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Symplectic topology of integrable Hamiltonian systems, I: Arnold–Liouville with singularities

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Abstract. The classical Arnold-Liouville theorem describes the geometry of an integrable Hamiltonian system near a regular level set of the moment map. Our results describe it near a nondegenerate singular level set: a tubular neighborhood of a connected singular nondegenerate level set, after a normal finite covering, admits a non-complete system of action-angle functions (the number of action functions is equal to the rank of the moment map), and it can be decomposed topologically, together with the associated singular Lagrangian foliation, to a direct product of simplest (codimension 1 and codimension 2) singularities. These results are essential for the global topological study of integrable Hamiltonian systems.

1. Introduction

The classical Arnold–Liouville theorem [1] describes the geometry of an integrable Hamiltonian system (IHS) with n degrees of freedom near regular level sets of the moment map: locally near every regular invariant torus there is a system of action-angle coordinates, and the corresponding foliation by invariant Lagrangian tori is trivial. In particular, there is a free symplectic n-dimensional torus action which preserves the moment map. The local base space of this trivial foliation is a disk equipped with a unique natural integral affine structure, given by the action coordinates. A good account on action-angle coordinates, and their generation for Mishchenko-Fomenko noncommuative integrable systems and Libermann symplectically complete foliations, can be found in the paper [15] by Dazord and Delzant, and references therein.

However, IHS's met in classical mechanics and physics always have singularities, and most often these singularities are nondegenerate in a natural sense. Thus the study of nondegenerate singularities (=tubular neighborhoods of singular level sets of the moment map) is one of the main step toward understanding the topological structure of general IHS's.

The question is: What can be said about the structure of nondegenerate singularities of IHS's? What remains true from Arnold–Liouville theorem in this case?

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In recent years, several significant results have been obtained concerning this question. Firstly, the local structure of nondegenerate singularities is now well-understood, due to the works by Birkhoff, Williamson, Rüssmann, Vey and Eliasson (cf. e.g. [8, 44, 41, 43, 23]). The second well-understood thing is elliptic singularities and systems 'de type torique', through the works by Dufour, Molino and others (cf. e.g. [12, 19, 20, 23, 37, 42]). In particular, Boucetta and Molino [12] observed that the orbit space of the integrable system in case of elliptic singularities is a singular integral affine manifold with a simple singular structure. Integrable systems with elliptic singularities are closely related to torus actions on symplectic manifolds, which were studied by Duistermaat, Heckman and others (cf. e.g. [22, 2, 16]). Third, codimension 1 nondegenerate singularities were studied by Fomenko [25] from the topological point of view, and was shown to admit a system of $(n-1)$ action and $(n-1)$ angle coordinates in my thesis [49]. Fourth, some simple cases of codimension 2 singularities were studied by Lerman, Umanskii and others (cf. e.g. [32, 33, 34, 9, 45]). However, for general nondegenerate singularities, the question had remained open. Here we listed only 'abstract' works. There are many authors who studied singularities of concrete well-known integrable systems, from different viewpoints, cf. e.g. [5, 10, 14, 18, 27, 26, 28, 29, 39, 40].

The aim of this paper is to give a more or less satisfactory answer to the question posed above, which incorporates cited results. Namely, we will show that, for a nondegenerate singularity of codimension k in an IHS with n degrees of freedom there is still a locally free symplectic torus group action (of dimension $n-k$), which preserves the moment map. There is a normal finite covering of the singularity, for which the above torus action becomes free, and the obtained torus bundle structure is topologically trivial. There is a coisotropic section to this trivial bundle, which gives rise to a set of $(n-k)$ action and $(n-k)$ angle coordinates. And the most important result, which completes the picture is as follows: the singular Lagrangian foliation associated to a nondegenerate singularity is, up to a normal finite covering, homeomorphic to a direct product of singular Lagrangian foliations of simplest (codimension 1 and/or focus-focus codimension 2) singularities.

Thus, we have the following topological classification of nondegenerate singularities of integrable Hamiltonian systems: every such singularity has a unique natural *canonical model* (or may be also called minimal model because of its minimality), which is a direct-product singularity with a free component-wise action of a finite group on it, cf. Theorem 7.3. This finite group is a topological invariant of the singularity, and may be called Galois group.

Canonical models for geodesic flows on multi-dimensional ellipsoids were computed in [50]. In [7] we will discuss how the method of separation of variables allows one to compute canonical model of singularities of integrable systems. It is an interesting problem to study singularities of systems which are integrable by other methods (Lie-group theoretic, isospectral deformation, etc.). In this direction, some results connected with Lie algebras were obtained by Bolsinov [10]. Isospectral deformation was deployed by Audin for some systems [3, 5, 4].

All the results listed above are true under a mild condition of topological stability. This condition is satisfied for all nondegenerate singularites of all known IHS's met in mechanics and physics. (Notice that this condition is rather a structural stability but not a stability in Lyapunov's sense). Without this condition, some of the above results still hold.

We suspect that our results (with some modifications) are also valid for integrable PDE. In this direction, some results were obtained by Ercolani and McLaughlin [24]. (Note: There seems to be an inacuracy in Theorem 1 of [24]).

The above results have direct consequences in the global topological study of IHS's, which are discussed in [51]. For example, one obtains that the singular orbit space of a nondegenerate IHS is a stratified integral affine manifold in a natural sense, and one can compute homologies by looking at the bifurcation diagram, etc. Our results may be useful also for perturbation theory of nearly integrable systems (see e.g. [30] for some results concerning the use of the topology of singularities of IHS's in Poincaré–Melnikov method), and for geometric quantization.

About the notations. In this paper, $\mathbf{T}$ denotes a torus, $\mathbf{D}$ a disk, $\mathbf{Z}$ the integral numbers, $\mathbf{R}$ the real numbers, $\mathbf{C}$ the complex numbers, $\mathbf{Z}_2$ the two element group. All IHS's are assumed to be smooth, nonresonant, and by abuse of language, they are identified with Poisson actions generated by n commuting first integrals. We refer to [15, 38] for a treatment of Arnold–Liouville theorem, and when we say that we use this theorem we will often in fact use the following statement: If X is a vector field with periodic flow such that on each Liouville gorus it equals a linear combination of Hamiltonian vector fields of first integrals, then X is a symplectic vector field.

Organization of the paper. In Section 2 we recall the main known results about the local structure of nondegenerate singularities of IHS's. Section 3 is devoted to some a priori non-local properties of nondegenerate singularities and singular Lagrangian foliations associated to IHS's. Then in Section 4 and Section 5 we describe the structure of nondegenerate singularities in two simplest cases: codimension 1 hyperbolic and codimension 2 focus-focus. The main results, however, are contained in the subsequent Sections, where we prove the existence of Hamiltonian torus actions in the general case, topological decomposition, and action-angle coordinates. In the last section we briefly discuss various notions of nondegenerate IHS's, and the use of integrable surgery in obtaining interesting symplectic structures and IHS's.

2. Local structure of nondegenerate singularities

Throughout this paper, the word IHS, which stands for integrable Hamiltonian system, will always mean a C^∞ smooth Poisson $\mathbf{R}^n$ action in a smooth symplectic $2n$-dimensional manifold (M^{2n}, ω), generated by a moment map $\mathbf{F}\colon\ M \to \mathbf{R}^n$. We will always assume that the connected components of the preimages of the moment map are compact.

Denote by $Q(2n)$ the set of quadratic forms on $\mathbf{R}^{2n}$. Under the standard Poisson bracket, $Q(2n)$ becomes a Lie algebra, isomorphic to the symplectic algebra $\mathrm{sp}(2n, \mathbf{R})$. Its Cartan subalgebras have dimension n.

Assume there is given an IHS with the moment map $\mathbf{F} = (F_1, \ldots, F_n)\colon M^{2n} \to \mathbf{R}^n$. Let $X_i = X_{F_i}$ denote the Hamiltonian vector fields of the components of the moment map.

A point $x \in M^{2n}$ is called a *singular point* for the above IHS, if it is singular for the moment map: the differential $D\mathbf{F} = (dF_1, \ldots, dF_n)$ at x has rank less than n. The *corank* of this singular point is corank $D\mathbf{F} = n -$ rank $D\mathbf{F}, D\mathbf{F} = (dF_1, \ldots, dF_n)$.

Let x be a singular point as above. Let $\mathcal{K}$ be the kernel of $D\mathbf{F}(x)$ and let $\mathcal{I}$ be the space generated by $X_i(x)(1 \leqslant i \leqslant n)$. $\mathcal{I}$ is a maximal isotropic subspace of $\mathcal{K}$ with respect to the symplectic structure $\omega_0 = \omega(x)$. Hence the quotient space $\mathcal{K}/\mathcal{I} = \overline{R}$ carries a natural symplectic structure $\overline{\omega_0}$. $\overline{R}$ is symplectically isomorphic to a subspace R of T_xM of dimension $2k$, k being the corank of the singular point. The quadratic parts of $F_1, \ldots, F_n$ at x generate a subspace $\mathcal{F}_R^{(2)}(x)$ of the space of quadratic forms on R. This subspace is a commutative subalgebra under the Poisson bracket, and is often called in the literature the *transversal linearization* of $\mathbf{F}$. With the above notations, the following definition is standard (see e.g. [17]).

DEFINITION 2.1. A singular point x of corank k is called *nondegenerate* if $\mathcal{F}_R^{(2)}(x)$ is a Cartan subalgebra of the algebra of quadratic forms on R.

The following classical result of Williamson [44] classifies linearized nondegenerate singular points (see also [1]).

THEOREM 2.2. (Williamson). *For any Cartan subalgebra $\mathcal{C}$ of $Q(2n)$ there is a symplectic system of coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $\mathbf{R}^{2n}$ and a basis $f_1, \ldots, f_n$ of $\mathcal{C}$ such that each f_i is one of the following*

$$f_i = x_i^2 + y_i^2 \textit{(elliptic type)}$$
$$f_i = x_i y_i \textit{(hyperbolic type)}$$
$$\left.\begin{array}{ll} f_i & = x_i y_{i+1} - x_{i+1} y_i \\ f_{i+1} & = x_i y_i + x_{i+1} y_{i+1} \end{array}\right\} \textit{(focus-focus type)}$$

For example, when $n = 2$ there are 4 possible combinations: (1) f_1, f_2 elliptic; (2) f_1 elliptic, f_2 hyperbolic; (3) f_1, f_2 hyperbolic and (4) f_1, f_2 focus-focus. In some papers, these 4 cases are also called center-center, center-saddle, saddle-saddle and focus-focus respectively (cf. [32]).

The above theorem is nothing but the classification up to conjugacy of Cartan subalgebras of real symplectic algebras. In his paper, Williamson also considered other subalgebras. Recall that a complex symplectic algebra has only one conjugacy class of Cartan subalgebras. That is to say, if we consider systems with analytic coefficients and complexify them, then the above 3 types become the same.

DEFINITION 2.3. If the transversal linearization of an IHS at a nondegenerate singular point x of corank k has $k_e = k_e(x)$ elliptic components, $k_h = k_h(x)$ hyperbolic components, $k_f = k_f(x)$ focus-focus components ($k_e, k_h, k_f \geqslant 0, k_e + k_h + 2k_f = k$), then we will say that point x has *Williamson type* (k_e, k_h, k_f). Point x is called *simple* if $k_e + k_h + k_f = 1$, i.e. if x is either codimension 1 elliptic, codimension 1 hyperbolic, or codimension 2 focus-focus.

It is useful to know the group of local linear automorphisms for simple fixed points of linear IHS's (with one or two degrees of freedom respectively), for these local linear automorphisms serve as components of the linearization of local automorphisms of nonlinear IHS's. We have:

PROPOSITION 2.4. *Let x be a simple fixed point of a linear IHS ($k_e+k_h+k_f = 1$).*

(a) *If x is elliptic then the group of local linear symplectomorphisms which preserve the moment map is isomorphic to a circle $\mathbf{S}^1$.*
(b) *If x is hyperbolic then this group is isomorphic to $\mathbf{Z}_2 \times \mathbf{R}^1$ = the multiplicative group of nonzero real numbers.*
(c) *If x is focus-focus then this group is isomorphic to $\mathbf{S}^1 \times \mathbf{R}^1$ = the multiplicative group of nonzero complex numbers.*

Proof. We use the canonical coordinates given in Williamson's Theorem 2.2. In elliptic case the group of local linear automorphisms is just the group of rotations of the plane $\{x_1, y_1\}$. In hyperbolic case, the group of endomorphisms of the plane $\{x_1, y_1\}$ which preserve the function x_1y_1 consists of elements $(x_1, y_1) \mapsto (ax_1, y_1/a)$ and $(x_1, y_1) \mapsto (ay_1, x_1/a)$ $(a \in \mathbf{R}\backslash\{0\})$. But the symplectic condition implies that only the maps $(x_1, y_1) \mapsto (ax_1, y_1/a)$ belong to our local automorphism group. In focus-focus case, introduce the following special complex structure in $\mathbf{R}^4$, for which $x_1 + ix_2$ and $y_1 - iy_2$ are holomorphic coordinates. Then $f_2 - if_1 = (x_1 + ix_2)(y_1 - iy_2)$ and the symplectic form $\omega = \sum \mathrm{d}x_i \wedge \mathrm{d}y_i$ is the real part of $d(x_1 + ix_2) \wedge d(y_1 - iy_2)$. After that the proof for the focus-focus case reduces to the proof for the hyperbolic case, with complex numbers instead of real numbers. □

The local structure of nondegenerate singular points of IHS has been known for some time. The main result is that locally near a nondegenerate singular point, the linearized and the nonlinearized systems are the same. More precisely, if for each IHS we call the singular foliation given by the connected level sets of the moment map the *associated singular Lagrangian foliation* (it is a regular Lagrangian foliation outside singular points of the moment map), then we have:

THEOREM 2.5 (Vey-Eliasson). *Locally near a nondegenerate singular point of an IHS, the associated singular Lagrangian foliation is diffeomorphic to that one given by the linearized system. In case of fixed points, i.e. singular points of maximal corank, the word 'diffeomorphic' can be replaced by 'symplectomorphic'.*

The above theorem was proved by Rüssmann [41] for the analytic case with $n = 2$, Vey [43] for the analytic case with any n, Eliasson [23] for the smooth case. What Rüssmann and Vey proved is essentially that Birkhoff formal canonical transformation converges if the system is integrable. Note that in Eliasson's paper there are some minor inaccuracies (e.g. Lemma 6), and there is only a sketch of the proof of symplectomorphism for cases other than elliptic. For elliptic case, a new proof is given by Dufour and Molino [20]. Lerman and Umanskii [32] also considered the smooth case with $n = 2$, but their results are somewhat weaker than the above theorem. Rigorously speaking, the cited papers proved the above theorem only for the case of fixed points. But the general case follows easily from the fixed point case.

