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# Residues and duality for algebraic schemes

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## 0. Introduction

0.1. Consider the Zariski site on the category $\mathcal{V}$ of connected algebraic schemes over a perfect field $k$. Denote the structure sheaf on this site by the symbol $\mathcal{O}$. We assume familiarity with the notion of $\mathcal{O}$-modules as laid out in [21] pp. 28–32. On the Zariski site we have a dualizing complex of $\mathcal{O}$-modules $(\Delta^\bullet, \{T_V\})$ (which is unique up to unique isomorphism), i.e.

1. For each $V \in \mathcal{V}$, $\Delta^\bullet_V$ is a residual complex (cf. [7] ch. VI, Sect. 1).
2. If $V$ is proper, there is a map of complexes $T_V\colon \Gamma(V, \Delta^\bullet_V) \to k$ such that the pair $(\Delta^\bullet_V, T_V)$ is a residue pair (cf. [28], p. 120, [29], 1.2 and also 1.6.1 of this paper).
3. For every cartesian square in $\mathcal{V}$ of the form

$$\begin{array}{ccc} V & \xrightarrow{\ i\ } & W_1 \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{\ j\ } & W_2 \end{array}$$

   with $i$, $j$ open immersions, $W_1$ and $W_2$ proper over $k$, the relation $j^*T_f = \beta_j^{-1} \circ \beta_i$ holds, where

   – $\beta_i\colon i^*\Delta^\bullet_{W_1} \to \Delta^\bullet_V$, $\beta_j\colon j^*\Delta^\bullet_{W_2} \to \Delta^\bullet_V$ are the natural restrictions of the Zariski sheaf $\Delta^\bullet$.

   – $T_f\colon f_*\Delta^\bullet_{W_1} \to \Delta^\bullet_{W_2}$ is the homotopy unique map such that $T_{W_2}$ is homotopic to $T_{W_2} \circ \Gamma(W_2, T_f)$.

   (cf. [29], 1.2.3).[1]

[1] In [29] the exposition is for the category of $k$-varieties, i.e. integral $k$ schemes of finite type. However the notions of (and results concerning) dualizing complexes on $\mathcal{V}$ carry over to our situation.

Dualizing complexes exist – indeed in R. Hartshorne's book [7] one is essentially constructed – and are unique up to unique isomorphism (cf. [29], 1.2.6).

*The principal aim of this paper is to realize* $(\Delta^\bullet, \{T_V\})$ *concretely in terms of differential forms and cohomological residues.* Cohomological residues have been developed by J. Lipman, E. Kunz, R. Hübl (cf. [21], [22], [19], [13], [15], [14]). We will also use, heavily, the modifications of these constructs as worked out by I.-C. Huang in [10] and [12]. Later in this introduction, we will discuss the difference between our approach and A. Yekutieli's approach in [31] where he constructs a concrete model for $(\Delta^\bullet, \{T_V\})$ on *reduced* algebraic $k$-schemes. Yekutieli relies heavily on the theory of residues of differential forms of local fields (rather than the residues of (local) cohomology classes, which we use). This theory was developed by A. N. Parshin, V. G. Lomadze, A. A. Beilinson and, in the case of topological local fields by Yekutieli himself (cf. [26], [27], [24], [1] and [31]). Yekutieli has extended his work to include all algebraic schemes over $k$ (cf. [33] and [32]).

## 0.2. TRACE STRUCTURES

Let $V \in \mathcal{V}$. Residue Complexes on $V$ are built out of the various injective hulls of the residue fields $k(v)$ (thought of as a $\mathcal{O}_{V,v}$-module) as $v$ varies in $V$. For a concrete model of the injective hull we follow Grothendieck [4] (as does Yekutieli in [31]). If $\sigma\colon L \to \widehat{\mathcal{O}}_{V,v}$ is a pseudo-coefficient field[2] then $\mathcal{K}(\sigma) := \mathrm{Hom}_\sigma^{\mathrm{cont}}(\widehat{\mathcal{O}}_{V,v}, \omega_\sigma)$ is an injective hull of $k(v)$, where $\widehat{\mathcal{O}}_{V,v}$ is given the $\mathfrak{m}_v$-adic topology and $\omega_\sigma = \Omega^d_{L/k}$ ($d = $ tr. $\deg(L/k)$). Given another pseudo-coefficient field $\sigma'$, how do we assign a canonical isomorphism between $\mathcal{K}(\sigma)$ and $\mathcal{K}(\sigma')$ ?

One immediate observation is that $(\mathcal{K}(\sigma), e_\sigma)$ represents the functor $\mathrm{Hom}_\sigma(-, \omega_\sigma)$ of $\mathcal{O}_{V,v}$-modules with zero-dimensional support.

In view of the above, if $v \in V$ is a smooth point, then there is a natural isomorphism between $\mathcal{K}(\sigma)$ and $\mathcal{K}(\sigma')$ described as follows: First, the pair $(H^d_v(\Omega^n_{V/k}), \mathrm{res}_\sigma)$ represents the same functor that $(\mathcal{K}(\sigma), e_\sigma)$ does, where $d = \dim \mathcal{O}_{V,v}$, $n$ is the dimension of the irreducible component containing $v$, and $\mathrm{res}_\sigma$ is the residue in 4.2.1 (cf. 4.2.2). Thus we have a canonical isomorphism $\mathcal{K}(\sigma) \xrightarrow{\sim} H^d_v(\Omega^n_{V/k})$. Similarly we have another isomorphism $\mathcal{K}(\sigma') \xrightarrow{\sim} H^d_v(\Omega^n_{V/k})$ induced by $\mathrm{res}_{\sigma'}$, and hence an isomorphism $\mathcal{K}(\sigma) \xrightarrow{\sim} \mathcal{K}(\sigma')$.

If $v \in V$ is not smooth, we achieve the isomorphism $\mathcal{K}(\sigma) \xrightarrow{\sim} \mathcal{K}(\sigma')$ by first shrinking $V$ around $v$ if necessary, and then imbedding $V$ into a smooth algebraic $k$-scheme $W$ as a closed subscheme. In greater detail, if $w$ is the image of $v$ in $W$, and $\tau$ a pseudo-coefficient field at $w$ which is a lift of $\sigma$, then $\mathcal{K}(\sigma)$ can be thought of as the submodule of $\mathcal{K}(\tau)$ annihilated by the kernel of the $k$-algebra surjection $\widehat{\mathcal{O}}_{W,w} \twoheadrightarrow \widehat{\mathcal{O}}_{V,v}$. If $\tau'$ is a lift of $\sigma'$, the isomorphism $\mathcal{K}(\tau) \xrightarrow{\sim} \mathcal{K}(\tau')$ described above

[2] In other words $\sigma$ is a $k$-algebra homomorphism such that $\sigma$ followed by the natural surjection $\widehat{\mathcal{O}}_{V,v} \to k(v)$ is a finite field extension.

restricts to an isomorphism $\mathcal{K}(\sigma)\xrightarrow{\sim}\mathcal{K}(\sigma')$. This last isomorphism is independent of the auxiliary lifts $\tau$ and $\tau'$ (cf. 4.5, 4.5.1, 4.6 and 4.6.1) and of $W$ (4.6). We also show in 4.7 that if $V$ is reduced, this isomorphism agrees with the one deduced by Yekutieli in [31], 4.3.13.

Let $\Psi_\sigma^{\sigma'}:\mathcal{K}(\sigma)\rightarrow\mathcal{K}(\sigma')$ be the above isomorphism. Then $\Psi_{\sigma'}^{\sigma''}\circ\Psi_\sigma^{\sigma'}=\Psi_\sigma^{\sigma''}$ for a third pseudo-coefficient field $\sigma''$ (cf. 4.6.2). Set $\mathcal{K}(v):=\varinjlim_\sigma \mathcal{K}(\sigma)$. $\mathcal{K}(v)$ is an injective hull of $k(v)$. Here are some functorial properties of $\mathcal{K}(v)$-axiomatized as a 'trace structure' in 5.1, and proved in 5.3 – which are crucial in constructing a concrete model of $\Delta^\bullet$:

1. For every pseudo-coefficient field $\sigma$ of $v$, there a $\sigma$-linear map $t_\sigma:\mathcal{K}(v)\rightarrow\omega_\sigma$ such that for an $\mathcal{O}_{V,v}$-module $M$ with zero-dimensional support, the natural map (induced by $t_\sigma$)

   $$\mathrm{Hom}_{\widehat{\mathcal{O}}_{V,v}}(M,\mathcal{K}(v))\rightarrow\mathrm{Hom}_\sigma(M,\omega_\sigma)$$

   is an isomorphism.
2. For a map $V\rightarrow W$ in $\mathcal{V}$ such that $v$ is closed in the fibre $f^{-1}f(v)$, there is an $\widehat{\mathcal{O}}_{W,f(v)}$-linear map (unique by 1. above)

   $$\theta_{f,v}:\mathcal{K}(v)\rightarrow\mathcal{K}(f(v)),$$

   such that $t_\tau\circ\theta_{f,v}=t_\sigma$ for every pseudo-coefficient field $\tau$ at $f(v)$, where $\sigma$ is the pseudo-coefficient field at $v$ obtained by composing $\tau$ with the natural map $\widehat{\mathcal{O}}_{W,f(v)}\rightarrow\widehat{\mathcal{O}}_{V,v}$. Further, $(\mathcal{K}(v),\theta_{f,v})$ represents the functor $\mathrm{Hom}_{\widehat{\mathcal{O}}_{V,v}}(M,\mathcal{K}(f(v)))$ of $\mathcal{O}_{V,v}$-modules $M$ with zero-dimensional support.
3. If $v$ is a smooth point, there is an $\widehat{\mathcal{O}}_{V,v}$-linear isomorphism (unique by 1. above)

   $$\phi_v:\mathcal{K}(v)\xrightarrow{\sim}H_v^d(\Omega_{V/k}^n),$$

   where $d=\dim\mathcal{O}_{V,v}$ and $n$ is the dimension of the irreducible component of $V$ containing $v$.

