{t_2}$.

Now assume that $K = \mathbb{Q}$, $S = \{\infty\}$ and let $l_1, \ldots, l_n$ be linearly independent linear forms in n variables with algebraic coefficients such that $H(l_i) \leqslant H$ and $[\mathbb{Q}(l_i):\mathbb{Q}] \leqslant D$ for $i = 1, \ldots, n$. Consider again the inequality

$$|l_1(\mathbf{x})\cdots l_n(\mathbf{x})| < |\det(l_1,\ldots,l_n)|\cdot|\mathbf{x}|^{-\delta} \quad \text{in } \mathbf{x}\in\mathbb{Z}^n, \tag{1.1}$$

where $0 < \delta < 1$. If $\mathbf{x} \in \mathbb{Z}^n$ is *primitive*, i.e. $\mathbf{x} = (x_1, \ldots, x_n)$ with $\gcd(x_1, \ldots, x_n) = 1$, then $H(\mathbf{x}) = |\mathbf{x}|$. Hence our Theorem implies at once the following improvement of Schmidt's result:

COROLLARY. *For every* δ *with* $0 < \delta < 1$ *there are proper linear subspaces* $T_1, \ldots, T_t$ *of* $\mathbb{Q}^n$ *with*

$$t \leqslant 2^{60n^2}\delta^{-7n}\log 4D\cdot\log\log 4D$$

such that every solution $\mathbf{x} \in \mathbb{Z}^n$ *of* (1.1) *with*

$$H(\mathbf{x}) \geqslant H, \quad \mathbf{x} \textit{ primitive}$$

lies in $T_1 \cup \cdots \cup T_t$.

Define the height of an algebraic number ξ by $H(\xi) := H(1, \xi)$. Let K, S be as in the Theorem and for $v \in S$, let α_v be an algebraic number of degree at most D over K and with $H(\alpha_v) \leqslant H$. Let $0 < \delta < 1$. Consider the inequality

$$\prod_{v \in S} \min\left(1, |\beta - \alpha_v|_v\right) < H(\beta)^{-2-\delta} \quad \text{in } \beta \in K. \tag{1.4}$$

By a generalisation of a theorem of Roth, (1.4) has only finitely many solutions. Bombieri and van der Poorten [1] (only for S consisting of one place) and Gross [9] (in full generality) derived good upper bounds for the number of solutions of (1.4). It is possible to derive a similar bound from our Theorem above. Namely, let $l_{1v}(\mathbf{x}) = x_1 - \alpha_v x_2$, $l_{2v}(\mathbf{x}) = x_2$ for $v \in S$ and put $\mathbf{x} = (\beta, 1)$ for $\beta \in K$. Then every solution β of (1.4) satisfies

$$\begin{aligned} \prod_{v \in S} \frac{|l_{1v}(\mathbf{x}) l_{2v}(\mathbf{x})|_v}{|\mathbf{x}|_v^2} &\leqslant \prod_{v \in S} \min\left(1, |\beta - \alpha_v|_v\right) \\ &< H(\beta)^{-2-\delta} = \prod_{v \in S} |\det(l_{1v}, l_{2v})|_v \cdot H(\mathbf{x})^{-2-\delta}. \end{aligned}$$

Now our Theorem with $n = 2$ implies that (1.4) has at most

$$\begin{aligned} &(24000 \cdot \delta^{-1})^{2s+1}(2 + \log\log 2H) \\ &\quad + (2^{240} \cdot \delta^{-14})^s \log 4D \cdot \log\log 4D \end{aligned} \tag{1.5}$$

solutions. The bounds of Bombieri and van der Poorten and Gross are of a similar shape, except that in their bounds the constants are better and the dependence on D is slightly worse, namely $(\log D)^2 \cdot \log\log D$. Our Theorem can also be used to derive good upper bounds for the numbers of solutions of norm form equations, S-unit equations and decomposable form equations; we shall derive these bounds in another paper. Schlickewei announced that he improved his own quantitative Subspace theorem in another direction and that he used this to show a.o. that the zero multiplicity of a linear recurrence sequence of order n with rational integral terms is bounded above in terms of n only. (lectures given at MSRI, Berkeley, 1993, Oberwolfach, 1993, Conference on Diophantine problems, Boulder, 1994).

Remarks about Roth's lemma. Following Roth [13], the generalisation of Roth's theorem mentioned above can be proved by contradiction. Assuming that (1.4) has infinitely many solutions, one constructs an auxiliary polynomial $F \in \mathbb{Z}[X_1, \ldots, X_m]$ which has large 'index' at some point $\underline{\beta} = (\beta_1, \ldots, \beta_m)$ where $\beta_1, \ldots, \beta_m$ are solutions of (1.4) with $H(\beta_1), \ldots, H(\beta_m)$ sufficiently large. Then one applies a non-vanishing result proved by Roth in [13], now known as Roth's lemma, implying that F cannot have large index at $\underline{\beta}$.

In his proof of the Subspace theorem [17], Schmidt applied the same Roth's lemma but in a much more difficult way, using techniques from the geometry of numbers. Schmidt used these same techniques but in a more explicit form in his proof of his quantitative Subspace theorem [19]. Schlickewei proved his results [14, 15, 16] by generalising Schmidt's arguments to the p-adic case. Very recently, Faltings and Wüstholz [8] gave a completely different proof of the (qualitative) Subspace theorem. They did not use geometry of numbers but instead a very powerful generalisation of Roth's lemma, discovered and proved by Faltings in [7], the *Arithmetic product theorem* ([7], Theorems 3.1, 3.3).

Our approach in the present paper is that of Schmidt. But unlike Schmidt we do not use Roth's lemma from [13] but a sharpening of this, which we derived in [6] by making explicit the arguments used by Faltings in his proof of the Arithmetic product theorem.[1] Further, in order to obtain an upper bound for the number of subspaces depending only exponentially on n we also had to modify the arguments from the geometry of numbers used by Schmidt. For instance, Schmidt applied a lemma of Davenport and it seems that that would have introduced a factor $(2^n)!$ in our upper bound which is doubly exponential in n. Therefore we wanted to avoid the use of Davenport's lemma and we did so by making explicit some arguments from [5].

A modified version of Roth's lemma is as follows. Let $F(X_1,\dots,X_m)\in\bar{\mathbb{Q}}[X_1,\dots,X_m]$ be a polynomial of degree $\leqslant d_h$ in X_h for $h=1,\dots,m$. Define the *index* of F at $\mathbf{x}=(x_1,\dots,x_m)$ to be the largest real number Θ such that $(\partial/\partial X_1)^{i_1}\cdots(\partial/\partial X_m)^{i_m}F(\mathbf{x})=0$ for all non-negative integers $i_1,\dots,i_n$ with $i_1/d_1+\cdots+i_m/d_m\leqslant\Theta$. As before, the height of $\xi\in\bar{\mathbb{Q}}$ is defined by $H(\xi)=H(1,\xi)$ and the height $H(F)$ of F is by definition the height of the vector of coefficients of F. By $c_1,c_2,\dots,$ we denote positive absolute constants. Now Roth's lemma states that there are positive numbers $\omega_1(m,\Theta)$ and $\omega_2(m,\Theta)$ depending only on m,Θ, such that if $m\geqslant 2, 0<\Theta<1$, if

$$\frac{d_h}{d_{h+1}}\geqslant\omega_1(m,\Theta)\quad\text{for } h=1,\dots,m-1 \tag{1.6}$$

and if $x_1,\dots,x_m$ are non-zero algebraic numbers with

$$H(x_h)^{d_h}\geqslant\left(c_1^{d_1+\cdots+d_m}H(F)\right)^{\omega_2(m,\Theta)}\quad\text{for } h=1,\dots,m, \tag{1.7}$$

then F has index $\leqslant\Theta$ at $\mathbf{x}=(x_1,\dots,x_m)$.

By modifying the arguments of Schmidt and Schlickewei one can show that the set of solutions $\mathbf{x}$ of (1.3) with $H(\mathbf{x})\geqslant H$ is contained in some union of proper linear subspaces of K^n, $T_1\cup\cdots\cup T_{t_1}$ with

$$t_1\leqslant c(n,\delta,s)\cdot\{m\log\omega_1(m,\Theta)+\log\omega_2(m,\Theta)\}, \tag{1.8}$$

[1] Wüstholz announced at the conference on Diophantine problems in Boulder, 1994, that his student R. Ferretti independently obtained a similar sharpening.

where

$$m = \delta^{-2} c_2^n s \log 4D, \quad \Theta = \delta c_3^{-n}, \quad c(n,\delta,s) = \left(c_4^{n^2} \delta^{-c_5 n}\right)^s; \tag{1.9}$$

the factor $c(n, \delta, s)$ comes from the techniques from the geometry of numbers, while the factor $m \log \omega_1(m, \Theta) + \log \omega_2(m, \Theta)$ comes from the application of Roth's lemma. Roth proved his lemma with

$$\omega_1(m,\Theta) = \omega_2(m,\Theta) = (\Theta^{-1})^{c_6^m}, \tag{1.10}$$

and Schmidt and Schlickewei applied Roth's lemma with (1.10). By substituting (1.9) and (1.10) into (1.8) one obtains

$$t_1 \leqslant c(n,\delta,s)(4D)^{c_7^n \delta^{-2}}.$$

In [6] we derived Roth's lemma with

$$\omega_1(m,\Theta) = m^{c_8}/\Theta, \quad \omega_2(m,\Theta) = (m^{c_9}/\Theta)^m$$

and by inserting this and (1.9) into (1.8) one obtains

$$t_1 \leqslant c(n,\delta,s) c_{10} m \log(m/\Theta) \leqslant \left(c_{11}^{n^2} \delta^{-c_{12} n}\right)^s \log 4D \cdot \log\log 4D.$$

An explicit computation of c_{11}, c_{12} yields the Theorem.

Recall that in Roth's lemma there is no restriction on the auxiliary polynomial F other than (1.6), but an arithmetic restriction (1.7) on F and the point $\mathbf{x}$. Bombieri and van der Poorten [1] and Gross [9] obtained their quantitative versions of Roth's theorem by using instead of Roth's lemma the Dyson–Esnault–Viehweg lemma [3]. This lemma states also that under certain conditions a polynomial F has small index at $\mathbf{x}$ but instead of the arithmetic condition (1.7) it has an algebraic condition on $F, \mathbf{x}$. It turned out that this algebraic condition could be satisfied by the auxiliary polynomial constructed in the proof of Roth's theorem but was too strong for the polynomial constructed in the proof of the Subspace theorem.

2. Preliminaries

In this section we have collected some facts about exterior products, inequalities related to heights and absolute values and results from the geometry of numbers over number fields.

We start with exterior products. Let F be any field. Further, let n, p be integers with $n \geqslant 2$, $1 \leqslant p \leqslant n$ and put $N := \binom{n}{p}$. Denote by $\sigma_1, \ldots, \sigma_N$ the subsets of $\{1, \ldots, n\}$ of cardinality p, ordered lexicographically: thus, $\sigma_1 = \{1, \ldots, p\}$, $\sigma_2 = \{1, \ldots, p-1, p+1\}, \ldots, \sigma_{N-1} = \{n-p, n-p+2, \ldots, n\}, \sigma_N = \{n-p+1, \ldots, n\}$. For vectors $\mathbf{x}_1 = (x_{11}, \ldots, x_{1n}), \ldots, \mathbf{x}_p = (x_{p1}, \ldots, x_{pn}) \in F^n$ put

$$\Delta_j = \Delta_j(\mathbf{x}_1, \ldots, \mathbf{x}_p) := \begin{vmatrix} x_{1,i_1} & x_{1,i_2} & \cdots & x_{1,i_p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p,i_1} & x_{p,i_2} & \cdots & x_{p,i_p} \end{vmatrix},$$

where $\sigma_j = \{i_1 < \cdots < i_p\}$, i.e. $\sigma_j = \{i_1, \ldots, i_p\}$ and $i_1 < \cdots < i_p$. Now define the vector in F^N

$$\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_n := (\Delta_1, \ldots, \Delta_N).$$

Note that $\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p$ is multilinear in $\mathbf{x}_1, \ldots, \mathbf{x}_p$. Further, $\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p = \mathbf{0}$ if and only if $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ is linearly dependent. For $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in F^n$ define the *scalar product* by $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$ and put

$$\mathbf{x}^* := (x_n, -x_{n-1}, x_{n-2}, \ldots, (-1)^{n-1} x_1).$$

Then for $\mathbf{x}_1, \ldots, \mathbf{x}_n \in F^n$ we have

$$\mathbf{x}_1 \cdot (\mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_n)^* = \det(\mathbf{x}_1, \ldots, \mathbf{x}_n). \tag{2.1}$$

Further, we have Laplace's identity

$$(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p) \cdot (\mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_p) = \det(\mathbf{x}_i \cdot \mathbf{y}_j)_{1 \leqslant i,j \leqslant p} \quad \text{for } \mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{y}_1, \ldots, \mathbf{y}_p \in F^n. \tag{2.2}$$

We use similar notation for linear forms. For the linear form $l(\mathbf{X}) = \mathbf{a} \cdot \mathbf{X} = \sum_{i=1}^n a_i X_i$, where $\mathbf{a} = (a_1, \ldots, a_n)$, we put $l^*(\mathbf{X}) = \mathbf{a}^* \cdot \mathbf{X}$. Further, for p linear forms $l_i(\mathbf{X}) = \mathbf{a}_i \cdot \mathbf{X}$ $(i = 1, \ldots, p)$ in n variables, we define the linear form in $\binom{n}{p}$ variables

$$(l_1 \wedge \cdots \wedge l_p)(\mathbf{X}) = (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_p) \cdot \mathbf{X}.$$

For instance (2.2) can be reformulated as

$$(l_1 \wedge \cdots \wedge l_p) \cdot (\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p) = \det(l_i(\mathbf{x}_j))_{1 \leqslant i,j \leqslant p}. \tag{2.3}$$

Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be two bases of F^n which are related by

$$\mathbf{b}_i = \sum_{j=1}^n \xi_{ij} \mathbf{a}_j \quad (i = 1, \ldots, n) \tag{2.4}$$

for certain $\xi_{ij} \in F$. For $j = 1, \ldots, \binom{n}{p}$ define

$$\mathbf{A}_j := \mathbf{a}_{i_1} \wedge \cdots \wedge \mathbf{a}_{i_{n-p}}, \quad \mathbf{B}_j := \mathbf{b}_{i_1} \wedge \cdots \wedge \mathbf{b}_{i_{n-p}},$$

where $\{i_1 < \ldots < i_{n-p}\} = \sigma_j$ is the jth subset of $\{1, \ldots, n\}$ of cardinality $n - p$. Then $\{\mathbf{A}_1, \ldots, \mathbf{A}_{\binom{n}{p}}\}$, $\{\mathbf{B}_1, \ldots, \mathbf{B}_{\binom{n}{p}}\}$ are two bases of $F^{\binom{n}{p}}$ and they are related by

$$\mathbf{B}_i = \sum_{j=1}^N \Xi_{ij} \mathbf{A}_j \quad (i = 1, \ldots, N), \tag{2.5}$$

where $\Xi_{ij} = \det(\xi_{i_k,j_l})_{1\leqslant k,l\leqslant n-p}$ with $\sigma_i = \{i_1 < \cdots < i_{n-p}\}$ and $\sigma_j = \{j_1 < \cdots < j_{n-p}\}$. We use this to establish a relationship between p-dimensional linear subspaces of F^n and $(\binom{n}{p} - 1)$-dimensional linear subspaces of $F^{\binom{n}{p}}$.

LEMMA 1. *Let $1 \leqslant p \leqslant n-1$. There is a well-defined injective mapping*

$$f_{pn} : \{p\text{-dimensional linear subspaces of } F^n\} \to$$

$$\{(\tbinom{n}{p} - 1)\text{-dimensional linear subspaces of } F^{\binom{n}{p}}\}$$

with the following property: given any p-dimensional linear subspace V of F^n, choose any basis $\{\mathbf{a}_1,\dots,\mathbf{a}_p\}$ of V and choose any vectors $\mathbf{a}_{p+1},\dots,\mathbf{a}_n$ such that $\{\mathbf{a}_1,\dots,\mathbf{a}_n\}$ is a basis of F^n. Then $\{\mathbf{A}_1,\dots,\mathbf{A}_{\binom{n}{p}-1}\}$ is a basis of $f_{pn}(V)$.

Proof. Put $N := \binom{n}{p}$. It suffices to prove that the K-vector space with basis $\{\mathbf{A}_1,\dots,\mathbf{A}_{N-1}\}$ is uniquely determined by the K-vector space with basis $\{\mathbf{a}_1,\dots,\mathbf{a}_p\}$ and vice versa. This follows by observing that if $\{\mathbf{a}_1,\dots,\mathbf{a}_n\}$, $\{\mathbf{b}_1,\dots,\mathbf{b}_n\}$ are any two bases of F^n then by (2.4), (2.5), $\{\mathbf{a}_1,\dots,\mathbf{a}_p\}$ and $\{\mathbf{b}_1,\dots,\mathbf{b}_p\}$ generate the same space $\iff$ $\xi_{ij} = 0$ for $i = 1,\dots,p$, $j = p+1,\dots,n$ $\iff$ $\Xi_{iN} = 0$ for $i = 1,\dots,N-1$ $\iff$ $\{\mathbf{A}_1,\dots,\mathbf{A}_{N-1}\}$ and $\{\mathbf{B}_1,\dots,\mathbf{B}_{N-1}\}$ generate the same space. □

We now mention some inequalities related to absolute values. Let K be an algebraic number field and $\{|\cdot|_v : v \in M_K\}$ the absolute values defined in Section 1. For every $v \in M_K$ there is a unique continuation of $|\cdot|_v$ to the algebraic closure $\bar{K}_v$ of the completion K_v of K at v which we denote also by $|\cdot|_v$. We fix embeddings $\alpha: K \hookrightarrow \bar{\mathbb{Q}}$, $\beta_v: K \hookrightarrow K_v$, $\gamma_v: K_v \hookrightarrow \bar{K}_v$, $\delta_v: \bar{\mathbb{Q}} \hookrightarrow \bar{K}_v$ such that $\delta_v\alpha = \gamma_v\beta_v$. Although formally incorrect, we assume for convenience that these embeddings are inclusions so that $K \subset K_v \subset \bar{K}_v$ and $K \subset \bar{\mathbb{Q}} \subset \bar{K}_v$. Thus, $\bar{\mathbb{Q}}$ is the algebraic closure of K and $|\cdot|_v$ is defined on $\bar{\mathbb{Q}}$.

We recall that the absolute values $|\cdot|_v$ $(v \in M_K)$ satisfy the Product formula $\prod_v |x|_v = 1$ for $x \in K^*$. For a finite subset S of M_K, containing all infinite places, we define the ring of S-integers

$$O_S = \{x \in K : |x|_v \leqslant 1 \quad \text{for } v \notin S\},$$

where we write $v \notin S$ for $v \in M_K \backslash S$. We will often use the immediate consequence of the Product formula that

$$\prod_{v\in S} |x|_v \geqslant 1 \quad \text{for } x \in O_S\backslash\{0\}. \tag{2.6}$$

In order to be able to deal with infinite and finite places simultaneously, we define for $v \in M_K$ the quantity $s(v)$ by

$$s(v) = \frac{1}{[K:\mathbb{Q}]} \quad \text{if } v \text{ is real infinite,}$$

$$s(v) = \frac{2}{[K:\mathbb{Q}]} \quad \text{if } v \text{ is complex infinite,}$$

$$s(v) = 0 \quad \text{if } v \text{ is finite.}$$

Thus,

$$\sum_{v \in M_K} s(v) = \sum_{v|\infty} s(v) = 1. \tag{2.7}$$

For $x_1, \ldots, x_n \in \bar{K}_v$, $a_1, \ldots, a_n \in \mathbb{Z}$ we have

$$|a_1 x_1 + \cdots + a_n x_n|_v \leqslant (|a_1| + \cdots + |a_n|)^{s(v)} \max(|x_1|_v, \ldots, |x_n|_v). \tag{2.8}$$

From the definitions of $|\mathbf{x}|_v$ one may immediately derive *Schwarz' inequality* for scalar products

$$|\mathbf{x} \cdot \mathbf{y}|_v \leqslant |\mathbf{x}|_v \cdot |\mathbf{y}|_v \quad \text{for } v \in M_K,\ \mathbf{x}, \mathbf{y} \in \bar{K}_v^n \tag{2.9}$$

and *Hadamard's inequality*

$$|\det(\mathbf{x}_1, \ldots, \mathbf{x}_n)|_v \leqslant |\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v \quad \text{for } v \in M_K,\ \mathbf{x}_1, \ldots, \mathbf{x}_n \in \bar{K}_v^n. \tag{2.10}$$

More generally, we have

$$|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p|_v \leqslant |\mathbf{x}_1|_v \cdots |\mathbf{x}_p|_v \quad \text{for } v \in M_K,\ \mathbf{x}_1, \ldots, \mathbf{x}_p \in \bar{K}_v^n. \tag{2.11}$$

By taking a number field K containing the coordinates of $\mathbf{x}_1, \ldots, \mathbf{x}_p$, applying (2.11) and taking the product over all v we obtain

$$H(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p) \leqslant H(\mathbf{x}_1) \cdots H(\mathbf{x}_p) \qquad \text{for } \mathbf{x}_1, \ldots, \mathbf{x}_p \in \bar{\mathbb{Q}}^n. \tag{2.12}$$

We need also a lower bound for $|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p|_v$ in terms of $|\mathbf{x}_1|_v \cdots |\mathbf{x}_p|_v$ when $\mathbf{x}_1, \ldots, \mathbf{x}_p \in \bar{\mathbb{Q}}^n$. For a field F and a non-zero vector $\mathbf{x} = (x_1, \ldots, x_n)$ with coordinates in some extension of F, define the field

$$F(\mathbf{x}) := F(x_1/x_j, \ldots, x_n/x_j) \quad \text{for any } j \text{ with } x_j \neq 0.$$

LEMMA 2. *Let $v \in M_K$ and let $\mathbf{x}_1, \ldots, \mathbf{x}_p$ be linearly independent vectors in $\bar{\mathbb{Q}}^n$ with $[K(\mathbf{x}_i): K] \leqslant D$, $H(\mathbf{x}_i) \leqslant H$ for $i = 1, \ldots, p$. Then*

$$H^{-pD^p} \leqslant \frac{|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p|_v}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_p|_v} \leqslant 1. \tag{2.13}$$

In particular, if $p = n$, *then*

$$H^{-nD^n} \leqslant \frac{|\det(\mathbf{x}_1, \ldots, \mathbf{x}_n)|_v}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v} \leqslant 1. \tag{2.14}$$

Remark. Obviously, in (2.10)–(2.14) we can replace the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_p$ by linear forms $l_1, \ldots, l_p$ in n variables.

Proof. The upper bound of (2.13) follows at once from (2.11). It remains to prove the lower bound. We assume that each of the $\mathbf{x}_i$ has a coordinate equal to 1 which is no restriction since (2.13) does not change when the $\mathbf{x}_i$ are multiplied by scalars. Thus, the composite L of the fields $K(\mathbf{x}_1), \ldots, K(\mathbf{x}_p)$ contains the coordinates of $\mathbf{x}_1, \ldots, \mathbf{x}_p$. Clearly, $[L\colon K] \leqslant D^p$. We recall that $|\cdot|_v$ has been extended to $\bar{\mathbb{Q}}$ hence to L. There are an integer g with $1 \leqslant g \leqslant [L\colon K] \leqslant D^p$ and a place w on L such that for every $x \in L$ we have $|x|_v = |x|_w^g$. Together with $H(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p) \geqslant 1$ and (2.10) this implies that

$$\begin{aligned}
&\frac{|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p|_v}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_p|_v} \\
&= \left(\frac{|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p|_w}{|\mathbf{x}_1|_w \cdots |\mathbf{x}_p|_w}\right)^g \geqslant \left(\frac{|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p|_w}{|\mathbf{x}_1|_w \cdots |\mathbf{x}_p|_w}\right)^{D^p} \\
&= (|\mathbf{x}_1|_w \cdots |\mathbf{x}_p|_w)^{-D^p} \left(\prod_{w' \in M_L \setminus \{w\}} |\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p|_{w'}\right)^{-D^p} \\
&\qquad \times H(\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p)^{D^p} \\
&\geqslant (|\mathbf{x}_1|_w \cdots |\mathbf{x}_p|_w)^{-D^p} \left(\prod_{w' \in M_L \setminus \{w\}} |\mathbf{x}_1|_{w'} \cdots |\mathbf{x}_p|_{w'}\right)^{-D^p} \\
&= (H(\mathbf{x}_1) \cdots H(\mathbf{x}_p))^{-D^p} \geqslant H^{-pD^p}.
\end{aligned}$$

□

Using the inequalities for exterior products mentioned above, we derive estimates for the height of a solution of a system of linear equations.

LEMMA 3. *Let* $\mathbf{a}_1, \ldots, \mathbf{a}_r \in \bar{\mathbb{Q}}^n$ *with* $H(\mathbf{a}_i) \leqslant H$ *for* $i = 1, \ldots, r$ *and let* $\mathbf{x} \in \bar{\mathbb{Q}}^n \setminus \{\mathbf{0}\}$ *be such that*

$$\mathbf{a}_i \cdot \mathbf{x} = 0 \quad \textit{for } i = 1, \ldots, r.$$

(i) *If rank*$\{\mathbf{a}_1, \ldots, \mathbf{a}_r\} = n - 1$, *then* $\mathbf{x}$ *is uniquely determined up to a scalar and*

$$H(\mathbf{x}) \leqslant H^{n-1}.$$

(ii) *Suppose that* rank$\{\mathbf{a}_1,\dots,\mathbf{a}_r\} \leqslant n-1$ *and that* $\mathbf{x} \in K^n$, *where* K *is a number field. Then there is an* $\mathbf{y} \in K^n$ *with* $\mathbf{y} \neq \mathbf{0}$, $\mathbf{a}_i \cdot \mathbf{y} = 0$ *for* $i = 1,\dots,r$ *and*

$$H(\mathbf{y}) \leqslant H^{n-1}.$$

Proof. (i) It is well-known from linear algebra that $\mathbf{x}$ is determined up to a scalar. Suppose that rank$\{\mathbf{a}_1,\dots,\mathbf{a}_{n-1}\} = n-1$ which is no restriction. Then $\mathbf{x}$ is also the up to a scalar unique solution of $\mathbf{a}_i \cdot \mathbf{x} = 0$ for $i = 1,\dots,n-1$. By (2.1), this system is satisfied by the non-zero vector $(\mathbf{a}_1 \wedge \dots \wedge \mathbf{a}_{n-1})^*$ hence $\mathbf{x}$ is a scalar multiple of this vector. Together with (2.12) this implies that

$$H(\mathbf{x}) = H(\mathbf{a}_1 \wedge \dots \wedge \mathbf{a}_{n-1}) \leqslant H(\mathbf{a}_1)\cdots H(\mathbf{a}_{n-1}) \leqslant H^{n-1}.$$

(ii) Let $G = \mathrm{Gal}(\bar{\mathbb{Q}}/K)$ be the group of automorphisms of $\bar{\mathbb{Q}}$ leaving K invariant. For $\mathbf{y} = (y_1,\dots,y_n) \in \bar{\mathbb{Q}}^n$, $\sigma \in G$, we put $\sigma(\mathbf{y}) = (\sigma(y_1),\dots,\sigma(y_n))$. Let $\mathbf{a}_1,\dots,\mathbf{a}_s$ be the vectors $\sigma(\mathbf{a}_i)$ with $i = 1,\dots,r$, $\sigma \in G$. Since $\mathbf{x} \in K^n$ we have $\mathbf{a}_i \cdot \mathbf{x} = 0$ for $i = 1,\dots,s$. Since $\mathbf{x} \neq \mathbf{0}$ we have rank$\{\mathbf{a}_1,\dots,\mathbf{a}_s\} \leqslant n-1$. If this rank is $< n-1$ we choose vectors $\mathbf{a}_{s+1},\dots,\mathbf{a}_t$ from $(1,0,\dots,0),\dots,(0,\dots,1)$ such that rank$\{\mathbf{a}_1,\dots,\mathbf{a}_t\} = n-1$. Note that $H(\mathbf{a}_i) \leqslant H$ and that $\sigma(\mathbf{a}_i) \in \{\mathbf{a}_1,\dots,\mathbf{a}_t\}$ for $i = 1,\dots,t$, $\sigma \in G$. Hence if $\mathbf{y}$ is a solution of the system $\mathbf{a}_i \cdot \mathbf{x} = 0$ for $i = 1,\dots,t$ then so is $\sigma(\mathbf{y})$ for $\sigma \in G$. By (i), this system has an up to a scalar unique non-zero solution $\mathbf{y}$. Choose $\mathbf{y}$ with one of the coordinates equal to one. Then $\sigma(\mathbf{y}) = \mathbf{y}$ for $\sigma \in G$ whence $\mathbf{y} \in K^n$. Further, by (i) we have $H(\mathbf{y}) \leqslant H^{n-1}$. □

Remark. In Lemma 3 we may replace $\mathbf{a}_i \cdot \mathbf{x} = 0$ by $l_i(\mathbf{x}) = 0$ for $i = 1,\dots,r$ where the l_i are linear forms in n variables with algebraic coefficients.

The discriminant of a number field K (over $\mathbb{Q}$) is denoted by Δ_K. The relative discriminant ideal of the extension of number fields L/K is denoted by $\mathfrak{d}_{L/K}$. Recall that $\mathfrak{d}_{L/K} \subseteq O_K$. We need the following estimates.

LEMMA 4. (i) *Let* K, L, M *be number fields with* $K \subseteq L \subseteq M$. *Then* $\mathfrak{d}_{M/K} = N_{L/K}(\mathfrak{d}_{M/L}) \cdot \mathfrak{d}_{L/K}^{[M:K]}$.
(ii) *Let* $K_1,\dots,K_r$ *be number fields and* $K = K_1\cdots K_r$ *their composite. Suppose that* $[K_i:\mathbb{Q}] = d_i > 1$ *for* $i = 1,\dots,r$ *and* $[K:\mathbb{Q}] = d$. *Then*

$$|\Delta_K|^{1/(d(d-1))} \leqslant \max_{1\leqslant i\leqslant r} |\Delta_{K_i}|^{1/(d_i(d_i-1))}.$$

Proof. (i) cf. [10], pp. 60, 66.

(ii) It suffices to prove this for $r = 2$. So let $K = K_1K_2$. If $K = K_1$ or $K = K_2$ then we are done. So suppose that $K \neq K_1$, $K \neq K_2$. Then by e.g. Lemma 7 of [21] we have

$$\Delta_K \mid \Delta_{K_1}^{d/d_1}\Delta_{K_2}^{d/d_2}.$$

Since $d \geqslant 2d_i$ we have $d-1 \geqslant 2(d_i-1)$ for $i=1,2$. Hence

$$\begin{aligned}|\Delta_K|^{1/(d(d-1))} &\leqslant |\Delta_{K_1}|^{1/(d_1(d-1))}|\Delta_{K_2}|^{1/(d_2(d-1))}\\ &\leqslant \left(|\Delta_{K_1}|^{1/(d_1(d_1-1))}|\Delta_{K_2}|^{1/(d_2(d_2-1))}\right)^{1/2}\\ &\leqslant \max_{i=1,2}|\Delta_{K_i}|^{1/(d_i(d_i-1))}. \end{aligned}$$ □

The next lemma is similar to an estimate of Silverman [20].

LEMMA 5. *Let* $\mathbf{x} \in \bar{\mathbb{Q}}^n\backslash\{\mathbf{0}\}$ *with* $\mathbb{Q}(\mathbf{x}) = K$, $[K:\mathbb{Q}] = d$. *Then*

$$H(\mathbf{x}) \geqslant |\Delta_K|^{1/(2d(d-1))}.$$

Proof. We assume that one of the coordinates of $\mathbf{x}$, the first, say, is equal to 1, i.e. $\mathbf{x} = (1, \xi_2, \dots, \xi_n)$. This is no restriction since $H(\lambda\mathbf{x}) = H(\mathbf{x}), \mathbb{Q}(\lambda\mathbf{x}) = \mathbb{Q}(\mathbf{x})$ for non-zero λ. Suppose we have shown that for $\xi \in \bar{\mathbb{Q}}^*$,

$$H(\xi) \geqslant |\Delta_F|^{1/(2f(f-1))} \quad \text{where } F = \mathbb{Q}(\xi), [F:\mathbb{Q}] = f \tag{2.15}$$

and $H(\xi) = H(1,\xi)$. Together with Lemma 4 this implies Lemma 5, since

$$H(\mathbf{x}) \geqslant \max_{2\leqslant i\leqslant n} H(\xi_i) \geqslant \max_{2\leqslant i\leqslant n} |\Delta_{K_i}|^{1/(2d_i(d_i-1))} \geqslant |\Delta_K|^{1/2d(d-1))},$$

where $K_i = \mathbb{Q}(\xi_i)$, $d_i = [K_i:\mathbb{Q}]$ for $i = 2,\dots,n$. Hence it remains to prove (2.15).

From the definitions of the $|\mathbf{x}|_v$ for $v \in M_K$ and $\mathbf{x} = (1,\xi)$ it follows that

$$H(\xi) = \left((N\mathfrak{a})^{-1}\prod_{i=1}^{f}(1+|\xi^{(i)}|^{1/2})\right)^{1/f}, \tag{2.16}$$

where $\mathfrak{a}$ is the fractional ideal in F generated by 1 and ξ, $N\mathfrak{a}$ is the norm of $\mathfrak{a}$ and $\xi^{(1)},\dots,\xi^{(f)}$ are the conjugates of ξ in $\mathbb{C}$. Let $\{\omega_1,\dots,\omega_f\}$ be a $\mathbb{Z}$-basis of the ideal $\mathfrak{a}^{f-1}$. The discriminant of this basis is

$$D_{K/\mathbb{Q}}(\omega_1,\dots,\omega_f) = D_{K/\mathbb{Q}}(\mathfrak{a}^{f-1}) = (N\mathfrak{a})^{2f-2}\Delta_K$$

(cf. [10], p. 66, Prop. 13). On the other hand we have $1,\xi,\dots,\xi^{f-1} \in \mathfrak{a}^{f-1}$, hence $D_{K/\mathbb{Q}}(1,\xi,\dots,\xi^{f-1}) = aD_{K/\mathbb{Q}}(\omega_1,\dots,\omega_f)$ for some positive $a \in \mathbb{Z}$. It follows that

$$|\Delta_K| \leqslant (N\mathfrak{a})^{-2(f-1)}|D_{K/\mathbb{Q}}(1,\xi,\dots,\xi^{f-1})| = (N\mathfrak{a})^{-2(f-1)}\Delta, \tag{2.17}$$

where

$$\Delta = \left(\det(\xi^{(i)})^j\right)_{\substack{1\leqslant i\leqslant f\\ 0\leqslant j\leqslant f-1}}^2$$

(cf. [10], p. 64). By Hadamard's inequality we have

$$|\Delta| \leqslant \prod_{i=1}^{f} \Big(\sum_{j=0}^{f-1} |\xi^{(i)}|^{2j} \Big) \leqslant \prod_{i=1}^{f} (1 + |\xi^{(i)}|^2)^{f-1}.$$

By inserting this into (2.17) and using (2.16) this gives

$$|\Delta_K| \leqslant \Big((N\mathfrak{a})^{-1} \prod_{i=1}^{f} (1 + |\xi^{(i)}|^2)^{1/2} \Big)^{2(f-1)} = H(\xi)^{2f(f-1)}$$

which is (2.15). □

McFeat [11] and Bombieri and Vaaler [2] generalised some of Minkowski's results on the geometry of numbers to adele rings of number fields. Below we recall some of their results.

Let K be a number field and $v \in M_K$. A subset C_v of K_v^n (n-fold topological product of K_v with the v-adic topology) is called a *symmetric convex body* in K_v^n if

(i) $\mathbf{0}$ is an interior point of C_v and C_v is compact;
(ii) if $\mathbf{x} \in C_v$, $\alpha \in K_v$ and $|\alpha|_v \leqslant 1$ then $\alpha\mathbf{x} \in C_v$;
(iii) if $v|\infty$ and if $\mathbf{x}, \mathbf{y} \in C_v$ then $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in C_v$ for all $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leqslant 1$; if $v \nmid \infty$ and if $\mathbf{x}, \mathbf{y} \in C_v$ then $\mathbf{x} + \mathbf{y} \in C_v$.