Thus the local topological structure of the singular Lagrangian foliation near a nondegenerate singular point is the same as of a linear Poisson action, up to diffeomorphisms. It is diffeomorphic to a direct product of a non-singular component and simple singular components. Here a non-singular component means a local regular foliation of dimension $n - k$ and codimension $n - k$, a simple singular component is a singular foliation associated to a simple singular fixed point. In particular, if x is a nondegenerate singular point, then we can describe the local level set of the moment map which contains x as follows: If x is a codimension 1 elliptic fixed point ($n = 1$), then this set is just one point x. If x is a codimension 1 hyperbolic fixed point ($n = 1$), then this set is a 'cross' (union of the local stable and nonstable curves). If x is a codimension 2 focus-focus fixed point ($n = 2$) then this set is a '2-dimensional cross', i.e. a union of 2 transversally intersecting 2-dimensional Lagrangian planes. In general, the local level set is diffeomorphic to a direct product of local level sets corresponding to the above simple cases, and a disk of dimension equal to the rank of the moment map at x. Note that the cross and 2-dimensional cross have a natural stratification (into one 0-dimensional stratum and four 1-dimensional strata; one 0-dimensional stratum and two 2-dimensional strata respectively). Thus by taking product, it follows that the local level set of x (i.e. containing x) also has a unique natural stratification. The proof of the following proposition is obvious:

PROPOSITION 2.6. *For the natural stratification of the local level set of the moment map at a nondegenerate singular point x we have:*

(a) *The stratification is a direct product of simplest stratifications discussed above.*
(b) *Each local stratum is also a domain in an orbit of the IHS.*
(c) *The stratum through x is the local orbit through x and has the smallest dimension among all local strata.*
(d) *If y is another point in this local level set, then $k_e(y) = k_e(x), k_h(y) \leqslant k_h(x), k_f(y) \leqslant k_f(x)$, and the stratum through y has dimension greater than the dimension of the stratum through x by $k_h(x) - k_h(y) + 2k_f(x) - 2k_f(y)$.*

(e) *There are points* $y^0 = y, y^1, \dots, y^s = x$, $s = k_h(x) - k_h(y) + k_f(x) - k_f(y)$, *in the local level set, such that the stratum of* y^i *contains the stratum of* y^{i+1} *in its closure, and* $k_h(y^{i+1}) - k_h(y^i) + k_f(y^{i+1}) - k_f(y^i) = 1$.

3. Associated singular Lagrangian foliation

To understand the topology of an integrable system, it is necessary to consider not only local structure of singularities, as in the previous section, but also non-local structure of them. By this I mean the structure near the singular leaves of the Lagrangian foliation associated to an IHS. We begin by recalling the following definition.

DEFINITION 3.1. Let $\mathbf{F}\colon\ (M^{2n}, \omega) \to \mathbf{R}^n$ be the moment map of a given IHS, and assume that the preimage of every point in $\mathbf{R}^n$ under $\mathbf{F}$ is compact and the differential $D\mathbf{F}$ is nondegenerate almost everywhere.

(a) The *leaf through a point* $x_0 \in M$ is the minimal closed invariant subset of M which contains x_0 and which does not intersect the closure of any orbit of the action, except the orbits contained in it.
(b) From (a) it is clear that every point is contained in exactly one leaf. The (singular) foliation given by these leaves is called the *Lagrangian foliation associated to the IHS*. The orbit space of the associated Lagrangian foliation (i.e. the topological space whose points correspond to the leaves of the foliation, and open sets correspond to saturated open sets) is called the orbit space of the IHS.
(c) A *singular leaf* is a leaf that contains a singular point. An *nondegenerate singular leaf* is a singular leaf whose singular points are all nondegenerate. A singular leaf is called *of codimension* k if k is the maximal corank of its singular points.

According to Arnold–Liouville theorem, non-singular orbits are Lagrangian tori of dimension n – hence the notion of associated Lagrangian foliation. For singular leaves, we use the term codimension instead of corank since they are nonlinear objects.

PROPOSITION 3.2. (a) *The moment map* $\mathbf{F}$ *is constant on every leaf of the associated Lagrangian foliation.*

(b) *If the singular point* x_0 *has corank* k*, then the orbit* $\mathcal{O}(x_0)$ *of the Poisson action through it has dimension* $n - k$.

(c) *If the closure of* $\mathcal{O}(x_0)$ *contains a nondegenerate singular point* y*, and* y *does not belong to* $\mathcal{O}(x_0)$*, then the corank of* y *is greater than the corank of* x_0.

(d) *If the point* x_0 *is nondegenerate, then there is only finitely many orbits whose closure contains* x_0.

(e) *In a nondegenerate leaf the closure of every orbit contains only itself and orbits of smaller dimensions.*

(f) *Every nondegenerate singular leaf contains only finitely many orbits.*

(g) *The orbits of dimension $n-k$ (smallest dimension) in a nondegenerate leaf of codimension k are diffeomorphic to the $(n-k)$-dimensional torus* $\mathbf{T}^{n-k}$.

Proof. Assertion (a) follows from the definitions. To prove (b), notice that, since the Poisson action preserves the symplectic structure and the moment map, every point in the orbit $\mathcal{O}(x_0)$ has the same corank k. We can assume, for example, $dF_1 \wedge dF_2 \wedge \cdots \wedge dF_{n-k} \neq 0$ at x_0. Then the vector fields $X_{F_1}, \ldots, X_{F_{n-k}}$ generate a $(n-k)$-dimensional submanifold through x_0. Because of the corank k, the other vector field $X_{F_{n-k+1}}, \ldots, X_{F_n}$ are tangent to this submanifold, so this submanifold coincides with $\mathcal{O}(x_0)$. b) is proved. Assertions (c) and (d) follow from Proposition 2.6. (e) is a consequence of (b). (f) follows from (d) and the compactness assumption. It follows from (e) that the orbits of smallest dimension in N must be compact. Note that, in view of (b), these orbits are orbits of a locally-free $\mathbf{R}^{n-k}$ action. Hence they are tori, and (g) is proved. □

In particular, a singular leaf of codimension k has dimension greater or equal to $n-k$: it must contain a compact orbit of dimension $n-k$ but it may also contain non-compact orbits of dimension greater than $n-k$. If a leaf is nonsingular or nondegenerate singular, then it is a connected component of the preimage of a point under the moment map, and a tubular neighborhood of it can be made saturated with respect to the foliation (i.e. it consists of the whole leaves only). Also, the local structure of nondegenerate singular leaves is given by Proposition 2.6. One can imagine a singular leaf, at least in case it is nondegenerate, as a stratified manifold – the stratification given by the orbits of various dimensions, and the codimension of the leaf equals n minus the smallest dimension of the orbits.

Later on, a *tubular neighborhood $\mathcal{U}(N)$ of a nondegenerate leaf N* will always mean an appropriately chosen sufficiently small saturated tubular neighborhood. The Lagrangian foliation in a tubular neighborhood of a nondegenerate singular leaf N will be denoted by $(\mathcal{U}(N), \mathcal{L})$. Remark that N is a deformation retract of $\mathcal{U}(N)$. Indeed, one can retract from $\mathcal{U}(N)$ to N by using an appropriate gradient vector field. Thus, throughout this paper, in topological arguments, where only the homotopy type matters, we can replace N by $\mathcal{U}(N)$ and vice versa.

By a *singularity of an IHS* we will mean (a germ of) a singular foliation $(\mathcal{U}(N), \mathcal{L})$. Two singularities are called *topologically equivalent* if there is a foliation preserving homeomorphism (not necessarily symplectomorphism) between them. In this paper all singularities are assumed to be nondegenerate.

Recall that, given an IHS in (M^{2n}, ω), its orbit space has a natural topology which we will use: the preimages of open subsets of the orbit space are saturated open subsets of M^{2n}. It is the strongest topology for which the projection map from M^{2n} to the orbit space is continuous.

PROPOSITION 3.3. *If all singularities of an IHS are nondegenerate, then the corresponding orbit space is a Hausdorff space.*

Proof. If all singularities are nondegenerate, then each leaf of the Lagrangian foliation is a connected level set of the moment map. The moment map can be factored through the orbit space (denoted by O^n):

$$M^{2n} \to O^n \to \mathbf{R}^n$$

From this the proposition follows easily. □

Of course, the converse statement to the above proposition is not true. For the orbit space to be Hausdorff, it is not necessary that all singularities are nondegenerate. In fact, IHS's met in mechanics sometimes admit so-called simply-degenerate singularities, and the orbit space is still Hausdorff.

Thus, the orbit space, under the condition of nondegeneracy, is a good topological space. A point in the orbit space O^n is called *singular* if it corresponds to a singular leaf. By Arnold–Liouville theorem, outside singular points the orbit space admits a unique natural integral affine structure. Later, from the results of Section 6, one will see that under the additional assumption of topological stability, the orbit space is a stratified affine manifold in a very natural sense. This assumption is called topological stability. Note that this stability is distinct from the stability in the sense of differential equations, which for nondegenerate singularities of IHS's is equivalent to the ellipticity condition.

Let N be a nondegenerate singular leaf. Then N consists of a finitely many orbits of our IHS. It is well-known that each orbit is of the type $\mathbf{T}^c \times \mathbf{R}^o$ for some nonnegative integers c, o. We would like to know more about these numbers, in order to understand the topology of N. Recall that $c + o$ is the dimension of the orbit, which is equal to the rank of the moment map on it.

DEFINITION 3.4. Let N be a nondegenerate singular leaf, $x \in N$. If the orbit $\mathcal{O}(x)$ is diffeomorphic to $\mathbf{T}^c \times \mathbf{R}^o$, then the numbers c and o are called *degree of closedness and degree of openness* of $\mathcal{O}(x)$ respectively. The 5-tuple (k_e, k_h, k_f, c, o) is called the *orbit type* of $\mathcal{O}(x)$, where (k_e, k_h, k_f) is the Williamson type of x.

It is clear that the orbit type of an orbit $\mathcal{O}$ does not depend on the choice of a point x in it. The orbit type of a regular Lagrangian torus is $(0, 0, 0, n, 0)$ where n is the number of degrees of freedom. In general, we have $k_e + k_h + 2k_f + c + o = n$. Moreover:

PROPOSITION 3.5. *The following three values: $k_e, k_f + c, k_f + k_h + o$ are invariants of a singular leaf N of the IHS. In other words, they don't depend on the choice of an orbit in N. They will be called* degree of ellipticity, closedness, and hyperbolicity *of N, respectively.*

Proof. Let x be an arbitrary point in N and y be a point sufficiently near x, not belonging to $\mathcal{O}(x)$. Since N is closed, it suffices to prove that $k_e(x) =$

$k_e(y), k_f(x) + c(x) = k_f(y) + c(y)$ and $k_f(x) + k_h(x) + o(x) = k_f(x) + k_h(y) + o(y)$.

Since y is near to x and N is nondegenerate, it follows that the closure of $\mathcal{O}(y)$ contains $\mathcal{O}(x)$ and y belongs to the local level set of the moment map through x. By Proposition 2.6, we can assume that $k_h(y) = k_h(x) - 1, k_f(y) = k_f(x)$ or $k_h(y) = k_h(x), k_f(y) = k_f(x) - 1$.

We will prove for the case $k_h(y) = k_h(x) - 1, k_f(y) = k_f(x)$. The other case is similar. The fact $k_e(y) = k_e(x)$ follows from Proposition 2.6. It remains to prove that $c(y) = c(x)$, since $c(y) + o(y) = c(x) + o(x) + 1$ by Proposition 2.6. On one hand, $\mathcal{O}(x) = \mathbf{T}^{c(x)} \times \mathbf{R}^{o(x)}$ implies that there is a free $\mathbf{T}^c(x)$ action on $\mathcal{O}(x)$, which is part of the Poisson action (our IHS). By using the Poincaré map (in $c(x)$ directions), one can approximate the generators of this part of the Poisson action by nearby generators (in the Abelian group $\mathbf{R}^n$) so that they give rise to a $\mathbf{T}^{c(x)}$ free action on $\mathcal{O}(y)$. It follows that $c(y) \geqslant c(x)$. On the other hand, since $\mathcal{O}(x)$ is in the closure of $\mathcal{O}(y)$, a set of $c(y)$ generators of the Poisson action, say $X_1, \ldots, X_{c(y)}$, which generate the free $\mathbf{T}^{c(y)}$ action on $\mathcal{O}(y)$, also generate a $\mathbf{T}^{c(y)}$ action on $\mathcal{O}(x)$. If this action on $\mathcal{O}(x)$ is locally free then $c(x) \geqslant c(y)$ and we are done. Otherwise we would have that $\sum a_i X_i(x) = 0$ for some nonzero linear combination $\sum a_i X_i$ with real coefficients. But then $\sum a_i X_i$ is small at y (in some metric), therefor it cannot give rise to period 1 flow on $\mathcal{O}(y)$, and we come to a contradiction. □

As a consequence of the above proposition, we have:

PROPOSITION 3.6. *If a point x in a singular leaf N has corank k equal to the codimension of N (maximal possible) and has Williamson type (k_e, k_h, k_f), then any other point x' in N with the same corank will have the same Williamson type. We will say also that N has the Williamson type (k_e, k_h, k_f).*

Proof. k_e is invariant because of the previous proposition. It suffices to proof that k_f is invariant. But we know that $k_f + c$ is invariant, and for points of maximal corank we have $c = n - k$. □

The local structure theorems of Williamson, Vey, and Eliasson suggest that to study the topology of singular Lagrangian foliations we should study the following simple cases first: codimension 1 elliptic singularities (i.e. of Williamson type (1, 0, 0)), codimension 1 hyperbolic singularities (Williamson type (0, 1, 0)), and codimension 2 focus-focus singularities (Williamson type (0, 0, 1)). The cases of codimension 1 hyperbolic and codimension 2 focus-focus singularities are considered in the next sections. The structure of elliptic singularities (Williamson type (k, 0, 0)) has been known for some time, and we will recall it now. The point is that since elliptic orbits are stable in the sense of differential equations, each elliptic singular leaf is just one orbit and the non-local problem is just a parametrized version of the local problem. In particular, for elliptic singularities the Lagrangian foliations associated to linearized and nonlinearized systems are the same. To be more precise:

DEFINITION 3.7. A nondegenerate singular point (singularity) of corank (codimension) k of an IHS is called *elliptic* if it has Williamson type $(k, 0, 0)$, i.e. if it has only elliptic components.