The residue machinery developed in Section 3 and Section 4 is used not just to define $\mathcal{K}(v)$, but also the maps $\theta_{f,v}$ mentioned above. Section 4 concentrates on the case of pseudo-coefficent fields, and works out the isomorphism $\mathcal{K}(\sigma)\xrightarrow{\sim}\mathcal{K}(\sigma')$ mentioned above.

## 0.3. Construction

If $V$ is smooth we set $\mathcal{K}_V^\bullet=E_V^\bullet(\Omega_{V/k}^n)[n]$, where $n=\dim V$. Now drop the assumption that $V$ is smooth. For $v\in V$ let $\bar{\mathcal{K}}(v)$ denote the sky-scraper sheaf on $V$ induced by $\mathcal{K}(v)$. Suppose there is a closed immersion $i:V\rightarrow X$ with $X$ smooth. Then using $\theta_{i,v}$, there is an isomorphism of graded quasi-coherent sheaves $\mathcal{K}_V^\bullet\xrightarrow{\sim}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_V,\mathcal{K}_X^\bullet)$, where $\mathcal{K}_V^p=\bigoplus_{d(v)=p}\bar{\mathcal{K}}(v)$. The right side of the above isomorphism is a complex, and hence induces a structure of a complex on the graded sheaf $\mathcal{K}_V^\bullet$. This structure is independent of the immersion $V \to X$, as we show in Section 6.

By construction, the coboundary map on $\mathcal{K}_V^\bullet$ localizes well. Hence if $V \in \mathcal{V}$ is arbitrary, we may cover $V$ by open subschemes $\{V_\alpha\}$ each of which admits a closed immersion $i_\alpha \longrightarrow X_\alpha$ with $X_\alpha$ smooth, and then define a complex $\mathcal{K}_V^\bullet$ by glueing together the $\mathcal{K}_{V_\alpha}^\bullet$. The collection $\{\mathcal{K}_V^\bullet\}_{V \in \mathcal{V}}$, with the obvious notion of a restriction, forms a complex of $\mathcal{O}$-modules $\mathcal{K}^\bullet$. From the construction, if $V$ is a closed subscheme of a smooth scheme, then $\Delta_V^\bullet \xrightarrow{\sim} \mathcal{K}_V^\bullet$. It turns out (cf. 8.7) that $\Delta^\bullet$ is isomorphic to $\mathcal{K}^\bullet$ as a complex of $\mathcal{O}$-modules.

## 0.4. Traces

Let $f\colon V \to W$ be a proper map in $\mathcal{V}$. We construct a map of complexes $\theta_f\colon f_*\mathcal{K}_V^\bullet \to \mathcal{K}_W^\bullet$ which is concrete realization of the trace map $T_f\colon f_*\Delta_V^\bullet \to \Delta_W^\bullet$ in Duality theory (cf. [7], VI, Section 4).

First, we extend the definition of $\theta_{f,v}\colon \mathcal{K}(v) \to \mathcal{K}(f(v))$ to points $v$ which are not closed in the fibre $f^{-1}f(v)$ by setting $\theta_{f,v}$ equal to zero in this case. Let $\bar{\theta}_{f,v}\colon f_*\bar{\mathcal{K}}(v) \to \bar{\mathcal{K}}(f(v))$ be the resulting map. We show that $\theta_f = \Sigma_{v \in V}\bar{\theta}_{f,v}\colon f_*\mathcal{K}_V^\bullet \to \mathcal{K}_W^\bullet$ is a map of complexes and that $\theta_f$ is a concrete realization of $T_f$ (cf. Section 8, Proposition 8.1, Theorem 8.6 and the remarks in 8.5). We do the above in two stages. In Section 7, we deal with the case where $W = \operatorname{Spec} k$. In Section 8 we tackle the relative case. We use the main theorem of Section 2, viz. 2.1, in an essential way to show that $\theta_f$ is a map of complexes. In fact if $W$ and $f$ are smooth, 2.1 applies immediately. This is used to show that if $f$ is finite and dominant then $\theta_f$ is a map of complexes. This last case enters in an essential way to take care of arbitrary proper $f$.

## 0.5. Comparison with Yekutieli's construction

If $\mathcal{V}^{\text{red}}$ is the full subcategory of $\mathcal{V}$ consisting of *reduced* algebraic schemes, then in [31] Yekutieli constructs a sheaf of $\mathcal{O}$-modules $\mathcal{C}^\bullet$ (denoted $\mathcal{K}^\bullet$ in *ibid.*) on the Zariski site on $\mathcal{V}^{\text{red}}$, which he calls the 'Grothendieck Residue Complex'. $\mathcal{C}^\bullet$ is isomorphic to $\Delta^\bullet$ (cf. [28], especially the exercise on p. 126).

As we mentioned in 0.1, the construction involves residues of meromorphic differentials on topological local fields. This (non-cohomological) residue is used (among other things) to establish the isomorphism $\mathcal{K}(\sigma) \xrightarrow{\sim} \mathcal{K}(\sigma')$ discussed in 0.2 (cf. [31], 4.3.13). Yekutieli defines a 'System of Residue Data' ([31], 4.3.10) – an axiomatization of what is required to construct $\mathcal{C}^\bullet$ – and using his non-cohomological residues, the existence of such a system is established (cf. [31], 4.3.16).

In our paper we eschew the residues of the Russian school ([26], [27], [24], [1]) as well as the topological-algebra machinery (semi-topological rings, topological local fields) of Yekutieli, and use instead the residue machinery developed over the years by Kunz, Lipman, Hubl and Huang cited in 0.1[3]. If $V \in \mathcal{V}^{\text{red}}$, then up to a sign our complex $\mathcal{K}_V^\bullet$ agrees with $\mathcal{C}_V^\bullet$ (cf. 9.3) provided $V$ is equidimensional.

While this work was in progress, Yekutieli developed a theory of continuous diferential operators on 'Beilinson Completion Algebras' (cf. [33]) using which he extends his results in [31] to the entire category of algebraic $k$-schemes and constructs a dualizing complex of $\mathcal{O}$-modules in $\mathcal{V}$ in [32]. He also has traces for proper morphisms. The interest then is in our techniques which are substantially different from Yekutieli's. A work of related interest is [11] of Huang.

## 1. Cousin complexes

We assume familiarity with the notion of a Cousin complex as laid out in [7], as well as the explicit description of Cousin complexes given by [17], [18] and [2]. (See also [14] Section 2). In this section we point out another description of the Cousin complex associated to a quasi-coherent sheaf $\mathcal{F}$ on a scheme $X$ (cf. [7], p. 232, (2.3) and the Definition in *ibid*, p. 235) in a very special situation (cf. Proposition 1.2 below).

1.1. Let $R$ be a noetherian ring of finite Krull dimension, which is equidimensional. In this case the 'height function' $h : \operatorname{Spec} R \to \mathbb{Z}$ (here $h(\mathfrak{p})$ is the height of $\mathfrak{p}$) is a codimension function, i.e. $h(\mathfrak{p}) = h(\mathfrak{q}) + 1$ for every immediate specialization $\mathfrak{q} \mapsto \mathfrak{p}$ in $\operatorname{Spec} R$. Let $M$ be an $R$-module such that $\operatorname{Supp}(M) = \operatorname{Spec} R$, i.e. $\operatorname{ann}_R M = 0$. For any prime ideal $\mathfrak{p}$ in $R$, we set $H_\mathfrak{p}(M) = H^{h(\mathfrak{p})}_{\{\mathfrak{p}R_\mathfrak{p}\}}(M_\mathfrak{p})$. Let $E_R^\bullet(M)$ be the Cousin complex of $M$ with respect to the filtration on $\operatorname{Spec} R$ given by $Z^p = \{\mathfrak{p} \in \operatorname{Spec} R \mid h(\mathfrak{p}) \leqslant p\}$. Then $E_R^\bullet(M) = C_\mathcal{S}^\bullet(M)$, the Cousin complex associated to the system of denominators $\mathcal{S} = \mathcal{S}(M) = \mathcal{S}(R)$ (cf. [2], [17] and [18] for details)[4]. By [2] (5.3), for any $\mathfrak{p} \in \operatorname{Spec} R$,

$$E_R^p(M) \xrightarrow{\sim} \bigoplus_{h(\mathfrak{p})=p} E_{R_\mathfrak{p}}^p(M_\mathfrak{p}).$$

One checks easily that $E_{R_\mathfrak{p}}^p(M_\mathfrak{p}) = H_\mathfrak{p}(M)$ for $p = h(\mathfrak{p})$. Moreover, under this identification, a generalised fraction

$$\begin{bmatrix} m/g \\ f_1, \ldots, f_p \end{bmatrix} \in E_{R_\mathfrak{p}}^p(M)$$

as defined for example in [2], p. 18 (cf. also Section 2 of [14]) gets identified with the corresponding generalized fraction as defined for e.g. in [21], p. 59. We use the convention that

$$\delta \begin{bmatrix} m/g \\ f_1, \ldots, f_p \end{bmatrix} = \begin{bmatrix} m/1 \\ g, f_1, \ldots, f_p \end{bmatrix}, \tag{1.1.1}$$

[3] For some time, the connection between the two kinds of residues was not clear (though El-Zein's work [3] was a potential bridge). However Yekutieli's work opened up possibilities, and a connection was recently established (cf. [14] and [29]).