Note that for finite v, C_v is an O_v-module of rank n, where O_v is the local ring $\{x \in K_v : |x|_v \leqslant 1\}$.

The *ring of K-adeles* V_K is the set of infinite tuples $(x_v : v \in M_K)$ ((x_v) for short) with $x_v \in K_v$ for $v \in M_K$ and $|x_v|_v \leqslant 1$ for all but finitely many v, endowed with componentwise addition and multiplication. The nth cartesian power V_K^n may be identified with the set of infinite tuples of vectors $(\mathbf{x}_v) = (\mathbf{x}_v : v \in M_K)$ with $\mathbf{x}_v \in K_v^n$ for all $v \in M_K$ and $\mathbf{x}_v \in O_v^n$ for all but finitely many v. There is a diagonal embedding

$$\phi : K^n \hookrightarrow V_K^n : \mathbf{x} \mapsto (\mathbf{x}_v) \quad \text{with } \mathbf{x}_v = \mathbf{x} \text{ for } v \in M_K.$$

A symmetric convex body in V_K^n is a cartesian product

$$C = \prod_{v \in M_K} C_v = \{(\mathbf{x}_v) \in V_K^n : \mathbf{x}_v \in C_v \text{ for } v \in M_K\},$$

where for every $v \in M_K$, C_v is a symmetric convex body in K_v^n and where for all but finitely many v, $C_v = O_v^n$ is the unit ball. For positive $\lambda \in \mathbb{R}$, define the inflated convex body

$$\lambda C := \prod_{v|\infty} \lambda C_v \times \prod_{v \nmid \infty} C_v,$$

where $\lambda C_v = \{\lambda \mathbf{x}_v : \mathbf{x}_v \in C_v\}$ for $v|\infty$. Now the ith *successive minimum* $\lambda_i = \lambda_i(C)$ is defined by

$$\lambda_i := \min\{\lambda \in \mathbb{R}_{>0} : \phi^{-1}(\lambda C) \text{ contains } i\ K\text{-linearly independent points}\}.$$

Note that $\phi^{-1}(\lambda C) \subset K^n$. This minimum does exist since $\phi(K^n)$ is a discrete subset of V_K^n, i.e. $\phi(K^n)$ has finite intersection with any set $\prod_v D_v$ such that each D_v is a compact subset of K_v^n and $D_v = O_v^n$ for all but finitely many v. There are n successive minima $\lambda_1, \ldots, \lambda_n$ and we have $0 < \lambda_1 \leqslant \cdots \leqslant \lambda_n < \infty$.

Minkowski's theorem gives a relation between the product $\lambda_1 \cdots \lambda_n$ and the volume of C. Similarly as in [2,10] we define a measure on V_K^n built up from local measures β_v on K_v for $v \in M_K$. If v is real infinite then $K_v = \mathbb{R}$ and we take for β_v the usual Lebesgue measure on $\mathbb{R}$. If v is complex infinite then $K_v = \mathbb{C}$ and we take for β_v two times the Lebesgue measure on the complex plane. If v is finite then we take for β_v the Haar measure on K_v (the up to a constant unique measure such that $\beta_v(a + C) = \beta_v(C)$ for $C \subset K_v$, $a \in K_v$), normalised such that

$$\beta_v(O_v) = |\mathfrak{D}_v|_v^{[K:\mathbb{Q}]/2};$$

here $\mathfrak{D}_v$ is the local different of K at v and $|\mathfrak{a}|_v := \max\{|x|_v : x \in \mathfrak{a}\}$ for an O_v-ideal $\mathfrak{a}$. The corresponding product measure on K_v^n is denoted by β_v^n. For instance, if ρ is a linear transformation of K_v^n onto itself, then $\beta_v^n(\rho D) = |\det \rho|_v^{[K:\mathbb{Q}]} \beta_v^n(D)$ for any β_v^n-measurable $D \subset K_v^n$. Now let $\beta = \prod_v \beta_v$ be the product measure on V_K and β^n the n-fold product measure of this on V_K^n. Thus, if for every $v \in M_K$, D_v is a β_v^n -measurable subset of K_v^n and $D_v = O_v^n$ for all but finitely many v, then $D := \prod_v D_v$ has measure

$$\beta^n(D) = \prod_v \beta_v^n(D_v). \tag{2.18}$$

In particular, symmetric convex bodies in V_K^n are β^n-measurable and have positive measure.

McFeat ([11], Thms. 5, p. 19 and 6, p. 23) and Bombieri and Vaaler ([2], Thms. 3,6) proved the following generalisation of Minkowski's theorem:

LEMMA 6. *Let K be an algebraic number field of degree d and r_2 the number of complex infinite places of K. Further, let $n \geqslant 1$, C be a symmetric convex body in V_K^n, and $\lambda_1, \ldots, \lambda_n$ its successive minima. Then*

$$\left(\frac{\pi^n n!}{2}\right)^{r_2/d} \cdot \frac{2^n}{n!} |\Delta_K|^{-n/2d} \leqslant \lambda_1 \cdots \lambda_n \cdot \beta^n(C)^{1/d} \leqslant 2^n.$$

Finally, we need an effective version of the Chinese remainder theorem over K. An *A-ceiling* is an infinite tuple $(A_v) = (A_v : v \in M_K)$ of positive real numbers such that A_v belongs to the value group of $|\cdot|_v$ on K_v^* for all $v \in M_K$, $A_v = 1$ for all but finitely many v, and $\prod_v A_v = A$.

LEMMA 7. *Let K be a number field of degree d, $A > 1$, (A_v) an A-ceiling, and (a_v) a K-adele.*

(i) *If $A \geqslant |\Delta_K|^{1/2d}$, then there is an $x \in K$ with*

$$|x|_v \leqslant A_v \quad \text{for } v \in M_K \text{ and } x \neq 0.$$

(ii) *If $A \geqslant (d/2)|\Delta_K|^{1/2}$, then there is an $x \in K$ with*

$$|x - a_v|_v \leqslant A_v \quad \text{for } v \in M_K.$$

Proof. Let r_1 be the number of real and r_2 the number of complex infinite places of K.

(i) The one-dimensional convex body $C = \{(x_v) \in V_K : |x|_v \leqslant A_v \text{ for } v \in M_K\}$ has measure

$$\begin{aligned}\beta(C) &= \Big(\prod_v A_v\Big)^d 2^{r_1}(2\pi)^{r_2} \prod_{v\nmid\infty} |\mathfrak{D}_v|_v^{d/2} \\ &= 2^d(\pi/2)^{r_2}A^d|\Delta_K|^{-1/2} \geqslant 2^d A^d |\Delta_K|^{-1/2},\end{aligned}$$

in view of the identity $\prod_{v\nmid\infty} |\mathfrak{D}_v|_v = |\Delta_K|^{-1/d}$. So if $A \geqslant |\Delta_K|^{1/2d}$ then $\beta(C) \leqslant 1$. Then by Lemma 6 the only successive minimum λ_1 of C is $\leqslant 1$ hence C contains $\phi(x)$ for some non-zero $x \in K$.

(ii) By [11], p. 29, Thm. 8, there is such an x if $A \geqslant (d/2)(2/\pi)^{r_2}|\Delta_K|^{1/2}$. This implies (ii). See [12], Thm. 3 for a similar estimate. □

3. A gap principle

Let K be an algebraic number field of degree d and S a finite set of places on K of cardinality s containing all infinite places. Further, let n be an integer $\geqslant 2$ and let δ, C be reals with $0 < \delta < 1$ and $C \geqslant 1$. For $v \in S$, let $l_{1v}, \ldots, l_{nv}$ be linearly independent linear forms in n variables with coefficients in $\bar{K}_v$. In this section, we consider the inequality

$$\prod_{v\in S}\prod_{i=1}^{n} \frac{|l_{iv}(\mathbf{x})|_v}{|\mathbf{x}|_v} \leqslant C \cdot \prod_{v\in S} |\det(l_{1v}, \ldots, l_{nv})|_v \cdot H(\mathbf{x})^{-n-\delta}$$
$$\text{in } \mathbf{x} \in K^n,\ \mathbf{x} \neq \mathbf{0}. \tag{3.1}$$

The *linear scattering* of a subset $\mathcal{S}$ of K^n is the smallest integer h for which there exist proper linear subspaces $T_1, \ldots, T_h$ of K^n such that $\mathcal{S}$ is contained in $T_1 \cup \cdots \cup T_h$; we say that $\mathcal{S}$ has infinite linear scattering if such an integer h does not exist. For instance, $\mathcal{S}$ contains n linearly independent vectors $\Longleftrightarrow$ $\mathcal{S}$ has linear scattering $\geqslant 2$. Clearly, the linear scattering of $\mathcal{S}_1 \cup \mathcal{S}_2$ is at most the sum of the linear scatterings of $\mathcal{S}_1$ and $\mathcal{S}_2$. In this section we shall prove:

LEMMA 8. (Gap principle). *Let A, B be reals with $1 \leqslant A < B$. Then the set of solutions of* (3.1) *with*

$$A \leqslant H(\mathbf{x}) < B$$

has linear scattering at most

$$C^{2d} \cdot \left(\frac{150n^4}{\delta}\right)^{ns+1} \left(1 + \log\left(\frac{\log 2B}{\log 2A}\right)\right).$$

Remark. This gap principle is similar to ones obtained by Schmidt and Schlickewei, except that we do not require a large lower bound for A. Thus, our gap principle can be used also to deal with 'very small' solutions of (3.1).

In the proof of Lemma 8 we need some auxiliary results which will be proved first. We put $e = 2.7182\ldots$ and denote by $|\mathcal{A}|$ the cardinality of a set $\mathcal{A}$.

LEMMA 9. *Let θ be a real with $0 < \theta \leqslant 1/2$ and q an integer $\geqslant 1$.*
(i) *There exists a set Γ_1 with the following properties:*
$|\Gamma_1| \leqslant (e/\theta)^{q-1}$;
Γ_1 *consists of tuples $\underline{\gamma} = (\gamma_1, \ldots, \gamma_q)$ with $\gamma_i \geqslant 0$ for $i = 1, \ldots, q$ and $\gamma_1 + \cdots + \gamma_q = 1 - \theta$;*
for all reals $F_1, \ldots, F_q, L$ with

$$0 < F_i \leqslant 1 \quad \text{for } i = 1, \ldots, q, \qquad F_1 \cdots F_q \leqslant L \tag{3.2.}$$

there is a tuple $\underline{\gamma} \in \Gamma_1$ with $F_i \leqslant L^{\gamma_i}$ for $i = 1, \ldots, q$.
(ii) *There exists a set Γ_2 with the following properties:*
$|\Gamma_2| \leqslant (e(2 + \theta^{-1}))^q$;
Γ_2 *consists of q-tuples of non-negative real numbers $\underline{\gamma} = (\gamma_1, \ldots, \gamma_q)$;*
for all reals $G_1, \ldots, G_q, M$ with

$$0 < G_i \leqslant 1 \quad \text{for } i = 1, \ldots, q, \qquad 0 < M < 1, \quad G_1 \cdots G_q \geqslant M \tag{3.3}$$

there is a tuple $\underline{\gamma} \in \Gamma_2$ with $M^{\gamma_i + \theta/q} < G_i \leqslant M^{\gamma_i}$ for $i = 1, \ldots, q$.

Proof. (i) is a special case of Lemma 4 of [4]. We prove only (ii). Put $h = [\theta^{-1}] + 1$, $g = qh$. There are reals $c_1, \ldots, c_q$ with

$$G_i = M^{c_i}, \qquad c_i \geqslant 0 \quad \text{for } i = 1, \ldots, q,\ c_1 + \cdots + c_q \leqslant 1.$$

Define the integers $f_1, \ldots, f_q$ by

$$f_i \leqslant gc_i < f_i + 1 \quad \text{for } i = 1, \ldots, q, \tag{3.4}$$

and put $\gamma_i = f_i/g$ for $i = 1, \ldots, q$. Then

$$0 \leqslant \gamma_i \leqslant c_i < \gamma_i + \frac{1}{hq} < \gamma_i + \frac{\theta}{q}$$

and therefore,

$$M^{\gamma_i+\theta/q} < M^{c_i} = G_i \leqslant M^{\gamma_i} \quad \text{for } i = 1,\ldots,q.$$

By (3.4) and $c_1 + \cdots + c_q \leqslant 1$ we have $f_1 + \cdots + f_q \leqslant g(c_1 + \cdots + c_q) \leqslant g$. This implies that $\underline{\gamma} = (\gamma_1, \ldots, \gamma_q)$ belongs to the set

$$\Gamma_2 := \{(f_1/g, \ldots, f_q/g) \colon f_1, \ldots, f_q \in \mathbb{Z}, \\ f_i \geqslant 0 \text{ for } i = 1, \ldots, q,\ f_1 + \cdots + f_q \leqslant g\}.$$

For integers $x > 0, y \geqslant 0$ we have

$$\binom{x+y}{y} \leqslant \frac{(x+y)^{x+y}}{x^x y^y} = \left(1 + \frac{y}{x}\right)^x \left(1 + \frac{x}{y}\right)^y \leqslant \left(e(1 + \frac{x}{y})\right)^y \tag{3.5}$$

where the expression at the right is 1 if $y = 0$. Hence

$$|\Gamma_2| = \binom{g+q}{q} = \binom{(h+1)q}{q} \leqslant (e(h+1))^q \leqslant (e(2+\theta^{-1}))^q. \qquad \square$$

LEMMA 10. *Let K, S, n have the same meaning as in Lemma 8 and put $d := [K:\mathbb{Q}]$, $s := |S|$. Further, let F be a real $\geqslant 1$ and let $\mathcal{V}$ be a subset of K^n of linear scattering*

$$\geqslant \max(2F^{2d}, 4 \times 7^{d+2s}).$$

Then there are $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathcal{V}$ with

$$0 < \prod_{v \notin S} \frac{|\det(\mathbf{x}_1, \ldots, \mathbf{x}_n)|_v}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v} \leqslant F^{-1}. \tag{3.6}$$

Proof. We assume that $\mathbf{0} \notin \mathcal{V}$ and $F > 1$ which are no restrictions by Hadamard's inequality. Denote by $[\mathbf{y}_1, \ldots, \mathbf{y}_m]$ the linear subspace of K^n generated by $\mathbf{y}_1, \ldots, \mathbf{y}_m$. Choose a prime ideal $\mathfrak{p}$ of K not corresponding to a place in S with minimal norm $N\mathfrak{p}$. Define the integer m by

$$(N\mathfrak{p})^{m-1} \leqslant F^d < (N\mathfrak{p})^m.$$

Then $m \geqslant 1$. We distinguish between the cases $m \geqslant 2$ and $m = 1$.

The case $m \geqslant 2$. Let v be the place corresponding to $\mathfrak{p}$ and let $R = \{x \in K \colon |x|_v \leqslant 1\}$ be the local ring at $\mathfrak{p}$. The maximal ideal $\{x \in K \colon |x|_v < 1\}$ of R is principal; let π be a generator of this maximal ideal. For $i = 0, \ldots, m$, let T_i be a full set of representatives for the residue classes of R modulo π^{m-i}. Note that

$$|T_i| = |R/(\pi^{m-i})| = |O_K/\mathfrak{p}^{m-i}| = (N\mathfrak{p})^{m-i}. \tag{3.7}$$

For $i = 0, \dots, m,\ a \in T_i$ define the $n \times n$-matrix

$$A_{i,a} = \begin{pmatrix} \pi^i & a & & & 0 \\ 0 & \pi^{m-i} & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}.$$

We claim that for every row vector $\mathbf{x} \in R^n$ there are $i \in \{0, \dots, m\},\ a \in T_i$ and $\mathbf{y} \in R^n$ with

$$\mathbf{x} = \mathbf{y}A_{i,a}.$$

Namely, let $\mathbf{x} = (x_1, \dots, x_n)$. If $x_1 \not\equiv 0 \pmod{\pi^m}$ then for some $i \in \{0, \dots, m-1\}$ we have $x_1 = \pi^i y_1$ with $y_1 \in R$, $|y_1|_v = 1$ and there is an $a \in T_i$ with $x_2 \equiv a y_1 \pmod{\pi^{m-i}}$. If $x_1 \equiv 0 \pmod{\pi^m}$ then we have $x_1 = \pi^i y_1$, $x_2 \equiv a y_1 \pmod{\pi^{m-i}}$ where $i = m$, $y_1 \in R$ and a is the only element of T_i. Define $y_2 \in R$ by $x_2 = a y_1 + \pi^{m-i} y_2$ and put $y_i = x_i$ for $i \geqslant 3$. Then clearly $\mathbf{x} = \mathbf{y}A_{i,a}$ where $\mathbf{y} = (y_1, \dots, y_n)$.

Let $B_1, \dots, B_r$ be the matrices $A_{i,a}$ $(i = 0, \dots, m,\ a \in T_i)$ in some order. We partition $\mathcal{V}$ into classes $\mathcal{V}_1, \dots, \mathcal{V}_r$ such that $\mathbf{x} \in \mathcal{V}$ belongs to class $\mathcal{V}_i$ if there are $\lambda \in K^*$ with $|\lambda|_v = |\mathbf{x}|_v$ and $\mathbf{y} \in R^n$ such that $\mathbf{x} = \lambda B_i \mathbf{y}$. By $m \geqslant 2$ and (3.7) we have

$$r = \sum_{j=0}^{m} |T_j| = \sum_{j=0}^{m} (N\mathfrak{p})^{m-j} < 2(N\mathfrak{p})^m < 2F^{2d}$$

and the latter number is at most the linear scattering of $\mathcal{V}$. Therefore, at least one of the classes $\mathcal{V}_i$ has linear scattering $\geqslant 2$, i.e. $\mathcal{V}_i$ contains n linearly independent vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$. For $j = 1, \dots, n$ there are $\lambda_j \in K^*$ with $|\lambda_j|_v = |\mathbf{x}_j|_v$ and $\mathbf{y}_j \in R^n$ such that $\mathbf{x}_j = \lambda_j B_i \mathbf{y}_j$. Therefore,

$$\begin{aligned} \frac{|\det(\mathbf{x}_1, \dots, \mathbf{x}_n)|_v}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v} &= |\det(\lambda_1 \mathbf{x}_1, \dots, \lambda_n \mathbf{x}_n)|_v \\ &= |\det B_i|_v \cdot |\det(\mathbf{y}_1, \dots, \mathbf{y}_n)|_v \\ &\leqslant |\det B_i|_v = |\pi^m|_v = (N\mathfrak{p})^{-m/d} < F^{-1}. \end{aligned}$$

By Hadamard's inequality we have for $w \in M_K \backslash (S \cup \{v\})$ that $|\det(\mathbf{x}_1, \dots, \mathbf{x}_n)|_w / (|\mathbf{x}_1|_w \cdots |\mathbf{x}_n|_w) \leqslant 1$. By taking the product over v and $w \in M_K \backslash (S \cup \{v\})$ we obtain (3.6).

The case $m = 1$**.** Suppose that there are no $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathcal{V}$ with (3.6). Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be any linearly independent vectors from $\mathcal{V}$. There is an ideal $\mathfrak{a} \subseteq O_K$, composed of prime ideals not corresponding to places in S, such that

$$\prod_{v \notin S} \frac{|\det(\mathbf{x}_1, \dots, \mathbf{x}_n)|_v}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v} = (N\mathfrak{a})^{-1/d}. \tag{3.8}$$

If $\mathfrak{a} \subsetneqq O_K$ then since $m = 1$ we have $N\mathfrak{a} \geqslant N\mathfrak{p} > F^d$ which together with (3.8) contradicts our assumption on $\mathcal{V}$. Therefore, $\mathfrak{a} = O_K$ and so the left-hand side of (3.8) is equal to 1. Together with Hadamard's inequality this implies that

$$|\det(\mathbf{x}_1, \ldots, \mathbf{x}_n)|_v = |\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v \quad \text{for } v \notin S. \tag{3.9}$$

Since $\mathcal{V}$ has linear scattering $\geqslant 3$ there are linearly independent $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in $\mathcal{V}$ and there is an $\mathbf{x}_{n+1} \in \mathcal{V}$ with

$$\mathbf{x}_{n+1} \notin [\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}], \quad \mathbf{x}_{n+1} \notin [\mathbf{x}_1, \ldots, \mathbf{x}_{n-2}, \mathbf{x}_n]. \tag{3.10}$$

We fix $\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}$. Let $\mathbf{y}$ be any vector in $\mathcal{V}$ with

$$\begin{aligned} &\mathbf{y} \notin [\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}], \quad \mathbf{y} \notin [\mathbf{x}_1, \ldots, \mathbf{x}_{n-2}, \mathbf{x}_n], \\ &\mathbf{y} \notin [\mathbf{x}_1, \ldots, \mathbf{x}_{n-2}, \mathbf{x}_{n+1}]. \end{aligned} \tag{3.11}$$

We have $\mathbf{x}_{n+1} = \sum_{i=1}^n a_i \mathbf{x}_i$, $\mathbf{y} = \sum_{i=1}^n y_i \mathbf{x}_i$ with $a_i, y_i \in K$. We repeatedly apply (3.9). We have $\det(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}, \mathbf{x}_{n+1}) = a_n \det(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ where $a_n \neq 0$ by (3.10). Together with (3.9) this implies

$$|a_n|_v = \frac{|\det(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}, \mathbf{x}_{n+1})|_v}{|\det(\mathbf{x}_1, \ldots, \mathbf{x}_n)|_v} = \frac{|\mathbf{x}_{n+1}|_v}{|\mathbf{x}_n|_v} \quad \text{for } v \notin S.$$

Similarly,

$$|a_{n-1}|_v = \frac{|\det(\mathbf{x}_1, \ldots, \mathbf{x}_{n-2}, \mathbf{x}_n, \mathbf{x}_{n+1})|_v}{|\det(\mathbf{x}_1, \ldots, \mathbf{x}_n)|_v} = \frac{|\mathbf{x}_{n+1}|_v}{|\mathbf{x}_{n-1}|_v} \quad \text{for } v \notin S.$$

By (3.11) we have similar properties for y_n, y_{n-1}. Summarising, we have

$$|a_i|_v = \frac{|\mathbf{x}_{n+1}|_v}{|\mathbf{x}_i|_v}, \qquad |y_i|_v = \frac{|\mathbf{y}|_v}{|\mathbf{x}_i|_v} \quad \text{for } i = n-1, n,\ v \notin S. \tag{3.12}$$

It is easy to see that by (3.11),

$$a_{n-1} y_n - a_n y_{n-1} = \frac{\det(\mathbf{x}_1, \ldots, \mathbf{x}_{n-2}, \mathbf{x}_{n+1}, \mathbf{y})}{\det(\mathbf{x}_1, \ldots, \mathbf{x}_n)} \neq 0.$$

Together with (3.9), (3.12) this implies that

$$\begin{aligned} |a_{n-1} y_n - a_n y_{n-1}|_v &= \frac{|\mathbf{x}_{n+1}|_v \cdot |\mathbf{y}|_v}{|\mathbf{x}_{n-1}|_v \cdot |\mathbf{x}_n|_v} \\ &= |a_{n-1} y_n|_v = |a_n y_{n-1}|_v \quad \text{for } v \notin S. \end{aligned}$$

This implies that

$$\frac{a_n y_{n-1}}{a_{n-1} y_n} \in O_S^*,\ 1 - \frac{a_n y_{n-1}}{a_{n-1} y_n} = \frac{a_{n-1} y_n - a_n y_{n-1}}{a_{n-1} y_n} \in O_S^*,$$

where O_S^* is the group of S-units $\{x \in K : |x|_v = 1 \text{ for } v \notin S\}$. By Theorem 1 of [4], there are at most $3 \times 7^{d+2s}$ elements $\xi \in O_S^*$ with $1 - \xi \in O_S^*$. As we have just seen, for every $\mathbf{y} \in \mathcal{V}$ with (3.11) there is such a ξ with $a_n y_{n-1}/a_{n-1}y_n = \xi$ or, which is the same,

$$\mathbf{y} \in \left[\mathbf{x}_1, \ldots, \mathbf{x}_{n-2}, \frac{a_{n-1}}{a_n} \cdot \xi \mathbf{x}_{n-1} + \mathbf{x}_n\right].$$

Taking into consideration that in (3.11) we excluded three linear subspaces for $\mathbf{y}$, it follows that $\mathcal{V}$ has linear scattering at most $3 + 3 \times 7^{d+2s} < 4 \times 7^{d+2s}$, contrary to our assumption on $\mathcal{V}$. Thus, our supposition that there are no $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in $\mathcal{V}$ with (3.6) leads to a contradiction. This completes the proof of Lemma 10. □

Proof of Lemma 8. We assume that $|l_{iv}|_v = 1$ for $i = 1, \ldots, n$, $v \in S$ which is clearly no restriction. Let D be any real with $2A \leqslant D < 2B$. Put

$$\zeta := \frac{\delta}{2n - 2}.$$

First we estimate the linear scattering of the set of solutions $\mathbf{x} \in K^n$ of

$$\prod_{v \in S} \prod_{i=1}^{n} \frac{|l_{iv}(\mathbf{x})|_v}{|\mathbf{x}|_v} \leq C \cdot \prod_{v \in S} |\det(l_{1v}, \ldots, l_{nv})|_v \cdot H(\mathbf{x})^{-n-\delta}, \tag{3.1}$$

with

$$\frac{D}{2} \leqslant H(\mathbf{x}) < \frac{D^{1+\zeta}}{2}. \tag{3.13}$$

For $i = 1, \ldots, n$, let $\mathcal{S}_1(i, D)$ be the set of $\mathbf{x} \in K^n$ with (3.1), (3.13) and

$$\prod_{v \in S} \frac{|l_{iv}(\mathbf{x})|_v}{|\mathbf{x}|_v} < H(\mathbf{x})^{-n-\delta}. \tag{3.14}$$

Further, let $\mathcal{S}_2(D)$ be the set of $\mathbf{x} \in K^n$ with (3.1), (3.13) and

$$\prod_{v \in S} \frac{|l_{iv}(\mathbf{x})|_v}{|\mathbf{x}|_v} \geqslant H(\mathbf{x})^{-n-\delta} \quad \text{for } i = 1, \ldots, n. \tag{3.15}$$

We first estimate the linear scattering of $\mathcal{S}_1(i, D)$ for $i = 1, \ldots, n$. Fix i and put

$$\theta := \frac{\delta}{2(n + \delta)}.$$

Note that by Schwarz' inequality we have

$$\frac{|l_{jv}(\mathbf{x})|_v}{|\mathbf{x}|_v} \leqslant \frac{|l_{jv}|_v \cdot |\mathbf{x}|_v}{|\mathbf{x}|_v} \leqslant 1 \quad \text{for } j = 1, \ldots, n, \ v \in S. \tag{3.16}$$

From (3.14) and (3.16) and from Lemma 9 (i) with the above choice of θ, with $q = s$ and with $L = H(\mathbf{x})^{-n-\delta}$, we infer that there is a set Γ_1 of s-tuples $\underline{\gamma} = (\gamma_v : v \in S)$ with $\gamma_v \geqslant 0$ for $v \in S$ and $\sum_{v \in S} \gamma_v = 1 - \theta$, of cardinality

$$|\Gamma_1| \leqslant (e/\theta)^{s-1} \leqslant \left(e\left(2 + \frac{2n}{\delta}\right)\right)^{s-1} \tag{3.17}$$

such that for every $\mathbf{x} \in \mathcal{S}_1(i, D)$ there is a $\underline{\gamma} \in \Gamma_1$ with

$$\frac{|l_{iv}(\mathbf{x})|_v}{|\mathbf{x}|_v} \leqslant \left(H(\mathbf{x})^{-n-\delta}\right)^{\gamma_v} \quad \text{for } v \in S. \tag{3.18}$$

For each $\underline{\gamma} \in \Gamma_1$, let $\mathcal{S}_1(i, D, \underline{\gamma})$ be the set of $\mathbf{x} \in \mathcal{S}_1(i, D)$ satisfying (3.18). We claim that $\mathcal{S}_1(i, D, \underline{\gamma})$ has linear scattering smaller than

$$A := \max\left(2 \times (2n^{3/2})^{2d}, 4 \times 7^{d+2s}\right).$$

Namely, suppose that for some $\underline{\gamma} \in \Gamma_1$ this is not true. Then by Lemma 10 with $F = 2n^{3/2}$ there are $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathcal{S}_1(i, D, \underline{\gamma})$ with

$$0 < \prod_{v \notin S} \frac{|\det(\mathbf{x}_1, \ldots, \mathbf{x}_n)|_v}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v} \leq (2n^{3/2})^{-1}. \tag{3.19}$$

We assume that

$$H(\mathbf{x}_1) \leqslant \cdots \leqslant H(\mathbf{x}_n) \tag{3.20}$$

which is obviously no restriction.

Let $\mathbf{x}_k = (x_{k1}, \ldots, x_{kn})$ for $k = 1, \ldots, n$. Take $v \in S$. Let $l_{iv}(\mathbf{X}) = \alpha_1 X_1 + \cdots + \alpha_n X_n$. After a permutation of coordinates if necessary, we may assume that $|\alpha_1|_v = \max_i |\alpha_i|_v$. Then, since $|l_{iv}|_v = 1$, we have $|\alpha_1|_v \geqslant n^{-s(v)/2}$. Denote by Δ_j the determinant of the $(n-1) \times (n-1)$-matrix obtained by removing the jth row from

$$\begin{pmatrix} x_{12} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n2} & \cdots & x_{nn} \end{pmatrix}.$$

By Hadamard's inequality, (3.18) and (3.20) we have

$$\frac{|\det(\mathbf{x}_1, \ldots, \mathbf{x}_n)|_v}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v} \leqslant \frac{n^{s(v)/2}}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v} \cdot \left| \det \begin{pmatrix} l_{iv}(\mathbf{x}_1) & x_{12} & \cdots & x_{1n} \\ \vdots & \vdots & & \vdots \\ l_{iv}(\mathbf{x}_n) & x_{n2} & \cdots & x_{nn} \end{pmatrix} \right|_v$$

$$= \frac{n^{s(v)/2}}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v} \cdot \left| \sum_{j=1}^{n} \pm l_{iv}(\mathbf{x}_j)\Delta_j \right|_v$$

$$\leqslant n^{3s(v)/2} \cdot \max_{1 \leqslant j \leqslant n} \frac{|l_{iv}(\mathbf{x}_j)|_v}{|\mathbf{x}_j|_v} \cdot \left(|\Delta_j|_v \prod_{k \neq j} |\mathbf{x}_k|_v^{-1} \right)$$

$$\leqslant n^{3s(v)/2} \max_{1 \leqslant j \leqslant n} \left(H(\mathbf{x}_j)^{-n-\delta} \right)^{\gamma_v}$$

$$\leqslant n^{3s(v)/2} \left(H(\mathbf{x}_1)^{-n-\delta} \right)^{\gamma_v}.$$

By taking the product over $v \in S$ we get

$$\prod_{v \in S} \frac{|\det(\mathbf{x}_1, \ldots, \mathbf{x}_n)|_v}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v} \leqslant n^{3/2} \left(H(\mathbf{x}_1)^{-n-\delta} \right)^{\sum_{v \in S} \gamma_v}$$

$$= n^{3/2} \left(H(\mathbf{x}_1)^{-n-\delta} \right)^{1-\theta} = n^{3/2} H(\mathbf{x}_1)^{-n-\delta/2}.$$

Together with (3.19) and the Product formula this implies

$$\frac{1}{H(\mathbf{x}_1) \cdots H(\mathbf{x}_n)} \leqslant \tfrac{1}{2} H(\mathbf{x}_1)^{-n-\delta/2}.$$

By (3.13) we have $H(\mathbf{x}_1) \geqslant \frac{1}{2} D$ and $H(\mathbf{x}_2), \ldots, H(\mathbf{x}_n) \leqslant \frac{1}{2} D^{1+\zeta}$, where $\zeta = \delta/(2n-2)$. By inserting these inequalities we obtain

$$\begin{aligned} 1 &\leqslant \tfrac{1}{2} H(\mathbf{x}_1)^{1-n-\delta/2} H(\mathbf{x}_2) \cdots H(\mathbf{x}_n) \\ &= \tfrac{1}{2} H(\mathbf{x}_1)^{-(n-1)(1+\zeta)} H(\mathbf{x}_2) \cdots H(\mathbf{x}_n) \\ &\leqslant \tfrac{1}{2} (D/2)^{-(n-1)(1+\zeta)} (D^{1+\zeta}/2)^{n-1} \\ &= 2^{\delta/2-1} < 1. \end{aligned}$$

Thus, our assumption that one of the sets $\mathcal{S}_1(i, D, \underline{\gamma})$ has linear scattering $\geqslant A$ leads to a contradiction. Now by (3.17), by $d \leqslant 2s$, and by the fact that the number of possibilities for $\underline{\gamma}$ is at most $|\Gamma_1| \leqslant (e(2+2n/\delta))^{s-1}$, the set $\mathcal{S}_1(i, D)$ has linear scattering

$$\begin{aligned} &< \max\left(2 \times (2n^{3/2})^{2d}, 4 \times 7^{d+2s}\right) \cdot (e(2+2n/\delta))^{s-1} \\ &< 4 \times (2n)^{6s} (9n/\delta)^{s-1} < \frac{1}{n} \left(\frac{600n^7}{\delta} \right)^s. \end{aligned}$$

Hence $\cup_{i=1}^{n} \mathcal{S}_1(i, D)$ has linear scattering

$$< \left(\frac{600n^7}{\delta} \right)^s. \tag{3.21}$$

We now estimate the linear scattering of $\mathcal{S}_2(D)$. By (3.15) we have for $\mathbf{x} \in \mathcal{S}_2(D)$ that

$$\prod_{v\in S}\prod_{i=1}^{n}\frac{|l_{iv}(\mathbf{x})|_v}{|\mathbf{x}|_v} \geqslant H(\mathbf{x})^{-n(n+\delta)}. \tag{3.22}$$

Put

$$\theta := \frac{\delta}{2n(n+\delta)}.$$

By (3.16) and (3.22) and by Lemma 9 (ii) with this value of θ, with $q = ns$ and with $M = H(\mathbf{x})^{-n(n+\delta)}$, there is a set Γ_2 of ns-tuples $\underline{\gamma} = (\gamma_{iv}: i = 1, \dots, n, v \in S)$ of non-negative reals, with

$$|\Gamma_2| \leqslant (2+e/\theta)^{ns} \leqslant \left(2+\frac{2en(n+\delta)}{\delta}\right)^{ns} \tag{3.23}$$

such that for every $\mathbf{x} \in \mathcal{S}_2(D)$ there is a tuple $\underline{\gamma} \in \Gamma_2$ with

$$H(\mathbf{x})^{-n(n+\delta)(\gamma_{iv}+\theta/ns)} < \frac{|l_{iv}(\mathbf{x})|_v}{|\mathbf{x}|_v} \leqslant H(\mathbf{x})^{-n(n+\delta)\gamma_{iv}}$$
$$\text{for } i = 1, \dots, n,\ v \in S. \tag{3.24}$$

Let $\mathcal{S}_2(D, \underline{\gamma})$ be the set of $\mathbf{x} \in \mathcal{S}_2(D)$ satisfying (3.24). We show that each set $\mathcal{S}_2(D, \underline{\gamma})$ has linear scattering smaller than

$$B := \max\left(2 \times (2n^{n/2}C)^{2d}, 4 \times 7^{d+2s}\right). \tag{3.25}$$