PROPOSITION 3.8. *A nondegenerate singular leaf of codimension k is elliptic if and only if it has dimension $n-k$. In this case it is an isotropic $(n-k)$-dimensional smooth torus, called* elliptic torus of codimension k.

THEOREM 3.9. ([23, 20]). *Let N be an elliptic singularity of codimension k of an IHS given by the moment map $\mathbf{F} : M \to \mathbf{R}^n$. Then in a tubular neighborhood of N there is a symplectic system of coordinates $(x_1, \ldots, x_n, y_1 (\text{mod } 1), \ldots, y_{n-k} (\text{mod } 1), y_{n-k+1}, \ldots, y_n)$, (i.e. $\omega = \sum \mathrm{d}x_i \wedge \mathrm{d}y_i$), for which $N = \{x_1 = \cdots = x_n = y_{n-k+1} = \cdots = y_n = 0\}$, such that $\mathbf{F}$ can be expressed as a smooth function of $x_1, \ldots, x_{n-k}, x_{n-k+1}^2 + y_{n-k+1}^2, \ldots, x_n^2 + y_n^2$:*

$$\mathbf{F} = \mathbf{F}(x_1, \ldots, x_{n-k}, x_{n-k+1}^2 + y_{n-k+1}^2, \ldots, x_n^2 + y_n^2).$$

In other words, the Lagrangian foliation given by an IHS near every elliptic singularity is symplectomorphic to the Lagrangian foliation associated to the linearized system.

See the paper of Dufour and Molino [20] for a proof of the above theorem. In [23], Eliasson simply says that it is just a parametrized version of his local structure theorem.

COROLLARY 3.10. *If N is an elliptic singular leaf of codimension k of an IHS with n degrees of freedom then there is a unique natural $\mathbf{T}^n$ symplectic action in $\mathcal{U}(N)$ which preserves the moment map, and which is free almost everywhere. There is also a highly non-unique $\mathbf{T}^{n-k}$ subgroup action of the above action, which is free everywhere in $\mathcal{U}(N)$.*

Using one of the free $\mathbf{T}^{n-k}$ actions in the above corollary, one can use the Marsden-Weinstein reduction procedure (cf. [35] to reduce codimension k elliptic singularities of IHS's with n degrees of freedom to codimension k elliptic singularities of IHS's with k degrees of freedom. □

4. Codimension 1 case

All known integrable systems from physics and mechanics exhibit codimension 1 nondegenerate singiularities. For these singularities, it follows from Williamson's classification that there are only 2 possible cases: elliptic and hyperbolic. Elliptic singularities were discussed in the previous section. In this section we assume all singularities to be hyperbolic, though the results will be also obviously true for elliptic singularities (except the uniqueness of the torus $\mathbf{T}^{n-1}$ action).

THEOREM 4.1. *Suppose N is a nondegenerate codimension 1 singularity of an IHS with n degrees of freedom. Then in $\mathcal{U}(\mathcal{N})$ there is a Hamiltonian $\mathbf{T}^{n-1}$ action such that:*

(a) *This group action preserves the moment map.*
(b) *This group action is locally free and it is free outside singular points of N.*
(c) *Each singular point can have at most one non-trivial element of $\mathbf{T}^{n-1}$ which preserves it. In other words, the isotropy group of each point is at most $\mathbf{Z}_2$.*

REMARK. In case $n = 2$, the above theorem appeared in [46].

Proof. Since N is hyperbolic, it contains not only singular points, but also regular points. Let y be a regular point in N, then the orbit $\mathcal{O}(y)$ will be diffeomorphic to $\mathbf{T}^{n-1} \times \mathbf{R}^1$, according to the previous section. We can choose $n - 1$ generators of the Poisson action of our IHS, in terms of $n - 1$ Hamiltonian vector fields $X_1, \ldots, X_{n-1}$, such that the flow of each X_i restricted to $\mathcal{O}(y)$ is periodic with period 1, and together they generate a free $\mathbf{T}^{n-1}$ action on $\mathcal{O}(y)$. Let $\mathcal{O}(x)$ be a singular orbit contained in the closure of $\mathcal{O}(y)$. Then of course the flow of X_i are also time 1 periodic in $\mathcal{O}(x)$. Let $\mathcal{O}(z)$ be another regular orbit in N which also contains $\mathcal{O}(x)$. Let $y' \in \mathcal{O}(y)$, $z' \in \mathcal{O}(z)$ are two points sufficiently close to x. We will assume that they lie in different n-disks of the cross (times a disk) in the local level set at x (see Proposition 2.6). Then it follows from the local structure of the Lagrangian foliation at x that there is a regular orbit $\mathcal{O}_1$ (not contained in N) passing arbitrarily near to both points y and z. It follows that there are generators $X_i^!$ near to X_i of our Poisson action such that the flows of X_i are periodic of period exactly 1 in $\mathcal{O}_1$. In turn, there are generators $X_i^!$ near to $X_i^!$ such that the flows of $X_i^!$ are periodic of period exactly 1 in $\mathcal{O}(z)$. By making $\mathcal{O}_1$ tend to pass through y and z, in the limit we obtain that the flows of X_i are periodic of period exactly 1 in $\mathcal{O}(z)$. Since N is connected, by induction we see that $X_1, \ldots X_{n-1}$ generates a $\mathbf{T}^{n-1}$ action on N, and this action is free on regular orbits (of type $\mathbf{T}^{n-1} \times \mathbf{R}$) in N. From the proof of Proposition 3.5 it follows that this action is also locally free on singular orbits of N.

To extend the above $\mathbf{T}^{n-1}$ action from N to $\mathcal{U}(N)$, one can use the same process of extending the vector fields from $\mathcal{O}(y)$ to $\mathcal{O}_1$ as before. As a result, we obtain a natural $\mathbf{T}^{n-1}$ action in $\mathcal{U}(N)$, unique up to automorphisms of $\mathbf{T}^{n-1}$. Obviously, this action preserves our IHS. Outside singular leaves, this action is Hamiltonian, because of Arnold-Liouville theorem. Since singular leaves in $\mathcal{U}(N)$ form a small subset of measure 0, it follows that the above action is symplectic in the whole $\mathcal{U}(N)$. Since the symplectic form ω is exact in $\mathcal{U}(N)$ (because it is obviously exact in N), this torus action is even Hamiltonian.

We have proved assertions (a) and (b). To prove (c), assume that ξ is a nonzero linear combination of X_i, which generates a periodic flow in $\mathcal{O}(x)$ of period 1, and denote by g_1 its time 1 map in N. Since y' is near to x, $g_1(y')$ is also near to x. If $g_1(y')$ belongs to the same local stratum at x as y', then it follows that $g_1(y')$ must coincide with y', and ξ is an integral linear combination of X_i. If not, $g_1(y')$ must line in the local stratum 'opposite' to the local stratum containing y', and it will follow that twice of ξ is an integral combination of X_i. Thus, any isotropic element of the $\mathbf{T}^{n-1}$ action on $\mathcal{O}(x)$ has order at most 2. To see that the isotropic subgroup has at most two elements, assume for example that $X_1/2$ and $X_2/2$ generate periodic flows of period 1 in $\mathcal{O}(x)$. Then the time 1 maps of these vector fields send y' to the opposite local stratum. It follows that the time 1 map of $(X_1+X_2)/2$ sends y' to itself, and $(X_1+X_2)/2$ is an integral linear combination of X_i, a contradiction.□

PROPOSITION 4.2. *If in the above theorem, there are points with non-trivial isotropy group, then there is a natural unique double covering of $\mathcal{U}(N)$, denoted by $\overline{\mathcal{U}(N)}$, such that everything (symplectic form, IHS, $\mathbf{T}^{n-1}$-action) can be lifted to this double covering, and the $\mathbf{T}^{n-1}$-action in this covering will be free.*

Proof. The above theorem, without the torus action, was in fact proved by Fomenko [25]. To prove it, one can write down a presentation for the fundamental group $\pi_1(N)=\pi_1(\mathcal{U}(N))$ as follows:

Generators (three types): $\alpha_1,\dots,\alpha_{n-1}$, which are generated by $X_1,\dots,X_{n-1}$ in a regular orbit in N. $\beta_1,\dots,\beta_s (s\geqslant 0)$, which are 'exceptional cycles': they lie in singular orbits and are not conjugate to an integral combination of α_i. $\gamma_1,\dots,\gamma_t (t\geqslant 1)$, which are 'base cycles': the fundamental group of the quotient space of N by the $\mathbf{T}^{n-1}$ action is generated by the image of these cycles.

Relations: α_i commute with all the other generators. Twice of β_i are integral combinations of α_i.

Let G be the subgroup of $\pi_1(N)$, consisting of the words which contain an even total number of β_i. For example, α_1 and $\beta_1\beta_2$ are elements of G. Then G is a subgroup of index 2, and it is easy to see that the double covering associated to G will satisfy our requirements. □

In the following theorem, which gives a canonical form for codimension 1 nondegenerate hyperbolic singularities, we assume that the torus action discussed above is free. Otherwise we will take the double covering first.

THEOREM 4.3. *There exist a system of coordinates $(x_1,y_1,\dots,x_n,y_n)$ in (an appropriately chosen) $\mathcal{U}(N)$, where $y_1,\dots,y_{n-1}$ are defined modulo 1 (angle coordinates), and (x_n,y_n) defines an immersion from a surface P^2 to $\mathbf{R}^2$, such that:*

(a) *These coordinates give a diffeomorphism from $\mathcal{U}(N)$ to $\mathbf{D}^{n-1}\times\mathbf{T}^{n-1}\times P^2$*

(b) *The symplectic form ω has the canonical form*

$$\omega=\sum_{i=1}^{n-1}dx_i\wedge dy_i+\pi^*(\omega_1)$$

where ω_1 is some area form on P^2, and π^* means the lifting.

(c) *$x_1,\dots,x_{n-1}$ are invariants of our IHS.*

Such a system of coordinates is not unique, but will be called canonical.

Proof. Let x_i be the Hamiltonian functions of Hamiltonian vector fields which generate the $\mathbf{T}^{n-1}$ action in Theorem 4.1, $x_i(N) = 0$. Denote their corresponding vector fields by ξ_i. Remark that the $\mathbf{T}^{n-1}$-group action in Theorem 4.1 gives rise to a trivial $\mathbf{T}^{n-1}$-foliation. Denote its base by B^{n+1}. Let L be a section of this foliation, and define functions z_i $(i = 1, \dots, n-1)$ by putting them equal to zero on L and setting $dz_i(\xi_i) = \{x_i, z_i\} = 1$. Set $\omega_1 = \omega - \sum_1^{n-1} \mathrm{d}x_i \wedge \mathrm{d}z_i$. Then one checks that $L_{\xi_i}\omega_1 = i_{\xi_i}\omega_1 = 0$. It means that ω_1 is a lift of some closed 2-form from B^{n+1} to $\mathcal{U}(N)$, which we will also denote by ω_1. Since ω is non-degenerate, it follows that ω_1 is non-degenerate on every 2-surface (with boundary) $P^2_{x_1,\dots,x_{n-1}} = B^{n+1} \cap \{x_1, \dots, x_{n-1} \text{fixed}\}$. Using Moser's path method [36], one can construct a diffeomorphism $\phi : B^{n+1} \to \mathbf{D}^{n-1} \times P^2$, under which ω_1 restricted to $P^2_{x_1,\dots,x_{n-1}}$ does not depend on the choice of $x_1, \dots, x_{n-1}$. In other words, there is an area form ω_2 on P^2 such that $\omega_1 - \omega_2$ vanishes on every $P^2_{x_1,\dots,x_{n-1}}$. Since $d(\omega_1 - \omega_2) = 0$, we can write it as $\omega_1 - \omega_2 = d(\sum_1^{n-1} a_i \, \mathrm{d}x_i + \beta)$, where β is some 1-form on B^{n+1} (which is not zero on $P^2_{x_1,\dots,x_{n-1}}$ in general). If we can eliminate β, i.e. write $\omega_1 - \omega_2 = d(\sum_1^{n-1} a_i \, \mathrm{d}x_i)$, then we will have $\omega = (\sum_1^{n-1} \mathrm{d}x_i \wedge \mathrm{d}z_i - \sum_1^{n-1} \mathrm{d}x_i \wedge \mathrm{d}a_i + \omega_2 = \sum_1^{n-1} \mathrm{d}x_i \wedge d(z_i - a_i) + \omega_2$, and the theorem will be proved by putting $y_i = z_i - a_i$. Let us show now how to eliminate β. β restricted on every $P^2_{x_1,\dots,x_{n-1}}$ is a closed 1-form, hence it represents a cohomology element $[\beta](x_1, \dots, x_{n-1}) \in H^1(P^2)$. If $[\beta](x_1, \dots, x_{n-1}) \equiv 0$ then $\beta = dF - b_1 \, \mathrm{d}x_1 - \dots - b_{n-1} \, \mathrm{d}x_{n-1}$ for some functions $F, b_1, \dots, b_n$, and we have $\omega_1 - \omega_2 = d(\sum(a_i - b_i x_i)dx_i)$. In general, we can achieve $[\beta](x_1, \dots, x_{n-1}) \equiv 0$ by induction on the number of generators of $H^1(P^2)$ as follows. Let γ be a simple curve in P^2 which represents a non-zero cycle. Set $b(x_1, \dots, x_{n-1}) = \langle [\beta], \gamma \rangle (x_1, \dots, x_{n-1})$. Immerse P^2 in an annulus so that only simple curves homotopic to γ go to non-zero cycle there, other simple curves go to vanishing cycles. By this immersion we have a (non single-valued) system of coordinates (u, v) on P^2, u – mod 1. ω_2 has the form $\omega_2 = adu \wedge dv$ for some positive function a. Change ω_2 for the following 2-form on $\mathcal{U}(N)$: $\omega_2' = \omega_2 + db \wedge du$. It is clear that ω_2' and ω_2 restricted on every $P^2_{x_1,\dots,x_{n-1}}$ are the same. Moreover, ω_2' is closed and of rank 2. Thus the distribution by its $(n-1)$-dimensional tangent zero-subspaces is integrable, and it gives rise again to a diffeomorphism $\phi' : B^{n+1} \to \mathbf{D}^{n-1} \times P^2$. Replacing ω_2 by ω_2', we have $\omega_1 - \omega_2' = d(\sum_1^{n-1} a_i dx_i + \beta')$, with $\beta' = \beta - bdu$, whence $\langle [\beta'], \gamma \rangle = 0$. □

Using the $\mathbf{T}^{n-1}$ action constructed in the previous theorems, we can apply the Marsden-Weinstein reduction procedure to obtain a $(n-1)$-dimensional family of codimension 1 hyperbolic singularities of IHS with one degree of freedom. These singularities are called *surface singularities* because they lie in 2-dimensional surfaces. They are so simple that we are not going to make them simpler. In [11], Fomenko called them 'letter-atoms'. These 'letter-atoms' were computed for many interesting examples of IHS arising in classical mechanics and physics (see e.g. 5; 26, 28, 39, 39, 40). Some of these codimension 1 singularities even have special names (see [11]).