[4] If Supp $(M) \neq \operatorname{Spec} R$, then $C^\bullet_{\mathcal{S}(M)} = E^\bullet_{R/I}(M)$ where $I = \operatorname{ann}_R M$.

where $\delta$ is the coboundary map in $C^\bullet_{\mathcal{S}}(M)$. This differs from the conventions adopted by [17], [18], [2] by a sign.

The complex $E^\bullet_R(M)$ gives us maps $\delta_{\mathfrak{p},\mathfrak{q}}: H_{\mathfrak{p}}(M) \to H_{\mathfrak{q}}(M)$ for $\mathfrak{p}$ and $\mathfrak{q}$ in Spec $R$. This map is zero unless $\mathfrak{p} \mapsto \mathfrak{q}$. Here $\mapsto$ denotes *immediate specialization*.

PROPOSITION 1.2. *Let* $\mathfrak{p} \mapsto \mathfrak{q}$, *so that* $\mathfrak{p}R_{\mathfrak{q}}$ *is a closed point of the punctured spectrum* $U = \operatorname{Spec} R_{\mathfrak{q}} \backslash \{\mathfrak{q}R_{\mathfrak{q}}\}$. *Then with* $\widetilde{M}$ *the quasi-coherent sheaf on* $U$ *induced by* $M$, $\delta_{\mathfrak{p},\mathfrak{q}}$ *is the natural composition*

$$H_{\mathfrak{p}}(M) \to H^p(U, \widetilde{M}) \to H_{\mathfrak{q}}(M),$$

*where* $p = h(\mathfrak{p})$.

*Proof.* An element $\xi \in H_{\mathfrak{p}}(M)$ can be represented by a generalised fraction, which for typographical convenience we denote by $m/g//\underline{f}$, where $\underline{f} = (f_1, \ldots, f_p)$ is in $\mathcal{S}^p$. Let $Y$ be the closed subscheme of $U$ given by the vanishing of the $f_i$. Then, by definition of a system of denominators, $Y$ consists of a finite number of closed points, which correspond to certain prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of $R$ of height $p$. Note that $\mathfrak{p}$ occurs in this list. Since $\xi \in H_{\mathfrak{p}}(M)$, therefore $\Sigma_i \delta_{\mathfrak{p}_i,\mathfrak{q}}\xi = \delta_{\mathfrak{p},\mathfrak{q}}\xi$. Let $U_0$ be the open subscheme of $U$ on which $g$ never vanishes, and for $i = 1, \ldots, p$, let $U_i$ be the open subscheme of $U$ on which $f_i$ never vanishes. We are precisely in the situation examined in [21], pp. 78–81 (with $\mathcal{F} = \widetilde{M}$) and we are done by *ibid*, (8.6). □

1.3. NOTATIONS. Let $X$ be a noetherian equidimensional scheme of finite Krull dimension. For $x \in X$, let $h(x)$ denote the dimension of the local ring $\mathcal{O}_{X,x}$ and for $p \geqslant 0$ set $Z^p = \{x \in X \mid h(x) \leqslant p\}$. For $\mathcal{F}$ quasi-coherent on $X$, $E^\bullet_X(\mathcal{F})$ will denote the Cousin complex of $\mathcal{F}$ with respect to the filtration $Z^\bullet$ as defined in [7], p. 235. Moreover, $E^\bullet_X(\mathcal{F}[d])$ will denote the complex $E^\bullet_X(\mathcal{F})[d]$.

Note that if $\operatorname{Supp}(\mathcal{F}) = X$, then $E^\bullet_X(\mathcal{F})$ can be described by 1.2.

1.4. Let $Z$ be a Noetherian scheme of finite Krull dimension. Let $z \in Z$, and suppose $M$ is a $\mathcal{O}_{Z,z}$-module with zero-dimensional support, i.e., every element of $M$ is annihilated by some power of the maximal ideal of $\mathcal{O}_{Z,z}$. Define a quasi-coherent $\mathcal{O}_Z$-module by

(a) For $U$ an open subset of $Z$,

$$\bar{M}(U) = \begin{cases} M & \text{if } z \in U \\ 0 & \text{otherwise} \end{cases}.$$

(b) For a pair of open sets $V \subset U$, with $z \in V$, the restriction map $\bar{M}(U) \to \bar{M}(V)$ is defined to be the identity.

If $z' \in Z$ is a specialization of $z$, and $N$ is an $\mathcal{O}_{Z,z'}$-module with zero-dimensional support then each $\mathcal{O}_{Z,z'}$ homomorphism $\phi: M \to N$ gives rise to an $\mathcal{O}_Z$-module map $\bar{\phi}: \bar{M} \to \bar{N}$. Conversely, an $\mathcal{O}_Z$-module map $\psi: \bar{M} \to \bar{N}$ determines a unique $\mathcal{O}_{Z,z'}$-homomorphism $\phi: M \to N$ such that $\psi = \bar{\phi}$.

## 1.5. Residual complexes

Recall that a complex of quasi-coherent $\mathcal{O}_Z$-modules $\mathcal{F}^\bullet$ on $Z$ is called *residual* if $\mathcal{F}^\bullet = \bigoplus_{z \in Z} \bar{J}(z)$, where $J(z)$ is an $\mathcal{O}_{Z,z}$ injective hull of the residue field at $z$, and if the cohomology sheaves of $\mathcal{F}^\bullet$ are coherent. In this case, there is an associated co-dimension function $d: X \to \mathbb{Z}$ such that $\mathcal{F}^p = \bigoplus_{d(z)=p} \bar{J}(z)$, $p \in \mathbb{Z}$ (cf. [7], p. 287, Remark 4 and [7], pp. 305–306, Proposition (1.1) (c)).

We refer the reader to [7], Chapter VI for further details about residual complexes.

### *1.5.1. Conventions*

(a) In this paper, for any residual complex $\mathcal{R}^\bullet$ on $Z$, and any $z \in Z$, $\bar{\mathcal{R}}(z)$ will denote the direct summand corresponding to $z$, and $\mathcal{R}(z)$ the corresponding $\mathcal{O}_{Z,z}$-module. In other words, if $d: Z \to \mathbb{Z}$ is the codimension function corresponding to $\mathcal{R}^\bullet$, then $\mathcal{R}(z) = (\Gamma_z \mathcal{R}^\bullet)[d(z)]$. Thus if $\mathcal{F}^\bullet$ is the complex in 1.5 above, then $\mathcal{F}(z) = J(z)$. Further, for $z \in Z$, with $d(z) = p$, $i_z: \bar{\mathcal{R}}(z) \to \mathcal{R}^p$ will denote the natural inclusion, and $\pi_z: \mathcal{R}^p \to \bar{\mathcal{R}}(z)$ the natural projection.

(b) As in [28], we say that a residual complex $\mathcal{R}^\bullet$ on $Z$ is *normalized* if $d(z) = -\dim(\overline{\{z\}})$ for every $z \in Z$, where $d$ is the codimension function associated with $\mathcal{R}^\bullet$.

1.6. Let $K$ be an Artin local ring; $\{p\} :=$ Spec $K$ and $I$ an injective hull of the $K$-module $K/\mathfrak{m}_K$ (where $\mathfrak{m}_K$ = the maximal ideal of $K$). Let $f: X \to \{p\}$ be a finite-type morphism of schemes. For each *closed point* $x \in X$, let $(X_x)_\#$ be the category of $\mathcal{O}_{X,x}$-modules with zero-dimensional support (in the notation of [10], Section 7, $(X_x)_\#$ is the category $(\mathcal{O}_{X,x})_\#$. Let $\mathrm{Mod}_x$ be the category of $\mathcal{O}_{X,x}$-modules. Define

$$F_x: (X_x)_\# \to \mathrm{Mod}_x$$

by

$$F_x = \mathrm{Hom}_K(-, I).$$

Let $Q$ denote both the localization functors $K(X) \to D(X)$ and $K(\{p\}) \to D(\{p\})$. In analogy with [28] we make the following definition:

1.6.1. DEFINITION. Let $f: X \to \{p\}$ be proper. A pair $(\mathcal{R}^\bullet, \theta)$ is called a *residue pair* if

(a) $\mathcal{R}^\bullet$ is a normalized residue complex on $X$, and $\theta: f_* \mathcal{R}^\bullet \to I$ is a map of complexes (here we identify quasi-coherent sheaves on $\{p\}$ with $K$-modules).