Suppose again that this is not true for some $\underline{\gamma} \in \Gamma_2$. Put $F := 2n^{n/2}C$. Then by Lemma 10 there are $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathcal{S}_2(D, \underline{\gamma})$ such that

$$0 < \prod_{v\notin S}\frac{|\det(\mathbf{x}_1, \dots, \mathbf{x}_n)|_v}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v} \leqslant F^{-1} \tag{3.26}$$

and

$$H(\mathbf{x}_1) \leq \cdots \leqslant H(\mathbf{x}_n). \tag{3.27}$$

Take $v \in S$. For $j = 1, \dots, n$, choose $\alpha_j \in K_v$ with $|\alpha_j|_v = |\mathbf{x}_j|_v$. For $i = 1, \dots, n$, put $\mathbf{y}_i := (\alpha_1^{-1}l_{iv}(\mathbf{x}_1), \dots, \alpha_n^{-1}l_{iv}(\mathbf{x}_n))$. Then by (3.24), (3.27),

$$|\mathbf{y}_i|_v \leqslant n^{s(v)/2}\max_{1\leqslant j\leqslant n}\frac{|l_{iv}(\mathbf{x}_j)|_v}{|\mathbf{x}_j|_v} \leqslant n^{s(v)/2}\max_{1\leqslant j\leqslant n} H(\mathbf{x}_j)^{-n(n+\delta)\gamma_{iv}}$$
$$\leqslant n^{s(v)/2}H(\mathbf{x}_1)^{-n(n+\delta)\gamma_{iv}} \quad \text{for } i = 1, \dots, n. \tag{3.28}$$

Put $\Delta_v := \det(l_{1v}, \ldots, l_{nv})$. Now Hadamard's inequality, (3.28) and the lower bound in (3.24) imply that

$$\begin{aligned} \frac{|\det(\mathbf{x}_1, \ldots, \mathbf{x}_n)|_v}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v} &= |\Delta_v|_v^{-1} \cdot \frac{|\det(l_{iv}(\mathbf{x}_j))|_v}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v} \\ &= |\Delta_v|_v^{-1} |\det(\mathbf{y}_1, \ldots, \mathbf{y}_n)|_v \leqslant |\Delta_v|_v^{-1} |\mathbf{y}_1|_v \cdots |\mathbf{y}_n|_v \\ &\leqslant (n^{n/2})^{s(v)} |\Delta_v|_v^{-1} H(\mathbf{x}_1)^{-n(n+\delta)(\sum_{i=1}^n \gamma_{iv})} \\ &\leqslant (n^{n/2})^{s(v)} |\Delta_v|_v^{-1} H(\mathbf{x}_1)^{n(n+\delta)\theta/s} \left(\prod_{i=1}^n \frac{|l_{iv}(\mathbf{x}_1)|_v}{|\mathbf{x}_1|_v} \right). \end{aligned}$$

By taking the product over $v \in S$ and inserting

$$\prod_{v \in S} \prod_{i=1}^n \frac{|l_{iv}(\mathbf{x}_1)|_v}{|\mathbf{x}_1|_v} \leqslant C \cdot \prod_{v \in S} |\Delta_v|_v \cdot H(\mathbf{x}_1)^{-n-\delta}$$

which follows since $\mathbf{x}_1$ satisfies (3.1), we get

$$\begin{aligned} &\prod_{v \in S} \frac{|\det(\mathbf{x}_1, \ldots, \mathbf{x}_n)|_v}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v} \\ &\quad < n^{n/2} \cdot \left(\prod_{v \in S} |\Delta_v|_v \right)^{-1} H(\mathbf{x}_1)^{n(n+\delta)\theta} \left(\prod_{v \in S} \prod_{i=1}^n \frac{|l_{iv}(\mathbf{x}_1)|_v}{|\mathbf{x}_1|_v} \right) \\ &\quad < n^{n/2} C \cdot H(\mathbf{x}_1)^{n(n+\delta)\theta - n - \delta} = n^{n/2} C H(\mathbf{x}_1)^{-n-\delta/2}. \end{aligned}$$

Together with (3.26) and the Product formula this gives

$$\begin{aligned} \frac{1}{H(\mathbf{x}_1) \cdots H(\mathbf{x}_n)} &= \prod_v \frac{|\det(\mathbf{x}_1, \ldots, \mathbf{x}_n)|_v}{|\mathbf{x}_1|_v \cdots |\mathbf{x}_n|_v} \\ &< n^{n/2} F^{-1} C \cdot H(\mathbf{x}_1)^{-n-\delta/2} < \tfrac{1}{2} H(\mathbf{x}_1)^{-n-\delta/2}. \end{aligned}$$

By inserting $H(\mathbf{x}_1) \geqslant D/2$, $H(\mathbf{x}_i) < D^{1+\zeta}/2$ for $i = 1, \ldots, n$ which follow from (3.13) we obtain

$$\begin{aligned} 1 &< \tfrac{1}{2} H(\mathbf{x}_1)^{1-n-\delta/2} H(\mathbf{x}_2) \cdots H(\mathbf{x}_n) \\ &\leqslant \tfrac{1}{2} (D/2)^{1-n-\delta/2} \cdot (D^{1+\zeta}/2)^{n-1} \\ &= 2^{\delta/2 - 1} D^{1-n-\delta/2 + (n-1)(1+\zeta)} = 2^{\delta/2 - 1} < 1 \end{aligned}$$

which is impossible. Thus, by assuming that some set $\mathcal{S}_2(D, \underline{\gamma})$ has linear scattering $\geqslant B$ we arrive at a contradiction. Hence each set $\mathcal{S}_2(D, \underline{\gamma})$ has linear scattering

$< B$ and together with (3.23), (3.25) this implies that $\mathcal{S}_2(D)$ has linear scattering at most

$$\begin{aligned} B \cdot |\Gamma_2| &\leqslant 4 \times (4n)^{2ns} C^{2d} \cdot \left(2 + \frac{2en(n+\delta)}{\delta}\right)^{ns} \\ &< 4C^{2d} \times (4n)^{2ns} (9n^2/\delta)^{ns} \\ &< 4C^{2d} \times \left(\frac{150n^4}{\delta}\right)^{ns}. \end{aligned}$$

Together with the upper bound for the linear scattering of $\cup_{i=1}^{n} \mathcal{S}_1(i, D)$ in (3.21), this implies that the set of solutions of (3.1) satisfying (3.13) has linear scattering at most

$$\left(\frac{600n^7}{\delta}\right)^s + 4C^{2d}\left(\frac{150n^4}{\delta}\right)^{ns} < 5C^{2d}\left(\frac{150n^4}{\delta}\right)^{ns}; \tag{3.29}$$

here we used that $n \geqslant 2$.

We now consider the solutions of (3.1) with $A \leqslant H(\mathbf{x}) < B$. Let k be the smallest integer with

$$(2A)^{(1+\zeta)^k} \geqslant 2B.$$

Then

$$\begin{aligned} k &< 1 + \frac{\log(\log 2B / \log 2A)}{\log(1+\zeta)} \\ &< \frac{2}{\zeta}\left(1 + \log\left(\frac{\log 2B}{\log 2A}\right)\right) < \frac{4n}{\delta}\left(1 + \log\left(\frac{\log 2B}{\log 2A}\right)\right). \end{aligned} \tag{3.30}$$

For every solution $\mathbf{x} \in K^n$ of (3.1) with $A \leqslant H(\mathbf{x}) < B$ there is a $j \in \{1, \ldots, k\}$ with

$$\tfrac{1}{2}(2A)^{(1+\zeta)^{j-1}} \leqslant H(\mathbf{x}) < \tfrac{1}{2}(2A)^{(1+\zeta)^j}.$$

Together with (3.29) (taking $D = (2A)^{(1+\zeta)^{j-1}}$) and (3.30) this implies that the set of solutions $\mathbf{x} \in K^n$ of (3.1) with $A \leqslant H(\mathbf{x}) < B$ has linear scattering at most

$$\begin{aligned} 5C^{2d}\left(\frac{150n^4}{\delta}\right)^{ns} \cdot k &< 5C^{2d}\left(\frac{150n^4}{\delta}\right)^{ns} \cdot \frac{4n}{\delta}\left(1 + \log\left(\frac{\log 2B}{\log 2A}\right)\right) \\ &< C^{2d}\left(\frac{150n^4}{\delta}\right)^{ns+1} \cdot \left(1 + \log\left(\frac{\log 2B}{\log 2A}\right)\right). \end{aligned}$$

This completes the proof of Lemma 8. □

Proof of part (ii) of the Theorem. Apply Lemma 8 with $C = 1$, $A = 1$, $B = H$. It follows that the set of solutions of

$$\prod_{v\in S}\prod_{i=1}^{n}\frac{|l_{iv}(\mathbf{x})|_v}{|\mathbf{x}|_v} \leqslant \prod_{v\in S}|\det(l_{1v},\ldots,l_{nv})|_v\cdot H(\mathbf{x})^{-n-\delta}$$

in $\mathbf{x}\in K^n$ with $H(\mathbf{x}) < H$, $\mathbf{x}\neq\mathbf{0}$ has linear scattering at most

$$\left(\frac{150n^4}{\delta}\right)^{ns+1}\left(1+\log\left(\frac{\log 2H}{\log 2}\right)\right) < \left(\frac{150n^4}{\delta}\right)^{ns+1}(2+\log\log 2H).$$

□

4. Reductions

In this section we reduce part (i) of the Theorem to a modified version Theorem A. After that, we derive Theorem A from Theorems B and C stated in this section. Theorem B will be proved in Sections 5–6 and Theorem C in Sections 7–9. As before, we use the following notation:

K is an algebraic number field of degree d with ring of integers O_K and discriminant Δ_K;
S is a finite set of places on K of cardinality s, containing all infinite places;
for $v\in S$, $\{l_{1v},\ldots,l_{nv}\}$ is a linearly independent set of linear forms in n variables with coefficients in $\bar{\mathbb{Q}}$;
δ is a real with $0<\delta<1$.

As before, for a field F and a non-zero vector $\mathbf{x} = (x_1,\ldots,x_n)$ with coordinates in some extension of F we define the extension $F(\mathbf{x}) = F(x_1/x_j,\ldots,x_n/x_j)$ for j with $x_j\neq 0$ and for a linear form l with vector of coefficients $\mathbf{a}$ we put $F(l) = F(\mathbf{a})$. Further we define

$D := \max\{[K(l_{iv})\colon K]\colon v\in S, i=1,\ldots,n\}$,
$H := \max\{H(l_{iv})\colon v\in S, i=1,\ldots,n\}$,
$\Delta := |\Delta_L|$, where L is the composite of the fields $K(l_{iv})$ ($v\in S$, $i=1,\ldots,n$).

We call a non-zero vector $\mathbf{x}\in\bar{\mathbb{Q}}^n$ *primitive* if whenever $\mathbb{Q}(\mathbf{x}) = K_0$ we have

$$\begin{cases}\mathbf{x}\in O_{K_0}^n, & |\mathbf{x}|_v \leqslant \left(|\Delta_{K_0}|^{1/2[K_0:\mathbb{Q}]}H(\mathbf{x})\right)^{s(v)} \quad \text{for } v\in M_{K_0},\ v|\infty,\\ & \displaystyle\prod_{\substack{v\in M_{K_0}\\ v\nmid\infty}}|\mathbf{x}|_v \geqslant |\Delta_{K_0}|^{-1/2[K_0:\mathbb{Q}]}.\end{cases} \tag{4.1}$$

For instance, $\mathbf{x}\in\mathbb{Q}^n$ is primitive if and only if its coordinates are coprime rational integers. For every non-zero $\mathbf{x}\in\bar{\mathbb{Q}}^n$ there is a $\lambda\in\bar{\mathbb{Q}}^*$ such that $\lambda\mathbf{x}$ is primitive.

Namely, suppose that $\mathbb{Q}(\mathbf{x}) = K_0$. Then there is a $\lambda_1 \in \bar{\mathbb{Q}}^*$ such that $\mathbf{x}' := \lambda_1\mathbf{x} \in K_0^n$. By Lemma 7 (i) and $\sum_{v|\infty} s(v) = 1$ there is a $\lambda_2 \in K_0^*$ such that

$$|\lambda_2|_v \leqslant |\mathbf{x}'|_v^{-1}(|\Delta_{K_0}|^{1/2[K_0:\mathbb{Q}]}H(\mathbf{x}'))^{s(v)} \quad \text{for } v \in M_{K_0},\ v|\infty,$$
$$|\lambda_2|_v \leqslant |\mathbf{x}'|_v^{-1} \quad \text{for } v \in M_{K_0},\ v \nmid \infty.$$

For $\mathbf{x}'' := \lambda_2\mathbf{x}' = \lambda_1\lambda_2\mathbf{x}$ this implies that for $v \in M_{K_0}, v|\infty$,

$$|\mathbf{x}''|_v \leqslant (|\Delta_{K_0}|^{1/2[K_0:\mathbb{Q}]}H(\mathbf{x}'))^{s(v)} = (|\Delta_{K_0}|^{1/2[K_0:\mathbb{Q}]}H(\mathbf{x}''))^{s(v)}$$

and that $|\mathbf{x}''|_v \leqslant 1$ for $v \nmid \infty$, whence $\mathbf{x}'' \in O_{K_0}^n$. Moreover,

$$\prod_{\substack{v\in M_{K_0}\\ v\nmid\infty}} |\mathbf{x}''|_v = H(\mathbf{x}'') \cdot \left(\prod_{\substack{v\in M_{K_0}\\ v|\infty}} |\mathbf{x}''|_v \right)^{-1} \mathfrak{e}|\Delta_{K_0}|^{-1/2[K_0:\mathbb{Q}]}.$$

Hence $\mathbf{x}''$ is primitive.

It will be convenient to consider instead of inequality (1.3) in the Theorem,

$$\prod_{v\in S}\prod_{i=1}^{n} |l_{iv}(\mathbf{x})|_v \leqslant \prod_{v\in S} |\det(l_{1v},\ldots,l_{nv})|_v \cdot H(\mathbf{x})^{-\delta}$$
$$\text{in primitive } \mathbf{x} \in K^n\backslash\{\mathbf{0}\}. \tag{4.2}$$

We shall derive part (i) of the Theorem from:

THEOREM A. *Put*

$$T := (2^{40n^2}\delta^{-5n})^s \log 4D \cdot \log\log 4D.$$

Assume that

for each infinite place v on K and for $i = 1, \ldots, n$, the linear form l_{iv} has its coefficients in $\bar{\mathbb{Q}} \cap K_v$. (4.3)

Then the set of solutions of (4.2) *with*

$$H(\mathbf{x}) \geqslant \tfrac{1}{2}(2H\Delta)^{e^T} \tag{4.4}$$

has linear scattering $\leqslant T$.

The lower bound in (4.4) has been chosen large enough to swallow the constants appearing in the proof of Theorem A. In particular, since we have to use geometry of numbers over number fields, in our estimates there will be constants depending on the discriminants of certain number fields and these are swallowed because of the Δ in the lower bound. In what follows we derive part (i) of the Theorem from Theorem A and we use an idea of Schlickewei [16] to deal with Δ.

As we want to derive part (i) of the Theorem, we consider the solutions of

$$\prod_{v\in S}\prod_{i=1}^{n}\frac{|l_{iv}(\mathbf{x})|_v}{|\mathbf{x}|_v} \leqslant \prod_{v\in S}|\det(l_{1v},\ldots,l_{nv})|_v\cdot H(\mathbf{x})^{-n-\delta}$$
$$\text{in } \mathbf{x}\in K^n\backslash\{\mathbf{0}\}, \tag{1.3}$$

where K, S, n, δ and the l_{iv} are as above but the l_{iv} do not necessarily satisfy (4.3) for $v|\infty$. By Lemma 8, the set of solutions of (1.3) with

$$H\leqslant H(\mathbf{x})<\tfrac{1}{2}(2H)^{200nD^ns^2/\delta}$$

has linear scattering at most

$$A:=\left(\frac{150n^4}{\delta}\right)^{ns+1}(1+\log(200nD^ns^2/\delta)). \tag{4.5}$$

If $\lambda\in K^*$ is such that $\mathbf{x}':=\lambda\mathbf{x}$ is primitive, then $H(\mathbf{x}')=H(\mathbf{x})$ and the left-hand side of (1.3) does not change when $\mathbf{x}$ is replaced by $\mathbf{x}'$. Hence the linear scattering of the set of solutions of (1.3) with $H(\mathbf{x})\geqslant H$ is at most $A+B$, where B is the linear scattering of the set of solutions of (1.3) with

$$\mathbf{x}\text{ is primitive},\quad H(\mathbf{x})\geqslant\tfrac{1}{2}(2H)^{200nD^ns^2/\delta}. \tag{4.6}$$

From now on, we consider only the solutions of (1.3) with (4.6). We need some lemmas.

LEMMA 11. *Every solution* $\mathbf{x}$ *of* (1.3) *with* (4.6) *satisfies an inequality*

$$\left\{\begin{array}{l}\displaystyle\prod_{w\in S_0}\prod_{i=1}^{n}\frac{|l'_{iw}(\mathbf{x})|_w}{|\mathbf{x}|_w}\leqslant\prod_{w\in S_0}|\det(l'_{1w},\ldots,l'_{nw})|_w\cdot H(\mathbf{x})^{-n-99\delta/100},\\ \mathbb{Q}(\mathbf{x})=K_0,\ \mathbf{x}\textit{ is primitive},\ H(\mathbf{x})\geqslant\tfrac{1}{2}(2H)^{200nD^ns^2/\delta},\end{array}\right. \tag{4.7}$$

where K_0 *is a subfield of* K, S_0 *is the set of places on* K *lying below those in* S *and for* $w\in S_0$, $\{l'_{1w},\ldots,l'_{nw}\}$ *is a linearly independent set of linear forms in* n *variables with algebraic coefficients, such that*

$$\begin{aligned}&l'_{iw}\ \textit{has its coefficients in the completion } K_{0,w} \textit{ of } K_0 \textit{ at } w\\ &\quad\textit{for each infinite place } w\in M_{K_0} \textit{ and for } i=1,\ldots,n,\\ &\quad D':=\max\{[K_0(l'_{iw})\colon K_0]\colon w\in S_0, i=1,\ldots,n\}\leqslant d^2D^2,\\ &\quad H':=\max\{H(l'_{iw})\colon w\in S_0, i=1,\ldots,n\}\leqslant 2H^2.\end{aligned} \tag{4.8}$$

Moreover, the tuple $(K_0;\ l'_{iw}\ (w\in S_0,\ i=1,\ldots,n))$ *belongs to a fixed set* $\mathcal{C}$ *of cardinality at most* 2^{3s} *independent of* $\mathbf{x}$.

Proof. Fix a solution $\mathbf{x}$ of (1.3) with (4.6) and put $K_0 := \mathbb{Q}(\mathbf{x})$. Clearly, K_0 is a subfield of K. For $w \in S_0$, let G_w denote the set of places in S lying above w and put $g_w := |G_w|$. The linear forms l'_{iw} in (4.7) will be determined uniquely by the linear forms l_{iv} we start with and by K_0 and the choice of a $v \in G_w$ for each $w \in S_0$. Thus, the number of possibilities for the tuple $(K_0;\ l'_{iw}\ (w \in S_0,\ i = 1, \dots, n))$ is at most the number of possiblities for K_0 and $v \in G_w$ for $w \in S_0$ which is

$$r := \sum_{K_0} \prod_{w \in S_0} g_w.$$

We estimate r from above. Let L be the normal closure of $[K:\mathbb{Q}]$, G the Galois group of $L/\mathbb{Q}$ and H the Galois group of L/K. The number of subfields of K is precisely the number of subgroups of G containing H. Each such subgroup is a union of (left) cosets of H in G. There are precisely $d = [K:\mathbb{Q}]$ cosets of H in G, hence there are at most 2^d unions of cosets. Therefore, K has at most 2^d subfields. Further, for a fixed subfield K_0 we have

$$\prod_{w \in S_0} g_w \leqslant 2^{\sum_{w \in S_0} g_w} = 2^s.$$

Hence

$$r \leqslant 2^{d+s} \leqslant 2^{3s}.$$

We now construct the linear forms l'_{iw}. In an intermediate step we will get linear forms l''_{iw}. For each $w \in S_0$ and each $v \in G_w$ there is a real number $f(v|w)$ such that

$$|\xi|_v = |\xi|_w^{f(v|w)} \quad \text{for all } \xi \in K_0.$$

We have

$$\frac{1}{[K:K_0]} \leqslant f(v|w) \leqslant 1 \quad \text{for } w \in S_0, v \in G_w, \quad \sum_{v \in G_w} f(v|w) \leqslant 1. \tag{4.9}$$

For $w \in S_0$ choose $v \in G_w$ such that

$$\left(\frac{|l_{1v}(\mathbf{x}) \cdots l_{nv}(\mathbf{x})|_v}{|\det(l_{1v}, \dots, l_{nv})|_v |\mathbf{x}|_v^n} \right)^{1/f(v|w)}$$

is minimal and put $l''_{iw} := l_{iv}$ for $i = 1, \dots, n$. Thus,

$$A_w(\mathbf{x}) \leqslant A_v(\mathbf{x})^{1/f(v|w)} \quad \text{for } v \in G_w,$$

where

$$A_w(\mathbf{x}) := \frac{|l''_{1w}(\mathbf{x}) \cdots l''_{nw}(\mathbf{x})|_w}{|\det(l''_{1w}, \dots, l''_{nw})|_w |\mathbf{x}|_w^n},$$

$$A_v(\mathbf{x}) := \frac{|l_{1v}(\mathbf{x}) \cdots l_{nv}(\mathbf{x})|_v}{|\det(l_{1v}, \dots, l_{nv})|_v |\mathbf{x}|_v^n}.$$

Hence

$$A_w(\mathbf{x}) \leqslant \left(\prod_{v\in G_w} A_v(\mathbf{x})\right)^{c_w} \quad \text{where } c_w := \left(\sum_{v\in G_w} f(v|w)\right)^{-1}. \tag{4.10}$$

By (4.9) we have $1 \leqslant c_w \leqslant [K:K_0] \leqslant [K:\mathbb{Q}] \leqslant 2s$. Further, by Schwarz' inequality we have $|l_{iv}(\mathbf{x})|_v \leqslant |l_{iv}|_v|\mathbf{x}|_v$ and so by Lemma 2,

$$A_v(\mathbf{x}) \leqslant \frac{|l_{1v}|_v \cdots |l_{nv}|_v}{|\det(l_{1v},\ldots,l_{nv})|_v} \leqslant H^{nD^n} \quad \text{for } v \in G_w.$$

By inserting this into (4.10) we get

$$A_w(\mathbf{x}) \leqslant \left(H^{nD^n}\right)^{(c_w-1)g_w} \prod_{v\in G_w} A_v(\mathbf{x}) \leqslant \left(H^{nD^n}\right)^{2sg_w} \prod_{v\in G_w} A_v(\mathbf{x}).$$

Now by taking the product over $w \in S_0$, using that $\sum_{w\in S_0} g_w = s$, we get

$$\prod_{w\in S_0} A_w(\mathbf{x}) \leqslant H^{2nD^n s^2} \cdot \prod_{v\in S} A_v(\mathbf{x}).$$

By (1.3) we have $\prod_{v\in S} A_v(\mathbf{x}) \leqslant H(\mathbf{x})^{-n-\delta}$. Hence

$$\prod_{w\in S_0} A_w(\mathbf{x}) \leqslant H^{2nD^n s^2} H(\mathbf{x})^{-n-\delta}$$

or, rewriting this,

$$\prod_{w\in S_0}\prod_{i=1}^{n} \frac{|l''_{iw}(\mathbf{x})|_w}{|\mathbf{x}|_w} \leq H^{2nD^n s^2} \prod_{w\in S_0} |\det(l''_{1w},\ldots,l''_{nw})|_w \cdot H(\mathbf{x})^{-n-\delta}. \tag{4.11}$$

We recall that $\mathbb{Q}(\mathbf{x}) = K_0$ and that $\mathbf{x}$ satisfies (4.6). Note that the l''_{iw} depend only on K_0 and certain choices of $v \in G_w$ for $w \in S_0$. Moreover,

$$[K_0(l''_{iw}): K_0] \leqslant dD,\ H(l''_{iw}) \leqslant H \quad \text{for } w \in S_0,\ i = 1,\ldots,n, \tag{4.12}$$

since $l''_{iw} = l_{iv}$ for some $v \in G_w$ and $[K_0(l''_{iw}): K_0] \leqslant [K(l_{iv}):\mathbb{Q}] \leqslant dD$.

We now construct the linear forms l'_{iw} from the l''_{iw}. The collection $\{l'_{iw}: w \in S_0,\ i = 1,\ldots,n\}$ will be determined uniquely by $\{l''_{iw}: w \in S_0,\ i = 1,\ldots,n\}$. For the finite places $w \in S_0$ and for the infinite places $w \in S_0$ with $K_{0,w} = \mathbb{C}$ we put $l'_{iw} := l''_{iw}$ for $i = 1,\ldots,n$. Note that if $K_{0,w} = \mathbb{C}$ then l'_{iw} has its coefficients in $\bar{\mathbb{Q}} \cap K_{0,w}$. Now suppose there are places $w \in S_0$ with $K_{0,w} = \mathbb{R}$ and take one of these. We assume that for $i = 1,\ldots,n$, one of the coefficients of l''_{iw} is 1 which is no restriction since (4.11) and (4.12) do not change when the l''_{iw} are multiplied with constants. For $i = 1,\ldots,n$ we write

$$l''_{iw} = m_{iw} + \sqrt{-1}n_{iw}, \qquad \bar{l}''_{iw} = m_{iw} - \sqrt{-1}n_{iw},$$

where m_{iw}, n_{iw} are linear forms with coefficients in $\mathbb{R} = K_{0,w}$ and $\bar{l}''_{iw}$ is the complex conjugate of l''_{iw}. Note that

$$\det(l''_{1w}, \dots, l''_{nw}) = \sum c_I \Delta_I,$$

where the sum is taken over all subsets I of $\{1, \dots, n\}$, c_I is a power of $\sqrt{-1}$ and Δ_I is the determinant of n linear forms, the i-th being m_{iw} if $i \in I$ and n_{iw} if $i \notin I$. Choose I such that $|\Delta_I|_w$ is maximal. Put $l'_{iw} := m_{iw}$ if $i \in I$ and $l'_{iw} := n_{iw}$ if $i \notin I$. Then

$$|\det(l''_{1w}, \dots, l''_{nw})|_w \leqslant 2^{ns(w)} |\det(l'_{1w}, \dots, l'_{nw})|_w,$$

$$|l'_{iw}(\mathbf{x})|_w \leqslant |l''_{iw}(\mathbf{x})|_w \quad \text{for } i = 1, \dots, n.$$

These inequalities hold for each $w \in S_0$ with $K_{0,w} = \mathbb{R}$ and clearly also for the other places in S_0. By inserting these into (4.11) and using that $H(\mathbf{x}) \geqslant \frac{1}{2}(2H)^{200nD^n s^2/\delta}$ we get

$$\prod_{w \in S_0} \prod_{i=1}^{n} \frac{|l'_{iw}(\mathbf{x})|_w}{|\mathbf{x}|_w} \leqslant 2^n H^{2nD^n s^2} \prod_{w \in S_0} |\det(l'_{1w}, \dots, l'_{nw})|_w \cdot H(\mathbf{x})^{-n-\delta}$$

$$\leqslant \prod_{w \in S_0} |\det(l'_{1w}, \dots, l'_{nw})|_w \cdot H(\mathbf{x})^{-n-99\delta/100}.$$

Now the proof of Lemma 11 is complete, except that we still have to verify (4.8). If $w \in S_0$ is finite or if $K_{0,w} = \mathbb{C}$ then (4.12) implies at once that $[K_0(l'_{iw}) \colon K_0] \leqslant d^2 D^2$, $H(l'_{iw}) \leqslant 2H^2$ for $i = 1, \dots, n$. Let $w \in S_0$ be a place with $K_{0,w} = \mathbb{R}$ (supposing there is any). Take $i \in \{1, \dots, n\}$. The linear form l'_{iw} is either the real or imaginary part of l''_{iw}, hence a constant multiple of $l''_{iw} \pm \bar{l}''_{iw}$. Therefore, $K_0(l'_{iw}) \subseteq K_1$, where K_1 is the composite of $K_0(l''_{iw})$, $K_0(\bar{l}''_{iw})$. By (4.12) and the fact that $\bar{l}''_{iw}$ is conjugate to l''_{iw} over K_0 we have $[K_0(\bar{l}''_{iw}) \colon K_0] = [K_0(l''_{iw}) \colon K_0] \leqslant dD$. Hence

$$[K_0(l'_{iw}) \colon K_0] \leqslant [K_1 \colon K_0] \leqslant d^2 D^2.$$

Since l'_{iw} is a constant multiple of $l''_{iw} \pm \bar{l}''_{iw}$ we have $H(l'_{iw}) = H(l''_{iw} \pm \bar{l}''_{iw})$. Further, since both l''_{iw} and $\bar{l}''_{iw}$ have a 1 among their coefficients, their coefficients belong to K_1 and

$$|l''_{iw} \pm \bar{l}''_{iw}|_v \leqslant 2^{s(v)} |l''_{iw}|_v \cdot |\bar{l}''_{iw}|_v \quad \text{for } v \in M_{K_1}.$$

We have $H(l''_{iw}) = H(\bar{l}''_{iw})$ since $l''_{iw}, \bar{l}''_{iw}$ are conjugate over $\mathbb{Q}$. Together with (4.12) this implies that

$$H(l'_{iw}) = H(l''_{iw} \pm \bar{l}''_{iw}) = \prod_{v \in M_{K_1}} |l''_{iw} \pm \bar{l}''_{iw}|_v \leqslant \prod_{v \in M_{K_1}} (2^{s(v)} |l''_{iw}|_v \cdot |\bar{l}''_{iw}|_v)$$

$$= 2H(l''_{iw}) H(\bar{l}''_{iw}) \leqslant 2H^2.$$

This completes the proof of Lemma 11. □

We now consider the solutions of a fixed system (4.7). Put

$$\Delta' := |\Delta_{L'}|,$$

where L' is the composite of the fields $K_0(l'_{iw})$ for $w \in S_0$, $i = 1, \dots, n$.

LEMMA 12. *For every solution* $\mathbf{x}$ *of* (4.7) *we have*

$$H(\mathbf{x}) \geqslant \tfrac{1}{2}(2H'\Delta')^{(8s^3D^2)^{-2ns}}.$$

Proof. Put $f(\mathbb{Q}) := 1$ and for a number field $M \neq \mathbb{Q}$, put $f(M) := |\Delta_M|^{1/2m(m-1)}$ where $m = [M:\mathbb{Q}]$. Let $\mathbf{x}$ be a solution of (4.7). By Lemma 5 we have $H(\mathbf{x}) \geqslant f(K_0)$. Further, for $w \in S_0$, $i = 1, \dots, n$ we have by (4.6) and (4.8),

$$H(\mathbf{x}) \geqslant 2H^2 \geqslant H' \geqslant H(l'_{iw}) \geqslant f(\mathbb{Q}(l'_{iw})).$$

Together with Lemma 4, (ii), noting that L' is also the composite of the fields K_0, $\mathbb{Q}(l'_{iw})$ $(w \in S_0,\ i = 1, \dots, n)$, this implies that

$$H(\mathbf{x}) \geqslant f(L') = |\Delta'|^{1/2a(a-1)} \tag{4.13}$$

where $a = [L':\mathbb{Q}]$. By (4.8) and $d \leqslant 2s$, $n \geqslant 2$ we have

$$a \leqslant [K_0:\mathbb{Q}] \prod_{w\in S_0} \prod_{i=1}^{n} [K_0(l'_{iw}):K_0] \leqslant d(d^2D^2)^{ns} \leqslant (4\sqrt{2}s^3D^2)^{ns}.$$

Further, by (4.6) we have $H(\mathbf{x}) \geqslant 4H^2 \geqslant 2H'$. Together with (4.13) this implies that

$$H(\mathbf{x}) \geqslant \max(2H', |\Delta'|^{1/2a^2}) \geqslant (2H'\Delta')^{1/4a^2} \geqslant \tfrac{1}{2}(2H'\Delta')^{(8s^3D^2)^{-2ns}}. \qquad \square$$

Derivation of Part (i) of the Theorem from Theorem A.
We first estimate the linear scattering of the set of solutions of (4.7). Put

$$T' := \left(2^{40n}(49\delta/50)^{-5}\right)^{ns} \log 4D' \cdot \log\log 4D'.$$

First consider the solutions of (4.7) with

$$H(\mathbf{x}) \geqslant \tfrac{1}{2}(2H'\Delta')^{e^{T'}}.$$

Put $R := \prod_{w\in S_0} |\det(l'_{1w}, \dots, l'_{nw})|_w$. Let $\mathbf{x}$ be a solution of (4.7) with (4.14). We know that $\mathbf{x}$ is primitive, i.e. satisfies (4.1). By Lemma 4 (i) we know that $|\Delta_{K_0}| \leqslant \Delta'$. Together with (4.14) this implies that

$$\prod_{w\notin S_0} |\mathbf{x}|_w \mathfrak{e} \prod_{w\nmid\infty} |\mathbf{x}|_w \geqslant |\Delta_{K_0}|^{-1/2[K_0:\mathbb{Q}]} \geqslant \Delta'^{-1} \geqslant H(\mathbf{x})^{-\delta/100}.$$

Hence

$$\prod_{w\in S_0}\prod_{i=1}^{n}|l'_{iw}(\mathbf{x})|_w \le R\cdot\left(\prod_{w\in S_0}|\mathbf{x}|_w\right)^n\cdot H(\mathbf{x})^{-n-99\delta/100}$$

$$= R\cdot\left(\prod_{w\notin S_0}|\mathbf{x}|_w\right)^{-n}\cdot H(\mathbf{x})^{-99\delta/100}\leqslant R\cdot H(\mathbf{x})^{-49\delta/50}.$$

By applying Theorem A to this inequality, with $D', H', \Delta', 49\delta/50$ replacing D, H, Δ, δ, we infer that the set of solutions of (4.7) satisfying (4.14) has linear scattering at most T'.

By Lemma 12, the solutions $\mathbf{x}$ of (4.7) for which (4.14) does not hold satisfy in fact

$$\tfrac{1}{2}(2H'\Delta')^{(8s^3D^2)^{-2ns}}\leqslant H(\mathbf{x})<\tfrac{1}{2}(2H'\Delta')^{e^{T'}}. \tag{4.15}$$

By Lemma 8, the set of solutions of (4.7) with (4.15) has linear scattering at most

$$\left(\frac{150n^4\times 100}{99\delta}\right)^{ns+1}\cdot(1+\log\{e^{T'}(8s^3D^2)^{2ns}\})$$

$$\leqslant(2T'-1)\left(\frac{152n^4}{\delta}\right)^{ns+1}.$$

Hence the linear scattering of the set of all solutions of (4.7) is at most $2T'(152n^4/\delta)^{ns+1}$. By Lemma 11, every solution $\mathbf{x}$ satisfying (4.6) of the inequality (1.3) we started with satisfies one of at most 2^{3s} systems (4.7). Hence the set of solutions of (1.3) satisfying (4.6) has linear scattering at most $2^{3s+1}T'(152n^4/\delta)^{ns+1}$. In view of $n\geqslant 2$, $ns+1\leqslant 3ns/2$, $D'\leqslant d^2D^2\leqslant 4s^2D^2$, $\log 4D'\log\log 4D'\leqslant 100\times 2^s\cdot\log 4D\log\log 4D$ this is at most

$$2^{3s+1}\left(\frac{152n^4}{\delta}\right)^{ns+1}\left((2^{40n}(50/49\delta)^5\right)^{ns}\log 4D'\log\log 4D'$$

$$<200\left(2^2 152^{3/2}(50/49)^5 n^{6n}2^{40n}\delta^{-7}\right)^{ns}\cdot\log 4D\log\log 4D$$

$$<(2^{54n^2}\delta^{-7n})^s\log 4D\log\log 4D =: B.$$

From an earlier observation we know that the linear scattering of the set of solutions of (1.3) not satisfying (4.6) is at most A where A is given by (4.5). Hence the linear scattering of the set of all solutions of (1.3) is at most

$$A+B=\left(\frac{150n^4}{\delta}\right)^{ns+1}\cdot(1+\log(200nD^ns^2/\delta))$$

$$+ \left(2^{54n^2}\delta^{-7n}\right)^s \log 4D \log\log 4D$$
$$< \left(2^{60n^2}\delta^{-7n}\right)^s \log 4D \log\log 4D.$$

This completes the proof of part (i) of the Theorem. □

We derive Theorem A from two other Theorems B and C. In the statements of these theorems we need some notation which is introduced below. As before, K is a number field and S a finite set of places on K of cardinality s containing all infinite places. Define the ring of S-integers

$$O_S = \{x \in K : |x|_v \leqslant 1 \quad \text{for } v \notin S\}.$$

In what follows, by a tuple $(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q)$ we always mean a tuple consisting of

- an integer $N \geqslant 2$;
- a tuple of reals $\underline{\gamma} = (\gamma_{iv} : v \in S,\ i = 1, \ldots, N)$;
- a system of linear forms $\widehat{\mathcal{L}} = \{\widehat{l}_{iv} : v \in S,\ i = 1, \ldots, N\}$ in N variables such that each $\widehat{l}_{iv}$ has algebraic coefficients and such that for $v \in S$, $\{\widehat{l}_{1v}, \ldots, \widehat{l}_{Nv}\}$ is linearly independent;
- a real $Q > 1$. (4.16)

Further, a tuple $(N, \underline{\gamma}, \widehat{\mathcal{L}})$ without Q will always consist of $N, \underline{\gamma}, \widehat{\mathcal{L}}$ as in (4.16). For a tuple $(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q)$ as above we define the set

$$\Pi(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q) := \{\mathbf{y} \in O_S^N : |\widehat{l}_{iv}(\mathbf{y})|_v \leqslant Q^{\gamma_{iv}} \quad \text{for } v \in S,\ i = 1, \ldots, N\}$$

and

$$V(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q) := \text{the } K\text{-vector space generated by } \Pi(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q).$$

Obviously, $V(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q) \subseteq K^N$.