After the Marsden-Weinstein reduction we have an $(n-1)$-dimensional family of surface singularities. There is no guarantee that surface singularities in this family must be topologically equivalent, although in all known examples of IHS arising in mechanics and physics they turn out to be so. This situation leads to the following:

PROPOSITION 4.4. *For a codimension 1 hyperbolic singularity $\mathcal{U}(N)$ of an IHS with n degrees of freedom, the following three conditions are equivalent:*

(a) *All singular leaves in $\mathcal{U}(N)$ are topologically equivalent.*
(b) *Under the Marsden-Weinstein reduction with respect to the torus action given by Theorem 4.1, the topological structure of surface singularities in the obtained $(n-1)$-dimensional family is constant (i.e. does not depend on the parameter).*
(c) *The singular value set of the moment map restricted to $\mathcal{U}(N)$ is a smooth $(n-1)$-dimensional disk containing the image of N in $\mathbf{R}^n$.*

DEFINITION 4.5. A non-degenerate codimension 1 singularity of an IHS is called *topologically stable* if it satisfies one of the equivalent conditions in Proposition 4.4.

Proof of Proposition 4.4. The proof is rather straightforward. We will prove that (c) implies (a). Without loss of generality, we can assume that the local singular value set in $\mathbf{R}^n$ is given by the equation $F_1 = 0$, where F_1 is one of the components of the moment map. Then $F_1 = 0$ on all singular leaves in $\mathcal{U}(N)$. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two hyperbolic orbits in N. It follows from the nondegeneracy that $F_1(y) \neq 0$ for any point y near to $\mathcal{O}_1$ and not belonging to a local singular leaf near to $\mathcal{O}_1$. Since by continuity, any singular leaf near to $\mathcal{O}_2$ passes nearby $\mathcal{O}_1$, it follows that each such a singular leaf must contain a local singular leaf near to $\mathcal{O}_1$. This fact implies the topological equivalence of all singular leaves near to N. □

REMARK. In case $n = 2$ the notion of topological stability (for codimension 1 hyperbolic singularities) appeared in [9].

Theorem 4.3 tells us that if the $\mathbf{T}^{n-1}$ action on $\mathcal{U}(\mathcal{N})$ is free then $\mathcal{U}(\mathcal{N})$ is symplectomorphic to $\mathbf{D}^{n-1} \times \mathbf{T}^{n-1} \times P^2$. If the action is not free then $\mathcal{U}(\mathcal{N})$ is symplectomorphic to the quotient of the above product by a free action of some finite group Γ. This group Γ may be taken to be $\mathbf{Z}_2$, according to Proposition 4.2. However, if we require that Γ act on $\mathbf{D}^{n-1} \times \mathbf{T}^{n-1} \times P^2$ component-wise (i.e. the action commutes with the projections onto the components of the product, then in general it must be bigger than $\mathbf{Z}_2$. Indeed, if $\mathbf{Z}_2$ acts on $\overline{\mathcal{U}(N)} = \mathbf{D}^{n-1} \times \mathbf{T}^{n-1} \times P^2$

with quotient equal to $\mathcal{U}(N)$, then exceptional cycles $\beta_1, \dots \beta_s$ of $\pi_1(N)$ (as in Proposition 4.2) must have the same image in $\mathbf{T}^{n-1}$. It is clearly not the case in general (except if $n = 2$), so we have to modify our construction of finite covering of $\mathcal{U}(N)$ as follows:

Remember that we have generators $\alpha_1, \dots, \alpha_{n-1}; \beta_1, \dots, \beta_s; \gamma_1, \dots, \gamma_t$ for $\pi_1(N)$ (cf. Proposition 4.2). An important observation is that there is a natural homomorphism from $\pi_1(N)$ to the group $\mathbf{T}^{n-1}$, associated to the action of $\mathbf{T}^{n-1}$ on N, whichs sends $\alpha_1, \dots, \alpha_{n-1}, \gamma_1, \dots, \gamma_t$ to 0, and $\beta_1, \dots, \beta_s$ to elements of order 2. Let $G_{\text{can}} \supset \pi_1(N)$ be the kernel of this homomorphism, and $\Gamma_{\text{can}} = \pi_1(N)/G_{\text{can}}$ be the image. So Γ_{can} is a subgroup of $\mathbf{T}^{n-1}$ (which contains only elements of order at most 2). Denote by $\overline{\mathcal{U}(N)}_{\text{can}}$ the normal finite covering of $\mathcal{U}(N)$ corresponding to G_{can}, and call it the canonical covering. Then we have:

THEOREM 4.6. *Let $(\mathcal{U}(N), \mathcal{L})$ be a nondegenerate codimension 1 singularity of an IHS with n degrees of freedom.*

(i) *The symplectic form, IHS, and $\mathbf{T}^{n-1}$-action can be lifted from $\mathcal{U}(N)$ to the canonical covering $\overline{\mathcal{U}(N)}_{\text{can}}$, and the $\mathbf{T}^{n-1}$-action on this canonical covering will be free.*
(ii) *$\overline{\mathcal{U}(N)}_{\text{can}}$ is symplectomorphic to $\mathbf{D}^{n-1} \times \mathbf{T}^{n-1} \times P^2$ with the canonical symplectic form as in Theorem 4.3. Under this symplectomorphism, Γ_{can} acts on $\mathbf{D}^{n-1} \times \mathbf{T}^{n-1} \times P^2$ component-wise, i.e. it commutes with the projections. We say that $\overline{\mathcal{U}(N)}_{\text{can}}$ admits an* equivariant canonical system of coordinates.

Proof. (i) The proof is immediate from the definition of $\overline{\mathcal{U}(N)}_{\text{can}}$.

(ii) $\overline{\mathcal{U}(N)}_{\text{can}}$ is a trivial principal $\mathbf{T}^{n-1}$-bundle, and Γ_{can} acts on its sections. By lifting a section from N, we can choose a section L_1 in $\overline{\mathcal{U}(N)}_{\text{can}}$ which is stable under Γ_{can}-action homotopically. (N is a singular $\mathbf{T}^{n-1}$ bundle with base space equal to a graph, so we can talk about its sections). Define

$$L_2(x) = 1/(\#\Gamma_{\text{can}}) \sum_{\gamma \in \Gamma_{\text{can}}} ((\gamma L_1)(x) - \gamma),$$

where x is a point in the base space of $\overline{\mathcal{U}(N)}_{\text{can}}$, γL_1 is the action of γ on L_1, $-\gamma$ means the action of $-\gamma$ in the torus over x. Then L_2 is a section which is Γ_{can}-equivariant, that is $\gamma L_2 = L_2 + \gamma$ for any $\gamma \in \Gamma_{\text{can}}$. Now applying arguments of the proof of Theorem 4.3 to L_2, keeping the equivariance of L_2 all the time, we will find an equivariant coisotropic section L_3. It is immediate that if we trivialize $\overline{\mathcal{U}(N)}_{\text{can}}$ via L_3, Γ_{can} will act on this trivialization component-wise. □

REMARK. It is also clear that Γ_{can} is the smallest group that acts component-wise on something of the type $\mathbf{D}^{n-1} \times \mathbf{T}^{n-1} \times P^2$ with $\mathcal{U}(N)$ as the quotient. We will call ($\overline{\mathcal{U}(N)}_{\text{can}}$, action of Γ_{can}) the *canonical model* of singularity $(\mathcal{U}(N), \mathcal{L})$ because of this property.

5. Codimension 2 focus-focus case

There are quite many integrable systems which exhibit focus-focus singularities. Topological classification of focus-focus singularities for integrable systems with two degrees of freedom was obtained in [48]. Let us recall the main result from there.

THEOREM 5.1 ([48]). *Let N be a nondegenerate singular leaf in an IHS with two degrees of freedom which contains a focus-focus (fixed) point. Then we have:*

(a) *There is a unique natural $\mathbf{S}^1$ Hamiltonian action in a neighborhood of N which preserves the IHS. This action is locally free outside focus-focus fixed points of N.*

(b) *If N does not contains closed 1-dimensional orbits, then the local orbit space is a disk with a 'removable' singular point at the image of N under the projection. However, it is not affinely equivalent to a regular affine disk. In fact, the monodromy obtained by parallel transportation around the image of N of the affine structure is represented by the matrix $\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)$, where m is the number of fixed focus-focus points in N. Moreover, any two such singularities with the same number of focus-focus points are topologically equivalent.*

(c) *If N contains closed 1-dimensional orbits, then there is an arbitrarily C^∞ small integrable perturbation of the IHS, under which N is replaced by a new singular leaf N' which is close to N, contains the same number of focus-focus points as N, and contains no 1-dimensional closed orbit.*

The above theorem describes the structure of codimension 2 focus-focus singularities in case of 2 degrees of freedom. Assertion (a) can be seen from the local structure of focus-focus singular points. Assertion (b) is called the phenomenon of non-triviality of the monodromy, which was first observed by Duistermaat and Cushman [21] for the spherical pendulum. This nontriviality of the monodromy was then found for various systems (see e.g. [14, 6]) before we observed that it is a common property of focus-focus singularities. The proof of (b) given in [48] is rather simple and purely topological. Independently, Zou [45] also proved (b), for the case $m = 1$, under some additional assumptions. Lerman and Umanskii [32, 34] also studied the topology of focus-focus singularities, but their description seems too complicated. Assertion (c) says that the condition in (b) can be always achived by a small perturbation. (c) can be proved easily by the use of the $\mathbf{S}^1$ action given in (a).

The condition given in (b) of the above theorem is called the *topological stability* condition. In other words, a focus-focus singular leaf (i.e. a singular leaf containing focus-focus points) in an IHS with two degrees of freedom is called topologically stable if it is 'purely focus-focus', i.e. if it does not contain codimension 1 hyperbolic singular points.

Like in previous section, we would like to reduce codimension 2 focus-focus singularities (i.e. singularities which contain codimension 2 focus-focus points) of

IHS's with many degrees of freedom to focus-focus singularities of IHS's with 2 degrees of freedom. First of all, we need some torus action. This action is provided by the following theorem.

THEOREM 5.2. *Let N be a codimension 2 nondegenerate focus-focus singular leaf in an IHS with n degrees of freedom, $n > 2$. Then we have:*

(a) *There is a natural $\mathbf{T}^{n-1}$ action in a tubular saturated neighborhood $\mathcal{U}(N)$ of N, which preserves the given IHS. This group action is unique up to isomorphisms of the Abelian group $\mathbf{T}^{n-1}$.*
(b) *If $\mathcal{O}$ is an orbit in N consisting of codimension 2 focus-focus points, then $\mathcal{O}$ is diffeomorphic to $\mathbf{T}^{n-2}$, and the above action is transitive on $\mathcal{O}$. Moreover, the isotropy group is $\mathbf{S}^1$, i.e. a connected 1-dimensional subgroup of $\mathbf{T}^{n-1}$.*
(c) *If $\mathcal{O}'$ is an orbit in N consisting of codimension 1 hyperbolic singular points, then the above action is transivite and locally free on $\mathcal{O}'$. Moreover, the isotropy group is at most $\mathbf{Z}_2$.*
(d) *All hyperbolic orbits in N can be taken away by a C^∞ small integrable perturbation of the IHS, like in assertion (c) of Theorem 5.1.*

Proof. If $\mathcal{O}$ is a regular orbit in N, then it follows from Proposition 3.5 that $\mathcal{O}$ is diffeomorphis to $\mathbf{T}^{n-1} \times \mathbf{R}^1$. Consequently, there is a natural $\mathbf{T}^{n-1}$ action in $\mathcal{O}$. By the same arguments as in Theorem 4.1, one can extend this action to a natural Hamiltonian $\mathbf{T}^{n-1}$ action in $\mathcal{U}(N)$, which preserves the moment map, and (a) is proved. Assertion (c) is also proved by the same arguments as in Theorem 4.1. From the existence of torus action, one can apply the canonical coordinates of Theorem 4.3 to easily prove assertion (d). Only assertion (b) needs a little more work.

To prove (b), let $\mathcal{O}(x)$ be a focus-focus singular orbit in N. Then it is diffeomorphic to $\mathbf{T}^{n-2}$. Since the $\mathbf{T}^{n-1}$ action is transivite in $\mathcal{O}(x)$, the isotropy group of (or at) $\mathcal{O}(x)$ is a closed 1-dimensional subgroup of the Abelian group $\mathbf{T}^{n-1}$. We have to show that this subgroup is connected.