(b) $(Q\mathcal{R}^\bullet, Q\theta)$ is a dualizing pair, i.e. $Q\theta: \mathbb{R}f_*\mathcal{R}^\bullet \to I$ induces an isomorphism

$$\mathbb{R}\mathrm{Hom}^\bullet_X(\mathcal{F}^\bullet, Q\mathcal{R}^\bullet) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}^\bullet_{\{p\}}(\mathbb{R}f_*\mathcal{F}^\bullet, I)$$

in $D(\{p\})$ for every $\mathcal{F}^\bullet$ in $D^+_{\mathrm{qc}}(X)$.

1.6.2. REMARK. Let $f: X \to \{p\}$ be proper. One can show that residue pairs exist. This is a consequence of the fact that dualizing pairs exist and [7], p. 304, (1.1) (cf. also p. 306, Remark 1 of *ibid.*).

Theorem 2 of [28] has the following generalisation:

THEOREM 1.7. *Let $f: X \to \{p\}$ be a proper morphism; $\mathcal{R}^\bullet$ a normalized residual complex and $\theta: f_*\mathcal{R}^\bullet \to I$ a map of complexes of $K$-modules. Then $(\mathcal{R}^\bullet, \theta)$ is a residue pair if and only if for every closed point $x \in X$, the pair $(\mathcal{R}(x), \theta(x))$ represents the functor $F_x$, where $\theta(x): \mathcal{R}(x) \to I$ is the natural inclusion $\mathcal{R}(x) \subset f_*\mathcal{R}^\bullet = \Gamma(X, \mathcal{R}^\bullet)$ followed by the map $\theta: f_*\mathcal{R}^\bullet \to I$.*

*Proof.* Suppose $(\mathcal{R}^\bullet, \theta)$ is a residue pair. Let $x \in X$ be a closed point; $M \in (X, x)_\#$ and $g: M \to I$ a member of $F_x(M)$. Let $\bar{M}$ be the quasi-coherent sheaf associated to $M$ as in 1.4. Since $x$ is a closed point it is not hard to see that $\bar{M}$ is a sky-scraper sheaf supported at $x$. Clearly $M$ is flasque, and hence $\mathbb{R}f_*\bar{M}$ can be identified with $f_*\bar{M}$. Let $\bar{g}: f_*\bar{M} \to I$ be the $\mathcal{O}_{\{p\}}$-map corresponding to $g: M \to I$. By Grothendieck duality, and the fact that $\bar{M}$ is a Cousin Complex, it is immediate that there is a unique $\mathcal{O}_X$-map $\bar{h}: \bar{M} \to \mathcal{R}^\bullet$ such that $\bar{g} = \theta \circ f_*(\bar{h})$ (cf. [7], p. 247, Lemma (3.2)). It is trivial to check that this gives rise to a unique $\mathcal{O}_{X,x}$-linear map $h: M \to \mathcal{R}(x)$ such that $\theta(x) \circ h = g$. Thus $(\mathcal{R}(x), \theta(x))$ represents the functor $F_x$.

For the converse we need analogues of Lemmas 1, 2 and 3 of [28].

Let $A$ be a local ring, essentially of finite type over $K$, such that $K/\mathfrak{m}_K \to A/\mathfrak{m}_A$ is finite. Let $\mathrm{Mod}_A$ denote (as usual) the category of $A$-modules, and $A_\#$ the full subcategory of $A$-modules with zero-dimensional support. Define

$$F = F_A \to \mathrm{Mod}_A$$

by

$$F = \mathrm{Hom}_X(-, I).$$

Set $\mathcal{K} = \mathrm{Hom}^c_K(A, I)$ where the superscript '$c$' denotes *continuous* $K$-homomorphisms with $A$ being endowed with its $\mathfrak{m}_A$-adic topology and $I$ with the $\mathfrak{m}_K$-adic topology. $\mathcal{K}$ is well-known to be an injective hull of the $A$-module $A/\mathfrak{m}_A$, and hence $\mathcal{K} \in A_\#$. Let $\tau \in F(\mathcal{K})$ be the $K$-map $\mathcal{K} \to I$ given by 'evaluation at 1'. We then have the following lemmas (compare with Lemmas 1, 2 and 3 of [28]):

LEMMA 1.7.1. *$(\mathcal{K}, \tau)$ represents the functor $F$.*

LEMMA 1.7.2. *Let $(J, q, \gamma)$ be a triple consisting of*

(a) *An injective hull $J$ of the $A$-module $A/\mathfrak{m}_A$.*

(b) *An element* $q \in F(J)$.
(c) *An A-linear map* $\gamma: \mathcal{K} \to J$ *such that* $q \circ \gamma = \tau$.

*Then* $\gamma$ *is an isomorphism and* $(J, q)$ *represents* $F$.

LEMMA 1.7.3. *Let* $\mathcal{S}^\bullet$ *and* $\mathcal{S}'^\bullet$ *be normalized residual complexes on* $A$ *(i.e. the corresponding complexes of quasi-coherent* $\mathcal{O}_{\mathrm{Spec}(A)}$ *complexes on* Spec$(A)$ *are normalized residual). Let* $\mathcal{S}(\mathfrak{m}) = \Gamma_{\mathfrak{m}}(\mathcal{S}^\bullet)$ *and* $\mathcal{S}'(\mathfrak{m}) = \Gamma_{\mathfrak{m}}(\mathcal{S}'^\bullet)$, *where* $\Gamma_{\mathfrak{m}}$ *is the functor 'sections supported in* $\mathfrak{m}$*'. Then*

(a) *A morphism* $\alpha: \mathcal{S}^\bullet \to \mathcal{S}'^\bullet$ *is an isomorphism if and only if the map* $\Gamma_{\mathfrak{m}}(\alpha): \mathcal{S}(\mathfrak{m}) \to \mathcal{S}'(\mathfrak{m})$ *is an isomorphism.*
(b) *If* $\mathcal{S}'^\bullet$ *is equal to* $\mathcal{S}^\bullet$ *in* (a), *then* $\alpha$ *is the identity map if and only if* $\Gamma_{\mathfrak{m}}(\alpha)$ *is the identity map.*

The proofs of Lemmas 1.7.1, 1.7.2, 1.7.3 are, *mutatis mutandis*, as in Lemmas 1, 2 and 3 of [28] (cf. pp.122–124 of *ibid.*).

Now suppose $(\mathcal{R}^\bullet, \theta)$ is a pair such that $\mathcal{R}^\bullet$ is normalized residual, and $\theta: f_*\mathcal{R}^\bullet \to I$ is a map of complexes such that $(\mathcal{R}(x), \theta(x))$ represents $F_x$ for every closed point $x \in X$. Let $(\mathcal{F}^\bullet, \psi)$ be a residue pair (cf. 1.6.2). Then we have a unique map $\alpha: \mathcal{R}^\bullet \to \mathcal{F}^\bullet$ such that $\psi \circ f_*\alpha = \theta$. Lemmas 1.7.2 and 1.7.3 then show that $\alpha$ is an isomorphism. □

## 2. Cousin complexes and equidimensional maps

Let $f: X \to Y$ be a dominant map of schemes. We assume throughout that $X$ and $Y$ satisfy the assumptions in 1.3 and that $f$ is *equidimensional* of dimension $d$. Let $h_X$ and $h_Y$ denote the height functions on $X$ and $Y$ respectively. When no confusion is likely to arise, we suppress the subscripts $X$ and $Y$ in $E_X^\bullet$, $E_Y^\bullet$, $h_X$ and $h_Y$. For $x \in X$, and $y \in Y$, write $H_x = H_x^{h(x)}$ and $H_y = H_y^{h(y)}$. Then $E_X^r = \bigoplus_{h(x)=r} H_x$ and $E_y^r = \bigoplus_{h(y)=r} H_y$.

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We define for $x \in X$, a map

$$\rho_{f,x}: H_x(\mathcal{F}) \to H_{f(x)}(R^d f_*\mathcal{F})$$

as follows (with $y = f(x)$)

(a) if $x$ is not closed in $f^{-1}(y)$, then $\rho_{f,x} = 0$
(b) if $x$ is closed in $f^{-1}(y)$, so that $h(x) = h(y) + d$ then with

$$\begin{aligned} X_y &= X \times_Y (\mathrm{Spec}(\mathcal{O}_{Y,y})) \\ \mathcal{F}_{X_y} &= \mathcal{F} \otimes_X X_y \\ r &= h(y) \end{aligned}$$

we let $\rho_{f,x}$ be the composition

$$H_x^{r+d}(\mathcal{F}) \xrightarrow{\text{natural}} H_{f^{-1}(y)}^{r+d}(X_y, \mathcal{F}_{X_y}) \xrightarrow{\sim} H_y^r(R^d f_* \mathcal{F})$$

the isomorphism arising from the Leray Grothendieck spectral sequence. Thus we obtain a map of graded-sheaves

$$\rho_f : f_* E_X^\bullet(\mathcal{F}[d]) \to E_Y^\bullet(R^d f_* \mathcal{F}).$$

The main result of this section is

PROPOSITION 2.1. *With above notations, suppose* $\mathrm{Supp}(\mathcal{F}) = X$ *and* $\mathrm{Supp}(R^d f_* \mathcal{F}) = Y$. *Then the map* $\rho_f$ *is a homomorphism of complexes.*

Write $\delta^X$ for the coboundary map in $E_X^\bullet(\mathcal{F}[d])$ and $\delta^Y$ for the coboundary map in $E_Y^\bullet(R^d f_* \mathcal{F})$. Let $x \in X$, $h(x) = r + d$, and $\xi \in H_x^{r+d}(\mathcal{F})$. We need to show

$$\delta^Y(\rho_{f,x}\xi) = \sum_{x \mapsto x'} (\rho_{f,x'} \delta^X_{x,x'} \xi). \tag{2.1.1}$$

We divide the proof of 2.1.1 into two cases: (a) when $x$ is closed in its fibre, and (b) when $x$ is not closed in its fibre.