The idea to prove Theorem A is as follows. We first show that for every solution $\mathbf{x}$ of (4.2) there is a proper linear subspace W of K^n and a tuple $(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q)$ with $N = \binom{n}{k}$ where $k = \dim_K W$ such that

$$\mathbf{x} \in W,\ f_{kn}(W) = V(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q),\ \dim_K V(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q) = N - 1,$$

where f_{kn} is the injective mapping defined in Lemma 1 from the k-dimensional linear subspaces of K^n to the $(N-1)$-dimensional linear subspaces of K^N; moreover, the tuple $(N, \underline{\gamma}, \widehat{\mathcal{L}})$ can be chosen from a finite set independent of $\mathbf{x}$. This is stated in a quantitative form in Theorem B. Second we show that for a fixed tuple $(N, \underline{\gamma}, \widehat{\mathcal{L}})$ and for varying Q with $\dim_K V(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q) = N - 1$ there are only finitely many possibilities for the space $V(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q)$; this is stated in a quantitative form in Theorem C. Now the injectivity of the map f_{kn} implies that there are only finitely many possibilities for W. Thus, it follows that the set of solutions of (4.2) is contained in the union of finitely many proper linear subspaces of K^n.

THEOREM B. *Let K, S, n, s, δ, the system of linear forms $\{l_{iv}: v \in S,\ i = 1,\dots,n\}$, D, H, Δ and T have the same meaning and satisfy the same conditions as in Theorem A, so that in particular $0 < \delta < 1$ and the linear forms l_{iv} satisfy (4.3). Then for every solution $\mathbf{x}$ of*

$$\prod_{v\in S}\prod_{i=1}^{n}|l_{iv}(\mathbf{x})|_v \leqslant \prod_{v\in S}|\det(l_{1v},\dots,l_{nv})|_v \cdot H(\mathbf{x})^{-\delta}$$

$$\text{in primitive } \mathbf{x} \in K^n \tag{4.2}$$

with

$$H(\mathbf{x}) \geqslant \tfrac{1}{2}(2H\Delta)^{e^T}, \tag{4.4}$$

there are a proper linear subspace W of K^n and a tuple $(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q)$ with $N = \binom{n}{k}$ where $k = \dim_K W$, such that the four conditions (4.17)–(4.20) *below are satisfied:*

$$\mathbf{x} \in W,\ f_{kn}(W) = V(N,\underline{\gamma},\widehat{\mathcal{L}};Q), \quad \dim_K V(N,\underline{\gamma},\widehat{\mathcal{L}};Q) = N-1; \tag{4.17}$$

$$\begin{aligned}\underline{\gamma} &= (\gamma_{iv}: v \in S,\ i = 1,\dots,N) \text{ with}\\ &\quad \gamma_{iv} \leqslant s(v) \text{ for } v \in S,\ i = 1,\dots,N \text{ and}\\ &\quad \sum_{v\in S}\sum_{i=1}^{N}\gamma_{iv} \leqslant -\delta/6n^3;\end{aligned} \tag{4.18}$$

$$\begin{aligned}\widehat{\mathcal{L}} &= \{\widehat{l}_{iv}: v \in S,\ i = 1,\dots,N\} \text{ with}\\ &\quad H(\widehat{l}_{iv}) \leqslant H^n,\ [K(\widehat{l}_{iv}):K] \leqslant D^n,\\ &\quad |\widehat{l}_{iv}|_v = 1 \text{ for } v \in S,\ i = 1,\dots,N;\end{aligned} \tag{4.19}$$

$$Q \geqslant \{\tfrac{1}{2}(2H\Delta)^{e^T}\}^{3n}, \tag{4.20}$$

and such that $(N, \underline{\gamma}, \widehat{\mathcal{L}}) \in \mathcal{C}$ where $\mathcal{C}$ is a fixed set independent of $\mathbf{x}$ of cardinality at most

$$C_1 := (30\cdot n^4 2^n \cdot \delta^{-1})^{ns+n}.$$

THEOREM C. *Let K, S be as in Theorem B, let $0 < \epsilon < 1$ and let $(N, \underline{\gamma}, \widehat{\mathcal{L}})$ be a tuple for which $N \geqslant 2$ and for which*

$$\underline{\gamma} = (\gamma_{iv}: v \in S, i = 1,\dots,N) \text{ with}$$

$$\gamma_{iv} \leqslant s(v) \text{ for } v \in S,\ i = 1,\dots,N, \quad \sum_{v\in S}\sum_{i=1}^{N}\gamma_{iv} \leqslant -\epsilon; \tag{4.21}$$

$$\widehat{\mathcal{L}} = \{\widehat{l}_{iv}: v \in S,\ i = 1, \dots, N\} \text{ with}$$
$$H(\widehat{l}_{iv}) \leqslant \hat{H},\ [K(\widehat{l}_{iv}): K] \leqslant \hat{D},$$
$$|\widehat{l}_{iv}|_v = 1 \text{ for } v \in S,\ i = 1, \dots, N. \tag{4.22}$$

Then there is a collection of $(N-1)$*-dimensional linear subspaces of* K^N *of cardinality at most*

$$C_2 := 2^{30} N^8 s^2 \epsilon^{-4} \log 4\hat{D} \cdot \log\log 4\hat{D}$$

such that for every Q *with*

$$\dim_K V(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q) = N - 1, \tag{4.23}$$

$$Q > (2\hat{H})^{e^{C_2}}, \tag{4.24}$$

the vector space $V(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q)$ *belongs to this collection.*

Qualitative forms of Theorems B, C were proved implicitly by Schmidt and Schlickewei. In the proof of Theorem B, which is in Sections 5 and 6, we use geometry of numbers over number fields; here we make explicit the arguments from [5], Section 3.3. In the proof of Theorem C which is in Sections 7–9 we use the 'Roth-machinery;' here we closely follow Schmidt, [18], [19].

Derivation of Theorem A from Theorems B and C.
Let $\mathbf{x}$ be a solution of (4.2) satisfying (4.4) and W the proper linear subspace of K^n and $(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q)$ the tuple from Theorem B. We show that W belongs to a collection independent of $\mathbf{x}$ of cardinality $\leqslant T$. Since by (4.17) we have $\mathbf{x} \in W$ this implies Theorem A.

By Theorem B we have at most

$$C_1 = \left(30n^4 2^n \delta^{-1}\right)^{ns+n} \leqslant \left(2^{11n^2} \delta^{-2n}\right)^s \tag{4.25}$$

possibilities for the tuple $(N, \underline{\gamma}, \widehat{\mathcal{L}})$. We apply Theorem C to each possible tuple. By $N = \binom{n}{k}$ for some $k \leqslant n-1$, (4.18) and (4.19) we must apply Theorem C with

$$\binom{n}{k}, \quad H^n, \quad D^n, \quad \delta/6n^3$$

replacing $N, \hat{H}, \hat{D}, \epsilon$, respectively. Let C_2' be the quantity obtained from C_2 by replacing $N, \hat{D}, \epsilon$ by $\binom{n}{k}$, D^n, $\delta/6n^3$, respectively. Since $\binom{n}{k} \leqslant 2^n$ we have

$$\begin{aligned} C_2' &\leqslant 2^{30} 2^{8n} s^2 6^4 n^{12} \delta^{-4} \log(4D^n) \log\log(4D^n) \\ &< \left(2^{29n^2} \delta^{-3n}\right)^s \log 4D \log\log 4D. \end{aligned} \tag{4.26}$$

Together with (4.20) this implies that

$$Q \geqslant \{\tfrac{1}{2}(2H\Delta)^{e^T}\}^{3n} \geqslant (2H^n)^{e^T} \geqslant (2H^n)^{e^{C_2'}}.$$

Hence Q satisfies (4.24) with C_2' replacing C_2 and H^n replacing $\hat{H}$. Therefore, by Theorem C, the number of possibilities for the vector space $V(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q)$ with *fixed* $N, \underline{\gamma}, \widehat{\mathcal{L}}$ and varying Q is at most C_2'. On combining this with (4.25), (4.26), we obtain that the number of possibilities for the space $V(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q)$ with *varying* $N, \underline{\gamma}, \widehat{\mathcal{L}}, Q$ is at most

$$C_1 C_2' < (2^{40n^2}\delta^{-5n})^s \log 4D \log\log 4D = T.$$

Because of the injectivity of the maps f_{kn}, the vector space W is uniquely determined by $V(N, \underline{\gamma}, \widehat{\mathcal{L}}; Q)$. Hence for W we have at most T possibilities. This implies Theorem A. □

5. Parallelepipeds

Let K be an algebraic number field and V_K its ring of adeles. We shall derive upper and lower bounds for the volume of a parallelepiped in V_K^n and then derive estimates for the product of the successive minima of this parallelepiped. This will be an important tool in the proof of Theorem B.

We use the following notation: $\mathrm{Gal}(F'/F)$ is the Galois group of a Galois field extension F'/F; for a linear form $l(\mathbf{X}) = \alpha_1 X_1 + \cdots + \alpha_n X_n$ with $\alpha_1, \ldots, \alpha_n \in F'$ and for $\sigma \in \mathrm{Gal}(F'/F)$ we put $\sigma(l)(\mathbf{X}) := \sigma(\alpha_1)X_1 + \cdots + \sigma(\alpha_n)X_n$; and a set of linear forms $\{l_1, \ldots, l_m\}$ with coefficients in F' is called *self-conjugate over* F if for every $i \in \{1, \ldots, m\}$ and $\sigma \in \mathrm{Gal}(F'/F)$ there is a $\lambda \in F'^*$ such that $\lambda\sigma(l_i) \in \{l_1, \ldots, l_m\}$.

Fix a place $v \in M_K$. As before, K_v denotes the completion of K at v. Let

$$\mathbf{x} \in K_v^n \backslash \{\mathbf{0}\}$$

and let

$$\mathcal{L}_v = \{l_{1v}, \ldots, l_{nv}\}$$

be a linearly independent set of linear forms in n variables such that for $i = 1, \ldots, n$,

l_{iv} has its coefficients in K_v if v is infinite,

l_{iv} has its coefficients in $\bar{K}_v$ if v is finite,

$l_{iv}(\mathbf{x}) \neq 0$.

Define the v-adic parallelepiped depending on $\mathbf{x}$,

$$\Pi_v(\mathbf{x}) := \{\mathbf{y} \in K_v^n : |l_{iv}(\mathbf{y})|_v \leqslant |l_{iv}(\mathbf{x})|_v \text{ for } i = 1, \ldots, n\}. \quad (5.1)$$

We need estimates for the volume $\beta_v^n(\Pi_v(\mathbf{x}))$ of $\Pi_v(\mathbf{x})$ where β_v^n is the measure on K_v^n defined in Section 2.

Let

$$\hat{\mathcal{L}}_v = \{l_{1v}, \ldots, l_{m_v,v}\}$$

be a minimal set of linear forms containing $\mathcal{L}_v$ that is self-conjugate over K_v. Such a set exists since by assumption the coefficients of $l_{1v}, \ldots, l_{nv}$ are algebraic over K_v. If v is infinite then $\hat{\mathcal{L}}_v = \mathcal{L}_v$, $m_v = n$ and if v is finite then $\hat{\mathcal{L}}_v \supseteq \mathcal{L}_v$, $m_v \geqslant n$.

Take $j \in \{1, \ldots, m_v\}$. There are $i \in \{1, \ldots, n\}$, $\lambda \in \bar{K}_v^*$, $\sigma \in \mathrm{Gal}(\bar{K}_v/K)$ such that $l_{jv} = \lambda\sigma(l_{iv})$. Then for $\mathbf{y} \in \Pi_v(\mathbf{x})$ we have, noting that $|\sigma(x)|_v = |x|_v$ for $x \in \bar{K}_v$,

$$\begin{aligned} |l_{jv}(\mathbf{y})|_v &= |\lambda|_v|\sigma(l_{iv}(\mathbf{y}))|_v = |\lambda|_v|l_{iv}(\mathbf{y})|_v \\ &\leqslant |\lambda|_v|l_{iv}(\mathbf{x})|_v = |\lambda\sigma(l_{iv}(\mathbf{x}))|_v = |l_{jv}(\mathbf{x})|_v. \end{aligned}$$

Hence

$$\Pi_v(\mathbf{x}) = \{\mathbf{y} \in K_v^n : |l_{iv}(\mathbf{y})|_v \leqslant |l_{iv}(\mathbf{x})|_v \text{ for } i = 1, \ldots, m_v\}. \tag{5.2}$$

Put

$$\begin{aligned} \beta_v(\mathbf{x}) &:= \{\beta_v^n(\Pi_v(\mathbf{x}))\}^{1/d} \quad \text{where } d := [K:\mathbb{Q}], \\ R_v(\mathbf{x}) &:= \max_{\{i_1,\ldots,i_n\}\subseteq\{1,\ldots,m_v\}} \frac{|\det(l_{i_1v}, \ldots, l_{i_nv})|_v}{|l_{i_1v}(\mathbf{x})\cdots l_{i_nv}(\mathbf{x})|_v}. \end{aligned}$$

As before, for a linear form $l(\mathbf{X}) = a_1X_1 + \cdots + a_nX_n$ with coefficients in $\bar{K}_v$ we define the field $K_v(l) := K_v(a_1/a_j, \ldots, a_n/a_j)$ where $a_j \neq 0$. We use the following notation if v is finite:

$\mathfrak{D}_v$ is the local different of K at v;

$O_v = \{x \in K_v : |x|_v \leqslant 1\}$ is the local ring of K_v;

$K_{iv} := K_v(l_{iv})$ for $i = 1, \ldots, m_v$;

$O_{iv} = \{x \in K_{iv} : |x|_v \leqslant 1\}$ is the local ring of K_{iv} for $i = 1, \ldots, m_v$;

$\mathfrak{d}_{iv}$ is the discriminant of the ring extension O_{iv}/O_v for $i = 1, \ldots, m_v$;

$$\delta_v := \min_{1\leqslant i\leqslant m_v} |\mathfrak{d}_{iv}|_v, \tag{5.3}$$

where $|\mathfrak{a}|_v := \max\{|x|_v : x \in \mathfrak{a}\}$ for any O_v-ideal $\mathfrak{a}$. Since O_v is a principal ideal domain, O_{iv} is a free O_v-module of rank $[K_{iv}:K_v]$. We recall that $\mathfrak{d}_{iv}$ is the O_v-ideal generated by the discriminant of any O_v-basis $\{\omega_1, \ldots, \omega_t\}$ of O_{iv}, that is $D_{K_{iv}/K_v}(\omega_1, \ldots, \omega_t) = \{\det(\sigma_i(\omega_j))\}^2$ where $\sigma_1, \ldots, \sigma_t$ are the K_v-isomorphisms of K_{iv}.

LEMMA 13. We have

$$\begin{cases} \beta_v(\mathbf{x}) = 2^{n/d}R_v(\mathbf{x})^{-1} & \textit{if } v \textit{ is real infinite},\\ \beta_v(\mathbf{x}) = (2\pi)^{n/d}R_v(\mathbf{x})^{-1} & \textit{if } v \textit{ is complex infinite},\\ \delta_v^{n/2}|\mathfrak{D}_v|_v^{n/2}R_v(\mathbf{x})^{-1} \leqslant \beta_v(\mathbf{x}) \leqslant |\mathfrak{D}_v|_v^{n/2}R_v(\mathbf{x})^{-1} & \textit{if } v \textit{ is finite}. \end{cases} \tag{5.4}$$

Proof. $\Pi_v(\mathbf{x}), R(\mathbf{x})$ and δ_v do not change when we replace l_{iv} by $l_{iv}(\mathbf{x})^{-1}l_{iv}$ for $i = 1, \dots, n$. Therefore, we may assume that $l_{iv}(\mathbf{x}) = 1$ for $i = 1, \dots, n$ and shall do so in the sequel. Then l_{iv} has its coefficients in K_{iv}; namely we know that for some $\lambda_i \in \bar{K}_v^*$, $\lambda_i l_{iv}$ has its coefficients in K_{iv} but then $\lambda_i = \lambda_i l_{iv}(\mathbf{x}) \in K_{iv}$. Similarly, if $l_{jv} = \lambda\sigma(l_{iv})$ for some $\lambda \in K_v^*, \sigma \in \mathrm{Gal}(\bar{K}_v/K_v)$ then this holds with $\lambda = 1$ i.e. $l_{jv} = \sigma(l_{iv})$. Hence we can extend $\{l_{1v}, \dots, l_{nv}\}$ to a minimal set of linear forms $\{l_{1v}, \dots, l_{m_v,v}\}$ such that for each $i \in \{1, \dots, m_v\}$ and $\sigma \in \mathrm{Gal}(\bar{K}_v/K_v)$, the linear form $\sigma(l_{iv})$ belongs also to this set. Put $\Pi_v := \Pi_v(\mathbf{x}), R_v := R_v(\mathbf{x})$. Then

$$\Pi_v = \{\mathbf{y} \in K_v^n : |l_{iv}(\mathbf{y})|_v \leqslant 1 \quad \text{for} \quad i = 1, \dots, m_v\}, \tag{5.5}$$

$$R_v = \max_{\{i_1,\dots,i_n\}\subseteq\{1,\dots,m_v\}} |\det(l_{i_1v}, \dots, l_{i_nv})|_v. \tag{5.6}$$

First we assume that for $i = 1, \dots, n$, l_{iv} has its coefficients in K_v. Then $m_v = n$ and

$$R_v = |\det(l_{1v}, \dots, l_{nv})|_v. \tag{5.7}$$

Let $\mathbf{a}_1, \dots, \mathbf{a}_n$ be the vectors given by $l_{iv}(\mathbf{a}_j) = \delta_{ij}$ for $i, j = 1, \dots, n$ and define the linear transformation $A\mathbf{y} = y_1\mathbf{a}_1 + \cdots + y_n\mathbf{a}_n$ for $\mathbf{y} = (y_1, \dots, y_n) \in K_v^n$. Thus,

$$\Pi_v = A(O_v^n) = \{A\mathbf{y} : \mathbf{y} \in O_v^n\}, \tag{5.8}$$

where we use O_v to denote the unit ball $\{x \in K_v : |x|_v \leqslant 1\}$ also if v is infinite.

First, let v be a real infinite place. Then $|\cdot|_v = |\cdot|^{1/d}$ where $|\cdot|$ is the usual absolute value on $\mathbb{R}$ and β_v^n the Lebesgue measure on $\mathbb{R}^n$. Further, $\beta_v(O_v) = 2$. Now from (5.8), a well-known property of the Lebesgue measure and (5.7) it follows that

$$\begin{aligned} \beta_v^n(\Pi_v) &= |\det A|\beta_v(O_v)^n = |\det(l_{1v}, \dots, l_{nv})|^{-1}\beta_v(O_v)^n \\ &= 2^n R_v^{-d} \end{aligned}$$

which is (5.4).

Second, let v be a complex infinite place. Then $|\cdot|_v = |\cdot|^{2/d}$, β_v is two times the Lebesgue measure on the complex plane, and $\beta_v(O_v) = 2\pi$. For $\mathbf{y} \in \mathbb{C}^n$ we define vectors $\mathbf{w}, \mathbf{z} \in \mathbb{R}^n$ by $\mathbf{y} = \mathbf{w} + \sqrt{-1}\mathbf{z}$ and we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by $\mathbf{y} \mapsto (\mathbf{w}, \mathbf{z})$. Further, we define real linear mappings M, N by $A = M + \sqrt{-1}N$. Thus,

$$A\mathbf{y} = (M\mathbf{w} - N\mathbf{z}, N\mathbf{w} + M\mathbf{z}).$$

Together with (5.8), (5.7) this implies that

$$\beta_v^n(\Pi_v) = \Delta\beta_v(O_v)^n = \Delta(2\pi)^n,$$

where

$$\begin{aligned}\Delta &= \left|\det\begin{pmatrix} M & -N \\ N & M\end{pmatrix}\right| = \left|\det\begin{pmatrix} M+\sqrt{-1}N & -N \\ -\sqrt{-1}(M+\sqrt{-1}N) & M\end{pmatrix}\right| \\ &= \left|\det\begin{pmatrix} M+\sqrt{-1}N & -N \\ 0 & M-\sqrt{-1}N\end{pmatrix}\right| = \left|\det A \cdot \overline{\det A}\right| \\ &= \left|\det A\right|^2 = R_v^{-d}.\end{aligned}$$

This implies (5.4).

Now assume that v is finite. Clearly, the vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ are linearly independent and belong to Π_v. Further, every $\mathbf{y} \in \Pi_v$ can be expressed as $\sum_{i=1}^n l_{iv}(\mathbf{y})\mathbf{a}_i$; since $l_{iv}(\mathbf{y}) \in O_v$ for $i = 1, \dots, n$, it follows that Π_v is a free O_v-module with basis $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Choose a non-zero $\alpha \in O_v$ such that $\alpha O_v^n = \{\alpha\mathbf{x} : \mathbf{x} \in O_v^n\} \subseteq \Pi_v$. Then by (5.8) αO_v^n has index (as an abelian group) $|\alpha^{-n}\det A|_v^d$ in Π_v. All cosets of αO_v^n in Π_v have the same β_v^n-measure since β_v^n is translation invariant. Hence

$$\beta_v^n(\Pi_v) = |\alpha^{-n}\det A|_v^d \cdot \beta_v^n(\alpha O_v^n).$$

Further, αO_v^n has index $|\alpha|_v^{-nd}$ in O_v^n, hence $\beta_v^n(O_v^n) = |\alpha|_v^{-nd}\beta_v^n(\alpha O_v^n)$. Therefore,

$$\begin{aligned}\beta_v^n(\Pi_v) &= |\det A|_v^d \beta_v^n(O_v^n) = |\det(l_{1v}, \dots, l_{nv})|_v^{-d} |\mathfrak{D}_v|_v^{n/d2} \\ &= R_v^{-d}|\mathfrak{D}_v|_v^{nd/2},\end{aligned} \tag{5.9}$$

where we used again (5.7). Since $\delta_v = 1$ this implies (5.4).

We now assume that at least one of the linear forms l_{iv} does not have its coefficients in K_v. Then v is finite. We shall reduce this case to the previous one, by using an argument from [2].

Partition $\{1, \dots, m_v\}$ into sets $C_1, \dots, C_t$ such that i, j belong to the same set if and only if $l_{jv} = \sigma(l_{iv})$ for some $\sigma \in \mathrm{Gal}(\bar{K}_v/K_v)$. Then for $i \in C_k$ $(k = 1, \dots, t)$ we have $[K_{iv} : K_v] = |C_k|$.

Fix $k \in \{1, \dots, t\}$, $p \in C_k$. Let $\{\omega_{ph} : h \in C_k\}$ be an O_v-basis of O_{pv}. Let $\xi_h \in K_v$ for $h \in C_k$. Then

$$\begin{aligned}\left|\sum_{h\in C_k} \omega_{ph}\xi_h\right|_v \leqslant 1 &\iff \sum_{h\in C_k} \omega_{ph}\xi_h \in O_{pv} \iff \xi_h \in O_v \text{ for } h \in C_k \\ &\iff |\xi_h|_v \leqslant 1 \text{ for } h \in C_k.\end{aligned} \tag{5.10}$$

For $i \in C_k$ let τ_i be the K_v-isomorphism from K_{pv} to K_{iv} with $\tau_i(l_{pv}) = l_{iv}$ and put $\omega_{ih} := \tau_i(\omega_{ph})$ for $h \in C_k$; then $\{\omega_{ih} : h \in C_k\}$ is an O_v-basis of O_{iv}. Hence (5.10) can be extended to

$$\left|\sum_{h\in C_k} \omega_{ih}\xi_h\right|_v \leqslant 1 \text{ for } i \in C_k \iff |\xi_h|_v \leqslant 1 \text{ for } h \in C_k. \tag{5.11}$$

We can express l_{pv} as

$$l_{pv} = \sum_{h\in C_k} \omega_{ph} f_h,$$

where f_h $(h \in C_k)$ is a linear form in n variables with coefficients in K_v. By applying τ_i we obtain

$$l_{iv} = \sum_{h\in C_k} \omega_{ih} f_h \quad \text{for } i \in C_k. \tag{5.12}$$

Now (5.11) implies that for $\mathbf{y} \in K_v^n$,

$$|l_{iv}(\mathbf{y})|_v \leqslant 1 \quad \text{for } i \in C_k \iff |f_h(\mathbf{y})|_v \leqslant 1 \quad \text{for } h \in C_k.$$

By combining the linear forms f_h $(h \in C_k)$ for $k = 1, \dots, t$ we obtain altogether m_v linear forms $f_1, \dots, f_{m_v}$ with coefficients in K_v such that for $\mathbf{y} \in K_v^n$,

$$\begin{aligned} &|l_{iv}(\mathbf{y})|_v \leqslant 1 \quad \text{for } i = 1, \dots, m_v \\ &\quad \iff |f_h(\mathbf{y})|_v \leqslant 1 \quad \text{for } h = 1, \dots, m_v. \end{aligned} \tag{5.13}$$

Together with (5.5) this implies that

$$\Pi_v = \{\mathbf{y} \in K_v^n : |f_h(\mathbf{y})|_v \leqslant 1 \quad \text{for } h = 1, \dots, m_v\}. \tag{5.14}$$

We assume that

$$|\det(f_1, \dots, f_n)|_v = \max_{\{i_1,\dots,i_n\}\subseteq\{1,\dots,m\}} |\det(f_{i_1}, \dots, f_{i_n})|_v$$

which is clearly no restriction. By (5.12) we have rank $\{f_1, \dots, f_{m_v}\}$ = rank $\{l_{1v}, \dots, l_{m_v,v}\} = n$. Hence $f_1, \dots, f_n$ are linearly independent. Therefore, there are $\alpha_{ij} \in K_v$ such that

$$f_i = \sum_{j=1}^{n} \alpha_{ij} f_j \quad \text{for } i = n+1, \dots, m_v. \tag{5.15}$$

By Cramer's rule we have

$$|\alpha_{ij}|_v = \frac{|\det(f_1, \dots, f_i, \dots, f_n)|_v}{|\det(f_1, \dots, f_n)|_v} \leqslant 1 \text{ for } i = n+1, \dots, m_v,\ j = 1, \dots, n.$$

Therefore, $|f_h(\mathbf{y})|_v \leqslant 1$ for $h = 1, \dots, n$ implies that $|f_h(\mathbf{y})|_v \leqslant 1$ for $h = 1, \dots, m_v$. Together with (5.14) this implies that

$$\Pi_v = \{\mathbf{y} \in K_v^n : |f_h(\mathbf{y})|_v \leqslant 1 \quad \text{for } h = 1, \dots, n\}.$$

Hence we have, similarly as in (5.9),

$$\beta_v^n(\Pi_v) = |\det(f_1, \dots, f_n)|_v^{-d} |\mathfrak{D}_v|_v^{nd/2}. \tag{5.16}$$

We have to compare $|\det(f_1, \dots, f_n)|_v$ with R_v. From (5.12), (5.15) and the inequalities $|\omega_{ij}|_v \leqslant 1$, $|\alpha_{ij}|_v \leqslant 1$ for $1 \leqslant i, j \leqslant m_v$ it follows that

$$l_{jv} = \sum_{h=1}^{v} \theta_{jh} f_h \quad \text{with } |\theta_{jh}|_v \leqslant 1 \quad \text{for } j = 1, \dots, m_v,\ h = 1, \dots, n.$$

This implies that for each subset $\{i_1, \dots, i_n\}$ of $\{1, \dots, m_v\}$ of cardinality n we have

$$|\det(l_{i_1 v}, \dots, l_{i_n v})|_v \leqslant |\det(f_1, \dots, f_n)|_v$$

which implies, together with (5.6),

$$R_v \leqslant |\det(f_1, \dots, f_n)|_v. \tag{5.17}$$

Fix again $k \in \{1, \dots, t\}$. Let $(\omega^{ij})_{i,j \in C_k}$ be the inverse of the matrix $(\omega_{ij})_{i,j \in C_k}$. To obtain an inequality reverse to (5.17) we need upper bounds for the numbers $|\omega^{ij}|_v$. Put $d_k := \det(\omega_{ij})_{i,j \in C_k}$. For $h, l \in C_k$ we have

$$|d_k \omega^{hl}|_v \leqslant 1$$

since $d_k \omega^{hl}$ is a determinant in some of the numbers ω_{ij}. Further, for each $i \in C_k$, the i-th row of the matrix $(\omega_{ij})_{i,j \in C_k}$ consists of an O_v-basis of O_{iv} while the other rows are the conjugates over K_v of the i-th row. Hence for each $i \in C_k$, d_k^2 generates the discriminant ideal $\mathfrak{d}_{iv}$ of O_{iv} over O_v. This implies that

$$|d_k|_v = |\mathfrak{d}_{iv}|_v^{1/2} \quad \text{for } i \in C_k.$$

Hence

$$|\omega^{hl}|_v \leqslant |d_k|_v^{-1} \leqslant |\mathfrak{d}_{iv}|_v^{-1/2} \text{ for } h, l, i \in C_k.$$

Putting $\omega^{hl} := 0$ if h, l do not belong to the same set C_k we obtain

$$|\omega^{hl}|_v \leqslant \Big(\min_{i=1,\dots,m_v} |\mathfrak{d}_{iv}|_v\Big)^{-1/2} \leqslant \delta_v^{-1/2} \quad \text{for } h, l = 1, \dots, m_v. \tag{5.18}$$

By (5.12) we have

$$f_i = \sum_{j \in C_k} \omega^{ij} l_{jv} \quad \text{for } h = 1, \dots, t,\ i, j \in C_k.$$

This implies that

$$\det(f_1,\ldots,f_n)=\sum_{\{i_1,\ldots,i_n\}\subseteq\{1,\ldots,m_v\}}\theta_{i_1,\ldots,i_n}\det(l_{i_1v},\ldots,l_{i_nv}),$$

where $\theta_{i_1,\ldots,i_n}$ is some $n\times n$ determinant with entries from the numbers ω^{hl} $(h,l\in\{1,\ldots,m_v\})$. So by (5.18), we have $|\theta_{i_1,\ldots,i_n}|_v\leqslant\delta_v^{-n/2}$. It follows that

$$|\det(f_1,\ldots,f_n)|_v\leqslant\delta_v^{-n/2}R_v.$$

Together with (5.16), (5.17) this implies that

$$\delta_v^{nd/2}R_v^{-d}|\mathfrak{D}_v|_v^{nd/2}\leqslant\beta_v^n(\Pi_v)\leqslant R_v^{-d}|\mathfrak{D}_v|_v^{nd/2}$$

which is equivalent to (5.4). This completes the proof of Lemma 13. □

Now let S be a finite set of places on K, containing all infinite places. For each $v\in S$, let $\{l_{1v},\ldots,l_{nv}\}$ be a linearly independent set of linear forms in n variables with algebraic coefficients such that if v is an infinite place then for $i=1,\ldots,n$ the coefficients of l_{iv} belong to $\bar{\mathbb{Q}}\cap K_v$. As before, let L be the composite of the fields $K(l_{iv})$ $(v\in S, i=1,\ldots,n)$ and put $\Delta:=|\Delta_L|$. Let

$$\mathbf{x}\in K^n$$

and define the parallelepiped in V_K^n,

$$\Pi(\mathbf{x}):=\prod_{v\in S}\Pi_v(\mathbf{x})\times\prod_{v\notin S}O_v^n,$$

where $\Pi_v(\mathbf{x})$ is defined by (5.1). According to the definition in Section 2 we have

$$\lambda\Pi(\mathbf{x})=\prod_{v|\infty}\lambda\Pi_v(\mathbf{x})\times\prod_{\substack{v\in S\\ v\nmid\infty}}\Pi_v(\mathbf{x})\times\prod_{v\notin S}O_v^n\quad\text{for }\lambda\in\mathbb{R},\lambda>0;$$

note that $\lambda\Pi(\mathbf{x})$ is precisely the set of adelic vectors $(\mathbf{y}_v)\in V_K^n$ satisfying

$$\begin{cases}|l_{iv}(\mathbf{y}_v)|_v\leqslant\lambda^{s(v)}|l_{iv}(\mathbf{x})|_v & \text{for } v\in S,\ i=1,\ldots,n,\\ |\mathbf{y}_v|_v\leqslant 1 \quad \text{for } v\notin S.\end{cases}$$

The set $\Pi(\mathbf{x})$ is convex symmetric. Denote the successive minima of $\Pi(\mathbf{x})$ by

$$\lambda_1(\mathbf{x}),\ldots,\lambda_n(\mathbf{x}).$$

Further, put

$$\begin{aligned}R(\mathbf{x}):&=\prod_{v\in S}R_v(\mathbf{x})\\ &=\prod_{v\in S}\max_{\{i_1,\ldots,i_n\}\subseteq\{1,\ldots,m_v\}}\frac{|\det(l_{i_1v},\ldots,l_{i_nv})|_v}{|l_{i_1v}(\mathbf{x})|_v\cdots|l_{i_nv}(\mathbf{x})|_v},\end{aligned}\tag{5.20}$$

where for each $v \in S$, $\{l_{1v}, \ldots, l_{m_v,v}\}$ is a minimal set of linear forms containing $\{l_{1v}, \ldots, l_{nv}\}$ which is self-conjugate over K_v.

LEMMA 14. $\frac{1}{n!}R(\mathbf{x}) \leqslant \lambda_1(\mathbf{x}) \cdots \lambda_n(\mathbf{x}) \leqslant \Delta^{n/2d} R(\mathbf{x})$ *where* $d = [K:\mathbb{Q}]$.