We can choose $n-1$ generators of the $\mathbf{T}^{n-1}$ action in terms of vector fields $X_1, \dots X_{n-1}$, such that the flows of these vector fields are periodic with minimal period 1 in N, and the vector field X_1 generates the connected component of the isotropy group of $\mathcal{O}(x)$.Assume by contradiction that there is a non-integral linear combination $\xi = \sum a_i X_i$, whose flow is time 1 periodic in $\mathcal{O}(x)$. Say a_2 is not an integral number. Denote the time 1 map of ξ by g. Of course, $g(x) = x$. Let y be a point in N which is near to x and not belonging to $\mathcal{O}(x)$. Then it follows from assertion (c) of Proposition 2.4 that $g(y)$ lies in the same stratum of the local level set (of the moment map at x) as y. Consequently, there is another linear combination $\xi' = \sum a'_i X_i$, with a'_i near to a_i if $i \neq 1$, such that the time one map of ξ' preserves y. But then on one hand, a'_i must be integral, and on the other hand, a'_2 is near to a_2 and cannot be integral. This contradiction shows that the isotropy group at $\mathcal{O}(x)$ of the $\mathbf{T}^{n-1}$ action must be connected. □

We can apply the Marsden-Weinstein reduction procedure to the above $\mathbf{T}^{n-1}$ action. But notice that, since the action is far from being free, in the result we will obtain singular symplectic 2-dimensional orbifolds, but not manifolds. It is what was done in [48] for the case $n = 2$. To reduce to focus-focus singularities of systems with 2 degrees of freedom, however, we need an appropriately chosen $\mathbf{T}^{n-2}$ subgroup action of our $\mathbf{T}^{n-1}$ group action. Denote such a subgroup in $\mathbf{T}^{n-1}$ by $\mathbf{T}_0^{n-2}$. First of all, the action of $\mathbf{T}_0^{n-2}$ on singular focus-focus orbits must be at least locally free. It means that $\mathbf{T}^{n-2}$ is transversal to $S_1, \dots, S_m$ in $\mathbf{T}^{n-1}$, where S_i are isotropy groups at singular focus-focus orbits, denoted by $\mathcal{O}_i$, m being the number of such orbits in N. Second, to get a regular reduction, the action of $\mathbf{T}_0^{n-2}$ on $\mathcal{O}_i$ must be free but not only locally free. If N does not contain hyperbolic orbits, then this condition is equivalent to the condition that $\mathbf{T}_0^{n-2}$ intersects each of S_i only at 0.

Notice that S_i represent vanishing cycles in N, and they can kill a big part of $\pi_1(\mathbf{T}^{n-1})$ in the fundamental group $\pi_1(\mathcal{U}(N))$ of $\mathcal{U}(N)$. As a consequence, if such a subgroup $\mathbf{T}_0^{n-2}$ as above does not exist, then in general there is no finite covering of $\mathcal{U}(N)$ admiting such a subgroup. In other word, there is no finite covering argument like in Proposition 4.2. However, the following condition of topological stability will ensure that a subgroup $\mathbf{T}_0^{n-2}$ of $\mathbf{T}^{n-1}$ which intersects S_i only at 0 does exist. In fact, this condition will imply that all 1-dimensional subgroups S_i coincide.

DEFINITION 5.3. A codimension 2 nondegenerate focus-focus singularity $\mathcal{U}(N)$ is called *topologically stable* if it has the following two properties:

(a) N does not contain hyperbolic singular orbits.
(b) All singular leaves in $\mathcal{U}(N)$ are topologically equivalent.

PROPOSITION 5.4. *Let $\mathcal{U}(N)$ be a codimension 2 nondegenerate focus-focus singularities.*

(a) *If $\mathcal{U}(N)$ satisfies the condition (b) in Definition 5.3 then all of the isotopy groups S_i at focus-focus orbits in N of the symplectic $\mathbf{T}^{n-1}$ action (given by Theorem 5.2 are the same.*
(b) *$\mathcal{U}(N)$ is topologically stable if and only if the singular value set of the moment map restricted to $\mathcal{U}(N)$ is a smooth $(n-2)$-dimensional disk containing the image of N in $\mathbf{R}^n$.*

Proof. (a) Suppose $\mathcal{U}(N)$ satisfies the condition (b) of Definition 5.3. Let $\mathcal{O}_1$ be a focus-focus orbit in N. Let $X_1, \dots, X_{n-1}$ be a system of generators of the $\mathbf{T}^{n-1}$ action as in Theorem 5.2, given in terms of Hamiltnonian vector fields, such that X_1 vanishes in $\mathcal{O}_1$. Then X_1 generates the isotropy group at $\mathcal{O}_1$. Let f_1 be the Hamiltonian function associated to X_1, defined by $f_1(N) = 0$. It is easy to be seen that X_1 also generates isotropy groups of focus-focus orbits near to $\mathcal{O}_1$ (these orbits lie in $\mathcal{U}(N)$ but outside N). Consequently, X_1 vanishes on all focus-focus orbits near to $\mathcal{O}_1$, and $f_1 = 0$ on all these orbits. Let $\mathcal{O}_2$ be another focus-focus orbit in N. The condition (b) of Definition 5.3 implies that singular leaves which contain focus-focus orbits near to $\mathcal{O}_2$ also contain focus-focus orbits near to $\mathcal{O}_1$. Since f_1 is invariant on each leaf, it follows that $f_1 = 0$ on all focus-focus orbits near to $\mathcal{O}_2$. Consequently, the symplectic gradient X_1 of f_1 vanishes on $\mathcal{O}_2$, and hence X_1 generates the isotropic group at $\mathcal{O}_2$. Thus the isotropic groups at $\mathcal{O}_1$ and $\mathcal{O}_2$ are the same.

(b) If $\mathcal{U}(N)$ is topologically stable then clearly the singular value set of the moment map restricted to $\mathcal{U}(N)$ is a smooth $(n-2)$-dimensional disk containing the image of N in $\mathbf{R}^n$. We will prove the inverse statement. Suppose that the local singular value set of the moment map in $\mathbf{R}^n$ is a disk of dimension $(n-2)$. Then N cannot contain hyperbolic codimension 1 singular orbits, otherwise this local singular value set must contain a subset diffeomorphic to $\mathbf{R}^{n-1}$. The rest of the proof resembles that of Proposition 4.4. □

Thus, for stable codimension 2 focus-focus singularities, one can use Marsden-Weinstein reduction with respect to some choice of $\mathbf{T}_0^{n-2}$ to reduce them to stable focus-focus singularities of IHS's with two degrees of freedom. Notice that, like the case of elliptic singularities and unlike the case of codimension 1 hyperbolic singularities, the free $\mathbf{T}^{n-2}$ exists but is not unique.

6. Torus action and topological stability

In this section we generalize the results obtained in the previous sections about torus actions and topological stability to nondegenerate singularities of any Williamson type.

Recall that if $(\mathcal{U}(N), \mathcal{L})$ is a nondegenerate singularity of codimension k and Williamson type (k_e, k_h, k_f), of an IHS with n degrees of freedom, then $k = k_e + k_h + 2k_f$, the ellipticity of N is k_e, the hyperbolicity of N is $k_h + k_f$, and the closedness of N is $c(N) = n - k + k_f = n - k_e - k_h - k_f$. Orbits of maximal dimension in N are of dimension $n - k_e$, diffeomorphic to $\mathbf{T}^{c(N)} \times \mathbf{R}^{k_h+k_f}$ and have orbit type $(k_e, 0, 0, c(N), k_h + k_f)$. It is also clear from the local structure that the union of orbits of maximal dimension is dense in N.

THEOREM 6.1. *Let $(\mathcal{U}(N), \mathcal{L})$ be a nondegenerate singularity of Williamson type (k_e, k_h, k_f) and codimension k, of an IHS with n degrees of freedom. Then there is a natural Hamiltonian torus $\mathbf{T}^{c(N)+k_e}$ action in $(\mathcal{U}(N), \mathcal{L})$ which preserves the moment map of the IHS. This action is unique, up to automorphisms of $\mathbf{T}^{c(N)+k_e}$, and it is free almost everywhere in $\mathcal{U}(N)$. The isotropy group of this action at N (i.e. the subgroup of $\mathbf{T}^{c(N)+k_e}$, consisting of elements which act trivially in N) is a torus $\mathbf{T}^{k_e}$ subgroup of $\mathbf{T}^{c(N)+k_e}$.*

Proof. The proof uses the same arguments as in Theorem 4.1 and Theorem 5.2. Let $\mathcal{O}(x)$ be an orbit maximal dimension in N. Then $\mathcal{O}(x)$ has orbit type

$(k_e, 0, 0, c(N), k_h + k_f)$. In particular, it consists elliptic singular points (if $k_e > 0$). The Poisson $\mathbf{R}^n$ action of our IHS on $\mathcal{O}(x)$ has a unique $\mathbf{R}^{n-k_h-k_f} = \mathbf{R}^{c(N)+k_e}$ subaction with compact orbits. Denote the corresponding subgroup of $\mathbf{R}^n$ by $\mathbf{R}_x^{c(N)+k_e}$. By considering the local structure of smaller orbits in the closure of $\mathcal{O}(x)$, we can see easily that the Poisson $\mathbf{R}_x^{c(N)+k_e}$ subaction has compact orbits on all ($\mathbf{R}^n-$) orbits of maximal dimension in N. Hence all orbits of this $\mathbf{R}_x^{c(N)+k_e}$ subaction in N are compact (they are compact subspaces of orbits of our IHS). We can redenote $\mathbf{R}_x^{c(N)+k_e}$ by $\mathbf{R}_N^{c(N)+k_e}$

If N' is a regular leaf in $\mathcal{U}(N)$, then N' passes nearby at least one of the orbits of maximal dimension in N. From the structure of elliptic singularities, it follows that there is a unique $(c(N) + k_e)$-dimensional subspace $\mathbf{R}_{N'}^{c(N)+k_e}$ of $\mathbf{R}^n$, which is near to $\mathbf{R}_x^{c(N)+k_e}$, such that the Poisson $\mathbf{R}_{N'}^{c(N)+k_e}$ subaction has compact orbits in $\mathcal{O}$ (and is locally free there).

By continuity, we can associate to each leaf N' in $\mathcal{U}(N)$ (regular or not) a subspace $\mathbf{R}_{N'}^{c(N)+k_e}$ of $\mathbf{R}^n$, in a continuous way, so that the Poisson $\mathbf{R}_{N'}^{c(N)+k_e}$ subaction has compact orbits in N', and is locally free if N' is regular. Using Arnold-Liouville theorem, we can derive from this family of subspaces a unique $\mathbf{T}^{c(N)+e_k}$ symplectic action which preserves the moment map. The isotropy group of this action at N corresponds to the k_e-dimensional subspace of $\mathbf{R}_x^{c(N)+k_e}$ which acts trivially on $\mathcal{O}_x$ (under the Poisson $\mathbf{R}^n$ action of our IHS). □

By the way, observe that the isotropic group at N discussed above arises from k_e degrees of ellipticity of N, and this group acts 'around' N with general orbits being small-sized k-tori.

For minimal orbits in N (i.e. closed orbits of dimension $n - k$), the associated isotropy groups of the $\mathbf{T}^{c(N)+k_e}$ action are closed subgroups of dimension $k_e + k_f$, containing the isotropy group at N. If we chose a subgroup $\mathbf{T}_0^{n-k}$ of $\mathbf{T}^{c(N)+k_e}$, which is transversal to all of the above isotropy groups, then the symplectic action of $\mathbf{T}_0^{n-k}$ (as a subaction of $\mathbf{T}^{c(N)+k_e}$) is locally free in $\mathcal{U}(N)$ and we can use Marsden–Weinstein reduction with respect to this group action. But in order to avoid orbifolds, we need this $\mathbf{T}_0^{n-k}$ action to be free. In particular, $\mathbf{T}_0^{n-k}$ should intersect each of the above $(k_e + k_f)$-dimensional isotropy groups only at 0.

In order to achieve the freeness of $\mathbf{T}_0^{n-k}$ action, we will use some covering argument like in Proposition 4.2 and some additional condition about isotropy groups like in Proposition 5.4. Recall that a *normal covering* of a topological space X is a covering $\overline{X}$ of X which corresponds to a normal subgroup of the fundamental group of X. In this case there is a free discrete group action in $\overline{X}$ with the quotient space equal to X.

THEOREM 6.2. *Let $\mathcal{U}(N)$ be as in theorem 6.1. Then if the zero components of the isotropy groups at minimal orbits of N coincide, then there is a natural finite* *normal covering* $\overline{\mathcal{U}(N)}_{\text{can}}$ *of* $\mathcal{U}(N)$, *such that everything (symplectic form, moment map,* $\mathbf{T}^{c(N)+k_e}$ *action) can be lifted from* $\mathcal{U}(N)$ *to* $\overline{\mathcal{U}(N)}_{\text{can}}$, *and the isotropy groups at minimal orbits of the preimage* $\overline{N}$ *of* N *are connected and coincide.*

Proof. The proof is exactly the same as that of Theorem 4.6, and is based on the following presentation of the fundamental group $\pi_1(N) = \pi_1(\mathcal{U}(N))$ of N:

Generators (three types): $\alpha_1, \ldots, \alpha_{n-k}$, which are generated by an orbit of the $\mathbf{T}^{c(N)+k_e}$ action. (Notice that there are only $n - k$ nonzero generating cycles of this type, because the other $k_e + k_f$ cycles are vanishing due to focus-focus and elliptic components). $\beta_1, \ldots, \beta_s (s \geqslant 0)$, which are 'exceptional cycles': they lie in closed orbits of N and are not conjugate to an integral combination of α_i. $\gamma_1, \ldots, \gamma_t (t \geqslant 1)$, which are 'base cycles': the fundamental group of the quotient space of N by the $\mathbf{T}^{c(N)+k_e}$ action is generated by the image of these cycles.

Relations: α_i commute with all the other generators. Twice of β_i are integral combinations of α_i. If say, β_1 and β_2 lie in the same closed orbit of N, then they commute. It may be that some of β_i which lie in different minimal orbits are conjugate (if, say, they are conjugate to a same cycle which lies in an non-closed orbit of N).

There is a natural homomorphism from $\pi_1(N)$ to $\mathbf{T}^{n-k}$ which maps $\alpha_1, \ldots, \alpha_{n-1}, \gamma_1, \ldots, \gamma_t$ to 0 and $\beta_1, \ldots, \beta_s$ to elements of order 2. Take G_{can} to be the kernel of this homomorphism, and $\overline{\mathcal{U}(N)}_{\text{can}}$ to be the normal finite covering of $\mathcal{U}(N)$ corresponding to G.

Like in Section 5, the condition about isotropy groups in Theorem 6.2 is a kind of topological stability condition. The following general definition of topological stability coincides with the ones given in the previous sections for that particular cases discussed there.

DEFINITION 6.3. A nondegenerate singularity $(\mathcal{U}(N), \mathcal{L})$ of an IHS is called *topologically stable* if the local singular value set of the moment map restricted to $\mathcal{U}(N)$ coincides with the singular value set of the moment map restricted to a small neighborhood of a singular point of maximal corank in N.