LEMMA 2.1.2. *Suppose* $x$ *is not closed in* $f^{-1}(y)$, *where* $y = f(x)$. *Then* $\Sigma \rho_{f,x'} \delta^X_{x,x'} \xi = 0$, *where the sum is taken over* $x \mapsto x'$ *with* $f(x') = y$ *and* $x'$ *closed in* $f^{-1}(y)$.

The lemma gives 2.1.1 for $x$ as in the lemma. Indeed, for such an $x$, by definition $\rho_{f,x} = 0$, and if $x \mapsto x'$ and $f(x') \neq y$, then $x'$ is not closed in $f^{-1} f(x')$ (cf. [7], p. 333, (3.4)).

Let $V = X_y$, $W = \mathrm{Spec}(\mathcal{O}_{Y,y})$, and $g : V \to W$ the equidimensional map induced by $f : X \to Y$. Let $v \in V$ be the point corresponding to $x$, and $w \in W$ the point corresponding to $y$, i.e., the closed point of $W$.

Let $\Sigma$ be a finite set of closed points of $g^{-1}(w)$. Let $G_\Sigma$ be the left exact functor:

$$G_\Sigma = \Gamma_{g^{-1}(w) - \Sigma}(V \backslash \Sigma, -)$$

and $F_\Sigma$ the left exact functor:

$$F_\Sigma = \Gamma_{\bar{v} - \Sigma}(V \backslash \Sigma, -).$$

There are natural maps $\phi_\Sigma : F_\Sigma \to G_\Sigma$. For $\Sigma \subset \Sigma'$ we have maps: $\mu^\Sigma_{\Sigma'} : \Gamma_\Sigma \leftarrow \Gamma_{\Sigma'}$, $\nu^\Sigma_{\Sigma'} : G_\Sigma \to G_{\Sigma'}$ and $\bar{\nu}^\Sigma_{\Sigma'} : F_\Sigma \to F_{\Sigma'}$, which make $\{\Gamma_\Sigma\}$; $\{G_\Sigma\}$ and $\{F_\Sigma\}$ into directed systems. Further,

$$0 \to \Gamma_\Sigma \to \Gamma_{g^{-1}(w)} \to G_\Sigma$$

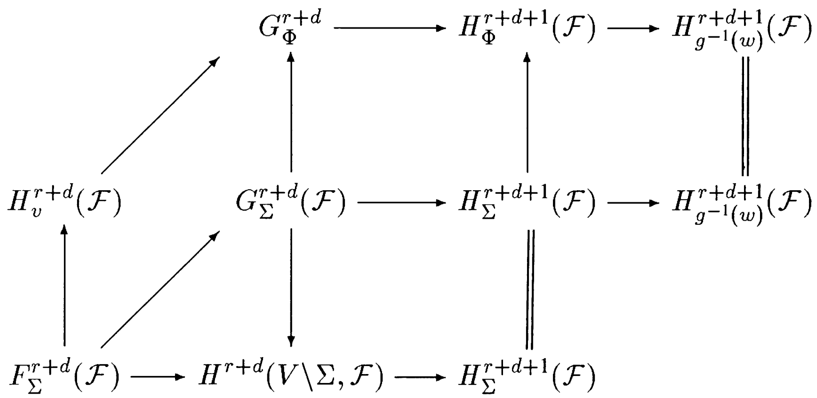
is an exact sequence of directed systems which is surjective on the right when evaluated at a flasque sheaf. Moreover, $\phi_{\Sigma'} \circ \bar{\nu} = \nu \circ \phi_{\Sigma}$. Let the direct limits of $\{\Gamma_\Sigma\}$, $\{G_\Sigma\}$, and $\{F_\Sigma\}$ be $\Gamma_\Phi$, $G_\Phi$, and $F_\Phi$ respectively. Note that $F_\Phi = \Gamma_v$; and we have a natural map $\Gamma_v \to G_\Phi$. Let the derived functors of $G_\Sigma$, $F_\Sigma$, $G_\Phi$, $\Gamma_\Phi$ be $G^j_\Sigma$, $F^j_\Sigma$, $G^j_\Phi$ and $H^j_\Phi$ respectively. Lemma 2.1.2 then follows from:

LEMMA 2.1.3. *Let $\Sigma$ be such that $\xi$ has a pre-image $\xi'$ under the natural map $F^{r+d}_\Sigma(\mathcal{F}) \to H^{r+d}_v(\mathcal{F})$. Then the image of $\xi'$ under*

$$F^{r+d}_\Sigma(\mathcal{F}) \to H^{r+d}(V\backslash\Sigma, \mathcal{F}) \to H^{r+d+1}_\Sigma(\mathcal{F})$$

*is* $\sum_{v' \in \Sigma} \delta^V_{v,v'}(\xi) \in H^{r+d+1}_\Sigma(\mathcal{F})$.

We deduce Lemma 2.1.2 from Lemma 2.1.3 as follows. A $\Sigma$ of the sort assumed in Lemma 2.1.3 always exists. Moreover the diagram below commutes and its top row is exact. Consequently, the image of $H^{r+d}_v(\mathcal{F})$ in $H^{r+d+1}_{g^{-1}(w)}$ (via the north-east pointing arrow and the top row) is zero. Using Lemma 2.1.3, it follows that the image of the sum $\Sigma_{v' \in \Sigma} \delta^V_{v,v'}$ in $H^{r+d+1}_{g^{-1}(w)}(\mathcal{F})$ is zero for $\Sigma$ 'sufficiently large'.




(2.1.4)

We now give the proof of Lemma 2.1.3.

*Proof.* Let $U \subset V$ be an open set containing $\Sigma$ and let $H_U = \Gamma_{\bar{v}\backslash(U-\Sigma)}(U\backslash\Sigma, -)$. Then the direct limit of $H_U$, as $U$ ranges over open sets containing $\Sigma$, is $\Gamma_v$. Let $\Gamma(U_\Sigma, -)$ and $\Gamma(U_\Sigma\backslash\Sigma, -)$ be the direct limits over $U \supset \Sigma$ of $\Gamma(U, -)$ and $\Gamma(U\backslash\Sigma, -)$ respectively. Let their respective derived functors be $H^i(U_\Sigma, -)$ and $H^i(U_\Sigma\backslash\Sigma, -)$. Then for $U \supset \Sigma$, we have an exact sequence

$$0 \to \Gamma_\Sigma \to \Gamma(U_\Sigma, -) \to \Gamma(U_\Sigma\backslash\Sigma, -) \tag{2.1.5}$$

with the right arrow being surjective when evaluated on flasque sheaves. The resulting connecting homomorphism $H^{r+d}(U_\Sigma\backslash\Sigma, -) \to H^{r+d+1}_\Sigma$ is compatible with the connecting homomorphism $H^{r+d}(U\backslash\Sigma, -) \to H^{r+d+1}_\Sigma$ in an obvious sense.