Proof. Put $\beta(\mathbf{x}) := \{\beta^n(\Pi(\mathbf{x}))\}^{1/d}$. Denote by r_1 the number of real and by r_2 the number of complex infinite places of K. For finite $v \in S$, let δ_v be the number defined by (5.3). By Lemma 13 and the identity

$$\prod_{v \nmid \infty} |\mathfrak{D}_v|_v = |\Delta_K|^{-1/d}$$

we have, for some constant F with

$$\left(\prod_{\substack{v \in S \\ v \nmid \infty}} \delta_v\right)^{n/2} \leqslant F \leqslant 1, \tag{5.21}$$

$$\begin{aligned} \beta(\mathbf{x}) &= \prod_{v \in S} \beta_v(\mathbf{x}) \prod_{v \notin S} |\mathfrak{D}_v|_v^{n/2} \\ &= 2^{r_1 n/d}(2\pi)^{r_2 n/d} F \prod_{v \nmid \infty} |\mathfrak{D}_v|_v^{n/2} \prod_{v \in S} R_v(\mathbf{x})^{-1} \\ &= 2^n (\pi/2)^{r_2 n/d} F |\Delta_K|^{-n/2d} R(\mathbf{x})^{-1}. \end{aligned} \tag{5.22}$$

Let $v \in S, v$ finite. For each $j \in \{1, \ldots, m_v\}$ there is an $i \in \{1, \ldots, n\}$ such that up to a constant the linear forms l_{iv}, l_{jv} are conjugate over K_v. Hence the local discriminants $\mathfrak{d}_{iv}$, $\mathfrak{d}_{jv}$ of O_{iv}/O_v and O_{jv}/O_v, respectively, are equal. Together with (5.3) this implies that

$$\delta_v = \min_{i=1,\ldots,n} |\mathfrak{d}_{iv}|_v.$$

Further, the local discriminant $\mathfrak{d}_{iv}$ divides the global discriminant $\mathfrak{d}_{K(l_{iv})/K}$ and by Lemma 4 (i), $\mathfrak{d}_{K(l_{iv})/K}$ divides $\mathfrak{d}_{L/K}$. Hence

$$|\mathfrak{d}_{L/K}|_v \leqslant \min_{i=1,\ldots,n} |\mathfrak{d}_{K(l_{iv})/K}|_v \leq \min_{i=1,\ldots,n} |\mathfrak{d}_{iv}|_v = \delta_v.$$

Therefore,

$$\prod_{\substack{v \in S \\ v \nmid \infty}} \delta_v \geqslant \prod_{\substack{v \in S \\ v \nmid \infty}} |\mathfrak{d}_{L/K}|_v \geqslant \prod_{v \nmid \infty} |\mathfrak{d}_{L/K}|_v = N_{K/\mathbb{Q}}(\mathfrak{d}_{L/K})^{-1/d}.$$

Together with (5.21) this implies that

$$N_K(\mathfrak{d}_{L/K})^{-n/2d} \leqslant F \leqslant 1.$$

By inserting this into (5.22), using that by Lemma 4 (i) we have

$$\Delta = |\Delta_L| \geqslant N_{K/\mathbb{Q}}(\mathfrak{d}_{L/K})|\Delta_K|,$$

we obtain

$$\begin{aligned} &2^n(\pi/2)^{r_2 n/d}\Delta^{-n/2d}R(\mathbf{x})^{-1} \\ &\quad \leqslant \beta(\mathbf{x}) \leqslant 2^n(\pi/2)^{r_2 n/d}|\Delta_K|^{-n/2d}R(\mathbf{x})^{-1}. \end{aligned} \tag{5.23}$$

Together with Lemma 6 this implies that

$$\begin{aligned} \lambda_1(\mathbf{x})\cdots\lambda_n(\mathbf{x}) &\geq \frac{2^n}{n!}\left(\frac{\pi^n n!}{2}\right)^{r_2/d}|\Delta_K|^{-n/2d}\beta(\mathbf{x})^{-1} \\ &\geqslant \frac{1}{n!}R(\mathbf{x})\left(\frac{2^n}{\pi^n}\cdot\frac{\pi^n n!}{2}\right)^{r_2/d} \geqslant \frac{1}{n!}R(\mathbf{x}) \end{aligned}$$

and

$$\lambda_1(\mathbf{x})\cdots\lambda_n(\mathbf{x}) \leqslant 2^n\beta(\mathbf{x})^{-1} \leqslant \left(\frac{2}{\pi}\right)^{r_2 n/d}\Delta^{n/2d}R(\mathbf{x}) \leqslant \Delta^{n/2d}R(\mathbf{x}).$$

This completes the proof of Lemma 14. □

6. Proof of Theorem B

We use the following lemma instead of Davenport's lemma used by Schmidt and Schlickewei.

LEMMA 15. *Let K be an algebraic number field of degree d and let $\mathbf{b}_1,\ldots,\mathbf{b}_n$ be linearly independent vectors in K^n. Further, for each infinite place v on K, let $l_{1v},\ldots,l_{nv}$ be linearly independent linear forms in n variables with coefficients in K_v and let $\mu_{1v},\ldots,\mu_{nv}$ be real numbers with*

$$0 < \mu_{1v} \leqslant \mu_{2v} \leqslant \cdots \leqslant \mu_{nv}.$$

Suppose

$$|l_{iv}(\mathbf{b}_j)|_v \leqslant \mu_{jv} \quad \text{for } 1 \leqslant i,j \leq n,\ v|\infty. \tag{6.1}$$

Then there are permutations κ_v of $(1,\ldots,n)$ for each infinite place v on K, and vectors

$$\mathbf{v}_1 = \mathbf{b}_1,\ \mathbf{v}_i = \sum_{j=1}^{i-1}\xi_{ij}\mathbf{b}_j + \mathbf{b}_i \quad \text{with } \xi_{ij} \in O_K \quad \text{for } 1 \leqslant j < i \leqslant n$$

such that

$$|l_{\kappa_v(i),v}(\mathbf{v}_j)|_v \leqslant \{2d|\Delta_K|^{1/2}\}^{s(v)(i+j)}\min(\mu_{iv},\mu_{jv}) \quad \text{for } 1 \leqslant i,j \leqslant n,\ v|\infty.$$

Proof. We proceed by induction on n. For $n = 1$ the assertion is trivial. Let $n \geqslant 2$ and suppose that Lemma 15 holds for $n-1$. Let V be the vector space with basis $\{\mathbf{b}_1, \dots, \mathbf{b}_{n-1}\}$. We identify V with K^{n-1}.

Take an infinite place v. There are $\alpha_{1v}, \dots, \alpha_{nv} \in K_v$, not all zero, such that

$$\sum_{k=1}^{n} \alpha_{kv} l_{kv}(\mathbf{b}_j) = 0 \quad \text{for } j = 1, \dots, n-1.$$

Choose $\kappa_v(n) \in \{1, \dots, n\}$ such that

$$|\alpha_{\kappa_v(n),v}|_v = \max\{|\alpha_{1v}|_v, \dots, |\alpha_{nv}|_v\}$$

and put $C_v := \{1, \dots, n\} \backslash \{\kappa_v(n)\}$. Then $\alpha_{\kappa_v(n),v} \neq 0$. Put

$$\beta_{iv} := -\alpha_{iv}/\alpha_{\kappa_v(n),v} \quad \text{for } i \in C_v.$$

Thus,

$$l_{\kappa_v(n),v}(\mathbf{b}_j) = \sum_{k \in C_v} \beta_{kv} l_{kv}(\mathbf{b}_j) \qquad \text{for } j = 1, \dots, n-1, \ v|\infty$$
$$\text{with } |\beta_{kv}|_v \leqslant 1 \ \text{ for } k \in C_v. \tag{6.2}$$

The restrictions of l_{iv} $(i = 1, \dots, n)$ to V form a system of linear forms of rank $n-1$ and the restriction of $l_{\kappa_v(n),v}$ to V is linearly dependent on the restrictions of l_{kv} $(k \in C_v)$ to V. Hence the restrictions of l_{kv} $(k \in C_v)$ to V are linearly independent. By applying the induction hypothesis to $\mathbf{b}_1, \dots, \mathbf{b}_{n-1}$ and the linear forms l_{kv} $(k \in C_v, v|\infty)$ we infer that there are a bijective function κ_v from $\{1, \dots, n-1\}$ to C_v for each $v|\infty$, and vectors

$$\mathbf{v}_1 = \mathbf{b}_1, \ \mathbf{v}_i = \sum_{j=1}^{i-1} \xi_{ij}\mathbf{b}_j + \mathbf{b}_i \quad \text{with } \xi_{ij} \in O_K$$
$$\text{for } 1 \leqslant j < i \leqslant n-1, \ v|\infty \tag{6.3}$$

such that

$$|l_{\kappa_v(i),v}(\mathbf{v}_j)|_v \leqslant \{2d|\Delta_K|^{1/2}\}^{(i+j)s(v)} \min(\mu_{iv}, \mu_{jv})$$
$$\text{for } 1 \leqslant j \leqslant i \leqslant n-1, \ v|\infty.$$

Recalling that $|\cdot|_v^{1/s(v)}$ is the usual absolute value, whence satisfies the triangle inequality, this implies together with (6.2) that

$$|l_{\kappa_v(n),v}(\mathbf{v}_j)|_v \leqslant \left(\sum_{k=1}^{n-1} \{2d|\Delta_K|^{1/2}\}^{k+j}\right)^{s(v)} \mu_{jv}$$
$$\leqslant \{2d|\Delta_K|^{1/2}\}^{s(v)(n+j)} \mu_{jv}$$

for $v|\infty$, $j = 1, \dots, n-1$. Therefore,

$$|l_{\kappa_v(i),v}(\mathbf{v}_j)|_v \leqslant \{2d|\Delta_K|^{1/2}\}^{s(v)(i+j)} \min(\mu_{iv}, \mu_{jv})$$
$$\text{for } i = 1, \dots, n,\ j = 1, \dots, n-1,\ v|\infty. \tag{6.4}$$

Because of (6.3), the proof of Lemma 15 is complete once we have shown that there is a vector

$$\mathbf{v}_n = \mathbf{b}_n + \xi_1\mathbf{v}_1 + \cdots + \xi_{n-1}\mathbf{v}_{n-1} \quad \text{with } \xi_1, \dots, \xi_{n-1} \in O_K$$

such that

$$|l_{\kappa_v(i),v}(\mathbf{v}_n)|_v \leqslant \{2d|\Delta_K|^{1/2}\}^{(n+i)s(v)}\mu_{iv} \quad \text{for } i = 1, \dots, n,\ v|\infty. \tag{6.5}$$

Write l'_{iv} for $l_{\kappa_v(i),v}$. Then $l'_{1v}, \dots, l'_{n-1,v}$ are linearly independent on V hence $\det(l'_{iv}(\mathbf{b}_j)_{i,j=1,\dots,n-1}) \neq 0$. Therefore, there are $\gamma_{jv} \in K_v$ such that

$$l'_{iv}(\mathbf{b}_n) = \sum_{j=1}^{n-1} \gamma_{jv} l'_{iv}(\mathbf{b}_j) \quad \text{for } i = 1, \dots, n-1,\ v|\infty. \tag{6.6}$$

Further, by (6.2), (6.6) we have

$$\begin{aligned}\sum_{j=1}^{n-1} \gamma_{jv} l'_{nv}(\mathbf{b}_j) &= \sum_{k=1}^{n-1} \beta_{\kappa_v(k),v} \left\{ \sum_{j=1}^{n-1} \gamma_{jv} l'_{kv}(\mathbf{b}_j) \right\} \\ &= \sum_{k=1}^{n-1} \beta_{\kappa_v(k),v} l'_{kv}(\mathbf{b}_n).\end{aligned} \tag{6.7}$$

By (6.3), $\mathbf{b}_1, \dots, \mathbf{b}_{n-1}$ can be expressed as linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$. Hence there are $\gamma'_{jv} \in K_v$ with $\sum_{j=1}^{n-1} \gamma_{jv}\mathbf{b}_j = \sum_{j=1}^{n-1} \gamma'_{jv}\mathbf{v}_j$. Together with (6.6), (6.7) this implies

$$l'_{iv}(\mathbf{b}_n) = \sum_{j=1}^{n-1} \gamma'_{jv} l'_{iv}(\mathbf{v}_j) + \alpha_{iv} \quad \text{for } i = 1, \dots, n,\ v|\infty, \tag{6.8}$$

where

$$\begin{aligned}\alpha_{iv} &= 0 \quad \text{for } i = 1, \dots, n-1, \\ \alpha_{nv} &= l'_{nv}(\mathbf{b}_n) - \sum_{k=1}^{n-1} \beta_{\kappa_v(k),v} l'_{kv}(\mathbf{b}_n).\end{aligned} \tag{6.9}$$

By Lemma 7 (ii), there are $\xi_1, \dots, \xi_{n-1} \in K$ with

$$|\xi_j + \gamma'_{jv}|_v \leqslant \left\{\frac{d}{2}|\Delta_K|^{1/2}\right\}^{s(v)} \quad \text{for } v|\infty,\ j = 1, \dots, n-1,$$

$$|\xi_j|_v \leq 1 \quad \text{for } v \nmid \infty, j = 1, \dots, n-1.$$

Hence $\xi_j \in O_K$ for $j = 1, \dots, n-1$. Put

$$\mathbf{v}_n = \mathbf{b}_n + \xi_1 \mathbf{v}_1 + \cdots + \xi_{n-1}\mathbf{v}_{n-1},$$
$$\delta_{jv} := \xi_j + \gamma'_{jv} \quad \text{for } j = 1, \dots, n-1,\ v|\infty.$$

Then, by (6.8),

$$l'_{iv}(\mathbf{v}_n) = \sum_{j=1}^{n-1} \delta_{jv} l'_{iv}(\mathbf{v}_j) + \alpha_{iv} \quad \text{with } |\delta_{jv}|_v \leqslant \left\{\frac{d}{2}|\Delta_K|^{1/2}\right\}^{s(v)}$$
$$\text{for } j = 1, \dots, n-1,\ v|\infty. \tag{6.10}$$

Take $v|\infty$. Using again that $|\cdot|_v^{1/s(v)}$ satisfies the triangle inequality we have by (6.9), (6.2),

$$|\alpha_{nv}|_v \leqslant \left\{|l'_{nv}(\mathbf{b}_n)|_v^{1/s(v)} + \sum_{k=1}^{n-1} |\beta_{\kappa_v(k),v}|_v^{1/s(v)} |l'_{kv}(\mathbf{b}_n)|_v^{1/s(v)}\right\}^{s(v)}$$
$$\leqslant n^{s(v)}\mu_{nv}$$

and clearly also $|\alpha_{jv}|_v \leqslant j^{s(v)}\mu_{jv}$ for $j = 1, \dots, n-1$. Together with (6.9), (6.4) this implies for $i = 1, \dots, n$,

$$|l'_{iv}(\mathbf{v}_n)|_v \leqslant \left\{\sum_{j=1}^{n} |\delta_{jv}|_v^{1/s(v)} |l'_{iv}(\mathbf{v}_j)|_v^{1/s(v)} + |\alpha_{iv}|_v^{1/s(v)}\right\}^{s(v)}$$
$$\leqslant \left\{\sum_{j=1}^{m} \frac{d}{2}|\Delta_K|^{1/2}(2d|\Delta_K|^{1/2})^{i+j}\min(\mu_{iv}, \mu_{jv})^{1/s(v)} + i\mu_{iv}^{1/s(v)}\right\}^{s(v)}$$
$$\leq \left(2d|\Delta_K|^{1/2}\right)^{(n+i)s(v)} \mu_{iv}$$

which is precisely (6.5). This proves Lemma 15. □

Let K be an algebraic number field of degree d, S a finite set of places on K containing all infinite places, $n \geqslant 2$ an integer, $0 < \delta < 1$ a real and for $v \in S$, let $\{l_{1v}, \dots, l_{nv}\}$ be a linearly independent set of linear forms in n variables with algebraic coefficients. Suppose that for each infinite place v and each $i \in \{1, \dots, n\}$, the coefficients of l_{iv} belong to $K_v \cap \bar{\mathbb{Q}}$, and that

$$[K(l_{iv})\colon K] \leqslant D, H(l_{iv}) \leqslant H \quad \text{for } v \in S,\ i = 1, \dots, n.$$

By Lemma 4 (i), we have $|\Delta_{K_0}|^{1/[K_0:\mathbb{Q}]} \leqslant |\Delta_K|^{1/d}$ for each subfield K_0 of K. This implies that if $\mathbf{x} \in K^n$ is primitive, i.e. satisfies (4.1), then

$$\mathbf{x} \in O_K^n, \qquad |\mathbf{x}|_v \leqslant (|\Delta_K|^{1/2d}H(\mathbf{x}))^{s(v)} \quad \text{for } v|\infty,$$
$$\prod_{v\nmid\infty} |\mathbf{x}|_v \geqslant |\Delta_K|^{-1/2d}. \tag{6.11}$$

Hence every primitive solution **x** of (4.2) with (4.4) is also a solution of

$$\prod_{v\in S}\prod_{i=1}^{n}|l_{iv}(\mathbf{x})|_v \leq \prod_{v\in S}|\det(l_{1v},\ldots,l_{nv})|_v\cdot H(\mathbf{x})^{-\delta}$$

$$\text{in } \mathbf{x}\in K^n \text{ with (6.11) and with } H(\mathbf{x})\geqslant \tfrac{1}{2}(2H\Delta)^{e^T}, \tag{6.12}$$

where

$$T=(2^{40n^2}\delta^{-5n})^s, \qquad \Delta=|\Delta_L|,$$

with L being the composite of the fields $K(l_{iv})$ ($v\in S$, $i=1,\ldots,n$). We will show that for every solution **x** of (6.12) there is a tuple $(N,\underline{\gamma},\hat{\mathcal{L}};Q)$ as in Theorem B.

Since (6.12) does not change when the l_{iv} are multiplied with constants we may assume that $|l_{iv}|_v=1$ for $v\in S, i=1,\ldots,n$ and we shall do so in the sequel. For $v\in S$, let $\{l_{1v},\ldots,l_{m_v,v}\}$ be a minimal set of linear forms containing $l_{1v},\ldots,l_{nv}$ which is self-conjugate over K_v. Thus, $m_v=n$ if $v|\infty$. We assume also w.l.o.g. that $|l_{iv}|_v=1$ for $i=n+1,\ldots,m_v$. Further, if l_{iv},l_{jv} are (up to a constant) conjugate over K_v then they are also conjugate over K, hence $[K(l_{iv})\colon K]=[K(l_{jv})\colon K]$, $H(l_{iv})=H(l_{jv})$. Summarising,

$$[K(l_{iv})\colon K]\leqslant D,\ H(l_{iv})\leqslant H,\ |l_{iv}|_v=1 \ \text{ for } v\in S,\ i=1,\ldots,m_v. \tag{6.13}$$

By inserting this into Lemma 2 we get

$$1\geqslant |l_{i_1v}\wedge\cdots\wedge l_{i_pv}|_v\geqslant H^{-pD^p} \tag{6.14}$$

for each $v\in S$ and each linearly independent subset $\{l_{i_1v},\ldots,l_{i_pv}\}$ of $\{l_{1v},\ldots,l_{m_v,v}\}$. Further, by Schwartz' inequality we have

$$|l_{iv}(\mathbf{y})|_v\leqslant |l_{iv}|_v|\mathbf{y}|_v\leqslant|\mathbf{y}|_v \quad \text{for } \mathbf{y}\in K^n,\ v\in S,\ i=1,\ldots,m_v. \tag{6.15}$$

We shall frequently use (6.14), (6.15).

Let **x** be a solution of (6.12). To **x** we associate the adelic parallelepiped

$$\Pi(\mathbf{x})=\left\{\begin{array}{l}(\mathbf{y}_v)\in V_K^n\colon |l_{iv}(\mathbf{y}_v)|_v\ \leq |l_{iv}(\mathbf{x})|_v \text{ for } v\in S,\ i=1,\ldots,n\\ \qquad\qquad\qquad |\mathbf{y}_v|_v\ \leq 1 \text{ for } v\notin S.\end{array}\right\}$$

Recall that by (5.2) we may extend the set of indices i from $\{1,\ldots,n\}$ to $\{1,\ldots,m_v\}$:

$$\Pi(\mathbf{x})=\left\{\begin{array}{l}(\mathbf{y}_v)\in V_K^n\colon |l_{iv}(\mathbf{y}_v)|_v\ \leq |l_{iv}(\mathbf{x})|_v \text{ for } v\in S,\ i=1,\ldots,m_v\\ \qquad\qquad\qquad |\mathbf{y}_v|_v\ \leq 1 \text{ for } v\notin S.\end{array}\right\} \tag{6.16}$$

As before, $\lambda_1(\mathbf{x}),\ldots,\lambda_n(\mathbf{x})$ denote the successive minima of $\Pi(\mathbf{x})$ and

$$R(\mathbf{x})=\prod_{v\in S}\max_{\{i_1,\ldots,i_n\}\subseteq\{1,\ldots,m_v\}}\frac{|\det(l_{i_1v},\ldots,l_{i_nv})|_v}{|l_{i_1v}(\mathbf{x})\cdots l_{i_nv}(\mathbf{x})|_v}.$$

In what follows we write $\lambda_1, \ldots, \lambda_n, R$ for $\lambda_1(\mathbf{x}), \ldots, \lambda_n(\mathbf{x}), R(\mathbf{x})$, remembering that these quantities depend on $\mathbf{x}$. From (6.12) it follows that

$$R \geqslant \prod_{v \in S} \frac{|\det(l_{1v}, \ldots, l_{nv})|_v}{|l_{1v}(\mathbf{x}) \cdots l_{nv}(\mathbf{x})|_v} \geqslant H(\mathbf{x})^{\delta}. \tag{6.17}$$

Hence

$$R \geqslant \{\tfrac{1}{2}(2H\Delta)^{e^T}\}^{\delta}. \tag{6.18}$$

(6.11) and (6.17) imply that

$$|\mathbf{x}|_v \leqslant \{|\Delta_K|^{1/2d} R^{1/\delta}\}^{s(v)} \quad \text{for } v \in M_K. \tag{6.19}$$

Let ϕ be the diagonal embedding of K^n into V_K^n. By (6.16) we have for $\mathbf{y} \in K^n$, $\lambda > 0$ that

$$\begin{aligned} \mathbf{y} \in \phi^{-1}(\lambda\Pi(\mathbf{x})) \iff & |l_{iv}(\mathbf{y})|_v \leqslant \lambda^{s(v)} |l_{iv}(\mathbf{x})|_v \\ & \quad \text{for } v \in S,\ i = 1, \ldots, m_v, \\ & |\mathbf{y}|_v \leqslant 1 \ \text{ for } v \notin S. \end{aligned} \tag{6.20}$$

LEMMA 16. (i) *Let $\lambda > 0$. Then for every $\mathbf{y} \in \phi^{-1}(\lambda\Pi(\mathbf{x}))$ we have*

$$|\mathbf{y}|_v \leqslant |\det(l_{1v}, \ldots, l_{nv})|_v^{-1} \{n|\Delta_K|^{1/2d} R^{1/\delta} \lambda\}^{s(v)} \quad \textit{for } v \in S.$$

(ii) $\lambda_1 \geqslant n^{-1} H^{-nsD^n} |\Delta_K|^{-1/2d} R^{-1/\delta}$.
(iii) *There is an $\alpha \in K^*$ such that for all $\lambda > 0$ and all $\mathbf{y} \in \phi^{-1}(\lambda\Pi(\mathbf{x}))$ we have*

$$\alpha\mathbf{y} \in O_K^n, \quad \alpha\mathbf{y} \in \phi^{-1}(\{|\Delta_K|^{1/2d} H^{nsD^{2n}} \lambda\}\Pi(\mathbf{x}))$$

Proof. Let $\mathbf{y} \in \phi^{-1}(\lambda\Pi(\mathbf{x}))$. Fix $v \in S$ and put $\Delta_v := \det(l_{1v}, \ldots, l_{nv})$. Let $\mathbf{a}_j$ be the coefficient vector of $(l_{1v} \wedge \cdots \wedge l_{j-1,v} \wedge l_{j+1,v} \wedge \cdots \wedge l_{nv})^*$ (cf. Section 2). Then

$$\mathbf{y} = \sum_{j=1}^{n} \Delta_v^{-1} l_{jv}(\mathbf{y}) \mathbf{a}_j. \tag{6.21}$$

By (6.20) and (6.15) we have for $j = 1, \ldots, n$,

$$|l_{jv}(\mathbf{y})|_v \leqslant |l_{jv}(\mathbf{x})|_v \lambda^{s(v)} \leqslant |\mathbf{x}|_v \lambda^{s(v)}.$$

Together with (6.19) this implies

$$|l_{jv}(\mathbf{y})|_v \leqslant \left(|\Delta_K|^{1/2d} R^{1/\delta} \lambda\right)^{s(v)}. \tag{6.22}$$

By (6.14) we have

$$|\mathbf{a}_j|_v = |l_{1v} \wedge \cdots \wedge l_{j-1,v} \wedge l_{j+1,v} \wedge \cdots \wedge l_{nv}|_v \leqslant 1.$$

Together with (6.21), (6.22) this implies that

$$|\mathbf{y}|_v \leqslant n^{s(v)} |\Delta_v|_v^{-1} \{|\Delta_K|^{1/2d} R^{1/\delta} \lambda\}^{s(v)}$$

which is (i).

(ii) Choose $\mathbf{y} \in \phi^{-1}(\lambda_1 \Pi(\mathbf{x}))$ with $\mathbf{y} \neq \mathbf{0}$. Then $|\mathbf{y}|_v \leqslant 1$ for $v \notin S$. Hence by (i) and (6.14),

$$1 \leqslant H(\mathbf{y}) \leqslant \prod_{v \in S} |\mathbf{y}|_v \leqslant \left(\prod_{v \in S} |\Delta_v|_v^{-1} \right) . n |\Delta_K|^{1/2d} R^{1/\delta} \lambda_1$$
$$\leqslant n H^{nsD^n} |\Delta_K|^{1/2d} R^{1/\delta} \lambda_1$$

which implies (ii).

(iii) Fix $v \in S$, v finite. Let M be the composite of the fields $K(l_{1v}), \dots, K(l_{nv})$. The value group of $|\cdot|_v$ on K, which is $G_{K,v} := \{|x|_v : : x \in K^*\}$ has finite index, e_v, say, in the value group $G_{M,v} := \{|x|_v : x \in M^*\}$ of $|\cdot|_v$ on M . Note that

$$1 \leqslant e_v \leqslant [M : K] \leqslant D^n. \tag{6.23}$$

For $i = 1, \dots, n$, there is a $\gamma_i \in \bar{\mathbb{Q}}^*$ such that $l'_{iv} := \gamma_i l_{iv}$ has its coefficients in M. Hence, putting again $\Delta_v := \det(l_{1v}, \dots, l_{nv})$,

$$|\Delta_v|_v = \frac{|\det(l'_{1v}, \dots, l'_{nv})|_v}{|l'_{1v}|_v \cdots |l'_{nv}|_v} \in G_{M,v}.$$

Put

$$C_v := |\Delta_v|_v^{e_v};$$

then $C_v \in G_{K,v}$. By Lemma 7 (i) there is an $\alpha \in K^*$ with

$$\begin{cases} |\alpha|_v \leqslant C_v & \text{for } v \in S, v \nmid \infty, \\ |\alpha|_v \leqslant 1 & \text{for } v \notin S, \\ |\alpha|_v \leqslant |\Delta_K|^{1/2d} \left(\prod_{\substack{w \in S \\ w \nmid \infty}} C_v \right)^{-s(v)} & \text{for } v \mid \infty. \end{cases} \tag{6.24}$$

Now let $\mathbf{y} \in \phi^{-1}(\lambda \Pi(\mathbf{x}))$, where $\lambda > 0$ and $\mathbf{y} \neq \mathbf{0}$. By (i) and $s(v) = 0$ for finite v we have $|\mathbf{y}|_v \leqslant C_v^{-1/e_v}$ for finite $v \in S$. Further, by (6.14) we have $C_v \leqslant 1$. Hence

$$|\alpha \mathbf{y}|_v \leqslant C_v^{1-1/e_v} \leqslant 1 \quad \text{for } v \in S, v \nmid \infty. \tag{6.25}$$

Since also $|\alpha\mathbf{y}|_v \leqslant 1$ for $v \notin S$ we have $\alpha\mathbf{y} \in O_K^n$. Further, by (6.23) we have $|\alpha|_v \leqslant 1$ for all finite places v while by (6.21), (6.14) and (6.23) we have for each infinite place v,

$$|\alpha|_v \leq \left\{ |\Delta_K|^{1/2d} \cdot \prod_{\substack{w \in S \\ w \nmid \infty}} |\det(l_{1w}, \ldots, l_{nw})|_w^{-e_w} \right\}^{s(v)}$$

$$\leqslant \left\{ |\Delta_K|^{1/2d} \prod_{\substack{w \in S \\ w \nmid \infty}} (H^{nD^n})^{e_w} \right\}^{s(v)} \leqslant (|\Delta_K|^{1/2d} H^{nsD^{2n}})^{s(v)}.$$

Hence

$$|l_{iv}(\alpha\mathbf{y})|_v \leqslant (|\Delta_K|^{1/2d} H^{nsD^{2n}} \lambda)^{s(v)} |l_{iv}(\mathbf{x})|_v \quad \text{for } v \in S,\ i = 1, \ldots, m_v.$$

Together with (6.25) and (6.20) this implies (iii). □

LEMMA 17. *There are linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in O_K^n$ and permutations κ_v of $(1, \ldots, n)$ for each $v|\infty$, all depending on $\mathbf{x}$, with the following properties:*

(i) *for $j = 1, \ldots, n$, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_j$ belong to the K-vector space generated by $\phi^{-1}(\lambda_j \Pi(\mathbf{x}))$;*

(ii) *we have*

$$\begin{cases} |l_{\kappa_v(i),v}(\mathbf{v}_j)|_v \leqslant |l_{\kappa_v(i),v}(\mathbf{x})|_v \{G\lambda_{\min(i,j)}\}^{s(v)} \\ \quad \text{for } v|\infty,\ i = 1, \ldots, n,\ j = 1, \ldots, n, \\ |l_{iv}(\mathbf{v}_j)|_v \leqslant |l_{iv}(\mathbf{x})|_v \quad \text{for } v \in S, v \nmid \infty,\ i = 1, \ldots, m_v,\ j = 1, \ldots, n, \end{cases} \tag{6.26}$$

where

$$G := |\Delta_K|^{1/2d} H^{nsD^{2n}} (2d|\Delta_K|^{1/2})^{2n}.$$

Proof. Choose linearly independent vectors $\mathbf{b}'_1, \ldots, \mathbf{b}'_n$ with $\mathbf{b}'_j \in \phi^{-1}(\lambda_j \Pi(\mathbf{x}))$ for $j = 1, \ldots, n$ and put $\mathbf{b}_j := \alpha \mathbf{b}'_j$, where α is the number from Lemma 16 (iii). Then for $j = 1, \ldots, n$, the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_j$ belong to the K-vector space generated by $\phi^{-1}(\lambda_j \Pi(\mathbf{x}))$. Moreover,

$$\mathbf{b}_j \in O_K^n,\ \ \mathbf{b}_j \in \phi^{-1}(\{|\Delta_K|^{1/2d} H^{nsD^{2n}} \lambda_j\} \Pi(\mathbf{x})) \quad \text{for } j = 1, \ldots, n.$$

Together with (6.20) this implies that

$$\begin{cases} |l_{iv}(\mathbf{b}_j)|_v \leqslant (|\Delta_K|^{1/2d} H^{nsD^{2n}} \lambda_j)^{s(v)} |l_{iv}(\mathbf{x})|_v \\ \quad \text{for } v|\infty,\ i = 1, \ldots, n,\ j = 1, \ldots, n, \\ |l_{iv}(\mathbf{b}_j)|_v \leqslant |l_{iv}(\mathbf{x})|_v \ \text{ for } v \in S, v \nmid \infty,\ i = 1, \ldots, m_v,\ j = 1, \ldots, n, \\ \mathbf{b}_j \in O_K^n \ \text{ for } j = 1, \ldots, n. \end{cases} \tag{6.27}$$

We apply Lemma 15 with the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$, the linear forms $l_{iv}(\mathbf{x})^{-1} l_{iv}$ ($v|\infty$, $i = 1, \ldots, n$) and the numbers $\mu_{jv} = (|\Delta_K|^{1/2d} H^{nsD^{2n}} \lambda_j)^{s(v)}$ ($v|\infty$, $j = 1, \ldots, n$). It follows that there are vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with

$$\mathbf{v}_1 = \mathbf{b}_1, \;\; \mathbf{v}_j = \sum_{k=1}^{j-1} \xi_{jk} \mathbf{b}_k + \mathbf{b}_j \quad \text{with } \xi_{jk} \in O_K \quad \text{for } 1 \leqslant k < j \leqslant n, \tag{6.28}$$

and permutations κ_v of $(1, \ldots, n)$ for $v|\infty$ such that

$$\begin{aligned} &|l_{\kappa_v(i)}(\mathbf{x})|_v^{-1} |l_{\kappa_v(i),v}(\mathbf{v}_j)|_v \\ &\leq (2d|\Delta_K|^{1/2})^{(i+j)s(v)} \min(\mu_{iv}, \mu_{jv}) \\ &\leq \{G\lambda_{\min(i,j)}\}^{s(v)} \text{ for } v|\infty, \; i, j = 1, \ldots, n. \end{aligned} \tag{6.29}$$

From (6.27) and the fact that the numbers ξ_{jk} in (6.28) belong to O_K it follows that

$$\begin{aligned} &\mathbf{v}_j \in O_K^n, \;\; |l_{iv}(\mathbf{v}_j)|_v \leqslant |l_{iv}(\mathbf{x})|_v \;\; \text{for } v \in S, v \nmid \infty, \\ &\quad i = 1, \ldots, m_v, \; j = 1, \ldots, n. \end{aligned}$$

Together with (6.29) this implies (6.26). Further, (6.28) implies that for $j = 1, \ldots, n$, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_j$ are linear combinations of $\mathbf{b}_1, \ldots, \mathbf{b}_j$, whence belong to the K-vector space generated by $\phi^{-1}(\lambda_j \Pi(\mathbf{x}))$. This completes the proof of Lemma 17. □

LEMMA 18. *There is a set* Γ *of cardinality*

$$|\Gamma| \leqslant (n!)^{-s} (30n^4 2^n \delta^{-1})^{ns+n}$$

consisting of tuples of real numbers

$$(\mathbf{c}; \mathbf{d}) = (c_{iv} : v \in S, i = 1, \ldots, m_v; \; d_i : i = 1, \ldots, n)$$

with

$$c_{iv} \leqslant \frac{11}{10\delta} s(v) \;\; \textit{for } v \in S, \; i = 1, \ldots, m_v \tag{6.30}$$

$$d_1 \leqslant 0, \qquad -\frac{11}{10\delta} \leqslant d_1 \leqslant \cdots \leqslant d_n \tag{6.31}$$

such that for every solution $\mathbf{x}$ *of* (6.12) *there is a tuple* $(\mathbf{c}; \mathbf{d}) \in \Gamma$ *with*

$$R^{c_{iv} - \{1/c(n)s\}} < |l_{iv}(\mathbf{x})|_v \leq R^{c_{iv}} \quad \textit{for } v \in S, \; i = 1, \ldots, m_v, \tag{6.32}$$

$$R^{d_j} \leqslant \lambda_j < R^{d_j + \{1/c(n)\}} \quad \textit{for } v \in S, \; j = 1, \ldots, n, \tag{6.33}$$

where

$$c(n) := 4n^3 2^n.$$

Proof. Put

$$u_{iv} := |l_{iv}(\mathbf{x})|_v (R^{-11/10\delta})^{s(v)} \quad \text{for } v \in S,\ i = 1, \dots, m_v.$$

By (6.15),(6.19), (6.18) we have for $v \in S, i = 1, \dots, m_v$,

$$u_{iv} \leqslant |\mathbf{x}|_v (R^{-11/10\delta})^{s(v)} \leqslant (|\Delta_K|^{1/2d} R^{-1/10\delta})^{s(v)} \leqslant 1. \tag{6.34}$$

We call two indices $i, k \in \{1, \dots, m_v\}$ *v-conjugate* if there are $\lambda \in \bar{\mathbb{Q}}^*$, $\sigma \in \mathrm{Gal}(\bar{K}_v/K_v)$ such that $l_{kv} = \lambda\sigma(l_{iv})$; then $|\lambda|_v = 1$ since $|l_{iv}|_v = |l_{kv}|_v = 1$. This implies that $|l_{iv}(\mathbf{x})|_v = |l_{kv}(\mathbf{x})|_v$ whence

$$u_{iv} = u_{kv} \quad \text{for } v \in S \text{ and for } v\text{-conjugate } i, k \in \{1, \dots, m_v\}. \tag{6.35}$$

Further, by the definition of R (cf. (5.20)) and by (6.14) and (6.18) we have

$$\begin{aligned} \prod_{v\in S}\prod_{i=1}^{n} u_{iv} &= \left(\prod_{v\in S}\prod_{i=1}^{n} |l_{iv}(\mathbf{x})|_v\right) R^{-11n/10\delta} \\ &\geqslant R^{-1}\left(\prod_{v\in S} |\det(l_{1v}, \dots, l_{nv})|_v\right) R^{-11n/10\delta} \\ &\geqslant H^{-nsD^n} R^{-1-11n/10\delta} \geqslant R^{-2n/\delta}. \end{aligned} \tag{6.36}$$

(Note that the product is taken over $i = 1, \dots, n$, not over $i = 1, \dots, m_v$). By Lemma 9 (ii), (6.34), (6.36) there is a set Γ_1 of ns-tuples of non-negative reals $\underline{\gamma} = (\gamma_{iv} : v \in S, i = 1, \dots, n)$ independent of $\mathbf{x}$ of cardinality

$$|\Gamma_1| \leqslant (2 + 2e \cdot c(n)\delta^{-1})^{ns}$$

such that for some tuple $\underline{\gamma} \in \Gamma_1$, we have

$$(R^{-2n/\delta})^{\gamma_{iv} + \{\delta/2c(n)ns\}} < u_{iv} \leqslant (R^{-2n/\delta})^{\gamma_{iv}} \quad \text{for } v \in S,\ i = 1, \dots, n. \tag{6.37}$$

For $i = 1, \dots, m_v$, let t_{iv} be the smallest index from $\{1, \dots, n\}$ that is v-conjugate to i. Put

$$\gamma'_{iv} := \gamma_{t_{iv}, v} \quad \text{for } v \in S,\ i = 1, \dots, m_v.$$

Clearly, $t_{iv} = t_{jv}$ and hence $\gamma'_{iv} = \gamma'_{jv}$ whenever i, j are v-conjugate. Together with (6.35) and (6.37) this implies that $(R^{-2n/\delta})^{\gamma'_{iv} + \{\delta/2c(n)ns\}} < u_{iv} \leqslant (R^{-2n/\delta})^{\gamma'_{iv}}$ for $v \in S,\ i = 1, \dots, m_v$. Putting

$$c_{iv} := \frac{11}{10\delta} s(v) - \frac{2n}{\delta} \gamma'_{iv} \quad \text{for } v \in S, i = 1, \dots, m_v,$$

we can rewrite this as

$$R^{c_{iv}-\{1/c(n)s\}} < |l_{iv}(\mathbf{x})|_v \leqslant R^{c_{iv}} \quad \text{for } v \in S,\ i = 1, \ldots, m_v,$$

which is (6.32). Since $\gamma_{iv} \geqslant 0$ we have $c_{iv} \leqslant (11/10\delta)s(v)$ for $v \in S, i = 1, \ldots, m_v$ which is (6.30). Finally, $\mathbf{c} = (c_{iv}\colon v \in S, i = 1, \ldots, m_v)$ depends only on $\underline{\gamma} \in \Gamma_1$. Hence for $\mathbf{c}$ we have at most $|\Gamma_1|$ possibilities.