Because of Vey-Eliasson theorem about the local structure of singular points, it is easy to describe the above local singular value sets. Examples: (1) x is a singular fixed point of Williamson type $(0, n, 0)$, then the local singular value set at x of the moment map is a subset in $\mathbf{R}^n(F_1, \ldots, F_n)$ diffeomorphic to the union of n hyperplanes $\{F_i = 0\}$. (2) If x is a fixed point of Williamson type $(0, 1, 1)$, then this set is diffeomorphic to the union of the line $\{F_2 = F_3 = 0\}$ with the plane $\{F_1 = 0\}$ in the 3-space. (3) If x is an codimension 1 elliptic point then this set is diffeomorphic to a closed half-space.

It is clear from the definition that elliptic singularities are automatically topologically stable. It is as well easy to construct examples of nondegenerate hyperbolic and focus-focus singularities, that are not topologically stable. In general, we have the following result, whose proof is straightforward:

PROPOSITION 6.4. *If $(\mathcal{U}(N), \mathcal{L})$ is a nondegenerate topologically stable singularity of codimension k of an IHS with n degrees of freedom then we have:*

(a) *All singular leaves of codimension k in $\mathcal{U}(N)$ are topologically equivalent.*
(b) *A (germ of a) tubular neighborhood of any singular leaf in $\mathcal{U}(N)$ is a nondegenerate topologically stable singularity.*
(c) *All closed orbits in N have the same (minimal) dimension $n - k$.*

We mention the following important consequence of Theorem 6.2 and Proposition 6.4:

COROLLARY 6.5. *If $(\mathcal{U}(N), \mathcal{L})$ is a nondegenerate singularity of codimension k of an IHS with n degrees of freedom, which is topologically stable or which has $k_f(N) = 0$, then in an appropriate finite covering $\overline{\mathcal{U}(N)}$ there is a free symplectic torus $\mathbf{T}^{n-k}$ action which preserves the moment map.*

7. Topological decomposition

In this section all singularities are assumed to be topologically stable nondegenerate. If N_1 and N_2 are two nondegenerate singular leaves in two different IHS's, of codimension k_1 and k_2, with the corresponding Lagrangian foliation $(\mathcal{U}(N_1), \mathcal{L}_1)$ and $(\mathcal{U}(N_2), \mathcal{L}_2)$, then the *direct product* of these singularities is the singular leaf $N = N_1 \times N_2$ of codimension $k_1 + k_2$ with the associated Lagrangian foliation equal to the direct product of the given Lagrangian foliations:

$$(\mathcal{U}(N), \mathcal{L}) = (\mathcal{U}(N_1), \mathcal{L}_1) \times (\mathcal{U}(N_2), \mathcal{L}_2).$$

DEFINITION 7.1. A nondegenerate singularity $(\mathcal{U}(N), \mathcal{L})$ of codimension k and Williamson type (k_e, k_h, k_f) of an IHS with n degrees of freedom is called of *direct-product type* topologically (or a *direct-product singularity*) if it is homeomorphic, together with the Lagrangian foliation, to a following direct product:

$$\begin{aligned}(\mathcal{U}(N), \mathcal{L}) &\overset{\text{homeo}}{=} (\mathcal{U}(\mathbf{T}^{n-k}), \mathcal{L}_r) \times (P^2(N_1), \mathcal{L}_1) \times \cdots \\ &\times (P^2(N_{k_e+k_h}), \mathcal{L}_{k_e+k_h}) \times (P^4(N_1'), \mathcal{L}_1') \times \cdots \times (P^4(N_{k_f}'), \mathcal{L}_{k_f}'),\end{aligned}$$

where $(\mathcal{U}(\mathbf{T}^{n-k}), \mathcal{L}_r)$ denotes the Lagrangian foliation in a tubular neighborhood of a regular Lagrangian $(n-k)$-torus of an IHS with $n - k$ degrees of freedom; $(P^2(N_i), \mathcal{L}_i)$ for $1 \leqslant i \leqslant k_e + k_h$ denotes a codimension 1 nondegenerate surface singularity (= singularity of an IHS with one degree of freedom); $(P^4(N_i'), \mathcal{L}_i')$ for $1 \leqslant i \leqslant k_f$ denotes a focus-focus singularity of an IHS with two degrees of freedom.

DEFINITION 7.2. A nondegenerate singularity of an IHS is called of *almost-direct-product type* topologically (or simply an *almost-direct-product singularity*) if a finite covering of it is homeomorphic, together with the Lagrangian foliation, to a direct-product singularity.

Our main result is that any topologically stable nondegenerate singularity is of almost direct product type. More precisely, we have:

THEOREM 7.3. *If* $(\mathcal{U}(N), \mathcal{L})$ *is a nondegenerate topologically stable singularity of Williamson type* (k_e, k_h, k_f) *and codimension* k *of an IHS with* n *degrees of freedom then it can be written homeomorphically in the form of a quotient of a direct product singularity*

$$(\mathcal{U}(\mathbf{T}^{n-k}), \mathcal{L}_r) \times (P^2(N_1), \mathcal{L}_1) \times \cdots$$
$$\times (P^2(N_{k_e+k_h}), \mathcal{L}_{k_e+k_h}) \times (P^4(N_1'), \mathcal{L}_1') \times \ldots \times (P^4(N_{k_f}'), \mathcal{L}_{k_f}'),$$

by a free action of a finite group Γ *with the following property:* Γ *acts on the above product component-wise (i.e. it commutes with the projections onto the components), and moreover, it acts trivially on elliptic components.*

REMARK. The above decomposition is not symplectic, i.e. in general we cannot decompose the symplectic form to a direct sum of symplectic forms of the components. However, one will see from the proof that we can replace the word 'homeomorphically' by the word 'diffeomorphically'.

A direct product with an action group in the above theorem will be called a *model* of a stable nondegenerate singularity $(\mathcal{U}(N), \mathcal{L})$. A model is called *canonical* if there does not exist a nontrivial element of Γ which acts trivially on all of the components except one.

PROPOSITION 7.4. *If* $(\mathcal{U}(N), \mathcal{L})$ *is a topologically stable nondegenerate singularity then there is a unique canonical model* $(\overline{\mathcal{U}(N)}_{\mathrm{CAN}}$, action of $\Gamma_{\mathrm{CAN}})$ *for it.*

The group Γ_{CAN} which enters in the canonical model will be called the *Galois group* of the singularity $(\mathcal{U}(N), \mathcal{L})$. remark that for codimension 1 singularities, $\overline{\mathcal{U}(N)}_{\mathrm{CAN}} = \overline{\mathcal{U}(N)}_{\mathrm{can}}$. In general, $\overline{\mathcal{U}(N)}_{\mathrm{CAN}}$ is a finite covering of $\overline{\mathcal{U}(N)}_{\mathrm{can}}$.

COROLLARY 7.5. *(i) If* N *is a topologically stable nondegenerate singular leaf then it is an Eilenberg-Maclane space* $K(\pi_1(N), 1)$.

(ii) If $\phi : N \to N$ *is a homeomorphism from* N *to itself whose induced automorphism on* $H^1(N, \mathbf{R})$ *is identity, then* ϕ *is isomorphic to identity.*

(iii) If two topologically stable nondegenerate singularities of codiemsnion greater than 1 are such that their nearby singularities of smaller codimensions are topologically equivalent in a natural way, then these two singularities are also topologically equivalent.

The above corollary answers a question posed in [47] and is usefull in the computation of the canonical model of singularities of well-known integrable systems.

Proof of Theorem 7.3 and Proposition 7.4. We will prove for the case $k = n$ (i.e. the case when N contains a fixed point). The case $k < n$ can be then easily proved by the use of the results in Section 6 and Section 8. (In case $k < n$, we will use finite covering twice, but the construction is canonical, and one can verify directly from the construction of subgroups of $\pi_1(N)$ that we still have a normal covering).

Let $(\mathcal{U}(N), \mathcal{L})$ have maximal possible codimension $k = n$. For simplicity, we will also assume that N has no elliptic component, i.e. $k_e(N) = 0$ (the case where $k_e(N) > 0$ can then be proved with the aid of the torus action around N which arises from elliptic components (see Section 6)). We will use the following notion of $l - type$ of $(\mathcal{U}(N), \mathcal{L})$, which was introduced by Bolsinov for the case of hyperbolic singularities of IHS with two degrees of freedom.

Let x be a fixed point in N. By changing the Poisson action but leaving the topological structure of $(\mathcal{U}(N), \mathcal{L})$ unchanged, we can assume that at point x the moment map is linear and its components are as in Williamson's Theorem 2.2. We will denote the hyperbolic components of the moment map by $F_1, \ldots, F_{k_h}$, its focus-focus components by $F'_1, F''_2, \ldots, F'_{k_f}, F''_{k_f}$ $(k_h + 2k_f = n)$. Then we have $F_i = x_i y_i, F'_i = x'_i y''_i - x''_i y'_i, F''_i = x'_i y'_i + x''_i y''_i$, where $(x_1, y_1, \ldots, x_{k_h}, y_{k_h}, x'_1, y'_1, x''_1, y''_1, \ldots, x'_{k_f}, y'_{k_f}, x''_{k_f}, y''_{k_f})$ is a canonical system of coordinates at x. The local singular value set of our moment map restricted to a small neighborhood of x will be a germ at zero of the union of codimension 1 hyperplanes $\{F_i = 0\}$ and codimension 2 hyperplanes $\{F'_i = F''_i = 0\}$ in $\mathbf{R}^n$. By definition of topological stability, this set is also the singular value set of the moment map restricted to $\mathcal{U}(N)$. Denote by $I_i, 1 \leqslant i \leqslant k_h$ (resp., $k_h < i \leqslant n$) the subset of this singular value set, which have all coordinates equal to zero except F_i (resp., F'_i and F''_i). Let V_i $(1 \leqslant i \leqslant n)$ denotes the preimage of I_i in $\mathcal{U}(N)$ of the projection. Then V_i are symplectic 2-manifolds $(1 \leqslant i \leqslant k_h)$ and symplectic 4-manifolds $(k_h < i \leqslant n)$, and the intersection of the Lagrangian foliation in $\mathcal{U}(N)$ with V_i gives a singular Lagrangian foliation to these manifolds. With this Lagrangian foliation structure, V_i become hyperbolic and focus-focus singularities of IHS's with one and two degrees of freedom, respectively. We will denote the singularities associated to V_i by V_i again. Strictly speaking, since V_i may be non-connected, it may be that they are not singularities but a finite set of singularities. The (unordered) n-tuple $(V_1, \ldots, V_n)$ will be called the *l-type* of the singularity $(\mathcal{U}(N), \mathcal{L})$. (In case $k_e > 0$, we add elliptic codimension 1 singularities of IHS with one degree of freedom to this l-type).

Denote the singular leaf (or more precisely, the union of singular leaves) in V_i by K_i. Then K_i belong to N. Recall that N has a natural stratification, given by its orbits. The union of K_i, $1 \leqslant i \leqslant k_h$, is the 1-skeleton of N. The union of all

K_i, which we will denote by Spine(N), is the two-skeleton of N minus open 2-dimensional orbits (orbits diffeomorphic to $\mathbf{R}^2$). Observe the following important fact: Spine(N) gives all generators of the fundamental group $\pi_1(N)$. Other orbits in N don't give any new cycles of $\pi_1(N)$, just new commutation relations.

By a *primitive orbit* in N we will mean an one-dimensional orbit, or a two-dimensional orbit of one degree of closedness and one degree of openness. In other words, a primitive orbit is an orbit that lies in Spine(N) and is not a fixed point. Let $\mathcal{O}_i (i = 1, 2)$ be a primitive orbit with two boundary (limit) points x_i and y_i, such that $x_2 = y_1$. For each i, x_i and y_i are fixed points in N, which may coincide. Provide $\mathcal{O}_i$ with the direction going from x_i to y_i so that they become oriented 1-cells. Suppose that $\mathcal{O}_1$ and $\mathcal{O}_2$ lie in different V_i (components of l-type). Then we will construct a mapping from the oriented orbit $\mathcal{O}_1$ to another oriented orbit $\mathcal{O}'_1$ which belongs to the same V_i $\mathcal{O}_1$. This mapping will be well defined, and we will call it the *moving of* $\mathcal{O}_1$ *along* $\mathcal{O}_2$.

We will define the above moving in the case when $\mathcal{O}_i$ are one-dimensional (i.e. they lie in V_i but not V'_i). The other cases are similar. (To see why, take the quotient of $\mathcal{U}(N)$ by the $\mathbf{T}^{k_f}$ action arisen from focus-focus components as in Section 6). It follows from the local structure theorems that there is an open 2-dimensional orbit (diffeomorphic to $\mathbf{R}^2$) in N, denoted by $\mathcal{O}$, which contains $\mathcal{O}_1$ and $\mathcal{O}_2$ in its closure. The algebraic boundary of $\mathcal{O}$ consists of four 1-cells (1-dimensional orbits) and four 0-cells. Denote the other two 1-cells by $\mathcal{O}'_1$, $\mathcal{O}'_2$, in such an order that $\partial\mathcal{O} = \mathcal{O}_1 + \mathcal{O}_2 - \mathcal{O}'_1 - \mathcal{O}'_2$ algebraically. Then it is easy to be seen (using the moment map) that $\mathcal{O}'_1$ belongs to the same V_i or V'_i as $\mathcal{O}_1$. The moving of $\mathcal{O}_1$ along $\mathcal{O}_2$ is by definition the oriented orbit $\mathcal{O}'_1$ (with the end point at y_2). This moving can also be understood as an orientation homeomorphism from $\mathcal{O}_1$ to $\mathcal{O}'_1$, which maps end points to end points, and which is defined up to isotopies. The pair oriented $(\mathcal{O}'_2, \mathcal{O}'_1)$ will be called the *elementary homotopy deformation* of the oriented pair $(\mathcal{O}_1, \mathcal{O}_2)$.

If $\mathcal{O}_i$ (i = 1,. . . ,s) is a chain of oriented primitive orbits in N with start and end points x_i, y_i such that $x_i = y_{i-1}, x_1 = y_s$, then they form a closed 1-cycle in N. (Here, say if $\mathcal{O}_1$ lies in some focus-focus V_i, they we replace $\mathcal{O}_1$ by a curve lying in $\mathcal{O}_1$ and going from x_1 to y_1). We know that any closed curve in N is homotopic to such a cycle. Furthermore, any two homotopic cycles can be obtained from one another by a finite number of elementary homotopy deformations. That is because orbits with three or more degrees of openness in N do not contribute any new generator or relation to $\pi_1(N)$.