Here is another description of $\Sigma_{v' \in \Sigma} \delta^V(v, v')$ in terms of this connecting homomorphism. On taking direct limits, the natural maps $H_U \to \Gamma(U \backslash \Sigma, -)$ give rise to a natural map $\Gamma_v = \varinjlim_U H_U \to \Gamma(U_\Sigma \backslash \Sigma, -)$ and hence maps $H^i_v \to H^i(U_\Sigma \backslash \Sigma, -)$. Then $\Sigma_{v' \in \Sigma} \delta^V_{v,v'}$ is also the composition

$$H^{r+d}_v(\mathcal{F}) \to H^{r+d}(U_\Sigma \backslash \Sigma, \mathcal{F}) \to H^{r+d+1}_\Sigma(\mathcal{F}) = \bigoplus_{v' \in \Sigma} H^{r+d+1}_{v'}(\mathcal{F}).$$

This follows from 1.2. Since the diagram

$$\begin{array}{ccccc} \Gamma(V \backslash \Sigma, -) & \longrightarrow & \Gamma(U \backslash \Sigma, -) & \longrightarrow & \Gamma(U_\Sigma - \Sigma, -) \\ \uparrow & & \uparrow & & \uparrow \\ H_V = F_\Sigma & \longrightarrow & H_U & \longrightarrow & \varinjlim_U H_U = \Gamma_v \end{array}$$

commutes, and the composition of the arrows in the bottom row is precisely the map $F_\Sigma \to \varinjlim_{\Sigma'} F_{\Sigma'} = \Gamma_v$, therefore the composition $F^{r+d}_\Sigma(\mathcal{F}) \to H^{r+d}_v(\mathcal{F}) \to H^{r+d}(U_\Sigma \backslash \Sigma, \mathcal{F})$ is also the composition $F^{r+d}_\Sigma(\mathcal{F}) \to H^{r+d}(V \backslash \Sigma, \mathcal{F}) \to H^{r+d}(U_\Sigma \backslash \Sigma, \mathcal{F})$. Now use the compatibility of the connecting homomorphisms $H^{r+d}(U_\Sigma \backslash \Sigma, \mathcal{F}) \to H^{r+d+1}_\Sigma(\mathcal{F})$ and $H^{r+d}(V \backslash \Sigma, \mathcal{F}) \to H^{r+d+1}_\Sigma(\mathcal{F})$ as well as the description of $\Sigma_{v' \in \Sigma} \delta^V_{v,v'}$ to reach the desired conclusion. □

LEMMA 2.1.6. *Suppose $x$ is closed in $f^{-1}(y)$. Then for an immediate specialization $y \mapsto y'$, the equation*

$$\Sigma_{x'} \rho_{f,x'} \circ \Sigma_{x'} \delta^X_{x,x'} = \delta^Y_{y,y'} \circ \rho_{f,x} \tag{2.1.7}$$

*holds, where the $x'$ run through $x \mapsto x'$ such that $x' \in f^{-1}(y')$.*

This lemma gives 2.1.1 for $x$ closed in $f^{-1}(y)$. Indeed if $x \mapsto x'$, then $x'$ is closed in $f^{-1}f(x')$ if and only if $y \mapsto f(x')$ (apply [7], p. 333, (3.4)).

*Proof.* Let $W = \operatorname{Spec}(\mathcal{O}_{Y,y'})$, $V = X \times_Y W$, and $g: V \to W$ the equidimensional map induced by $f$. Let $w, w' \in W$ be the points corresponding to $y, y' \in Y$ respectively (so that $w'$ is the unique closed point of $W$). Let $v \in V$ be the point corresponding to $x \in X$. Note that $\bar{v}$ contains only closed points of $V$ other than $v$ itself, and hence $\bar{v}$ has dimension 1. Thus $\bar{v} \cap g^{-1}(w)$ is a finite set, say $v_1, \ldots, v_n$. Set $V' = V \backslash \{v_1, \ldots, v_n\}$. Then $v$ is a closed point of $V'$. For simplicity denote $\mathcal{F}_V$ by $\mathcal{F}$. One checks that if $\mu: H^{r+d}(V \backslash g^{-1}(w'), \mathcal{F}) \to H^{r+d+1}_{g^{-1}(w')}(\mathcal{F})$ and $\nu: H^r(w \backslash \{w'\}, R^d g_* \mathcal{F}) \to H^{r+1}_{w'}(R^d g_* \mathcal{F})$ are the corresponding connecting homomorphisms (cf. [8], p. 9, Prop. 1.9), then the diagram

$$\begin{array}{ccccc} H^{r+d}_{g^{-1}(w)}(\mathcal{F}) & \longrightarrow & H^{r+d}(V\backslash g^{-1}(w'),\mathcal{F}) & \xrightarrow{(-1)^d\mu} & H^{r+d+1}_{g^{-1}(w')}(\mathcal{F}) \\ \uparrow & & \uparrow & & \uparrow \\ H^r_w(R^dg_*\mathcal{F}) & \longrightarrow & H^r(W\backslash\{w'\},R^d_{g_*}\mathcal{F}) & \xrightarrow{\nu} & H^{r+1}_{w'}(R^dg_*\mathcal{F}) \end{array} \tag{2.1.8}$$

commutes, where the vertical isomorphisms arise, as usual, from the Grothendieck–Leray spectral sequences. The sign $(-1)^d$ arises from comparing the various spectral sequences arising from a Cartan–Eilenberg resolution of $g_*J^\bullet$ where $J^\bullet$ is an injective resolution of $\mathcal{F}$.

On the other hand we have a commutative diagram

$$\begin{array}{ccccc} H^{r+d}_v(\mathcal{F}) & \longrightarrow & H^{r+d}(V',\mathcal{F}) & \xrightarrow{\bar{\mu}} & \bigoplus_{i=1}^n H^{r+d+1}_{v_i}(\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow \\ H^{r+d}_{g^{-1}(w)}(\mathcal{F}) & \longrightarrow & H^{r+d}(V-g^{-1}(w'),\mathcal{F}) & \xrightarrow{\mu} & H^{r+d+1}_{g^{-1}(w')}(\mathcal{F}) \end{array} \tag{2.1.9}$$

where $\bar{\mu}$ is the connecting homomorphism of [8], p. 9, Prop. 1.9 . Since $E^\bullet_V(\mathcal{F}[d]) = E^\bullet_V(\mathcal{F})[d]$, therefore $\bar{\mu} = (-1)^d\Sigma_{i=1}^n\delta^V_{v,v_i}$. Putting together 2.1.8 and 2.1.9 we see that 2.1.7 holds. □

## 2.2. Proper smooth schemes over an Artin local ring

Let $f\colon X\to\{p\}$ be a proper smooth morphism of schemes where $\{p\} = \mathrm{Spec}(K)$ is the spectrum of an Artin local ring $K$. Fix an injective hull $I$ of the residue field of $K$ (thought of as a $K$-module). Assume $X$ is connected and has dimension $d$. Define $\Omega^d_f$ to be the top exterior power of the sheaf of relative of forms of the morphism $f$. Let $\mathcal{K}^\bullet_X = E^\bullet(\Omega^d_f\otimes f^*I[d])$. It is well-known that $\mathcal{K}^\bullet_X$ is a normalized residual complex. For a closed point $x\in X$ let $\mathrm{res}_x\colon H^d_x(\Omega^d_f\to K)$ be the map on the bottom of page 119 of [23] (cf. also [15], Definiton (2.1)). One application of 2.1 and 1.7 is the following

THEOREM 2.2.1. *The map of graded $\mathcal{O}_{\{p\}}$-modules*

$$\int^\bullet_{f,I} = \sum \mathrm{res}_x\otimes\mathrm{id}_I\colon f_*\mathcal{K}^\bullet_X\to K$$

*(where the sum runs over* closed points $x\in X$*) is a map of complexes. Moreover the pair $(\mathcal{K}^\bullet_X, \int^\bullet_{f,I})$ is a residue pair.*

EXPLANATION. It is not very difficult to see that for a closed point $x \in X$, the $\mathcal{O}_{X,x}$-module $\mathcal{K}(x) = H^d_x(\Omega^d_f) \otimes I$.

*Proof.* Let $\int_{X/\{p\}} : R^d f_* \Omega^d_f \to K$ be the integral in the main theorem of [16] (pp. 750–752 of *loc cit.*). Consider the map of complexes given by the composition

$$f_* \mathcal{K}^\bullet_X \xrightarrow{\rho} E^\bullet(R^d f_*(\Omega^d_f) \otimes I) \xrightarrow{\int_{X/\{p\}} \otimes \mathrm{id}} E^\bullet(\mathcal{O}_{\{p\}} \otimes I) = I.$$

By the definition of $\rho$ and by the Residue Theorem of [16], p. 752, we see that this composition is $\int_{f,I}$. (See also bottom of p. 119, and top of p. 120 of [23]). This gives the first half of the theorem. The remaining part follows from 1.7 and from the results in the next section (cf. 3.11 and 3.12(a)). □

## 3. Residues

### 3.1. Definitions, notations and remarks

All rings considered are commutative noetherian.

(a) For a field extension $L \to K$, we write $\mathrm{tr\,deg}_L K$ for the transcendence degree of $K$ over $L$.

(b) If $A \to B$ is a local homomorphism between local rings, then we write $\mathrm{rel\,dim}(B/A)$ for the *relative dimension* of $B/A$, i.e. for $\dim B - \dim A$.

(c) For any local ring $A$, we set $\mathfrak{m}_A$ = the maximal ideal of $A$; and $k_A = A/\mathfrak{m}_A$. Let $A \to B$ be a local homomorphism of local rings, then the homomorphism is said to be *residually finite* (resp. *residually finitely generated*) if $k_A \to k_B$ is a finite (resp. finitely generated) extension of fields.

(d) $\mathfrak{C}_{rf}$ (resp. $\mathfrak{C}_{rfg}$) will denote the category whose objects are complete local rings, and whose morphisms are residually finite (resp. residually finitely generated). $\mathfrak{C}_{fs}$ will denote the subcategory of $\mathfrak{C}_{rfg}$ whose objects are complete local rings, and whose morphisms are formally smooth ring homomorphisms.

(e) Let $A \to B$ be in $\mathfrak{C}_{fs}$.

  (i) We say that $b_1, \dots, b_r \in B$ is a *regular system of parameters* (resp. *system of parameters*) if $\mathfrak{m}_B = Bb_1 + \cdots + Bb_r + \mathfrak{m}_A B$ (resp. the images of $b_1, \dots, b_r$ in $B/\mathfrak{m}_A B$ are a system of parameters for $B/\mathfrak{m}_A B$).