Define the numbers

$$v_j := R^{-11/10\delta}\lambda_j^{-1} \quad \text{for } j = 1, \ldots, n.$$

By Lemma 16 (ii) and (6.18) we have $\lambda_1 \geqslant R^{-11/10\delta}$. Hence

$$1 \geqslant v_1 \geqslant v_2 \geqslant \cdots \geqslant v_n. \tag{6.38}$$

Further, by Lemma 14 and (6.18),

$$v_1 \cdots v_n = \frac{R^{-11n/10\delta}}{\lambda_1 \cdots \lambda_n} \geqslant R^{-11n/10\delta}\Delta^{-n/2d}R^{-1} > R^{-2n/\delta}. \tag{6.39}$$

By Lemma 9 (ii), (6.38) and (6.39) there is a set Γ_2 of n-tuples of non-negative reals $\underline{\delta} = (\delta_1, \ldots, \delta_n)$ independent of $\mathbf{x}$, of cardinality

$$|\Gamma_2| \leqslant (2 + 2e \cdot c(n)\delta^{-1})^n$$

such that for some tuple $\underline{\delta} \in \Gamma_2$ we have

$$\begin{aligned}(R^{-2n/\delta})^{\delta_j+\{\delta/2c(n)n\}} &< v_j = R^{-11/10\delta}\lambda_j^{-1} \\ &\leqslant (R^{-2n/\delta})^{\delta_j} \quad \text{for} \quad j = 1, \ldots, n.\end{aligned} \tag{6.40}$$

By (6.38), the inequalities (6.40) remain valid after replacing $\delta_1, \ldots, \delta_n$ by

$$\delta_1' := \min(\delta_1, \ldots, \delta_n),\ \delta_2' := \min(\delta_2, \ldots, \delta_n), \ldots, \delta_n' := \delta_n,$$

respectively. Putting

$$d_j := -\frac{11}{10\delta} + \frac{2n}{\delta}\delta_j' \quad \text{for } j = 1, \ldots, n,$$

we infer from (6.40) that

$$R^{d_j} \leqslant \lambda_j < R^{d_j+1/c(n)} \quad \text{for } j = 1, \ldots, n,$$

which is (6.33). From the definitions of $d_1, \ldots, d_n$ it follows at once that $-11/10\delta \leqslant d_1 \leqslant \cdots \leqslant d_n$. Further, since $\mathbf{x} \in \phi^{-1}(\Pi(\mathbf{x}))$ we have $\lambda_1 \leqslant 1$. Hence $d_1 \leqslant 0$. This implies (6.31). Finally, the tuple $\mathbf{d} = (d_1, \ldots, d_n)$ depends only on $\underline{\delta} \in \Gamma_2$. Hence for $\mathbf{d}$ we have at most $|\Gamma_2|$ possibilities. It follows that the number of possibilities for $(\mathbf{c}, \mathbf{d})$ is at most

$$\begin{aligned} |\Gamma_1| \cdot |\Gamma_2| &\leqslant (2 + 2e \cdot c(n)\delta^{-1})^{ns+n} \\ &< (n!)^{-s}\{n(2 + 8en^3 2^n \delta^{-1})\}^{ns+n} \\ &< (n!)^{-s}(30n^4 2^n \delta^{-1})^{ns+n}. \end{aligned}$$

This completes the proof of Lemma 18. □

Let $\mathbf{x}$ be a solution of (6.12) and $(\mathbf{c}; \mathbf{d})$ the corresponding tuple from Lemma 18. Let $\kappa_v(v|\infty)$ be the permutations from Lemma 17. Further, for each finite place $v \in S$, choose $\kappa_v(1), \ldots, \kappa_v(n)$ from $\{1, \ldots, m_v\}$ such that

$$\begin{aligned} &l_{\kappa_v(1)}, \ldots, l_{\kappa_v(n)} \text{ are linearly independent,} \\ &c_{\kappa_v(1),v} + \cdots + c_{\kappa_v(n),v} \text{ is minimal.} \end{aligned} \tag{6.41}$$

Define the linear forms

$$l'_{iv}(\mathbf{X}) := l_{\kappa_v(i),v}(\mathbf{X}) \quad \text{for } v \in S,\ i = 1, \ldots, n$$

and the numbers

$$e_{iv} := c_{\kappa_v(i),v} \quad \text{for } v \in S,\ i = 1, \ldots, n.$$

Thus, for every solution $\mathbf{x}$ of (6.12) we have constructed a tuple

$$\begin{aligned} \mathcal{T} := (&l'_{iv}\colon v \in S, i = 1, \ldots, n;\ e_{iv}\colon v \in S, i = 1, \ldots, n; \\ &d_i\colon i = 1, \ldots, n). \end{aligned} \tag{6.42}$$

By (6.32) we have

$$R^{e_{iv} - \{1/c(n)s\}} < |l'_{iv}(\mathbf{x})|_v \leqslant R^{e_{iv}} \quad \text{for} \quad v \in S,\ i = 1, \ldots, n. \tag{6.43}$$

We recall that

$$R^{d_j} \leqslant \lambda_j < R^{d_j + \{1/c(n)\}} \quad \textit{for } v \in S,\ j = 1, \ldots, n. \tag{6.33}$$

We derive some other properties of $\mathcal{T}$.

LEMMA 19. (i) *$\mathcal{T}$ belongs to a set independent of $\mathbf{x}$ of cardinality at most*

$$C_1 = (30n^4 2^n \delta^{-1})^{ns+1}.$$

(ii) *For $v \in S$, $l'_{1v}, \ldots, l'_{nv}$ are linearly independent linear forms with*

$$H(l'_{iv}) \leqslant H,\ [K(l'_{iv})\colon K] \leqslant D,\ |l'_{iv}|_v = 1 \ \textit{for } i = 1, \ldots, n.$$

(iii) $e_{iv} \leqslant \frac{11}{10\delta}s(v)$ *for* $v \in S, i = 1, \ldots, n$.

(iv) $\sum_{i=1}^{n} \sum_{v \in S} e_{iv} < -1 + (4n^2 2^n)^{-1}$.

(v) $d_1 \leqslant 0, -\frac{11}{10\delta} \leqslant d_1 \leqslant \cdots \leqslant d_n$.

(vi) $1 - (3n^2 2^n)^{-1} < d_1 + \cdots + d_n < 1 + (3n^2 2^n)^{-1}$.

Proof. (i) By (6.41), for finite places $v \in S$ the indices $\kappa_v(i)$, and hence the linear forms l'_{iv} and the numbers e_{iv} are uniquely determined by the tuple $\mathbf{c}$ from Lemma 18. For infinite places v, the linear forms l'_{iv} are uniquely determined by the permutations κ_v of $(1, \ldots, n)$ from Lemma 17, while the numbers e_{iv} depend only on κ_v and $\mathbf{c}$. Therefore, $\mathcal{T}$ is uniquely determined by κ_v $(v|\infty)$ and $(\mathbf{c}; \mathbf{d})$. It follows that for $\mathcal{T}$ we have at most

$$(n!)^r (n!)^{-s} (30n^4 2^n \delta^{-1})^{ns+1} \leqslant C_1$$

possibilities where r is the number of infinite places of K.

(ii) Let $v \in S$, $i = 1, \ldots, n$. From the definition of $l'_{1v}, \ldots, l'_{nv}$ it follows at once that these linear forms are linearly independent. Further, we have $l'_{iv} = l_{jv}$ for some $j \in \{1, \ldots, m_v\}$. Now (ii) follows at once from (6.13).

(iii) This follows at once from (6.30) and the fact that $e_{iv} = c_{jv}$ for some $j \in \{1, \ldots, m_v\}$.

(iv) We recall that $R = \prod_{v \in S} R_v$, where

$$R_v = \frac{|\det(l_{1v}, \ldots, l_{nv})|_v}{|l_{1v}(\mathbf{x}) \cdots l_{nv}(\mathbf{x})|_v} \quad \text{for } v|\infty$$

$$R_v = \frac{|\det(l_{i_1 v}, \ldots, l_{i_n v})|_v}{|l_{i_1 v}(\mathbf{x}) \cdots l_{i_n v}(\mathbf{x})|_v} \quad \text{for } v \in S, v \nmid \infty,$$

where $\{i_1, \ldots, i_n\}$ is a subset of $\{1, \ldots, m_v\}$ for which the right-hand side is maximal.

Fix $v \in S$. First let v be finite. By (6.41) and the definition of $e_{1v}, \ldots, e_{nv}$ we have $c_{i_1 v} + \cdots + c_{i_n v} \geqslant e_{1v} + \cdots + e_{nv}$. Together with (6.32) and (6.14) this gives

$$\begin{aligned} R_v^{-1} &\geqslant |l_{i_1 v}(\mathbf{x}) \cdots l_{i_n v}(\mathbf{x})|_v > R^{c_{i_1 v} + \cdots + c_{i_n v} - \{n/c(n)s\}} \\ &\geqslant R^{e_{1v} + \cdots + e_{nv} - \{n/c(n)s\}}. \end{aligned}$$

If v is infinite then $e_{1v}, \ldots, e_{nv}$ is a permutation of $c_{1v}, \ldots, c_{nv}$ whence by (6.32) we have also

$$\begin{aligned} R_v^{-1} &\geqslant |l_{1v}(\mathbf{x}) \cdots l_{nv}(\mathbf{x})|_v > R^{c_{1v} + \cdots + c_{nv} - \{n/c(n)s\}} \\ &= R^{e_{1v} + \cdots + e_{nv} - \{n/c(n)s\}}. \end{aligned}$$

Hence

$$R^{-1} = \prod_{v \in S} R_v^{-1} > R^{\sum_{i,v} e_{iv} - \{n/c(n)\}},$$

which implies that $\sum_{i,v} e_{iv} < -1 + 1/4n^2 2^n$.

(v) This is (6.31).

(vi) By (6.33), Lemma 14 and (6.18) we have

$$R^{d_1+\cdots+d_n} \leqslant \lambda_1 \cdots \lambda_n < \Delta^{n/2d} R < R^{1+\{1/3n^2 2^n\}}$$

and

$$R^{d_1+\cdots+d_n} > \lambda_1 \cdots \lambda_n R^{-n/c(n)} \geqslant \frac{1}{n!} R^{1-n/c(n)} > R^{1-\{1/3n^2 2^n\}}.$$

This implies (vi). □

LEMMA 20. *There are linearly independent vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n \in O_K^n$ *such that*

$$|l'_{iv}(\mathbf{v}_j)|_v \leqslant G^{s(v)} R^{e_{iv} + s(v) d_{\min(i,j)} + \{s(v)/c(n)\}}$$
$$\text{for } v \in S,\ i, j = 1, \ldots, n, \tag{6.44}$$

where

$$G = |\Delta_K|^{1/2d} H^{nsD^{2n}} (2d|\Delta_K|^{1/2})^{2n}$$

and such that $\mathbf{x}$ *lies in the* K*-vector space generated by* $\mathbf{v}_1, \ldots, \mathbf{v}_r$ *where* r *is the largest integer with* $d_r \leqslant 0$.

Proof. We take the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ from Lemma 17. These belong to O_K^n and are linearly independent. By (6.26), (6.43), (6.31), (6.33) we have, recalling that $l'_{iv} = l_{\kappa_v(i),v}$ for infinite places v and $l'_{iv} = l_{jv}$ for some $j \in \{1, \ldots, m_v\}$ for finite places v,

$$|l'_{iv}(\mathbf{v}_j)|_v \leqslant |l'_{iv}(\mathbf{x})|_v \cdot \{G\lambda_{\min(i,j)}\}^{s(v)}$$
$$\leqslant G^{s(v)} R^{e_{iv} + s(v) d_{\min(i,j)} + \{s(v)/c(n)\}} \quad \text{for } v \in S,\ i, j = 1, \ldots, n,$$

which is (6.44).

Let t be the largest integer with $\lambda_t \leqslant 1$ (which exists since $\mathbf{x} \in \phi^{-1}(\Pi(\mathbf{x}))$, whence $\lambda_1 \leqslant 1$) and let V be the K-vector space generated by $\phi^{-1}(\lambda_t \Pi(\mathbf{x}))$. We have $\mathbf{x} \in V$ since otherwise $\lambda_{t+1} \leqslant 1$. By Lemma 17 we have $\mathbf{v}_1, \ldots, \mathbf{v}_t \in V$. Since $\lambda_{t+1} > 1 \geqslant \lambda_t$ we have $\dim V = t$; hence $\{\mathbf{v}_1, \ldots, \mathbf{v}_t\}$ is a basis of V and therefore $\mathbf{x}$ is in the space generated by $\mathbf{v}_1, \ldots, \mathbf{v}_t$. By (6.33) we have $R^{d_t} \leq \lambda_t \leqslant 1$, whence $d_t \leqslant 0$ and therefore $r \geqslant t$. This proves Lemma 20. □

Let again $\mathbf{x}$ be a solution of (6.12) and let $\mathcal{T}$ be a tuple as in (6.42) for which (6.43), (6.33), Lemma 19 and Lemma 20 hold. Below we construct a tuple $(N, \underline{\gamma}, \hat{\mathcal{L}}; Q)$

satisfying (4.17)–(4.20) of Theorem B such that $(N, \underline{\gamma}, \hat{\mathcal{L}})$ depends only on $\mathcal{T}$. This implies Theorem B since by Lemma 19 (i) the number of possibilities for $\mathcal{T}$ is at most the number C_1 from Theorem B.

Put

$$Q := R^{3n/\delta}. \tag{6.45}$$

Note that by (6.18) we have (4.20), i.e. $Q \geqslant \{\frac{1}{2}(2H\Delta)^{e^T}\}^{3n}$.

There is an integer k with

$$1 \leqslant k \leqslant n-1, \qquad d_{k+1} > 0, \qquad d_{k+1} - d_k > \frac{1}{n^2}. \tag{6.46}$$

Namely, by Lemma 19 (v) we have $d_1 \leqslant 0$ and by Lemma 19 (vi) we have

$$d_n \geqslant \frac{1}{n}(d_1 + \cdots + d_n) > \frac{1}{n} - \frac{1}{3n^3 2^n} > 0.$$

Therefore, there is an $r \in \{1, \ldots, n-1\}$ with $d_r \leqslant 0, d_{r+1} > 0$. Let k be the integer from $\{r, r+1, \ldots, n-1\}$ for which $d_{k+1} - d_k$ is maximal. Then clearly $d_{k+1} > 0$ and

$$\begin{aligned} d_{k+1} - d_k &\geqslant \frac{1}{n-r}\Big\{(d_n - d_{n-1}) + \cdots + (d_{r+1} - d_r)\Big\} \\ &= \frac{1}{n-r}(d_n - d_r) \geqslant \frac{1}{n-1}\left(\frac{1}{n} - \frac{1}{3n^3 2^n}\right) \\ &> \frac{1}{n^2}. \end{aligned}$$

Put

$$N := \binom{n}{k}. \tag{6.47}$$

As before, let $\sigma_1, \ldots, \sigma_N$ be the sequence of subsets of $\{1, \ldots, n\}$ of cardinality $n-k$, ordered lexicographically. Thus, $\sigma_1 = \{1, \ldots, n-k\}, \ldots, \sigma_{N-1} = \{k, k+2, \ldots, n\}, \sigma_N = \{k+1, k+2, \ldots, N\}$. Define the set of linear forms

$$\hat{\mathcal{L}} = \{\hat{l}_{iv} : v \in S,\ i = 1, \ldots, N\} \tag{6.48}$$

with

$$\hat{l}_{iv} := \alpha_{iv} l'_{i_1,v} \wedge \cdots \wedge l'_{i_{n-k},v} \quad \text{for } v \in S,\ i = 1, \ldots, N,$$

where $\{i_1 < \cdots < i_{n-k}\} = \sigma_i$ and $\alpha_{iv} \in \bar{\mathbb{Q}}^*$ is chosen such that

$$|\hat{l}_{iv}|_v = 1 \quad \text{for } v \in S,\ i = 1, \ldots, N. \tag{6.49}$$

By Lemma 19(ii), the fact that $K(\hat{l}_{iv})$ is contained in the composite of the fields $K(l'_{i_1 v}),\dots,K(l'_{i_{n-k}v})$ and by (6.14) we have

$$[K(\hat{l}_{iv})\colon K]\leqslant D^n,\quad H(\hat{l}_{iv})\leqslant H^n \text{ for } v\in S,\ i=1,\dots,N. \tag{6.50}$$

Further, $\hat{l}_{1v},\dots,\hat{l}_{Nv}$ are linearly independent since $l'_{1v},\dots,l'_{nv}$ are linearly independent. Hence $\hat{\mathcal{L}}$ satisfies condition (4.19) of Theorem B. Note that by Lemma 19 (ii), Lemma 2 we have

$$1\leqslant|\alpha_{iv}|_v=|l'_{i_1,v}\wedge\dots\wedge l'_{i_{n-k},v}|_v^{-1}\leqslant H^{nD^n}\quad \text{for } v\in S,\ i=1,\dots,N. \tag{6.51}$$

For $i=1,\dots,N, v\in S$ define the numbers

$$\hat{e}_{iv}:=e_{i_1 v}+\cdots+e_{i_{n-k}v},\quad \hat{d}_i=d_{i_1}+\cdots+d_{i_{n-k}}, \tag{6.52}$$

where again $\{i_1<\cdots<i_{n-k}\}=\sigma_i$. Define the tuple

$$\underline{\gamma}=(\gamma_{iv}\colon v\in S,\ i=1,\dots,N) \tag{6.53}$$

with

$$\gamma_{iv}:=\frac{\delta}{3n}\left\{\frac{1}{c(n)s}+\hat{e}_{iv}+s(v)\left(\hat{d}_i+\frac{n}{c(n)}\right)\right\}\ \text{for } v|\infty,\ i=1,\dots,N-1,$$

$$\gamma_{Nv}:=\frac{\delta}{3n}\left\{\frac{1}{c(n)s}+\hat{e}_{Nv}+s(v)\left(\hat{d}_{N-1}+\frac{n}{c(n)}\right)\right\}\ \text{for } v|\infty,$$

$$\gamma_{iv}:=\frac{\delta}{3n}\left\{\min\left(0,\frac{1}{c(n)s}+\hat{e}_{iv}\right)\right\}\ \text{for } v\in S,\ v\nmid\infty, i=1,\dots,N,$$

where $c(n)=1/4n^3 2^n$. The special choices for γ_{Nv} $(v|\infty)$ will turn out to be crucial. It is easily verified that indeed $(N,\underline{\gamma},\hat{\mathcal{L}})$ depends only on the tuple $\mathcal{T}$ in (6.42).

We show that $\underline{\gamma}$ satisfies (4.18):

LEMMA 21. (i) $\gamma_{iv}\leqslant s(v)$ *for* $v\in S, i=1,\dots,N$.
(ii) $\sum_{v\in S}\sum_{i=1}^N\gamma_{iv}\leqslant-\delta/6n^3$.

Proof. (i) Obviously, $\gamma_{iv}\leqslant 0=s(v)$ if v is finite. Let v be an infinite place and $i\in\{1,\dots,N\}$. First we have $1/s\leqslant 2s(v)$. Second, by Lemma 19 (iii) (with $\sigma_i=\{i_1<\cdots<i_{n-k}\}$),

$$\hat{e}_{iv}=e_{i_1 v}+\cdots+e_{i_{n-k}v}\leqslant(n-k)\frac{11}{10\delta}s(v)\leqslant\frac{11n}{10\delta}s(v).$$

Third, by Lemma 19 (v), (vi),

$$\hat{d}_i=d_{i_1}+\cdots+d_{i_{n-k}}\leqslant d_1+\cdots+d_n-kd_1\leqslant 1+\frac{1}{3n^2 2^n}+\frac{11n}{10\delta}.$$

By inserting this and $1/c(n)s \leqslant 2s(v)/c(n) = 1/2n^3 2^n s(v)$ into (6.53) we obtain

$$\gamma_{iv} \leqslant \frac{\delta}{3n}\left\{\frac{1}{2n^3 2^n} + \frac{11n}{10\delta} + 1 + \frac{1}{3n^2 2^n} + \frac{11n}{10\delta}\right\} s(v) \leqslant s(v).$$

(ii) By (6.53) we have, taking into consideration the special choices for $\gamma_{Nv}(v|\infty)$, $\hat{d}_N - \hat{d}_{N-1} = d_{k+1} - d_k$ by (6.52), and $\sum_{v|\infty} s(v) = \sum_{v\in S} s(v) = 1$,

$$\begin{aligned}
&\sum_{v\in S}\sum_{i=1}^{N} \gamma_{iv} \\
&\leqslant \frac{\delta}{3n}\left\{\sum_{v\in S}\sum_{i=1}^{N}\left(\frac{1}{c(n)s} + \hat{e}_{iv} + s(v)\left(\hat{d}_i + \frac{n}{c(n)}\right)\right)\right. \\
&\qquad \left. -(\hat{d}_N - \hat{d}_{N-1})\left(\sum_{v|\infty} s(v)\right)\right\} \\
&= \frac{\delta}{3n}\left\{\frac{N}{c(n)} + \left(\sum_{v\in S}\sum_{i=1}^{N}\hat{e}_{iv}\right) + (\hat{d}_1 + \cdots + \hat{d}_N)\right. \\
&\qquad \left. + \frac{Nn}{c(n)} - (d_{k+1} - d_k)\right\}. \qquad (6.54)
\end{aligned}$$

Note that by (6.52),

$$\begin{aligned}
\sum_{v\in S}\sum_{i=1}^{N}\hat{e}_{iv} &= \binom{n-1}{k}\left(\sum_{v\in S}\sum_{i=1}^{n} e_{iv}\right), \\
\hat{d}_1 + \cdots + \hat{d}_N &= \binom{n-1}{k}(d_1 + \cdots + d_n).
\end{aligned}$$

Together with Lemma 19 (iv), (vi) this implies that

$$\begin{aligned}
\sum_{v\in S}\sum_{i=1}^{N}\hat{e}_{iv} &\leqslant \binom{n-1}{k}\left(-1 + \frac{1}{4n^2 2^n}\right), \\
\hat{d}_1 + \cdots + \hat{d}_N &\leqslant \binom{n-1}{k}\left(1 + \frac{1}{3n^2 2^n}\right).
\end{aligned}$$

By inserting these inequalities and also (6.46), i.e. $d_{k+1} - d_k > 1/n^2$, and $c(n) = 1/4n^3 2^n$, $N \leqslant 2^{n-1}$, $\binom{n-1}{k} \leqslant 2^{n-2}$ into (6.54) we obtain

$$\sum_{v\in S}\sum_{i=1}^{N}\gamma_{iv}$$

$$< \frac{\delta}{3n}\left\{\frac{(n+1)N}{c(n)} + \binom{n-1}{k}\left(-1+\frac{1}{4n^2 2^n}+1+\frac{1}{3n^2 2^n}\right) - \frac{1}{n^2}\right\}$$
$$\leqslant \frac{\delta}{3n}\left\{\frac{(n+1).2^{n-1}}{4n^3.2^n} + 2^{n-2}.\frac{7}{12n^2.2^n} - \frac{1}{n^2}\right\}$$
$$< -\frac{\delta}{3n}\cdot\frac{1}{2n^2} = -\frac{\delta}{6n^3}.$$

□

We have shown that $(N, \underline{\gamma}, \hat{\mathcal{L}}; Q)$ satisfies (4.20), (4.19), (4.18). We complete the proof of Theorem B by showing that there is a vector space W for which (4.17) holds.

LEMMA 22. *Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be the linearly independent vectors from Lemma 20 and let W be the K-vector space generated by $\mathbf{v}_1, \dots, \mathbf{v}_k$. Then*

$$\dim W = k, \ \ \mathbf{x} \in W, \ \ f_{kn}(W) = V(N, \underline{\gamma}, \hat{\mathcal{L}}; Q).$$

Proof. It is obvious that $\dim W = k$. Further, by (6.46) we have $k \geqslant r$ where r is the largest integer with $d_r \leqslant 0$ and this implies together with Lemma 20 that $\mathbf{x} \in W$.

It remains to prove that $\hat{W} = V$, where $\hat{W} := f_{kn}(W)$, $V := V(N, \underline{\gamma}, \hat{\mathcal{L}}; Q)$. For $i = 1, \dots, N$ define the vector

$$\hat{\mathbf{v}}_i := \mathbf{v}_{i_1} \wedge \cdots \wedge \mathbf{v}_{i_{n-k}},$$

where $\{i_1 < \cdots < i_{n-k}\} = \sigma_i$. Then $\hat{W}$ has basis $\{\hat{\mathbf{v}}_1, \dots, \hat{\mathbf{v}}_{N-1}\}$. Further, V is the K-vector space generated by

$$\begin{aligned}\Pi : &= \Pi(N, \underline{\gamma}, \hat{\mathcal{L}}; Q) \\ &= \{\hat{\mathbf{y}} \in O_S^N : |\hat{l}_{iv}(\hat{\mathbf{y}})|_v \leq Q^{\gamma_{iv}} \quad \text{for } v \in S,\ i = 1, \dots, N\}.\end{aligned}$$

We show that $\hat{\mathbf{v}}_1, \dots, \hat{\mathbf{v}}_{N-1} \in \Pi$ and that every vector $\hat{\mathbf{v}}_0 \in \Pi$ is linearly dependent on $\hat{\mathbf{v}}_1, \dots, \hat{\mathbf{v}}_{N-1}$. This clearly implies that $\hat{W} = V$.

Take $v \in S$ and $i, j \in \{1, \dots, N\}$. Suppose that $\sigma_i = \{i_1, \dots, i_{n-k}\}, \sigma_j = \{j_1, \dots, j_{n-k}\}$. By Laplace's rule (2.3) and by (6.48) we have

$$|\hat{l}_{iv}(\hat{\mathbf{v}}_j)|_v = |\alpha_{iv}|_v |\det((l'_{pv}(\mathbf{v}_q))_{p\in\sigma_i, q\in\sigma_j})_v. \tag{6.55}$$

By (6.51) we have $|\alpha_{iv}|_v \leqslant H^{nD^n}$ and by (6.44), taking the maximum over all permutations κ of σ_j,

$$\begin{aligned}&|\det((l'_{pv}(\mathbf{v}_q))_{p\in\sigma_i, q\in\sigma_j})|_v \\ &\quad \leqslant (n!)^{s(v)} \max_{\kappa} \prod_{t=1}^{n-k} |l_{i_t v}(\mathbf{v}_{\kappa(j_t)})|_v\end{aligned}$$

$$\leqslant (n!)^{s(v)} \max_{\kappa} \prod_{t=1}^{n-k} (G^{s(v)} R^{e_{i_t v} + s(v) d_{\min(i_t, \kappa(j_t))} + s(v)/c(n)}).$$

Together with (6.55) this implies

$$|\hat{l}_{iv}(\hat{\mathbf{v}}_j)|_v \leqslant (n! G^{n-k})^{s(v)} H^{nD^n} R^{\hat{e}_{iv} + (d_0 + n/c(n)) s(v)}, \tag{6.56}$$

where

$$d_0 := \max_{\kappa} \sum_{t=1}^{n-k} d_{\min(i_t, \kappa(j_t))}.$$

Since $d_1 \leqslant \cdots \leqslant d_n$ we have $\hat{d}_1 \leqslant \cdots \leqslant \hat{d_N}$, whence

$$\begin{aligned} d_0 &\leqslant \max_{\kappa} \{\min(d_{i_1} + \cdots + d_{i_{n-k}},\ d_{\kappa(j_1)} + \cdots + d_{\kappa(j_{n-k})})\} \\ &= \min(\hat{d}_i, \hat{d}_j) = \hat{d}_{\min(i,j)}. \end{aligned} \tag{6.57}$$

Further, by (6.18) and $s(v) \leqslant 1$, we have

$$\begin{aligned} (n! G^{n-k})^{s(v)} H^{nD^n} &\leqslant n! \{|\Delta_K|^{1/2d} H^{nsD^{2n}} (2d|\Delta_K|^{1/2})^{2n}\}^{n-k} H^{nD^n} \\ &\leqslant R^{1/c(n)s}. \end{aligned} \tag{6.58}$$

By inserting (6.57) and (6.58) into (6.56) we obtain

$$\begin{aligned} |\hat{l}_{iv}(\hat{\mathbf{v}}_j)|_v &\leqslant R^{\hat{e}_{iv} + \{\hat{d}_{\min(i,j)} + n/c(n)\} s(v) + 1/c(n)s} \\ &\qquad \text{for } v \in S,\ i, j = 1, \ldots, N. \end{aligned} \tag{6.59}$$

We are interested only in $\mathbf{v}_1, \ldots, \mathbf{v}_{N-1}$. (6.59) implies that for infinite places v and for $i = 1, \ldots, N,\ j = 1, \ldots, N-1$,

$$|\hat{l}_{iv}(\hat{\mathbf{v}}_j)|_v \leqslant R^{\hat{e}_{iv} + \{\hat{d}_{\min(i,N-1)} + n/c(n)\} s(v) + 1/c(n)s} = Q^{\gamma_{iv}}. \tag{6.60}$$

Further, since $\mathbf{v}_1, \ldots, \mathbf{v}_n \in O_K^n$ we have $\hat{\mathbf{v}}_j \in O_K^N$ i.e. $|\hat{\mathbf{v}}_j|_v \leqslant 1$ for every finite place v and for $j = 1, \ldots, N-1$. Together with Schwarz' inequality and (6.49) this implies that for finite $v \in S$ and for $i = 1, \ldots, N,\ j = 1, \ldots, N-1$,

$$|\hat{l}_{iv}(\hat{\mathbf{v}}_j)|_v \leqslant |\hat{l}_{iv}|_v |\hat{\mathbf{v}}_j|_v \leqslant 1,$$

which implies, together with (6.59),

$$|\hat{l}_{iv}(\hat{\mathbf{v}}_j)|_v \leqslant R^{\min(0, \hat{e}_{iv} + 1/c(n)s)} = Q^{\gamma_{iv}}.$$

It follows that indeed $\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_{N-1} \in \Pi$.

Take $\hat{\mathbf{v}}_0 \in \Pi$. We show that $\hat{\mathbf{v}}_0$ is linearly dependent on $\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_{N-1}$ or, which is the same, $\det(\hat{\mathbf{v}}_0, \hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_{N-1}) = 0$.

Fix $v \in S$. Then

$$\det(\hat{\mathbf{v}}_0, \ldots, \hat{\mathbf{v}}_{N-1}) = \det(\hat{l}_{1v}, \ldots, \hat{l}_{Nv})^{-1}\beta_v \quad \text{with } \beta_v := \det\big((\hat{l}_{iv}(\hat{\mathbf{v}}_j))_{\substack{1\leqslant i\leqslant N\\ 0\leqslant j\leqslant N-1}}\big). \tag{6.61}$$

By (6.49), (6.50) and Lemma 2 we have

$$|\det(\hat{l}_{1v}, \ldots, \hat{l}_{Nv})|_v^{-1} \leqslant (H^n)^{N(D^n)^N} \leqslant H^{(2D)^{n2^n}}.$$

Further, since $\mathbf{v}_0, \ldots, \mathbf{v}_{N-1} \in \Pi$ we have

$$\begin{aligned} |\beta_v|_v &\leqslant (N!)^{s(v)} \max_{\kappa} |\hat{l}_{1v}(\hat{\mathbf{v}}_{\kappa(0)}) \cdots \hat{l}_{Nv}(\hat{\mathbf{v}}_{\kappa(N)})|_v \\ &\leqslant 2^{n2^n s(v)} Q^{\gamma_{1v}+\cdots+\gamma_{Nv}}, \end{aligned} \tag{6.63}$$

where the maximum is taken over all permutations κ of $0, \ldots, N-1$. By combining (6.61), (6.62), (6.63) we obtain

$$|\det(\hat{\mathbf{v}}_0, \ldots, \hat{\mathbf{v}}_{N-1})|_v \leqslant (2H)^{(2D)^{n2^n}} Q^{\gamma_{1v}+\cdots+\gamma_{Nv}}.$$

By taking the product over $v \in S$ and using Lemma 21 (ii) and (4.20) we obtain

$$\begin{aligned} \prod_{v\in S} |\det(\hat{\mathbf{v}}_0, \ldots, \hat{\mathbf{v}}_{N-1})|_v &\leqslant (2H)^{s(2D)^{n2^n}} Q^{\sum_{v\in S}\sum_{i=1}^{N}\gamma_{iv}} \\ &\leqslant (2H)^{s(2D)^{n2^n}} Q^{-\delta/6n^3} < 1. \end{aligned}$$

But since $\hat{\mathbf{v}}_0, \ldots, \hat{\mathbf{v}}_{N-1} \in \Pi$ we have $\hat{\mathbf{v}}_j \in O_S^N$ for $j = 0, \ldots, N-1$, whence $\det(\hat{\mathbf{v}}_0, \ldots, \hat{\mathbf{v}}_{N-1})$ is an S-integer. Recalling that by the Product formula, $\prod_{v\in S} |a|_v \geqslant 1$ for every non-zero S-integer a, we infer that

$$\det(\hat{\mathbf{v}}_0, \ldots, \hat{\mathbf{v}}_{N-1}) = 0.$$

This completes the proof of Lemma 22 and hence of Theorem B. □

7. Non-vanishing Results

We derive a non-vanishing result for polynomials which is crucial in the proof of Theorem C.