From now on, in our homotopy arguments by a closed curve in N we will always mean a cycle consisting of consecutive primitive orbit. Let x be a fixed point in N. Let W_i, $1 \leqslant i \leqslant n$, be the connected component of V_i that contains x. Let γ be a closed curve going from x to x and lying entirely in V_1. Let $\mathcal{O}$ be an oriented primitive orbit with the start point at x and not lying in V_1. Then we can move $\mathcal{O}$ along γ by moving it step by step along orbits in γ. It is clear that the end result will depend only on the homotopy type of γ as an element in $\pi_1(W_1, x)$.

Denote this end result by $\mathcal{O}^{\gamma}$. It is equally clear that x is also the start point of $\mathcal{O}^{\gamma}$. If $\mathcal{O}$ belongs to W_i ($i \neq 1$), say, then there are 4 local primitive orbits lying in W_i having x as the start point. One can move all of these local orbits along γ. In other words, one can move a local singular leaf of x in W_i along γ. Recall that W_i has a natural orientation given by the induced symplectic structure. In can be easily seen by induction along primitive orbits in γ that the above moving is orientation-preserving local homeomorphism of the local singular leaf of x in W_i, which is isotopic either to identity or to an involution that maps each local primitive orbit in W_i at x to its opposite. It follows that the subset of $\pi_1(W_1)$ consisting of (homotopy type of) closed curves whose action on each of the local singular leaf of x in W_i is identity (in isotopy category) for all $i \neq 1$, is a normal subgroup of $\pi_1(W_1)$ of index at most 2^{n-1}. Denote the image of this subgroup in $\pi_1(N, x)$ by G_1.

By replacing W_1 by other W_i, we can construct subgroups G_i ($1 \leqslant i \leqslant n$) of $\pi_1(N, x)$ in the same way. A very important property of G_i that we will show below can be stated roughly as follows: the moving along elements of G_1 (1 can be replaced by any number from 1 to n) gives the trivial action of G_1 not only on the local singular leaf of x in W_i ($i \neq 1$), but also on the whole singular leaf through x in W_i, i.e. the intersection of W_i and K_i.

Let $\mathcal{O}_1, \ldots, \mathcal{O}_s$ be a chain of consecutive primitive orbits lying in W_i, $i \neq 1$, and $x_1 = x, y_1 = x_2, \ldots, y_s$ their corresponding start and end points. Let γ be a closed curve in W_1 which represents an element in G_1. Then γ moves $\mathcal{O}_1$ to itself. We can also move γ, considered not as a closed curve but simply a curve with the start point at x, along $\mathcal{O}_1$. Let the image be called γ_1. Then γ_1 will be in fact a closed curve starting and ending at x_2 (exactly because γ moves $\mathcal{O}_1$ to itself). Moreover, γ_1 moves $-\mathcal{O}_1$ (considered with the inverse direction) to itself. As a consequence, γ_1 also moves $\mathcal{O}_2$ to itself, and we can continue like this. In the end, we have that the curve $(\mathcal{O}_1, \ldots, \mathcal{O}_s)$ is moved to itself along γ. Here we should note that the motion of $(\mathcal{O}_1, \ldots, \mathcal{O}_s)$ along any curve γ (closed or not) with the starting point at x and with no component belonging to W_i is always well defined by applying step by step the elementary homotopy deformations and using the definition of moving for primitive orbits.

The main consequence of the above trivial action property of G_i is as follows: if $\alpha \in G_i$ and $\beta \in G_j$ with $i \neq j$ then α and β commute. In other words, the subgroups G_i of $\pi_1(N)$ commute pairwise. Indeed, let α be a closed curve in W_1 representing an element in G_1, β a closed curve in W_2 representing an element in G_2. Then α moves β to a curve β' which is in fact equal to β, and β moves α to a curve α' which is in fact equal to α, and we have $\alpha\beta = \beta'\alpha'$ (because by definition, $\beta'\alpha'$ is obtained from $\beta\alpha$ by a finite number of elementary homotopy deformation).

Let G denote the product of G_i in $\pi_1(N)$. We will now show that G is a normal subgroup of finite index in $\pi_1(N)$.

First, about the normality of G. We will call it $G(x)$, to remember that it is a subgroup of $\pi_1(N,x)$. If y is another fixed point of N, then similarly we can construct the subgroup $G(y)$ of $\pi_1(N,y)$. Let $\beta = (\mathcal{O}_1,\dots,\mathcal{O}_s)$ be a chain of consecutive primitive orbits starting at x and ending at y, we will construct a natural homomorphism from $G(x)$ to $G(y)$, called *moving from* $G(x)$ *to* $G(y)$ *along* β. By going back from y to x via another chain β', we will get a similar homomorphism from $G(y)$ to $G(x)$. Combining these two homomorphisms, we will get an automorphism of $G(x)$, which is obtained in fact by the conjugation of $G(x)$ with the cycle $\beta\beta'$. Thus the conjugation of $G(x)$ with any cycle in $\pi_1(N,x)$ is $G(x)$ itself, i.e. $G(x)$ is a normal subgroup of $\pi_1(N,x)$.

To construct the above moving from $G(x)$ to $G(y)$, we can assume for simplicity that the chosen chain β from x to y consists of only one orbit $\mathcal{O}_1$. $\mathcal{O}_1$ lies in some W_i, say W_1 for definiteness. Then if $\gamma \in G(x)$ is represented by a closed curve in $W_i, i \neq 1$, it is moved to an element in $\pi_1(N,y)$ by moving of the curve along $\mathcal{O}_1$ in the usual way as before. If γ is represented by a closed curve in W_1, then it is moved to the element of $\pi_1(N)$ which is represented by the closed curve $(-\mathcal{O}_1+\gamma+\mathcal{O}_1)$. If γ is a product of the above generators of $G(x)$, then it is moved to the product of the associated images. Because of the trivial action property of $G(x)$, one can easily check that the above moving is well defined and will be a homomorphism from $G(x)$ to $G(y)$.

Now, about the finiteness of the index of G in $\pi_1(N)$. Let γ be an arbitrary element of $\pi_1(N,x)$, which is represented by a closed curve (consisting of primitive orbits). Using a finite number of elementary homotopy deformations, we see that γ can be represented as $\beta_1+\cdots+\beta_n$, where each β_i is a chain of consecutive primitive orbits lying entirely in W_i. Because of the number of primitive orbits in N is finite, and the index of G_i in W_i (for each fixed x) is finite, we can chose for each i a finite number of curves (i.e. chains of consecutive primitive orbits) $\beta_i^1,\dots,\beta_I^i$, such that any curve β_i in V_i is homotopic (rel. end points) to $\gamma_i\beta_i^{s(i)}$, where γ_i is an element in some group $G(z)$ (i.e. freely conjugate to an element of $G(x)$), and $1 \leqslant s(i) \leqslant I$. Then γ is homotopic to $\gamma_1+\beta_1^s(1)+\cdots+\gamma_n+\beta_n^{s(n)}$. Now, because γ_n has the trivial action property, $\beta_{n-1}^{s(n-1)}+\gamma_n$ is homotopic (rel. end points) to $\gamma_n'+\beta_{n-1}^{s(n-1)}$, where γ_n' again belongs to some $G(z)$. By induction, we obtain that γ is homotopic to $\gamma_1'+\beta_1^{s(1)}+\cdots+\beta_n^{s(n)}$, where γ_1' is some element of $G(x)$. This proves the finiteness of the index of G in $\pi_1(N)$.

Let $\overline{\mathcal{U}(\mathbf{n})}$ denotes the normal finite covering of $\mathcal{U}(N)$ associated to the above normal subgroup G of the fundamental group $\pi_1(\mathcal{U}(N))=\pi_1(N)$. It is obvious that the symplectic structure, the singular Lagrangian foliation and the moment map can be lifted from $\mathcal{U}(\mathbf{n})$ to $\overline{\mathcal{U}(\mathbf{n})}$. $\overline{\mathcal{U}(\mathbf{n})}$ has the fundamental group equal to G, and this group has the trivial action property inherited from N. From this one can easily show that $\overline{\mathcal{U}(\mathbf{n})}$ has the direct-product type. Let Γ be the quotient group $\pi_1(N,x)/G(x)$. Then Γ acts freely on $\overline{\mathcal{U}(N)}$, with the quotient being the singularity $(\mathcal{U}(N),\mathcal{L})$. This action can be made component-wise, because it Γ acts on the associated finite covering of Spine(N) component-wisely. (It is an easy exercise to see why). Theorem 7.3 is proved.

It is also clear that only elements of G have the trivial action property. As a consequence, if G' is another subgroup of $\pi_1(N, x)$ such that the associated finite covering is of product type, then G' must be a subgroup of G. It follows that the model constructed above (for $k = n$) is the unique canonical model. Proposition 7.4 is proved. □

EXAMPLES. (1) Lerman and Umanskii [33] (see also [9]) classified topologically stable nondegenerate hyperbolic codimension 2 singularities of IHS's with two degrees with freedom, which contain only one fixed point. The result is that, topologically there are four different cases. But their description is rather complicated, in terms of cell decomposition. It is an easy exercise to write down explicitly the canonical model for all of these four cases (it was done in [49]). (2) Computation of the canonical model for singularities of the geodesic flow on multi-dimensional ellipsoids was done in [50].

8. Action-angle coordinates

The following theorem gives a non-complete system of action-angle coordinates to each topologically stable nondegenerate singularity. We will consider only the case when there is no elliptic component. In case when there are some elliptic components, there will be even more action-angle coordinates (some of which are polar). This case is left to the reader.

THEOREM 8.1. *Let $(\mathcal{U}(N), \mathcal{L})$ be a topologically stable nondegenerate singularity of codimension k and Williamson type $(0, k_h, k_f)$ of an IHS with n degrees of freedom. Suppose that a moment map preserving hamiltonian $\mathbf{T}^{n-k}$ action is free in $\mathcal{U}(N)$. Then we have:*

(a) *The above action provides $\mathcal{U}(N)$ with a principal $\mathbf{T}^{n-k}$ bundle structure, which is topologically trivial.*
(b) *There is a coisotropic section to this trivial bundle.*
(c) *$\mathcal{U}(N)$ is symplectomorphic to the direct product $\mathbf{D}^{n-k} \times \mathbf{T}^{n-k} \times P^{2k}$ with the symplectic form*

$$\omega = \sum_{1}^{n-k} \mathrm{d}x_i \wedge \mathrm{d}y_i + \pi^*(\omega_1)$$

where x_i are Euclidean coordinates on $\mathbf{D}^{n-k}$, y_i (mod 1) are coordinates on $\mathbf{T}^{n-k}$, ω_1 is a symplectic form on a $2k$-dimensional symplectic manifold P^{2k}, and π means the projection. Under this symplectomorphism, the moment map does not depend on y_i. The set of functions x_i, y_i will be called a non-complete system of action-angle coordinates *for singularity $(\mathcal{U}(N), \mathcal{L})$.*

(d) *If* $(\mathcal{U}(N), \mathcal{L})$ *is any topologically stable nondegenerate singularity of codimension k, then* $\mathcal{U}(N)_{\mathrm{can}}$ *(as in Theorem 6.2) is simplectomorphic to the direct product* $\mathbf{D}^{n-k} \times \mathbf{T}^{n-k} \times P^{2k}$ *with the canonical symplectic form as before, and moreover* Γ_{can} *acts on this product component-wise. In other words, we have an* equivariant non-complete system of action-angle coordinates *on* $\mathcal{U}(N)_{\mathrm{can}}$ *for each topologically stable nondegenerate singularity.*

Proof. (a) By changing the moment map but leaving the Lagrangian foliation unchanged, we can assume that the components $F_i, i = 1, \ldots, n$ of the moment map are chosen so that the Hamiltonian vector fields $X_{n-k+1} = X_{F_{n-k+1}}, \ldots, X_n = X_{F_n}$ generate a free $\mathbf{T}^{n-k}$ action in $\mathcal{U}(N)$, which give rise to the principal bundle in question, and the local singular value set of the moment map is given as the union of local codimension 1 hyperplanes $\{F_i = 0\}$, $1 \leqslant i \leqslant k_h$ and codimension 2 hyperplanes $\{F_i = F_i + 1 = 0\}$, $k_h < i \leqslant k, (k - i - 1) \vdots 2$.

Because the classifying space of $\mathbf{T}^{n-k}$ is the product of $(n - k)$ samples of $\mathbf{CP}^{\infty}$, a principal $\mathbf{T}^{n-k}$ bundle is trivial if and only if a finite covering of it is also trivial. Thus, in view of Theorem 7.3 (for the case $n = k$), we can assume that the base space of $\mathcal{U}(N)$ is a union of a $(n - k)$-dimensional family of topologically equivalent singularities of direct product type. Namely, we will assume that the base space of $\mathcal{U}(N)$, denoted by B, is homeomorphic to the direct product $\mathbf{D}^{n-k} \times W_1 \times \cdots \times W_{k_h+k_f}$, where $\mathbf{D}^{n-k}$ is coordinated by $F_1, \ldots, F_{n-k}$; each W_i $(1 \leqslant i \leqslant k_h)$ is a hyperbolic singularity of an IHS with one degree of freedom; and each W_i $(k_h \leqslant i \leqslant k_h + k_f)$ is a topologically stable focus-focus singularity of an IHS with two degrees of freedom. Under this homeomorphism, F_i, $1 \leqslant i \leqslant k_h$, becomes a function depending only on W_i, and F_{k_h+2i-1}, F_{k_h+2i}, $1 \leqslant i \leqslant k_f$, become functions depending only on W_{k_h+i}.

Because of homotopy, it is enough to prove the triviality of the bundle in N. Denote the base space of this bundle by Spine(B) (it is the spine of B in the same sense as in Section 7). We will define a parallel transprotation along the curves of Spine(B) of the elements of N. This parallel transportation will be locally flat (zero curvature), from which follows easily the triviality of the $\mathbf{T}^{n-k}$ bundle structure of N. To define this transportation, we simply use the vector fields $X_i = X_{F_i}$, $1 \leqslant i \leqslant k$. First of all, notice that these vector fields commute. Second, where they are different from zero, they are transversal to the tori of the bundle. Third, suppose for example that $X_1(x) = 0$ for some singular point $x \in N$. Then because of our choice of the moment map (more precisely, because we choose the local singular value set to be 'canonical'), the direction of X_i near x is transversal to the orbit through x. The second property means that we have a parallel transportation in regular orbits of N. The third property means that we can extend this parallel transportation over the singular orbits of N, to obtain a well defined parallel transportation (connection) in N. The first property means that this connection is locally flat. Assertion (a) is proved.