  (ii) Let $\widetilde{\Omega}^1_{B/A}$ be the universally separated differential module of $B/A$, i.e. $\widetilde{\Omega}^1_{B/A} := \Omega^1_{B/A} / \bigcap_i \mathfrak{m}^i_B \Omega^1_{B/A}$. For each integer $p \geqslant 0$, define the module of universally separated $p$-forms of $B/A$ thus:

$$\widetilde{\Omega}^p_{B/A} := \wedge^p_B \widetilde{\Omega}^1_{B/A}.$$

(f) If $A \to B$ is a morphism in $\mathfrak{C}_{fs}$ is as in (b) and $k_A \to k_B$ is of transcendence degree $q$, then $\widetilde{\Omega}^1_{B/A}$ is finite and free of rank $r + q$ (cf. [10], p. 14, (3.9)). Consequently $\widetilde{\Omega}^{r+q}_{B/A}$ is finite, free of rank 1 (and hence isomorphic, non-canonically, to $B$).

(g) We assume familiarity with the generalized fraction notation for elements in cohomology modules with supports in an ideal, and rules for manipulating these generalized fractions (cf. for e.g. [15] Sect. 3, pp. 71–72 or [10], Sect. 2).

(h) Let $A \to R$ be in $\mathfrak{C}_{fs}$. Let $t_1, \ldots, t_d \in R$ be a system of parameters for $R/A$. Let $q$ be the transcendence degree of $A/\mathfrak{m}_A \to R/\mathfrak{m}_R$. Let $J = (t_1, \ldots, t_d)R$. Set $M_i = R/\mathfrak{m}_A^i$. Then there is a canonical isomorphism

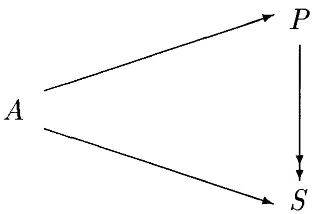
$$H_J^d(\tilde{\Omega}_{R/A}^{d+q}) \xrightarrow{\sim} \varinjlim_j \varprojlim_i \operatorname{Hom}_R(R/J^j, H_{\mathfrak{m}_R}^d(\tilde{\Omega}_{R/A}^{d+q} \otimes_A M_i)). \tag{3.1.1}$$

The map is given via the identification

$$H_J^d(\tilde{\Omega}_{R/A}^{d+q}) \otimes_A M_i = H_{\mathfrak{m}_R}^d(\tilde{\Omega}_{R/A}^{d+q} \otimes_A M_i).$$

### 3.2. Residues for surjective maps

Suppose we have a commutative diagram



such that every arrow is residually finitely generated; $S/A$ and $P/A$ are formally smooth algebras; and as indicated in the diagram $P \xrightarrow{\pi} S$ is surjective. Let the relative dimensions of $P/A$ (resp $S/A$) be $n$ (resp. $d$). Let $q$ be the transcendence degree of $A/\mathfrak{m}_A \to S/\mathfrak{m}_S$. It is well-known that $I = \ker(\pi)$ is generated by a $P$-regular sequence $t_1, \ldots, t_{n-d}$ (cf. for example [20], p. 314, C. 4). Let $y_1, \ldots, y_d \in S$ be a regular system of parameters for $S/A$. If $y_1', \ldots, y_d' \in P$ are pre-images of $y_1, \ldots, y_d$ then $t_1, \ldots, t_{n-d}, y_1', \ldots, y_d'$ are a regular system of parameters for $P/A$. For an $A$-module $M$ of zero-dimensional support, we define a $P$-linear residue map

$$\operatorname{Res}_{S/P,M}\colon H_{\mathfrak{m}_S}^d(\tilde{\Omega}_{S/A}^{d+q} \otimes M) \to H_{\mathfrak{m}_P}^n(\tilde{\Omega}_{P/A}^{n+q} \otimes M), \tag{3.2.1}$$

via the formula

$$\operatorname{Res}_{S/P,M}\begin{bmatrix} s\, \mathrm{d}y_1 \ldots \mathrm{d}y_d\, \mathrm{d}\xi_1, \ldots, \mathrm{d}\xi_q \otimes \alpha \\ y_1^{a_1}, \ldots, y_d^{a_d} \end{bmatrix}$$

$$= \begin{bmatrix} s'\, \mathrm{d}t_1 \ldots \mathrm{d}t_{n-d}\, \mathrm{d}y_1' \ldots \mathrm{d}y_d'\, \mathrm{d}\xi_1' \ldots \mathrm{d}\xi_q' \otimes \alpha \\ t_1, \ldots, t_{n-d}, y_1'^{a_1}, \ldots, y_d'^{a_d} \end{bmatrix}, \tag{3.2.2}$$

where $s'$ is any pre-image of $s \in S$; $\xi_1, \dots, \xi_q \in S$ are such that $\xi_1 + \mathfrak{m}_S, \dots, \xi_q + \mathfrak{m}_S$ form a differential basis for $S/\mathfrak{m}_S$ over $A/\mathfrak{m}_A$, and $\xi'_1, \dots, \xi'_q \in P$ are any pre-images of $\xi_1, \dots, \xi_q$ (cf. [25], p. 201, for a definition of differential basis). Since

$$\left[\begin{matrix} x\,\mathrm{d}t_1 \dots \mathrm{d}t_{n-d}\,\mathrm{d}y'_1 \dots \mathrm{d}y'_d\,\mathrm{d}\xi'_1 \dots \mathrm{d}\xi'_q \otimes \alpha \\ t_1, \dots, t_{n-d}, y_1'^{a_1}, \dots, y_d'^{a_d} \end{matrix}\right] = 0,$$

for $x \in I$ (in view of the $t_1, \dots, t_{n-d}$ occurring in the denominator of the generalized fraction), therefore the right side of 3.2.2 is well defined. Using the calculus for generalized fractions (enunciated for e.g. in [15], Sect. 3, pp. 71–72, [10], Sect. 2 or [21], p. 60, 7.2) it is not hard to see that the map $\mathrm{Res}_{S/P,M}$ is independent of all choices involved (i.e. of $t_1, \dots, t_{n-d}, y'_1, \dots, y'_d, \xi_1, \dots, \xi_q, \xi'_1, \dots, \xi'_q$).

We will write $\mathrm{Res}_{S/P}$ (or even simply Res ) for $\mathrm{Res}_{S/P,M}$ when no confusion is likely to arise.

Let $J_S = (y_1, \dots, y_d)S$ and $J_P = (t_1, \dots, t_{n-d}, y'_1, \dots, y'_d)P$.

Now

$$H^d_{\mathfrak{m}_S}(\tilde{\Omega}^{d+q}_{S/A} \otimes M) = H^d_{J_S}(\tilde{\Omega}^{d+q}_{S/A} \otimes M) = H^d_{J_S}(\tilde{\Omega}^{d+q}_{S/A}) \otimes M,$$

and

$$H^n_{\mathfrak{m}_P}(\tilde{\Omega}^{n+q}_{P/A} \otimes M) = H^n_{J_P}(\tilde{\Omega}^{n+q}_{P/A} \otimes M) = H^n_{J_P}(\tilde{\Omega}^{n+q}_{P/A}) \otimes M.$$

Set $M_i := A/\mathfrak{m}^i_A$. The maps $\mathrm{Res}_{S/P,M_i}$ give us a map

$$\widehat{\mathrm{Res}}_{S/P,J_S,J_P,A}\colon H^d_{J_S}(\tilde{\Omega}^{d+q}_{S/A}) \to H^n_{J_P}(\tilde{\Omega}^{n+q}_{P/A}), \tag{3.2.3}$$

via the isomorphism 3.1.1. We will write $\widehat{\mathrm{Res}}_{S/P,A}$ (or $\widehat{\mathrm{Res}}_{S/P}$, and sometimes even $\widehat{\mathrm{Res}}$ ) for $\widehat{\mathrm{Res}}_{S/P,J_S,J_P,A}$ when no confusion is likely to arise.

It is not hard to check that for an $A$-module with zero-dimensional support

$$\mathrm{Res}_{S/P,M} = \widehat{\mathrm{Res}} \otimes \mathrm{id}_M. \tag{3.2.4}$$

PROPOSITION 3.2.5. *Let $P \to Q \to S$ be a pair of surjective homomorphisms of formally smooth, complete local $A$-algebras ($A$ a complete local ring), residually finitely generated over $A$. Then*

(a) *For all $A$-modules $M$ with zero-dimensional support*

$$\mathrm{Res}_{Q/P,M} \circ \mathrm{Res}_{S/Q,M} = \mathrm{Res}_{S/P,M},$$

(b)

$$\widehat{\mathrm{Res}}_{Q/P} \circ \widehat{\mathrm{Res}}_{S/Q} = \widehat{\mathrm{Res}}_{S/P}.$$

*Proof.* The proposition is an obvious consequence of the definitions. □

3.2.6. REMARK. It is not hard to see that $\mathrm{Res}_{S/P,M}$ gives an isomorphism

$$H^d_{\mathfrak{m}_S}(\tilde{\Omega}^{d+q}_{S/A} \otimes M) \xrightarrow{\sim} \mathrm{Hom}_P(S, H^n_{\mathfrak{m}_P}(\tilde{\Omega}^{n+q}_{P/A} \otimes M)).$$

### 3.3. FACTORIZATIONS

Let $R \xrightarrow{f} T$ be a morphism in $\mathfrak{C}_{rfg}$. We follow I-C. Huang (cf. [10](6.1)) and call a triple $(S, \eta, \pi)$ a *factorization* of $R \xrightarrow{f} T$, if $\eta: R \to S$ is in $\mathfrak{C}_{fs}$; $\pi: S \to T$ is *surjective* and $\pi \circ \eta = f$. A factorization $(S_1, \eta_1, \pi_1)$ of $f$ is *dominated* by another factorization $(S_2, \eta_2, \pi_2)$ of $f$ if

(i) $\tau \circ \eta_1 = \eta_2$ ; $\pi_2 \circ \tau = \pi_1$
(ii) $\tau$ is in $\mathfrak{C}_{fs}$.