Let m, N be integers $\geqslant 2$. For $h = 1, \ldots, m$ denote by $\mathbf{X}_h$ the block of N variables $(X_{h1}, \ldots, X_{hN})$. $\bar{\mathbb{Q}}[\mathbf{X}_1, \ldots, \mathbf{X}_m]$ is the ring of polynomials in the mN variables $\mathbf{X}_1, \ldots, \mathbf{X}_m$ with coefficients from $\bar{\mathbb{Q}}$. We use $\mathbf{i}$ to denote a tuple of non-negative integers $(i_{hj}: h = 1, \ldots, m,\ j = 1, \ldots, N)$. For such a tuple $\mathbf{i}$ we define the partial derivative of $F \in \bar{\mathbb{Q}}[\mathbf{X}_1, \ldots, \mathbf{X}_m]$,

$$F_{\mathbf{i}} := \prod_{h=1}^{m}\prod_{j=1}^{N}\left(\frac{1}{i_{hj}!}\frac{\partial^{i_{hj}}}{\partial X_{hj}^{i_{hj}}}\right)F.$$

Let $\mathbf{d} = (d_1, \dots, d_m)$ be a tuple of positive integers and for a tuple $\mathbf{i}$ as above, put

$$(\mathbf{i}/\mathbf{d}) := \sum_{h=1}^{m} \frac{1}{d_h}\{i_{h1} + \cdots + i_{hN}\}.$$

DEFINITION. Let $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_m) \in \bar{\mathbb{Q}}^{mN}$, where $\mathbf{x}_h = (x_{h1}, \dots, x_{hN}) \in \bar{\mathbb{Q}}^N$ and let $F \in \bar{\mathbb{Q}}[\mathbf{X}_1, \dots, \mathbf{X}_m]$. If $F \neq 0$ then the *index* of F at $\mathbf{x}$ with respect to $\mathbf{d}$, notation $\mathrm{Ind}_{\mathbf{x},\mathbf{d}}(F)$, is defined as the largest number σ such that

$$F_{\mathbf{i}}(\mathbf{x}) = 0 \ \text{ for all } \mathbf{i} \text{ with } (\mathbf{i}/\mathbf{d}) \leqslant \sigma;$$

if $F = 0$ then we define $\mathrm{Ind}_{\mathbf{x},\mathbf{d}}(F) = \infty$. It is easy to verify that for $F, G \in \bar{\mathbb{Q}}[\mathbf{X}_1, \dots, \mathbf{X}_m]$, $\mathbf{x} \in \bar{\mathbb{Q}}^{mN}$ we have

$$\mathrm{Ind}_{\mathbf{x},\mathbf{d}}(FG) = \mathrm{Ind}_{\mathbf{x},\mathbf{d}}(F) + \mathrm{Ind}_{\mathbf{x},\mathbf{d}}(G). \tag{7.1}$$

We say that $F \in \bar{\mathbb{Q}}[\mathbf{X}_1, \dots, \mathbf{X}_m]$ is *homogeneous of degree* d_h *in* $\mathbf{X}_h$ for $h = 1, \dots, m$ if F is a linear combination of monomials

$$\prod_{h=1}^{m} \prod_{j=1}^{N} X_{hj}^{i_{hj}} \quad \text{with } i_{h1} + \cdots + i_{hN} = d_h \text{ for } h = 1, \dots, m.$$

For a tuple of positive integers $\mathbf{d} = (d_1, \dots, d_m)$, let $\Gamma_N(\mathbf{d})$ be the set of polynomials $F \in \bar{\mathbb{Q}}[\mathbf{X}_1, \dots, \mathbf{X}_m]$ homogeneous of degree d_h in $\mathbf{X}_h$ for $h = 1, \dots, m$.

For $F \in \bar{\mathbb{Q}}[\mathbf{X}_1, \dots, \mathbf{X}_m]$, define the height

$$H(F) := H(\mathbf{a}_F),$$

where $\mathbf{a}_F$ is the vector of coefficients of F. Further, for a number field K and a place v on K, put

$$|F|_v := |\mathbf{a}_F|_v;$$

thus, if F has its coefficients in K, then $H(F) = \prod_{v \in M_K} |F|_v$. We have

$$\begin{aligned} &H(F_{\mathbf{i}}) \leqslant 2^{d_1 + \cdots + d_m} H(F), \\ &|F_{\mathbf{i}}|_v \leqslant 2^{(d_1 + \cdots + d_m)s(v)} |F|_v \ \text{ for } F \in \Gamma_N(\mathbf{d}), \end{aligned} \tag{7.2}$$

since $F_{\mathbf{i}}$ is obtained by multiplying the coefficients of F by certain products $\binom{d_1}{j_1} \cdots \binom{d_m}{j_m}$.

We recall Theorem 3 of [6] ($e = 2.7182\dots$):

LEMMA 23. (Roth's lemma). *Let m be an integer $\geqslant 2$, $\mathbf{d} = (d_1, \dots, d_m)$ a tuple of positive integers and Θ a real with $0 < \Theta \leqslant 1$. Suppose that*

$$\frac{d_h}{d_{h+1}} \geqslant \frac{2m^2}{\Theta} \quad \textit{for } h = 1, \dots, m-1. \tag{7.3}$$

Further, let $F \in \bar{\mathbb{Q}}[X_{11}, X_{12}, \ldots, X_{m1}, X_{m2}]$ *be a non-zero polynomial in* $2m$ *variables which is homogeneous of degree* d_h *in* (X_{h1}, X_{h2}) *for* $h = 1, \ldots, m$ *and let* $\mathbf{x}_h = (x_{h1}, x_{h2})$ $(h = 1, \ldots, m)$ *be non-zero elements of* $\bar{\mathbb{Q}}^2$ *with*

$$H(\mathbf{x}_h)^{d_h} \geqslant \{e^{d_1+\cdots+d_m} H(F)\}^{(3m^2/\Theta)^m} \quad \text{for} \quad h = 1, \ldots, m. \tag{7.4}$$

Then F *has index* $< m\Theta$ *at* $\mathbf{x} = (x_1, \ldots, x_m)$ *w.r.t.* $\mathbf{d}$.

We need a generalisation of this for polynomials in $\Gamma_N(\mathbf{d})$ where $N \geqslant 2$. The next non-vanishing result is a sharpening of a result of Schmidt, cf. [18], p. 191, Theorem 10B. The height of an $(N-1)$-dimensional linear subspace of $\bar{\mathbb{Q}}^N$

$$\begin{aligned} V &= \{\mathbf{x} \in \bar{\mathbb{Q}}^N : a_1 x_1 + \cdots + a_N x_N = 0\} \\ &\quad \text{with } \mathbf{a} = (a_1, \ldots, a_N) \in \bar{\mathbb{Q}}^N \backslash \{\mathbf{0}\} \end{aligned}$$

is defined by

$$H(V) := H(\mathbf{a}). \tag{7.5}$$

LEMMA 24. *Let* m, N *be integers* $\geqslant 2$, $\mathbf{d} = (d_1, \ldots, d_m)$ *a tuple of positive integers and* Θ *a real with* $0 < \Theta \leqslant 1$. *Suppose again that*

$$\frac{d_h}{d_{h+1}} \geqslant \frac{2m^2}{\Theta} \quad \text{for } h = 1, \ldots, m-1. \tag{7.6}$$

Further, let F *be a non-zero polynomial from* $\Gamma_N(\mathbf{d})$ *and let* $V_1, \ldots, V_m$ *be* $(N-1)$-*dimensional linear subspaces of* $\mathbb{Q}$ *with*

$$H(V_h)^{d_h} \geqslant \left\{e^{d_1+\cdots+d_m} H(F)\right\}^{(N-1)(3m^2/\Theta)^m} \qquad \text{for } h = 1, \ldots, m. \tag{7.7}$$

Then there is a $\mathbf{x}_h \in V_h$ *for* $h = 1, \ldots, m$ *such that for* $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_m)$ *we have*

$$\mathrm{Ind}_{\mathbf{x},\mathbf{d}}(F) < m\Theta. \tag{7.8}$$

Proof. For $N = 2$ this is precisely Lemma 23 (note that the space $V_h = \{\lambda \mathbf{x}_h : \lambda \in \bar{\mathbb{Q}}\}$ has height $H(V_h) = H(\mathbf{x}_h)$) so we assume that $N \geqslant 3$. We use Schmidt's argument [18], pp. 192-194 to reduce this to $N = 2$.

Suppose that

$$\begin{aligned} V_h &= \{\mathbf{x} \in \bar{\mathbb{Q}}^N : b_{h1} x_1 + \cdots + b_{hN} x_N = 0\} \\ &\quad \text{where } \mathbf{b}_h = (b_{h1}, \ldots, b_{hN}) \in \mathbb{Q}^N \backslash \{\mathbf{0}\}. \end{aligned}$$

After permuting the variables if need be, we may assume that

$$b_{h1} \neq 0 \quad \text{for } h = 1, \ldots, m. \tag{7.9}$$

Let K be a number field containing b_{hj} for $h = 1, \dots, m,\ j = 1, \dots, N$. Put

$$\mathbf{c}_h := (1, b_{h2}/b_{h1}, \dots, b_{hN}/b_{h1}), \qquad \mathbf{c}_{hj} := (1, b_{hj}/b_{h1})$$

for $h = 1, \dots, m, j = 1, \dots, N$. Then

$$\begin{aligned} H(V_h) = H(\mathbf{c}_h) &= \prod_{v \in M_K} |\mathbf{c}_h|_v \leqslant \prod_{v \in M_K} (|\mathbf{c}_{h2}|_v \cdots |\mathbf{c}_{hN}|_v) \\ &= H(\mathbf{c}_{h2}) \cdots H(\mathbf{c}_{hN}). \end{aligned}$$

Hence, again after a permutation of the variables if necessary we may assume that

$$H(b_{h1}, b_{h2}) = H(\mathbf{c}_{h2}) \geqslant H(V_h)^{1/(N-1)} \quad \text{for } h = 1, \dots, m. \tag{7.10}$$

Now suppose that there are no $\mathbf{x}_h \in V_h$ $(h = 1, \dots, m)$ with (7.8). The idea is to arrive at a contradiction by applying Lemma 23 to $F^* := F(X_{11}, X_{12}, 0, \dots, 0; \dots; X_{m1}, X_{m2}, 0, \dots, 0)$ but this fails if $F^* = 0$. Therefore we proceed completely similarly to Schmidt [18], pp. 192–194. Since our terminology is different, we give the argument for convenience of the reader.

Let I be the set of tuples

$$\mathbf{i} = (i_{11}, i_{12}, 0, \dots, 0; \dots; i_{m1}, i_{m2}, 0, \dots, 0) \quad \text{with } (\mathbf{i}/\mathbf{d}) \leqslant m\Theta. \tag{7.11}$$

We write $\mathbf{x} \in \bar{\mathbb{Q}}^{mN}$ as $(\mathbf{x}_1, \dots, \mathbf{x}_m)$ where $\mathbf{x}_h = (x_{h1}, \dots, x_{hN})$ for $h = 1, \dots, m$. For each $\mathbf{i} \in I$, $F_{\mathbf{i}}$ vanishes identically on the vector space

$$V := V_1 \times \cdots \times V_m = \{\mathbf{x} \in \bar{\mathbb{Q}}^{mN} \colon l_h(\mathbf{x}) = 0 \text{ for } h = 1, \dots, m\},$$

where

$$l_h = b_{h1} X_{h1} + \cdots + b_{hN} X_{hN}.$$

We use that the linear forms

$$X_{hj}\ (h = 1, \dots, m, j = 3, \dots, N) \text{ are linearly independent on } V. \tag{7.12}$$

Namely, otherwise we have an identity

$$\sum_{h=1}^{m} \sum_{j=3}^{N} \alpha_{hj} X_{hj} = \sum_{h=1}^{m} \beta_h l_h = \sum_{h=1}^{m} \beta_h (b_{h1} X_{h1} + \cdots + b_{hm} X_{hm})$$

for certain $\alpha_{hj} \in \bar{\mathbb{Q}}$, not all zero, and certain $\beta_h \in \bar{\mathbb{Q}}$, not all zero, but this is impossible by (7.9).

Rename the variables X_{hj} $(h = 1, \dots, m, j = 1, 2)$ as $Y_1, \dots, Y_{2m}$ and the variables Y_{hj} $(h = 1, \dots, m, j = 3, \dots, N)$ as $Y_{2m+1}, \dots, Y_{mN}$. We can express F as

$$F = Y_{mN}^{s_0} (F^{(1)}(Y_1, \dots, Y_{mN-1}) + Y_{mN} G^{(1)}(Y_1, \dots, Y_{mN}))$$

where $s_0 \geqslant 0$, $F^{(1)} \in \bar{\mathbb{Q}}[Y_1, \ldots, Y_{mN-1}]$ is non-zero and $G^{(1)} \in \bar{\mathbb{Q}}[Y_1, \ldots, Y_{mN}]$. The coefficients of $F^{(1)}$ are among the coefficients of F, hence $H(F^{(1)}) \leqslant H(F)$. By (7.11), for each $\mathbf{i} \in I$ $F_{\mathbf{i}}$ is obtained by partially differentiating F to variables from $Y_1, \ldots, Y_{2m}$. Therefore,

$$F_{\mathbf{i}} = Y_{mN}^{s_0}(F_{\mathbf{i}}^{(1)} + Y_{mN}G_{\mathbf{i}}^{(1)}) \quad \text{for } \mathbf{i} \in I.$$

Each $F_{\mathbf{i}}$ ($\mathbf{i} \in I$) vanishes identically on V whereas by (7.12) Y_{mN} does not vanish identically on V; hence $F_{\mathbf{i}}^{(1)} + Y_{mN}G_{\mathbf{i}}^{(1)}$ vanishes identically on V. But then $F_{\mathbf{i}}^{(1)}$ vanishes identically on $V_1 := V \cap (Y_{mN} = 0)$ for $\mathbf{i} \in I$.

Similarly, $F^{(1)}$ can be expressed as

$$F^{(1)} = Y_{mN-1}^{s_1}(F^{(2)}(Y_1, \ldots, Y_{mN-2}) + Y_{mN-1}G^{(2)}(Y_1, \ldots, Y_{mN-1})),$$

with $F^{(2)} \neq 0$, $H(F^{(2)}) \leqslant H(F^{(1)}) \leqslant H(F)$, and we have

$$F_{\mathbf{i}}^{(1)} = Y_{mN-1}^{s_1}(F_{\mathbf{i}}^{(2)} + Y_{mN-1}G_{\mathbf{i}}^{(2)}) \quad \text{for } \mathbf{i} \in I.$$

Each $F_{\mathbf{i}}^{(1)}$ ($\mathbf{i} \in I$) vanishes identically on V_1 and by (7.12) Y_{mN-1} does not vanish identically on V_1. Hence we may conclude as above that for each $\mathbf{i} \in I$, $F_{\mathbf{i}}^{(2)}$ vanishes identically on $V_2 := V_1 \cap (Y_{mN-1} = 0) = V \cap (Y_{mN-1} = Y_{mN} = 0)$.

Continuing like this we arrive at a non-zero polynomial $F^{(m(N-2))}(Y_1, \ldots, Y_{2m})$ with $H(F^{(m(N-2))}) \leqslant H(F)$ such that for each $\mathbf{i} \in I$, $F_{\mathbf{i}}^{(m(N-2))}$ vanishes identically on

$$\begin{aligned} V_{m(N-2)} &= V \cap (Y_{2M+1} = \cdots = Y_{mN} = 0) \\ &= \{\mathbf{x} \in \bar{\mathbb{Q}}^{mn} : b_{h1}x_{h1} + b_{h2}x_{h2} = 0, \\ &\qquad x_{hj} = 0 \text{ for } h = 1, \ldots, m,\ j = 3, \ldots, N\}. \end{aligned}$$

This means that

$$\mathrm{Ind}_{\mathbf{x},\mathbf{d}}(F^{(m(N-2))}) \geqslant m\Theta \text{ for every } \mathbf{x} \in V_{m(N-2)}. \tag{7.13}$$

Define

$$\begin{aligned} V^* &:= \{(x_{11}, x_{12}; \ldots; x_{m1}, x_{m2}) \in \bar{\mathbb{Q}}^{2m} : b_{h1}x_{h1} + b_{h2}x_{h2} = 0 \\ &\qquad \text{for } h = 1, \ldots, m\}, \\ &F^*(X_{11}, X_{12}, \ldots, X_{m1}, X_{m2}) := \left(\prod_{h=1}^{m} X_{h1}^{a_h}\right) F^{(m(N-2))}, \end{aligned}$$

where $a_h \in \mathbb{Z}_{\geqslant 0}$ is chosen such that F^* is homogeneous of degree d_h in (X_{h1}, X_{h2}) for $h = 1, \ldots, m$. By (7.1), (7.13) F^* has index $\geqslant m\Theta$ w.r.t $\mathbf{d}$ at each point of V^*, so in particular at the point

$$\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, \mathbf{x}_m^*) \quad \text{with } \mathbf{x}_h^* = (b_{h2}, -b_{h1}) \text{ for } h = 1, \ldots, m.$$

We have $H(F^*) = H(F^{(m(N-2))}) \leqslant H(F)$. Together with (7.10), (7.7) this implies that

$$H(\mathbf{x}_h^*)^{d_h} = H(b_{h1}, b_{h2})^{d_h} \geqslant H(V_h)^{d_h/(N-1)} \geqslant \{e^{d_1+\cdots+d_m} H(F)\}^{(3m^2/\Theta)^m}$$
$$\geqslant \{e^{d_1+\cdots+d_m} H(F^*)\}^{(3m^2/\Theta)^m}$$

which is condition (7.4) of Lemma 23. Further, condition (7.3) of Lemma 23 follows from (7.6). This implies that F^* has index $< m\Theta$ at $\mathbf{x}^*$, contrary to what we showed above. Thus, the assumption that Lemma 24 is false leads to a contradiction. This completes our proof. □

We need another simple non-vanishing result which is a special case of [18], p. 184, Lemma 8A. For convenience of the reader we give a short proof.

LEMMA 25. *Let K be a field of characteristic* 0 *and $F \in K[X_1, \ldots, X_r]$ a non-zero polynomial with* $\deg_{X_i} F \leqslant s_i$ *for $i = 1, \ldots, r$. Further, let $B_1, \ldots, B_r$ be positive reals. Then there are rational integers $x_1, \ldots, x_r, i_1, \ldots, i_r$ with*

$$|x_j| \leqslant B_j,\ 0 \leqslant i_j \leqslant s_j/B_j \quad \text{for } j = 1, \ldots, r, \tag{7.14}$$

$$\frac{\partial^{i_1+\cdots+i_r}}{\partial X_1^{i_1} \cdots \partial X_r^{i_r}} F\,(x_1, \ldots, x_r) \neq 0. \tag{7.15}$$

Proof. We proceed by induction on r. First let $r = 1$ and put $a := [B_1]$, $b := [s_1/B_1]$. F cannot be divisible by $\prod_{j=-a}^{a}(X - j)^{b+1}$ which is a polynomial of degree $(2a + 1)(2b + 1) > s_1 = \deg F$. Therefore there are integers x_1, i_1 with $|x_1| \leqslant a \leqslant B_1$, $0 \leqslant i_1 \leqslant b \leqslant s_1/B_1$ such that $(\mathrm{d}/\mathrm{d}X_1)^{i_1} F(x_1) \neq 0$.

Now suppose that $r \geqslant 2$ and that Lemma 25 holds for polynomials in fewer than r variables. By applying Lemma 25 with $r = 1$ and the field $K(X_2, \ldots, X_r)$ replacing K it follows that there are integers x_1, i_1 with $|x_1| \leqslant B_1, 0 \leqslant i_1 \leqslant s_1/B_1$ such that

$$G(X_2, \ldots, X_r) := \left(\frac{\partial^{i_1}}{\partial X_1^{i_1}} F\right)(x_1, X_2, \ldots, X_r) \not\equiv 0.$$

Now the induction hypothesis applied to G implies that there are rational integers $x_2, \ldots, x_r$, $i_2, \ldots, i_r$ with $|x_j| \leqslant B_j$, $0 \leqslant i_j \leqslant (\deg_{X_j} G)/B_j \leqslant s_j/B_j$ for $j = 2, \ldots, r$, such that

$$\left(\frac{\partial^{i_2+\cdots+i_r}}{\partial X_2^{i_2} \cdots \partial X_r^{i_r}} G\right)(x_2, \ldots, x_r) \neq 0.$$

This implies (7.15). □

Let V be an $(N-1)$-dimensional linear subspace of $\bar{\mathbb{Q}}^N$. A *grid of size* A in V is a set

$$\Gamma = \{x_1\mathbf{a}_1 + \cdots + x_{N-1}\mathbf{a}_{N-1} : x_1, \dots, x_{N-1} \in \mathbb{Z},\ |x_i| \leqslant A \\ \text{for } i = 1, \dots, N-1\},$$

where $\{\mathbf{a}_1, \dots, \mathbf{a}_{N-1}\}$ is any basis of V. We call $\{\mathbf{a}_1, \dots, \mathbf{a}_{N-1}\}$ also a basis of Γ. The next lemma is our final non-vanishing result:

LEMMA 26. *Let $m, N, d_1, \dots, d_m, F, V_1, \dots, V_m, \Theta$ have the meaning of Lemma 24 and satisfy the conditions of Lemma 24, i.e. $m, N \geqslant 2$, $0 < \Theta \leqslant 1$, (7.6) and (7.7). Further, for $h = 1, \dots, m$, let Γ_h be any grid in V_h of size N/Θ. Then there are $\mathbf{x}_1 \in \Gamma_1, \dots, \mathbf{x}_m \in \Gamma_m$ such that for $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_m)$ we have*

$$\mathrm{Ind}_{\mathbf{x},\mathbf{d}}(F) < 2m\Theta.$$

Proof. For $h = 1, \dots, m$ let $\{\mathbf{a}_{h1}, \dots, \mathbf{a}_{h,N-1}\}$ be a basis of Γ_h. By Lemma 24 there is a tuple $\mathbf{i} = (i_{11}, \dots, i_{mN})$ of non-negative integers with $(\mathbf{i}/\mathbf{d}) < m\Theta$, such that $F_{\mathbf{i}}$ does not vanish identically on $V_1 \times \cdots \times V_m$. But then, the polynomial

$$G(Y_{11}, \dots, Y_{m,N-1}) := F_{\mathbf{i}}\left(\sum_{j=1}^{N-1} Y_{1j}\mathbf{a}_{1j}, \dots, \sum_{j=1}^{N-1} Y_{mj}\mathbf{a}_{mj}\right)$$

is not identically zero. Since G is of degree $\leqslant d_h$ in the variable Y_{hj} and by Lemma 25, there are integers y_{hj}, k_{hj} with

$$|y_{hj}| \leqslant N/\Theta,\ 0 \leqslant k_{hj} \leqslant d_h\Theta/N \\ \text{for } h = 1, \dots, m,\ j = 1, \dots, N-1,$$

such that

$$g := \left(\prod_{h=1}^{m}\prod_{j=1}^{N-1}\frac{\partial^{k_{hj}}}{\partial Y_{hj}^{k_{hj}}}\right) G(y_{11}, \dots, y_{m,N-1}) \neq 0.$$

Put

$$\mathbf{x}_h := \sum_{j=1}^{N-1} y_{hj}\mathbf{a}_{hj} \quad \text{for } h = 1, \dots, m.$$

Then $\mathbf{x}_h \in \Gamma_h$ for $h = 1, \dots, m$. Further, g is a linear combination with algebraic coefficients of numbers $F_{\mathbf{i}+\mathbf{e}}(\mathbf{x})$, where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_m)$ and $\mathbf{e}$ is a tuple of non-negative integers $(e_1, \dots, e_{m,N})$ with

$$\sum_{j=1}^{N} e_{hj} \leqslant \sum_{j=1}^{N-1} k_{hj} \quad \text{for } h = 1, \dots, m.$$

Hence there is such a tuple $\mathbf{e}$ with $F_{\mathbf{i}+\mathbf{e}}(\mathbf{x}) \neq 0$. Together with (7.16) this implies that

$$\begin{aligned} \mathrm{Ind}_{\mathbf{x},\mathbf{d}}(F) &< ((\mathbf{i}+\mathbf{e})/\mathbf{d}) = (\mathbf{i}/\mathbf{d}) + (\mathbf{e}/\mathbf{d}) \\ &\leqslant m\Theta + \sum_{h=1}^{m} \frac{1}{d_h} \sum_{j=1}^{N} e_{hj} \leqslant m\Theta + \sum_{h=1}^{m} \frac{1}{d_h} \sum_{j=1}^{N-1} k_{hj} \\ &\leqslant m\Theta + \sum_{h=1}^{m} \left(\frac{1}{d_h}(N-1) d_h \Theta / N \right) < 2m\Theta. \end{aligned}$$

This completes the proof of Lemma 26. □

8. Auxiliary Results for the Proof of Theorem C

We use the notation from Theorem C. Thus, K is a number field of degree d, S a finite set of places on K of cardinality s containing all infinite places, ϵ a real with $0 < \epsilon < 1$, N an integer $\geqslant 2$, $\underline{\gamma} = (\gamma_{iv} : v \in S, i = 1, \ldots, N)$ a tuple of reals with

$$\gamma_{iv} \leqslant s(v) \quad \text{for } v \in S,\ i = 1, \ldots, N, \qquad \sum_{v \in S} \sum_{i=1}^{N} \gamma_{iv} \leqslant -\epsilon, \tag{4.21}$$

and $\hat{\mathcal{L}} = \{\hat{l}_{iv} : v \in S, i = 1, \ldots, N\}$ a system of linear forms in N variables with algebraic coefficients, such that for each $v \in S$, $\{\hat{l}_{1v}, \ldots, \hat{l}_{Nv}\}$ is linearly independent and such that

$$H(\hat{l}_{iv}) \leqslant \hat{H},\ \ [K(\hat{l}_{iv}) : K] \leqslant \hat{D},\ \ |\hat{l}_{iv}|_v = 1 \text{ for } v \in S,\ i = 1, \ldots, N. \tag{4.22}$$

We shall frequently use that by Lemma 2,

$$\hat{H}^{-N\hat{D}^N} \leqslant |\det(\hat{l}_{1v}, \ldots, \hat{l}_{Nv})|_v \leqslant 1 \quad \text{for } v \in S. \tag{8.1}$$

As the tuple $(N, \underline{\gamma}, \hat{\mathcal{L}})$ will be kept fixed, we write $\Pi(Q)$, $V(Q)$ for $\Pi(N, \underline{\gamma}, \hat{\mathcal{L}}; Q)$, $V(N, \underline{\gamma}, \hat{\mathcal{L}}; Q)$ respectively. Thus,

$$\Pi(Q) = \{\mathbf{y} \in O_S^N : |\hat{l}_{iv}(\mathbf{y})|_v \leqslant Q^{\gamma_{iv}} \quad \text{for } v \in S,\ i = 1, \ldots, N\}$$

and $V(Q)$ is the K-vector space generated by $\Pi(Q)$. We assume that Q satisfies

$$\dim_K V(Q) = N - 1, \tag{4.23}$$

$$Q > (2\hat{H})^{e^{C_2}}, \quad \text{with } C_2 = 2^{30} N^8 s^2 \epsilon^{-4} \log 4\hat{D} \cdot \log\log 4\hat{D}. \tag{4.24}$$

Our first auxiliary result is an inequality between Q and the height $H(V(Q))$ of $V(Q)$. Our proof is similar to Schmidt [19], Lemma 7.3 except that we do not use reciprocal parallelepipeds.

LEMMA 27. *There is an* $(N-1)$*-dimensional linear subspace* V *of* K^N *with the following property:*
for every Q *with* (4.23), (4.24) *we have*

$$V(Q) = V$$

or

$$H(V(Q)) \geqslant Q^{\epsilon/3s\hat{D}^N}. \tag{8.2}$$

Proof. Fix Q with (4.23), (4.24). By (4.23), there are linearly independent vectors $\mathbf{g}_1, \dots, \mathbf{g}_{N-1}$ in $\Pi(Q)$. Put

$$\mathbf{g}^* := (\mathbf{g}_1 \wedge \cdots \wedge \mathbf{g}_{N-1})^*$$

(cf. Section 2). Then by (2.1),

$$V(Q) = \{\mathbf{x} \in K^N : \mathbf{g}^* \cdot \mathbf{x} = 0\}. \tag{8.3}$$

Define the linear forms

$$l_{kv}^* = (\hat{l}_{1v} \wedge \cdots \wedge \hat{l}_{k-1,v} \wedge \hat{l}_{k+1,v} \wedge \cdots \wedge \hat{l}_{Nv})^* \tag{8.4}$$

and put

$$D_{kv} := l_{kv}^*(\mathbf{g}^*) \quad \text{for } v \in S,\ k = 1, \dots, N. \tag{8.5}$$

By Laplace's rule (2.3) we have

$$D_{kv} = \det\left((l_{iv}(\mathbf{g}_j))_{\substack{1 \leqslant i \leqslant N, i \neq k \\ 1 \leqslant j \leqslant N-1}} \right) \quad \text{for } v \in S,\ k = 1, \dots, N.$$

Since $\mathbf{g}_1, \dots, \mathbf{g}_{N-1} \in \Pi(Q)$ we have $|l_{iv}(\mathbf{g}_j)|_v \leqslant Q^{\gamma_{iv}}$ for $v \in S$, $i = 1, \dots, N$, $j = 1, \dots, N-1$. Hence

$$\begin{aligned} |D_{kv}|_v &\leqslant (N!)^{s(v)} \max_{\kappa} \prod_{i \neq k} |l_{iv}(\mathbf{g}_{\kappa(i)})|_v \\ &\leqslant (N!)^{s(v)} Q^{\gamma_{1v} + \cdots + \gamma_{Nv} - \gamma_{kv}} \quad \text{for } v \in S,\ k = 1, \dots, N, \end{aligned} \tag{8.6}$$

where the maximum is taken over all bijective mappings κ from $\{1, \dots, N\} \backslash \{k\}$ to $\{1, \dots, , N-1\}$.

Suppose for the moment that there is a tuple $(i_v : v \in S)$ with

$$i_v \in \{1, \dots, N\},\ D_{i_v, v} \neq 0 \ \text{ for } v \in S, \tag{8.7}$$

$$\sum_{v \in S} \gamma_{i_v, v} \geqslant -\frac{\epsilon}{2}. \tag{8.8}$$

By (8.6), (4.21) and (8.8) we have

$$\prod_{v\in S} |D_{i_v,v}|_v \leqslant N!Q^{\left(\sum_{v\in S}\sum_{i=1}^{N}\gamma_{iv}\right)-\left(\sum_{v\in S}\gamma_{i_v,v}\right)} \leqslant N!Q^{-\epsilon/2}. \tag{8.9}$$

We estimate the left-hand side of (8.9) from below. Fix $v \in S$ and put $k := i_v$. Choose $\lambda \in \bar{\mathbb{Q}}^*$ such that the linear form λl^*_{kv} has its coefficients in the field $K(l^*_{kv}) = : L$. There is a place w on L such that $|x|_v = |x|_w^g$ for $x \in L$, where by (4.22) we have

$$1 \leqslant g \leqslant [L\colon K] \leqslant \hat{D}^{N-1}.$$

Note that by (8.7) we have $\lambda D_{kv} \in L^*$. Now the Product formula applied to λD_{kv} and Schwarz' inequality applied to (8.5) give

$$\begin{aligned}
1 &= \left(\prod_{q\in M_L} |\lambda D_{kv}|_q\right)^g = |\lambda D_{kv}|_w^g \left(\prod_{q\neq w} |\lambda D_{kv}|_q\right)^g \\
&\leqslant |\lambda D_{kv}|_w^g \left(\prod_{q\neq w} |\lambda l^*_{kv}|_q \cdot |\mathbf{g}^*|_q\right)^g \\
&= \left(\frac{|\lambda D_{kv}|_w}{|\lambda l^*_{kv}|_w \cdot |\mathbf{g}^*|_w}\right)^g \left(\prod_{q\in M_L} |\lambda l^*_{kv}|_q \cdot |\mathbf{g}^*|_q\right)^g \\
&= \left(\frac{|\lambda D_{kv}|_v}{|\lambda l^*_{kv}|_v |\mathbf{g}^*|_v}\right) \left(H(\lambda l^*_{kv}) \cdot H(\mathbf{g}^*)\right)^g \\
&\leqslant \left(\frac{|D_{kv}|_v}{|l^*_{kv}|_v \cdot |\mathbf{g}^*|_v}\right) \left(H(l^*_{kv}) \cdot H(\mathbf{g}^*)\right)^{\hat{D}^{N-1}},
\end{aligned}$$

and this implies that

$$|D_{kv}|_v \geqslant |l^*_{kv}|_v H(l^*_{kv})^{-\hat{D}^{N-1}} \cdot |\mathbf{g}^*|_v H(\mathbf{g}^*)^{-\hat{D}^{N-1}}. \tag{8.10}$$

By (8.4), Lemma 2 and (4.22) we have

$$|l^*_{kv}|_v \geqslant \hat{H}^{-(N-1)\hat{D}^{N-1}},$$

while by (8.4), (2.13) and (4.22) we have

$$H(l^*_{kv}) \leqslant \prod_{i\neq k} H(\hat{l}_{iv}) \leqslant \hat{H}^{N-1}. \tag{8.11}$$

By inserting this into (8.10) we get, recalling that $k = i_v$,

$$|D_{i_v,v}|_v \geqslant \hat{H}^{-2N\hat{D}^N} |\mathbf{g}^*|_v H(\mathbf{g}^*)^{-\hat{D}^N} \quad \text{for } v \in S.$$

Further, since $\mathbf{g}_1, \ldots, \mathbf{g}_{N-1} \in \Pi(Q)$ we have $\mathbf{g}_1, \ldots, \mathbf{g}_{N-1} \in O_S^N$. Hence $\mathbf{g}^* \in O_S^N$, i.e.

$$|\mathbf{g}^*|_v \leqslant 1 \quad \text{for } v \notin S.$$

Together with (8.3), i.e. $H(\mathbf{g}^*) = H(V(Q))$, these inequalities imply that

$$\begin{aligned}\prod_{v \in S} |D_{i_v,v}|_v &\geqslant \hat{H}^{-2Ns\hat{D}^N} \Big(\prod_{v \in S} |\mathbf{g}^*|_v \Big) H(\mathbf{g}^*)^{-s\hat{D}^N} \\ &\geqslant \hat{H}^{-2Ns\hat{D}^N} H(\mathbf{g}^*) H(V(Q))^{-s\hat{D}^N} \\ &\geqslant \hat{H}^{-2Ns\hat{D}^N} H(V(Q))^{-s\hat{D}^N}.\end{aligned}$$

By combining this with (8.9) and (4.24) we obtain

$$H(V(Q))^{-s\hat{D}^N} \leqslant N! \hat{H}^{2Ns\hat{D}^N} Q^{-\epsilon/2} \leqslant Q^{-\epsilon/3},$$

which is equivalent to (8.2).

We now assume that there is no tuple $(i_v : v \in S)$ satisfying both (8.7) and (8.8). We show that there is a fixed $(N-1)$-dimensional linear subspace V of K^N, independent of Q, such that $V(Q) = V$. For $v \in S$, let

$$I_v := \{i \in \{1, \ldots, N\} : D_{iv} \neq 0\}.$$

In view of (8.5) we have

$$l_{iv}^*(\mathbf{g}^*) = 0 \quad \text{for } v \in S,\ i \in \{1, \ldots, N\} \backslash I_v. \tag{8.12}$$

By (8.4), (4.22) and (2.13) we have

$$H(l_{iv}^*) \leqslant \hat{H}^{N-1} \quad \text{for } v \in S,\ i = 1, \ldots, N.$$

Together with Lemma 3(ii) this implies that there is a non-zero vector $\mathbf{h} \in K^N$ with

$$l_{iv}^*(\mathbf{h}) = 0 \quad \text{for } v \in S,\ i \in \{1, \ldots, N\} \backslash I_v, \tag{8.13}$$

$$H(\mathbf{h}) \leqslant \Bigg(\max_{\substack{v \in S \\ i=1,\ldots,N}} H(l_{iv}^*) \Bigg)^{N-1} \leqslant \hat{H}^{(N-1)^2}. \tag{8.14}$$

(If $I_v = \{1, \ldots, N\}$ for each $v \in S$ then (8.13) is an empty condition and (8.14) is satisfied by for instance $\mathbf{h} = (1, 0, \ldots, 0)$). Fix a non-zero $\mathbf{h} \in K^N$ with (8.13), (8.14) and put

$$V := \{\mathbf{x} \in K^N : \mathbf{x} \cdot \mathbf{h} = 0\}.$$

Our aim is to show that $V(Q) = V$. Since $V(Q)$ is the vector space generated by $\Pi(Q)$ and both $V(Q)$ and V have dimension $N - 1$, it suffices to show that $\Pi(Q) \subset V$ or which is the same $\mathbf{x} \cdot \mathbf{h} = 0$ for every $\mathbf{x} \in \Pi(Q)$.