Assertion (b) is a direct consequence of assertion (c), because if one have a non-complete system of action-angle coordinates then the submanifold $\{y_1 = \cdots =$

$y_{n-k} = 0\}$ is obviously a coisotropic section to the $\mathbf{T}^{n-k}$ bundle. In fact, one can see easily that assertions (b) and (c) are equivalent.

(c) The proof of assertion (c) is similar to that of 4.3. For completeness, we will recall it here.

Put $x_i = F_{i+k}, 1 \leqslant i \leqslant n-k$, where we suppose that the moment map is chosen as in the proof of assertion (a), and denote the Hamiltonian vector field of x_i by ξ_i.. Recall from assertion (a) that the $\mathbf{T}^{n-k}$ bundle structure of $\mathcal{U}(N)$ is trivial. Let L be an arbitrary section to this bundle (L is diffeomorphic to the base space B), and define functions z_i $(i = 1, \dots, n-k)$ by putting them equal to zero on L and setting $1 = \mathrm{d}z_i(\xi_i) = \{x_i, z_i\}$ (Poisson bracket). Set $\omega_1 = \omega - \sum_1^{n-k} \mathrm{d}x_i \wedge \mathrm{d}z_i$. Then one checks that $L_{\xi_i}\omega_1 = i_{\xi_i}\omega_1 = 0$. It means that ω_1 is a lift of some closed 2-form from B to $\mathcal{U}(N)$, which we will also denote by ω_1. Since ω is non-degenerate, it follows that ω_1 is non-degenerate on every $2k$-dimensional manifold (with boundary) $P^{2k}_{x_1,\dots,x_{n-1}} = B \cap \{x_1, \dots, x_{n-1}\mathtt{fixed}\}$. Using Moser's path method [36], one can construct a diffeomorphism $\phi : B \to \mathbf{D}^{n-k} \times P^{2k}$, under which ω_1 restricted to $P^{2k}_{x_1,\dots,x_{n-k}}$ does not depend on the choice of $x_1, \dots, x_{n-1}$. In other words, there is a symplectic form ω_2 on P^{2k} such that $\omega_1 - \omega_2$ vanishes on every $P^{2k}_{x_1,\dots,x_{n-k}}$. Since $d(\omega_1 - \omega_2) = 0$, we can write it as $\omega_1 - \omega_2 = d(\sum_1^{n-k} a_i \; \mathrm{d}x_i + \beta)$, where β is some 1-form on B^{n+1} (which is not zero on $P^{2k}_{x_1,\dots,x_{n-k}}$ in general). If we can eliminate β, i.e. write $\omega_1 - \omega_2 = d(\sum_1^{n-k} a_i \; \mathrm{d}x_i)$, then we will have $\omega = (\sum_1^{n-k} \mathrm{d}x_i \wedge \mathrm{d}z_i - \sum_1^{n-k} \mathrm{d}x_i \wedge \mathrm{d}a_i + \omega_2 = \sum_1^{n-k} \mathrm{d}x_i \wedge d(z_i - a_i) + \omega_2$, and the theorem will be proved by putting $y_i = z_i - a_i$. Let us show now how to eliminate β. β restricted on every $P^{2k}_{x_1,\dots,x_{n-k}}$ is a closed 1-form, hence it represents a cohomology element $[\beta](x_1, \dots, x_{n-k}) \in H^1(P^{2k})$. If $[\beta](x_1, \dots, x_{n-k}) \equiv 0$ then $\beta = dF - b_1 \,\mathrm{d}x_1 - \dots - b_{n-k}\,\mathrm{d}x_{n-k}$ for some functions $F, b_1, \dots, b_n$, and we have $\omega_1 - \omega_2 = d(\sum(a_i - b_i x_i)\mathrm{d}x_i)$. In general, we can achieve $[\beta](x_1, \dots, x_{n-k}) \equiv 0$ by induction on the number of generators of $H^1(P^{2k}, \mathbf{R})$ as follows. Let γ be an element in a fixed system of generators of $H^1(P^{2k}, \mathbf{R})$. Set $b(x_1, \dots, x_{n-k}) = \langle [\beta], \gamma \rangle (x_1, \dots, x_{n-k})$. Chose a closed 1-form δ on P^{2k} such that $\delta(\gamma) = 1$ and the action of δ on other generators from the chosen system is 0. Change ω_2 for the following 2-form on $\mathcal{U}(N)$: $\omega_2' = \omega_2 + db \wedge \delta$. It is clear that ω_2' and ω_2 restricted on every $P^{2k}_{x_1,\dots,x_{n-k}}$ are the same. Moreover, ω_2' is closed and of rank $2k$. Thus the distribution by its $(n-k)$-dimensional tangent zero-subspaces is integrable, and it gives rise again to a diffeomorphism $\phi' : B \to \mathbf{D}^{n-k} \times P^{2k}$. Replacing ω_2 by ω_2', we have $\omega_1 - \omega_2' = d(\sum_1^{n-k} a_i) \,\mathrm{d}x_i + \beta')$, with $\beta' = \beta - b\delta$, whence $\langle [\beta'], \gamma \rangle = 0$.

(d) The proof of (d) is similar to that of Theorem 4.6. □

9. Nondegenerate IHS's

There has been no commonly accepted notion of nondegenerate IHS, though intuitively one understands that such a notion should be connected to the nondegeneracy of singularities. In this last Section we propose two extreme notions: strong nondegeneracy and weak nondegeneracy for IHS's. We also suggest a middle condition, which will be simply called nondegeneracy, and which seems to us most suitable for practical perposes. Most, if not all, known IHS's in mechanics and physics satisfy the (middle) condition of nondegeneracy. Some, but not all, known IHS's satisfy the strong condition of nondegeneracy.

DEFINITION 9.1. An IHS is called *strongly nondegenerate* if all of its singularities are stable nondegenerate.

Examples of systems satisfying the strong nondegeneracy are the Euler and Lagrange tops, and the geodesic flow on the multi-dimensional ellipsoid where one cuts out the zero section in the cotangent bundle. Examples of systems non satisfying strong nondegeneracy condition include the Kovalevskaya top.

DEFINITION 9.2. An IHS is called *weakly nondegenerate* if the orbit space of its associated singular Lagrangian foliation is Hausdorff, and almost all of its singular points belong to nondegenerate singular leaves.

In the above definition, the word 'almost all' means a dense subset in the set of singular points. Since all singular leaves near a nondegenerate singular leaf are also nondegenerate, we have in fact an open dense subset.

Let $x \in M^{2n}$ be a singular point of corank an IHS with the moment map $\mathbf{F} = (F_1, \dots, F_n)$. We can assume that $dF_1 \wedge \cdots \wedge dF_{n-k}(x) \neq 0$. Applying a local Marsden-Weistein reduction with respect to $F_1, \dots, F_{n-1}$, we obtain a local integrable system with k degrees of freedom, for which x becomes a fixed point. x is called a *clean* singular point, if it becomes an isolated fixed point under this local reduction. Obviously, all nondegenerate singular points are clean.

DEFINITION 9.3. An IHS is called *nondegenerate* if it is weakly nondegenerate, all of its nondegenerate singularities are topologically stable, and all of its singular points are clean.

In a nondegenerate IHS, singular leaves which are not nondegenerate will be called *simply-degenerate*. Thus each singular leaf of a nondegenerate IHS is either nondegenerate or simply-degenerate. It can be shown easily by continuation that torus actions discussed in Section 6 still exist for simply-degenerate singularities: in a saturated tubular neighborhood of a simply-degenerate singularity of codimension k, there is a symplectic $(n-k)$-dimensional torus action which preserves the moment map. Moreover, an (equivariant) system of $(n-k)$ action and $(n-k)$

angle coordinates still exist. It is perhaps the most important property of simply-degenerate singularities.

One way to obtain (strongly or weakly) nondegenerate IHS's is to use *integrable surgery*, which will be discussed in more detail in [51]. Integrable surgery simply means the cutting and gluing of pieces of symplectic manifolds together with IHS's on them so that to obtain new symplectic structures and new IHS's. In other words, one does a surgery on the level of n-dimensional orbit spaces and tries to lift this surgery to the level of symplectic manifolds. In principle, by this way one should be able to see if a stratified affine manifold is an orbit space of any IHS, and how many IHS's does it correspond to. Surprisingly or not, this method gives a very simple way to construct many interesting symplectic manifolds, including nonstandard symplectic $\mathbf{R}^{2n}$ and Kodaira–Thurston example of a non-Kählerian closed symplectic manifold. For example, consider a simplest case, where the orbit space is a closed annulus, whose boundary corresponds to elliptic singularities. Interestingly enough, there is a discrete 2-dimensional family of IHS's admitting this orbit space, most of which lie in non-Kälerian closed symplectic 4-manifolds. Another simple example: Take $\frac{1}{4}$ of the plane to be the orbit space, so that the two half-lines correspond to elliptic singularities. Equip this $\frac{1}{4}$ plane with some affine structure, and we obtain an IHS lying on a symplectic manifold diffeomorphic to $\mathbf{R}^4$. Most of the times this $\mathbf{R}^4$ will be an exotic symplectic space.

To conclude this paper, we will give a physical example to illustrate our results. Namely, we will discuss here one of the most famous IHS's ever known: the Kovalevskaya's top. This top has attracted so many people, from different points of view, since its appearance in [31]. Kharlamov was first to study the bifurcation of Liouville tori of this system (see e.g. [29]). Here, for our purposes, we will follow Oshemkov [39]. Remark that codimension 1 singularities of this system were reexamined by Audin and Silhol [5] by the use of algebro-geometric methods.

Consider the Lie algebra $e(3)$ of the transformation group of the three-dimensional Euclidean space. Let $S_1, S_2, S_3, R_1, R_2, R_3$ be a system of coordinate functions on its dual $e(3)^*$, for which the Lie-Poisson bracket is of the form:

$$\{S_i, S_j\} = \epsilon_{ijk} S_k, \{R_i, R_j\} = 0, \{S_i, R_j\} = \{R_i, S_j\} = \epsilon_{ijk} R_k,$$

where $\{i, j, k\} = \{1, 2, 3\}$ and

$$\epsilon_{ijk} = \begin{cases} \text{sign of the transposition of } (i,j,k) \text{ if all the } i,j,k \text{ are different} \\ 0 \text{ otherwise.} \end{cases}$$

The above Lie-Poisson structure has 2 Casimir functions (i.e. functions which commute with all other smooth functions, with respect to the Lie-Poisson bracket):

$$f_1 = R_1^2 + R_2^2 + R_3^2, f_2 = S_1 R_1 + S_2 R_2 + S_3 R_3.$$

The restriction of the Poisson structure on the level sets $\{f_1 = \text{const.}) > 0, f_2 = \text{const.}\}$ is non-degenerate. These level sets are symplectic 4-manifolds, diffeomorphic to TS^2.

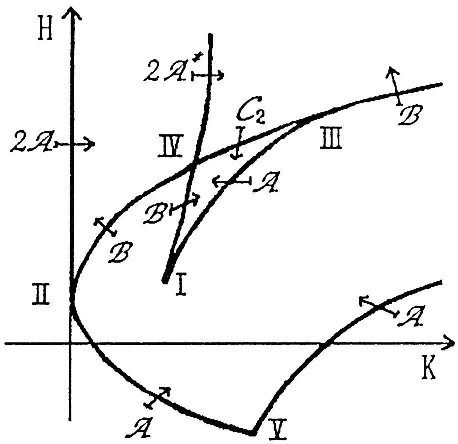
Motion of a rigid body with a fixed point in a gravity field can often be written in the form of a Hamiltonian system on $e(3)^*$, whence it admits f_1, f_2 as first integrals, and is Hamiltonian when restricted to the above symplectic 4-manifolds. In case of the Kovalevskaya's top, the corresponding Hamiltonian can be written in the form

$$H = 1/2(S_1^2 + S_2^2 + 2S_3^2) + R_1.$$

This Hamiltonian admits another first integral:

$$K = (S_1^2/2 - S_2^2/2 - R_1)^2 + (S_1S_2 - R_2)^2.$$

Let us consider, for example, the restriction of this Kovalevskaya's system to a particular symplectic 4-manifold $M^4 = \{f_1 = 1, f_2 = g\}$, where g is a constant with $0 < |g| < 1$. Then the bifurcation diagram of the moment map $(H, K) : M^4 \to \mathbf{R}^2$ is shown in the following figure.



Together with the bifurcation curves, in this figure we also show the type of codimension 1 nondegenerate singularities and point out simply-degenerate and codimension 2 singularities. In the figure, codimension 1 singularities are denoted by calligraphic letters, and these notations and description are taken from [11, 39]. In particular, $\mathcal{A}$ means an elliptic codimension 1 singularity, $2\mathcal{A}$ means a disjoint union of 2 such singularities, $\mathcal{B}$ means the codimension 1 hyperbolic singularity for which the $\mathbf{T}^{n-1}$ action is free (here $n - 1 = 1$), and which contains only one hyperbolic orbit in the singular leaf. $\mathcal{A}^*$ is obtained from $\mathcal{B}$ by a free $\mathbf{Z}_2$ action. $\mathcal{C}_2$ is the codimension 1 hyperbolic singularity for which the $\mathbf{T}^{n-1}$ action is free, and whose reduction is a surface singularity which can be embedded in the 2-sphere in such a way that the singular leaf is the union of two big circles (intersecting at 2 points).

The Kovalevskaya's top is nondegenerate but not strongly nondegenerate, and there are 3 simply-degenerate singularities, which are denoted in the above figure by I, II, III. Only singularity I (which corresponds to the cusp in the bifurcation diagram) is simply-degenerate in the sense of Lerman and Umanskii [32]. Singularity V is codimension 2 elliptic, and singularity IV is codimension 2 hyperbolic. Based on the structure of codimension 1 singularities around singularity IV, and the decomposition Theorem 7.3, we can easily compute singularity IV. The result is: IV = $(\mathcal{B} \times \mathcal{C}_2)/\mathbf{Z}_2$.

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