Fix $\mathbf{x} \in \Pi(Q)$. For $v \in S$, let A_v be the $N \times N$-matrix whose ith row consists of the coefficients of $\hat{l}_{iv}$ and let A_v^* the $N \times N$-matrix whose ith row consists of the coefficients of l_{iv}^*. Then by (2.1), (8.4) we have

$$^tA_v^* \cdot A_v = \Delta_v I,$$

where $^tA_v^*$ is the transpose of A_v^*, $\Delta_v = \det(\hat{l}_{1v}, \ldots, \hat{l}_{Nv})$ and I is the unit matrix. This implies that

$$\mathbf{x} \cdot \mathbf{h} = \Delta_v^{-1} \sum_{i=1}^{N} \hat{l}_{iv}(\mathbf{x}) l_{iv}^*(\mathbf{h})$$

so in view of (8.13),

$$\mathbf{x} \cdot \mathbf{h} = \Delta_v^{-1} \sum_{i \in I_v} \hat{l}_{iv}(\mathbf{x}) l_{iv}^*(\mathbf{h}) \quad \text{for } v \in S. \tag{8.15}$$

By (8.1) we have

$$|\Delta_v|_v^{-1} \leqslant \hat{H}^{N\hat{D}^N} \quad \text{for } v \in S. \tag{8.16}$$

Further, by (2.12) and (4.24) we have

$$|l_{iv}^*|_v \leqslant \prod_{j \neq i} |\hat{l}_{jv}|_v = 1 \quad \text{for } v \in S,\ i = 1, \ldots, N$$

and together with Schwarz' inequality this implies

$$|l_{iv}^*(\mathbf{h})|_v \leqslant |l_{iv}^*|_v |\mathbf{h}|_v \leqslant |\mathbf{h}|_v \quad \text{for } v \in S,\ i \in I_v. \tag{8.17}$$

For $v \in S$, choose $j_v \in I_v$ such that $\gamma_{j_v,v} = \max_{i \in I_v} \gamma_{iv}$. Since $\mathbf{x} \in \Pi(Q)$ we have

$$|\hat{l}_{iv}(\mathbf{x})|_v \leqslant Q^{\gamma_{iv}} \leqslant Q^{\gamma_{j_v,v}} \quad \text{for } v \in S,\ i \in I_v.$$

Together with (8.15), (8.16), (8.17) this implies that

$$\begin{aligned} |\mathbf{x} \cdot \mathbf{h}|_v &= |\Delta_v|_v^{-1} \left| \sum_{i \in I_v} \hat{l}_{iv}(\mathbf{x}) l_{iv}^*(\mathbf{h}) \right|_v \\ &\leqslant N^{s(v)} \hat{H}^{N\hat{D}^N} \max_{i \in I_v} |l_{iv}^*(\mathbf{h})|_v |\hat{l}_{iv}(\mathbf{x})|_v \\ &\leqslant N^{s(v)} \hat{H}^{N\hat{D}^N} |\mathbf{h}|_v Q^{\gamma_{j_v,v}} \quad \text{for } v \in S. \end{aligned} \tag{8.18}$$

Further, since $\mathbf{x} \in \Pi(Q)$ we have $\mathbf{x} \in O_S^N$, whence $|\mathbf{x}|_v \leqslant 1$ for $v \notin S$. Together with Schwarz' inequality this implies that

$$|\mathbf{x} \cdot \mathbf{h}|_v \leqslant |\mathbf{x}|_v \cdot |\mathbf{h}|_v \leqslant |\mathbf{h}|_v \quad \text{for } v \notin S. \tag{8.19}$$

Since $j_v \in I_v$ for $v \in S$, the tuple $(j_v : v \in S)$ satisfies (8.7), so by our assumption it does not satisfy (8.8). This means that

$$\sum_{v \in S} \gamma_{j_v, v} < -\epsilon/2. \tag{8.20}$$

Now assume that $\mathbf{x} \cdot \mathbf{h} \neq 0$. Then, by the Product formula and (8.18), (8.19), (8.20), (8.14) we have

$$\begin{aligned} 1 &= \prod_{v \in M_K} |\mathbf{x} \cdot \mathbf{h}|_v = \prod_{v \in S} |\mathbf{x} \cdot \mathbf{h}|_v \prod_{v \notin S} |\mathbf{x} \cdot \mathbf{h}|_v \\ &\leqslant N \cdot \hat{H}^{N s \hat{D}^N} \prod_{v \in S} |\mathbf{h}|_v Q^{\sum_{v \in S} \gamma_{j_v, v}} \cdot \prod_{v \notin S} |\mathbf{h}|_v \\ &\leqslant N \cdot \hat{H}^{N s \hat{D}^N} H(\mathbf{h}) Q^{-\epsilon/2} \\ &\leqslant N \cdot \hat{H}^{N s \hat{D}^N + (N-1)^2} Q^{-\epsilon/2} \end{aligned}$$

but this contradicts (4.24). Hence $\mathbf{x} \cdot \mathbf{h} = 0$. This completes the proof of Lemma 27. □

We need another, easier, gap principle, which is similar to [19], Lemma 7.6.

LEMMA 28. *Let A, B be reals with*

$$B > A > (2\hat{H})^{e^{C_2}},$$

where C_2 is the constant in (4.24). *There is a collection of $(N-1)$-dimensional linear subspaces of K^N of cardinality at most*

$$T(A, B) := 1 + 4\epsilon^{-1} \log(\log B / \log A)$$

such that for every Q with (4.23) *and with*

$$A \leqslant Q < B$$

the vector space $V(Q)$ belongs to this collection.

Proof. Let $E > (2\hat{H})^{e^{C_2}}$. Suppose there are Q with (4.23) and with

$$E \leqslant Q < E^{1+\epsilon/2}. \tag{8.21}$$

Let Q_E be the smallest such Q and put $V_E := V(Q_E)$. Then Q_E satisfies (4.24). We first show that for all Q with (4.23) and (8.21) we have

$$V(Q) = V_E. \tag{8.22}$$

Take linearly independent $\mathbf{x}_1,\dots,\mathbf{x}_{N-1} \in \Pi(Q_E)$. (8.22) follows once we have shown that for every $Q \in [Q_E, E^{1+\epsilon/2})$ and every $\mathbf{x}_N \in \Pi(Q)$ we have $\mathbf{x}_N \in V_E$ or, which is the same, $\det(\mathbf{x}_1,\dots,\mathbf{x}_N) = 0$.

Take $\mathbf{x}_N \in \Pi(Q)$. Fix $v \in S$. By (8.1) we have

$$\begin{aligned} |\det(\mathbf{x}_1,\dots,\mathbf{x}_N)|_v &= |\det(\hat{l}_{1v},\dots,\hat{l}_{Nv})|_v^{-1}|\det(\hat{l}_{iv}(\mathbf{x}_j))|_v \\ &\leqslant \hat{H}^{N\hat{D}^N}|\det(\hat{l}_{iv}(\mathbf{x}_j))|_v. \end{aligned} \tag{8.23}$$

Further, we have $|\hat{l}_{iv}(\mathbf{x}_j)|_v \leqslant Q_E^{\gamma_{iv}}$ for $i = 1,\dots,N$, $j = 1,\dots,N-1$ and also, by (4.21) we have

$$\begin{aligned} |\hat{l}_{iv}(\mathbf{x}_N)|_v &\leqslant Q^{\gamma_{iv}} = Q_E^{\gamma_{iv}}(Q/Q_E)^{\gamma_{iv}} \leqslant Q_E^{\gamma_{iv}}(Q/Q_E)^{s(v)} \\ &\leqslant Q_E^{\gamma_{iv}}(E^{1+\epsilon/2}/Q_E)^{s(v)} \leqslant Q_E^{\gamma_{iv}+s(v)\epsilon/2} \end{aligned}$$

for $i = 1,\dots,N$. Therefore (taking again the maximum over all permutations κ of $(1,\dots,N)$),

$$|\det(\hat{l}_{iv}(\mathbf{x}_j))|_v \leqslant (N!)^{s(v)}\prod_{i=1}^{N}|\hat{l}_{iv}(\mathbf{x}_{\kappa(i)})|_v \leqslant (N!)^{s(v)}Q_E^{\gamma_{1v}+\dots+\gamma_{Nv}+s(v)\epsilon/2}.$$

By inserting this into (8.23) we get

$$|\det(\mathbf{x}_1,\dots,\mathbf{x}_N)|_v \leqslant (N!)^{s(v)}\hat{H}^{N\hat{D}^N}Q_E^{\gamma_{1v}+\dots+\gamma_{Nv}+s(v)\epsilon/2} \quad \text{for } v \in S.$$

By taking the product over $v \in S$ and using (4.21) and that Q_E satisfies (4.24) we obtain

$$\begin{aligned} \prod_{v\in S}|\det(\mathbf{x}_1,\dots,\mathbf{x}_N)|_v &\leqslant N!\hat{H}^{Ns\hat{D}^N}Q_E^{\left(\sum_{v\in S}\sum_{i=1}^{N}\gamma_{iv}\right)+\epsilon/2} \\ &\leqslant N!\hat{H}^{Ns\hat{D}^N}Q_E^{-\epsilon/2} < 1. \end{aligned}$$

Further, we know that $\mathbf{x}_1,\dots,\mathbf{x}_N \in O_S^N$, whence $\det(\mathbf{x}_1,\dots,\mathbf{x}_N) \in O_S$ and that $\prod_{v\in S}|a|_v \geqslant 1$ for non-zero $a \in O_S$. Hence $\det(\mathbf{x}_1,\dots,\mathbf{x}_N) = 0$ which is what we wanted to show.

Now let k be the smallest integer with $(1+\epsilon/2)^k > \log B/\log A$. Put $E_i := A^{(1+\epsilon/2)^i}$ for $i = 0,\dots,k-1$. Then $E_i \geqslant A > (2\hat{H})^{e^{C_2}}$. Let I be the set of indices $i \in \{0,\dots,k-1\}$ for which there is a Q with (4.23) and with $E_i \leqslant Q < E_i^{1+\epsilon/2}$. Then I has cardinality at most

$$k < 1 + \frac{\log(\log B/\log A)}{\log(1+\epsilon/2)} < 1 + \frac{4}{\epsilon}\log(\log B/\log A) = T(A,B).$$

For every $Q \in [A,B)$ with (4.23) there is an $i \in I$ such that $Q \in [A^{(1+\epsilon/2)^i}, A^{(1+\epsilon/2)^{i+1}}) = [E_i, E_i^{1+\epsilon/2})$. Above we proved that $V(Q) = V_{E_i}$. Hence the spaces $V(Q)$ with Q satisfying (4.23) and $A \leqslant Q < B$ belong to the collection $\{V_{E_i} : i \in I\}$ which has cardinality at most $T(A, B)$. This proves Lemma 28. □

In the proof of Theorem C we need an auxiliary polynomial with certain properties, to which Lemma 26 in Section 7 will be applied. Let $m \geqslant 2$. For $h = 1, \ldots, m$, denote as before by $\mathbf{X}_h$ the block of variables $(X_{h1}, \ldots, X_{hN})$. For $v \in S$, we introduce new variables

$$U_{hiv} := \hat{l}_{iv}(\mathbf{X}_h) \quad (h = 1, \ldots, m,\ i = 1, \ldots, N).$$

Let $\mathbf{d} = (d_1, \ldots, d_N)$ be a tuple of positive integers. Denote by $\mathcal{R}(\mathbf{d})$ the set of non-zero polynomials in $\mathbb{Z}[\mathbf{X}_1, \ldots, \mathbf{X}_m]$ which are of degree d_h in the block $\mathbf{X}_h$ for $h = 1, \ldots, m$ and whose coefficients have gcd 1. In what follows, $\mathbf{i}$, $\mathbf{j}$ denote tuples of non-negative integers $(i_{hk} : h = 1, \ldots, m,\ k = 1, \ldots, N)$, $(j_{hk} : h = 1, \ldots, m,\ k = 1, \ldots, N)$, respectively. For $F \in \mathcal{R}(\mathbf{d})$ we put as usual

$$F_{\mathbf{i}} := \left(\prod_{h=1}^{m} \prod_{k=1}^{N} \frac{1}{i_{hk}!} \frac{\partial^{i_{hk}}}{\partial X_{hk}^{i_{hk}}} \right) F.$$

For each $v \in S$, $\{\hat{l}_{1v}, \ldots, \hat{l}_{Nv}\}$ is linearly independent, whence $F_{\mathbf{i}}$ can be expressed as

$$F_{\mathbf{i}} = \sum_{\mathbf{j}} c(\mathbf{i}, \mathbf{j}, v) U_{11v}^{j_{11}} \cdots U_{mNv}^{j_{mN}},$$

where the sum is taken over tuples $\mathbf{j}$ with

$$\sum_{k=1}^{N} j_{hk} = d_h - \sum_{k=1}^{N} i_{hk} \quad \text{for } h = 1, \ldots, m. \tag{8.24}$$

As before, we put

$$(\mathbf{i}/\mathbf{d}) := \sum_{h=1}^{m} \frac{1}{d_h} \cdot \sum_{k=1}^{N} i_{hk}.$$

LEMMA 29. (Polynomial theorem). *Let Θ be a real with $0 < \Theta < 1/N$, m an integer with*

$$m > 4\Theta^{-2} \log(2Nsd\hat{D}^{Ns}) \tag{8.25}$$

and $\mathbf{d} = (d_1, \ldots, d_m)$ any m-tuple of positive integers. Then there is a polynomial $F \in \mathcal{R}(\mathbf{d})$ with the following properties:

(i) $H(F) \leqslant (2^{mN} 3 N^{1/2} \hat{H})^{d_1 + \cdots + d_m}$;
(ii) *for all $v \in S$ and all tuples $\mathbf{i}, \mathbf{j}$ with* (8.24) *and with*

$$(\mathbf{i}/\mathbf{d}) < 2m\Theta,$$

$$\max_{k=1,\dots,N} \left| \sum_{h=1}^{m} \frac{j_{hk}}{d_h} - \frac{m}{N} \right| > 3mN\Theta, \tag{8.26}$$

we have $\mathbf{c}(\mathbf{i}, \mathbf{j}, v) = 0$*;*

(iii) *for all tuples* $\mathbf{i}$ *we have*

$$\prod_{v \in S} \max_{\mathbf{j}} |c(\mathbf{i}, \mathbf{j}, v)|_v \leqslant (2^{4mN} \hat{H}^{2Ns\hat{D}^N})^{d_1+\cdots+d_m}.$$

Proof. Let K_1 be the composite of the fields $K(\hat{l}_{iv})$ $(v \in S,\ i = 1, \dots, N)$. Then each $\hat{l}_{iv}$ is proportional to a linear form with coefficients in K_1. By $[K:\mathbb{Q}] = d$ and (4.22) we have $[K_1:\mathbb{Q}] \leqslant d\hat{D}^{Ns}$. Let t be the maximal number of pairwise non-proportional linear forms among $\hat{l}_{iv}$ $(v \in S,\ i = 1, \dots, N)$. Then $t \leqslant Ns$. By (8.25) we have

$$m > 4\Theta^{-2} \log(2t[K_1:\mathbb{Q}]). \tag{8.27}$$

This is precisely the condition on m in the Index theorem and the Polynomial theorem of [19], Section 9, and from these theorems we infer that there is a polynomial $F \in \mathcal{R}(\mathbf{d})$ with (i) and (ii). This is proved by using Siegel's lemma from [2]: the equations $c(\mathbf{i}, \mathbf{j}, v) = 0$ can be translated into a system of linear equations in the unknown integer coefficients of F, (8.27), (8.26) guarantee that the number of unknowns is larger than the number of equations, and then Siegel's lemma implies that this system of linear equations has a non-zero integral solution whose coordinates have absolute values bounded above by the right-hand side of (i).

We prove (iii). Fix $v \in S$. Since the coefficients of F have gcd 1 and by (i) we have

$$|F|_v = H(F)^{s(v)} \leqslant (2^{mN} \cdot 3N^{1/2}\hat{H})^{(d_1+\cdots+d_m)s(v)}.$$

Together with (7.2) this implies that for each tuple $\mathbf{i}$,

$$\begin{aligned} |F_{\mathbf{i}}|_v &\leqslant \left(2^{mN+1} \cdot 3N^{1/2}\hat{H}\right)^{(d_1+\cdots+d_m)s(v)} \\ &\leqslant \left(2^{mN+3}N^{1/2}\hat{H}\right)^{(d_1+\cdots+d_m)s(v)}. \end{aligned} \tag{8.28}$$

We have

$$X_{hi} = \sum_{k=1}^{N} \eta_{ik} U_{hkv} \quad (h = 1, \dots, m,\ i = 1, \dots, N), \tag{8.29}$$

where (η_{ij}) is the inverse matrix of the coefficient matrix A_v of $\hat{l}_{1v}, \ldots, \hat{l}_{Nv}$. We have $\eta_{ik} = \pm\Delta_{ik} \cdot \Delta_v^{-1}$, where Δ_{ik} is the determinant of the matrix obtained by removing the i-th row and the k-th column from A_v, and $\Delta_v = \det A_v = \det(\hat{l}_{1v}, \ldots, \hat{l}_{Nv})$. By (4.22) and Hadamard's inequality we have $|\Delta_{ij}|_v \leqslant 1$ for $i = 1, \ldots, N$, $j = 1, \ldots, N$. Together with (8.1) this implies that

$$|\eta_{ik}|_v \leqslant |\det(\hat{l}_{1v}, \ldots, \hat{l}_{Nv})|_v^{-1} \leqslant \hat{H}^{N\hat{D}^N} \quad \text{for } i = 1, \ldots, N,\ j = 1, \ldots, N. \tag{8.30}$$

Write

$$F_{\mathbf{i}}(\mathbf{X}_1, \ldots, \mathbf{X}_m) = \sum_{\mathbf{j}} p(\mathbf{i}, \mathbf{j}) X_{11}^{j_{11}} \cdots X_{mN}^{j_{mN}},$$

where the summation is over tuples $\mathbf{j}$ with (8.24). By inserting (8.29) we get

$$F_{\mathbf{i}} = \sum_{\mathbf{j}} p(\mathbf{i}, \mathbf{j}) \prod_{h=1}^{m} \prod_{k=1}^{N} \left(\sum_{l=1}^{N} \eta_{kl} U_{hlv} \right)^{j_{hk}}. \tag{8.31}$$

Put

$$A := \max_{\mathbf{j}} |p(\mathbf{i}, \mathbf{j})|_v, \quad B := \max(1, \max_{k,l} |\eta_{kl}|_v).$$

We have $F_{\mathbf{i}} = \sum_{\mathbf{p}} c(\mathbf{i}, \mathbf{p}, v) \prod_{h=1}^{m} \prod_{l=1}^{N} U_{hlv}^{p_{hl}}$ where the summation is over tuples $\mathbf{p} = (p_{hl})$. If v is an infinite place then we have, recalling that $|\cdot|_v^{1/s(v)}$ satisfies the triangle inequality,

$$\begin{aligned} |c(\mathbf{i}, \mathbf{p}, v)|_v^{1/s(v)} &\leqslant \sum_{\mathbf{j}} A^{1/s(v)} \prod_{h=1}^{m} \prod_{k=1}^{N} \Big(\sum_{l=1}^{N} B^{1/s(v)}\Big)^{j_{hk}} \\ &\leqslant N^{2(d_1+\cdots+d_m)} \left(AB^{d_1+\cdots+d_m}\right)^{1/s(v)} \end{aligned}$$

since $\mathbf{j}$ runs through tuples with (8.24). If v is a finite place then

$$|c(\mathbf{i}, \mathbf{p}, v)|_v \leqslant A \cdot \max_{\mathbf{j}} B^{\sum_{h,k} j_{hk}} \leqslant AB^{d_1+\cdots+d_m}.$$

So for both cases v infinite, v finite we have

$$|c(\mathbf{i}, \mathbf{p}, v)|_v \leqslant N^{2s(v)(d_1+\cdots+d_m)} AB^{d_1+\cdots+d_m}.$$

By estimating A from above using $A \leqslant |F_{\mathbf{i}}|_v$ and (8.28), and B from above using (8.30) we obtain

$$\begin{aligned} |c(\mathbf{i}, \mathbf{p}, v)|_v &\leqslant \left(N^{2s(v)} \cdot \{2^{mN+3} N^{1/2} \hat{H}\}^{s(v)} \cdot \hat{H}^{N\hat{D}^N}\right)^{d_1+\cdots+d_m} \\ &\leqslant (2^{4mNs(v)} \hat{H}^{2N\hat{D}^N})^{d_1+\cdots+d_m} \quad \text{for } v \in S. \end{aligned}$$

By taking the product over $v \in S$ we get

$$\prod_{v \in S} \max_{\mathbf{p}} |c(\mathbf{i}, \mathbf{p}, v)|_v \leqslant \left(2^{4mN} \hat{H}^{2Ns\hat{D}^N}\right)^{d_1+\cdots+d_m}$$

which is (iii). This completes the proof of Lemma 29. □

9. Proof of Theorem C

Let $(N, \underline{\gamma}, \hat{\mathcal{L}})$ be a tuple as in Theorem C satisfying $N \geqslant 2$, (4.21), (4.22). Put

$$\Theta := \frac{\epsilon}{30N^3}, \tag{9.1}$$

and let m be the smallest integer satisfying the condition of Lemma 29, i.e.

$$m > 4\Theta^{-2} \log(2Nsd\hat{D}^{Ns}). \tag{8.25}$$

Then by (9.1) we have

$$m < 4000N^7 s\epsilon^{-2} \log 4\hat{D}. \tag{9.2}$$

We assume that the collection of subspaces $V(Q)$ with Q satisfying (4.23), (4.24) has cardinality $> C_2$ and shall derive a contradiction from that. Then this collection consists of more than

$$1 + (m-1)t, \quad \text{with } t = 2 + [4\epsilon^{-1} \log(4m^2\Theta^{-1})]$$

subspaces, since

$$\begin{aligned} 1 + (m-1)t &\leqslant 5m\epsilon^{-1} \log(4m^2\Theta^{-1}) < 5m\epsilon^{-1} \log(120N^3m^2\epsilon^{-1}) \\ &< 5 \times 4000 \cdot N^7 s\epsilon^{-3} \log 4\hat{D} \\ &\quad \cdot \log(120 \times 4000^2 N^{17} s^2 \epsilon^{-5} (\log 4\hat{D})^2) \\ &< 2^{30} N^8 s^2 \epsilon^{-4} \log 4\hat{D} \cdot \log\log 4\hat{D} = C_2. \end{aligned}$$

Let V be the subspace from Lemma 27. Then there are reals $Q'_1, Q'_2, \ldots, Q'_{1+(m-1)t}$ with (4.23), (4.24) and $Q'_1 < Q'_2 < \cdots < Q'_{1+(m-1)t}$ such that the spaces $V(Q'_1), \ldots, V(Q'_{1+(m-1)t})$ are different and different from V. Put

$$Q_1 := Q'_1,\ Q_2 := Q'_{t+1}, \ldots, Q_m := Q'_{(m-1)t+1}$$

and

$$V_h := V(Q_h) \quad \text{for } h = 1, \ldots, m.$$

There are $t > 1 + 4\epsilon^{-1}\log\{4m^2/\Theta\}$ different spaces $V(Q)$ with $Q_h \leqslant V(Q) < Q_{h+1}$; together with Lemma 28 this implies that

$$Q_{h+1} \geqslant Q_h^{4m^2/\Theta} \quad \text{for } h = 1, \ldots, m-1. \tag{9.3}$$

Define positive integers $d_1, \ldots, d_m$ by

$$d_1 := 1 + \left[\frac{\log Q_m}{\Theta \log Q_1}\right], \tag{9.4}$$

$$d_1 \log Q_1 \leqslant d_h \log Q_h < d_1 \log Q_1 + \log Q_h \quad \text{for } h = 1, \ldots, m. \tag{9.5}$$

Thus,

$$d_1 \log Q_1 \leqslant d_h \log Q_h < d_1 \log Q_1 \cdot (1+\Theta) \quad \text{for } h = 1, \ldots, m. \tag{9.6}$$

Let $\mathbf{d} = (d_1, \ldots, d_m)$ and let $F \in \mathcal{R}(\mathbf{d})$ be the polynomial from Lemma 29 which exists since m satisfies (8.25). We want to apply Lemma 26. We have $N \geqslant 2$ and $m \geqslant 2, 0 < \Theta \leqslant 1$ by (9.1), (8.25), respectively. We verify that $d_1, \ldots, d_m$, F, $V_1, \ldots, V_m$ satisfy the other conditions of Lemma 26, i.e. (7.6), (7.7).

By (9.6), (9.3), (9.1) we have

$$\begin{aligned} \frac{d_h}{d_{h+1}} &= \frac{d_h \log Q_h}{d_{h+1}\log Q_{h+1}} \cdot \frac{\log Q_{h+1}}{\log Q_h} \geqslant (1+\Theta)^{-1} \cdot 4m^2/\Theta \\ &> 2m^2/\Theta \quad \text{for } h = 1, \ldots, m-1, \end{aligned}$$

which is (7.6).

By Lemma 27, $V_h = V(Q_h) \neq V$, (9.5) and the fact that Q_1 satisfies (4.24) we have

$$\begin{aligned} H(V_h)^{d_h} &\geqslant Q_h^{d_h \cdot \epsilon/3s\hat{D}^N} \geqslant Q_1^{d_1 \cdot \epsilon/3s\hat{D}^N} \geqslant (2\hat{H})^{d_1 e^{C_2}\epsilon/3s\hat{D}^N} \\ &\geqslant (2\hat{H})^{d_1 \cdot e^{C_2/2}} \quad \text{for } h = 1, \ldots, m. \end{aligned}$$

On the other hand, by Lemma 29 (i), $d_1 + \cdots + d_m \leqslant md_1$, (9.1) and (9.2) we have

$$\begin{aligned} &\{e^{d_1+\cdots+d_m} H(F)\}^{(N-1)(3m^2/\Theta)^m} \\ &\leqslant \{e \cdot 2^{mN} \cdot 3N^{1/2}\hat{H}\}^{(N-1)(3m^2/\Theta)^m(d_1+\cdots+d_m)} \\ &\leqslant (2\hat{H})^{2mN^2 \cdot (3m^2/\Theta)^m \cdot md_1} \leq (2\hat{H})^{d_1 \cdot (3m^2/\Theta)^{2m}} \\ &= (2\hat{H})^{d_1 \cdot \exp\{2m \log(90m^2N^2/\epsilon)\}} \\ &\leqslant (2\hat{H})^{d_1 \cdot \exp\{8000N^7 s\epsilon^{-2} \log 4\hat{D} \cdot \log(10^9 N^{16}\epsilon^{-3}(\log 4\hat{D})^2)\}} \\ &< (2\hat{H})^{d_1 \cdot e^{C_2/2}}. \end{aligned}$$

Therefore,

$$H(V_h)^{d_h} > \{e^{d_1+\cdots+d_m} H(F)\}^{(N-1)(3m^2/\Theta)^m} \quad \text{for } h = 1, \ldots, m,$$

which is (7.7). Hence indeed, m, N, Θ, $d_1, \ldots, d_m$, F, $V_1, \ldots, V_m$ satisfy the conditions of Lemma 26.

For $h = 1, \ldots, m$, choose a linearly independent set of vectors $\{\mathbf{g}_{h1}, \ldots, \mathbf{g}_{h,N-1}\}$ from $\Pi(Q_h)$ (which exists by (4.23)) and let Γ_h be the grid of size N/Θ,

$$\Gamma_h := \{x_1\mathbf{g}_{h1} + \cdots + x_{N-1}\mathbf{g}_{h,N-1} \colon x_1, \ldots, x_{N_1} \in \mathbb{Z},$$
$$|x_1|, \ldots, |x_{N-1}| \leqslant N/\Theta\}.$$

Now Lemma 26 implies that there are $\mathbf{x}_1 \in \Gamma_1, \ldots, \mathbf{x}_m \in \Gamma_m$ and a tuple of non-negative integers $\mathbf{i}$ with $(\mathbf{i}/\mathbf{d}) < 2m\Theta$, such that

$$f := F_{\mathbf{i}}(\mathbf{x}_1, \ldots, \mathbf{x}_m) \neq 0. \tag{9.7}$$

From $\mathbf{g}_{hj} \in \Pi(Q_h)$ it follows that $\mathbf{g}_{hj} \in O_S^N$ for $j = 1, \ldots, N$, hence $\mathbf{x}_h \in O_S^N$ for $h = 1, \ldots, N$. Further, $F_{\mathbf{i}}$ has its coefficients in $\mathbb{Z}$. Hence $f \in O_S \backslash \{0\}$ which implies $\prod_{v \in S} |f|_v \geqslant 1$. Below, we show that

$$\prod_{v \in S} |f|_v < 1.$$

Thus, the assumption that there are more than C_2 different subspaces among $V(Q)$ with Q running through the reals with (4.23), (4.24) does indeed lead to a contradiction.

Fix $v \in S$. Put

$$u_{hiv} := \hat{l}_{iv}(\mathbf{x}_h) \quad \text{for } h = 1, \ldots, m, \ i = 1, \ldots, N.$$

Since $\mathbf{g}_{hj} \in \Pi(Q_h)$, i.e. $|\hat{l}_{iv}(\mathbf{g}_{hj})|_v \leqslant Q_h^{\gamma_{iv}}$ for $i = 1, \ldots, N$ and $j = 1, \ldots, N-1$, and since $\mathbf{x}_h$ is in the grid Γ_h of size N/Θ, we have, using (2.8),

$$|u_{hiv}|_v \leqslant (N^2/\Theta)^{s(v)} Q_h^{\gamma_{iv}} \quad \text{for } h = 1, \ldots, m, \ i = 1, \ldots, N. \tag{9.8}$$

By Lemma 29 (ii) we have

$$f = \sum_{\mathbf{j}}^{*} c(\mathbf{i}, \mathbf{j}, v) u_{11v}^{j_{11}} \cdots u_{mNv}^{j_{mN}}, \tag{9.9}$$

where the summation is over all tuples of non-negative integers $\mathbf{j} = (j_{11}, \ldots, j_{mN})$ with

$$\begin{cases} \left| \sum_{h=1}^{m} \frac{j_{hk}}{d_h} - \frac{m}{N} \right| \leqslant 3mN\Theta & \text{for } k = 1, \ldots, N, \\ \sum_{k=1}^{N} j_{hk} = d_h - \sum_{k=1}^{N} i_{hk} & \text{for } h = 1, \ldots, N. \end{cases} \tag{9.10}$$

Now by (9.8), (9.9), by the trivial fact that there are at most $N^{d_1+\cdots+d_m}$ tuples $\mathbf{j}$ with (9.10) and by $\sum_{h,k} j_{hk} \leqslant d_1 + \cdots + d_m \leqslant md_1$, we have

$$\begin{aligned} |f|_v &\leqslant N^{(d_1+\cdots+d_m)s(v)} \cdot \max_{\mathbf{j}}{}^{*} |c(\mathbf{i},\mathbf{j},v)|_v |u_{11v}|_v^{j_{11}} \cdots |u_{mNv}|_v^{j_{mN}} \\ &\leqslant \{(N^3/\Theta)^{s(v)} \cdot A_v \cdot Q_1^{c_v}\}^{md_1}, \end{aligned} \tag{9.11}$$

where

$$A_v := \left(\max_{\mathbf{j}} |c(\mathbf{i},\mathbf{j},v)|_v\right)^{1/md_1},$$

$$c_v := \frac{1}{m} \max_{\mathbf{j}}{}^{*} \sum_{h=1}^{m} \sum_{k=1}^{N} \gamma_{kv} \frac{j_{hk}}{d_h} \cdot \frac{d_h \log Q_h}{d_1 \log Q_1},$$

and the maximum is taken over all tuples $\mathbf{j}$ with (9.10).

We estimate c_v from above. For each tuple $\mathbf{j}$ with (9.10) we have, recalling that $\gamma_{kv} \leqslant s(v)$ by (4.21) and $1 \leqslant d_h \log Q_h / d_1 \log Q_1 \leqslant 1 + \Theta$ by (9.6),

$$\begin{aligned} &\sum_{h=1}^{m} \sum_{k=1}^{N} \gamma_{kv} \frac{j_{hk}}{d_h} \cdot \frac{d_h \log Q_h}{d_1 \log Q_1} \\ &= \sum_{k=1}^{N} \left\{ (\gamma_{kv} - s(v)) \sum_{h=1}^{m} \frac{j_{hk}}{d_h} \cdot \frac{d_h \log Q_h}{d_1 \log Q_1} \right\} \\ &\quad + s(v) \left\{ \sum_{k=1}^{N} \sum_{h=1}^{m} \frac{j_{hk}}{d_h} \cdot \frac{d_h \log Q_h}{d_1 \log Q_1} \right\} \\ &\leqslant \sum_{k=1}^{N} \left\{ (\gamma_{jv} - s(v)) \sum_{h=1}^{m} \frac{j_{hk}}{d_h} \right\} + s(v)(1+\Theta) \left\{ \sum_{k=1}^{N} \sum_{h=1}^{m} \frac{j_{hk}}{d_h} \right\} \\ &\leqslant \left\{ \sum_{k=1}^{N} (\gamma_{kv} - s(v)) \right\} \left(\frac{m}{N} - 3mN\Theta \right) \\ &\qquad\qquad + s(v)(1+\Theta) N \left(\frac{m}{N} + 3mN\Theta \right) \\ &< m \left(\sum_{k=1}^{N} \gamma_{kv} \right) \left(\frac{1}{N} - 3N\Theta \right) + s(v) \cdot 7mN^2\Theta; \end{aligned}$$

here we used that by (9.1) and $0 < \epsilon < 1$ we have $\Theta < 1/30N^3$. Hence

$$c_v \leqslant \left(\sum_{k=1}^{N} \gamma_{kv} \right) \left(\frac{1}{N} - 3N\Theta \right) + s(v) \cdot 7N^2\Theta \quad \text{for } v \in S.$$

Together with (4.21), (9.1) this implies that

$$\begin{aligned}\sum_{v\in S} c_v &\leqslant \left(\sum_{v\in S}\sum_{k=1}^{N}\gamma_{kv}\right)\left(\frac{1}{N}-3N\Theta\right)+7N^2\Theta\\ &\leqslant -\epsilon\left(\frac{1}{N}-\frac{\epsilon}{10N^2}\right)+\frac{7}{30}\frac{\epsilon}{N}<-\frac{\epsilon}{2N}.\end{aligned}\tag{9.12}$$

Further, by Lemma 29 (iii) and $d_1+\cdots+d_m \leqslant md_1$ we have

$$\prod_{v\in S} A_v \leqslant 2^{4mN}\hat{H}^{2Ns\hat{D}^N}.\tag{9.13}$$

Now (9.11), (9.13), (9.12), (9.1), (9.2) and the fact that Q_1 satisfies (4.24) imply that

$$\begin{aligned}\prod_{v\in S}|f|_v &\leqslant \left\{(N^3/\Theta)\cdot\left(\prod_{v\in S}A_v\right)\cdot Q_1^{\sum_{v\in S}c_v}\right\}^{md_1}\\ &\leqslant \{(N^3/\Theta)\cdot 2^{4mN}\hat{H}^{2Ns\hat{D}^N}\cdot Q_1^{-\epsilon/2N}\}^{md_1}\\ &\leqslant \left\{\left(\frac{30N^6}{\epsilon}\cdot 2^{16000N^8s\epsilon^{-2}\log 4\hat{D}}\hat{H}^{2Ns\hat{D}^N}\right)^{2N/\epsilon}\cdot Q_1^{-1}\right\}^{md_1\epsilon/2N}\\ &< 1.\end{aligned}$$

This completes the proof of Theorem C. □

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