(cf. [*ibid.*], (6.2)). In this case $S_2$ as an $S_1$-algebra (via $\tau$) is a power series ring over $S_1$ ([*ibid.*] (6.3)). We collect together some results of I-C. Huang.

LEMMA 3.3.1. [I-C. Huang]

(a) *Every morphism in $\mathfrak{C}_{rfg}$ possesses a factorization.*
(b) *Any two factorizations of a morphism in $\mathfrak{C}_{rfg}$ are dominated by a third.*
(c) *If $(S, \eta, \pi)$ is a factorization of $f: R \to T$ then there is an $R$-algebra isomorphism $S \xrightarrow{\sim} R[X_1, \ldots, X_n]^{\wedge}_{\mathfrak{p}}/\mathcal{I}$ where $\mathfrak{p} \in \mathrm{Spec}(R[X_1, \ldots, X_n])$ and $\mathcal{I}$ is an ideal of $R[X_1, \ldots, X_n]^{\wedge}_{\mathfrak{p}}$.*

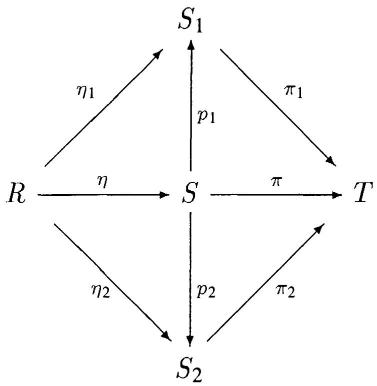
*Proof.* Proof of (a) is as in [10] (3.11), of (b) in [*ibid.*] (6.4). The assertion in [*ibid.*] (3.11) is proved by producing a factorization with $S = R[X_1, \ldots, X_n]^{\wedge}_{\mathfrak{p}}$. In view of this and (b), given any factorization $(S, \eta, \pi)$, it is dominated by $(S', \eta', \pi')$ where $S' = R[X_1, \ldots, X_n]^{\wedge}_{\mathfrak{p}}[|Y_1, \ldots, Y_m|]$. Clearly $S' = R[X_1, \ldots, X_n, Y_1, \ldots, Y_m]^{\wedge}_{\mathfrak{q}}$ for some $\mathfrak{q} \in \mathrm{Spec}\, R[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$. Since $S'$ is a power series over $S$, there is a surjective map $S' \to S$ and hence (c) follows.

For $f: R \to T$ a morphism in $\mathfrak{C}_{rfg}$, denote the collection of all factorizations of $f$ of the form $R \to R[X_1, \ldots, X_n]^{\wedge}_{\mathfrak{p}} \to T$ by $F_f$.

3.3.2. REMARK

(a) Given two factorizations $(S_1, \eta_1, \pi_1)$ and $(S_2, \eta_2, \pi_2)$ of $f: R \to T$, by 3.3.1. (b) there is a factorization $(S', \eta', \pi')$ dominating both (by say $\tau_1: S_1 \to S'$ and $\tau_2: S_2 \to S'$). Since $S'$ is a power series algebra over $S_1$ (resp. $S_2$), therefore there is a surjective map $p'_1: S' \to S_1$ (resp. $p'_2: S' \to S_2$) such that $p'_1 \circ \tau_1 = \mathrm{id}_{S_1}$ (resp. $p'_2 \circ \tau_2 = \mathrm{id}_{S_2}$). Using 3.3.1. (c) $S'$ is the homomorphic image of $S = R[X_1, \ldots, X_n]^{\wedge}_{\mathfrak{p}}$. Hence we can find $(S, \eta, \pi) \in F_f$ and surjective maps

$p_1: S \to S_1$ and $p_2: S \to S_2$ such that $p_i \circ \eta = \eta_i$, $\pi_i \circ p_i = \pi$ for $i = 1, 2$. In other words we have a commutative diagram




with $p_1$, $p_2$ surjective. Diagrams like 3.3.3. will be useful when comparing residues etc.

(b) Any two factorizations $(S_1, \eta_1, \pi_1)$ $(S_2, \eta_2, \pi_2)$ in $F_f$ are dominated by a third factorization in $F_f$. Indeed, if $S_1 = R[X_1^{(1)}, \ldots, X_n^{(1)}]^\wedge_{\mathfrak{p}_1}$ and $S_2 = R[X_1^{(2)}, \ldots, X_m^{(2)}]^\wedge_{\mathfrak{p}_2}$, then set $S := R[X_1^{(1)}, \ldots, X_n^{(1)}, X_1^{(2)}, \ldots, X_m^{(2)}]^\wedge_{\mathfrak{p}}$ where $\mathfrak{p}$ is the inverse image of $\mathfrak{m}_T$ under the natural map $R[X_1^{(1)}, \ldots, X_n^{(1)}, X_1^{(2)}, \ldots, X_m^{(2)}] \to T$.

THEOREM 3.4. *There exists a unique family of maps*

$$\operatorname{res}_{S/A,M}: H^d_{\mathfrak{m}_A}(\tilde{\Omega}^d_{S/A} \otimes_A M) \to M$$

*one for each morphism $f: A \to S$ in $\mathfrak{C}_{fs} \cap \mathfrak{C}_{rf}$ of relative dimension d, and for each $M \in A_\#$ such that*

(a) $\operatorname{res}_{S/A,M}$ *is A-linear, and functorial in $M \in A_\#$.*

(b) *If $S = A[X_1, \ldots, X_d]^\wedge_{\mathfrak{p}}$, then* $\operatorname{res}_{S/A,M}$ *equals the map* $\operatorname{res}_{X_1,\ldots,X_d;S/A,M}$ *of* [10] (7.1).

(c) *If $(P, \eta, \pi)$ is a factorization of $f$ then* $\operatorname{res}_{S/A,M} = \operatorname{res}_{P/A,M} \circ \operatorname{Res}_{S/P,M}$ *where* $\operatorname{Res}_{S/P,M}$ *is the map in* 3.2.1.

*Proof.* Since any $f: A \to S$ in $\mathfrak{C}_{fs} \cap \mathfrak{C}_{rf}$ has a factorization $(P, \eta, \pi) \in F_f$ (i.e. $P = A[X_1, \ldots, X_n]^\wedge_{\mathfrak{p}}$), therefore (b) and (c) give uniqueness of the family $\operatorname{res}_{S/A,M\,S/A,M}$. Since $\operatorname{res}_{S/A,M}$ is functorial in $M \in A_\#$ if $S = A[X_1, \ldots, X_d]^\wedge_{\mathfrak{p}}$, and $\operatorname{Res}_{S/P,M}$ is also functorial in $M \in A_\#$ for any factorization $(P, \eta, \pi)$ of $A \to S$, therefore functoriality in $M \in A_\#$ will follow from the remaining assertions.

For general $f: A \to S$ in $\mathfrak{C}_{fs} \cap \mathfrak{C}_{rf}$ we define $\operatorname{res}_{S/A,M}$ by taking a factorization $(P, \eta, \pi) \in F_f$ and setting $\operatorname{res}^P_{S/A,M} := \operatorname{res}_{P/A,M} \circ \operatorname{Res}_{S/A,M}$. If $(P', \eta', \pi')$ is another member of $F_f$, we claim that $\operatorname{res}^{P'} = \operatorname{res}^P$. In view of 3.3.2(b) we may assume $P' = P[|Y_1,\ldots,Y_t|]$, and that $\eta'$ is the composition $A\xrightarrow{\eta}P\xrightarrow{\text{nat}}P[|Y_1,\ldots,Y_t|]$ and $\pi'$ is the composition $P[|Y_1,\ldots,Y_t|]\xrightarrow{\text{nat}}P\xrightarrow{\pi}S$. One checks readily that $\text{res}_{P/A} = \text{res}_{P'/A}\circ\text{Res}_{P/P'}$. Since $\text{Res}_{S/P'} = \text{Res}_{P/P'}\circ\text{Res}_{S/P}$, the assertion follows. Define $\text{res}_{S/A,M}$ to be the common value of $\text{res}^P_{S/A,M}$ for $(P,\eta,\pi)\in F_f$. Assertion (c) above follows from 3.3.1(c), and the transitivity property of $\text{Res}_{S/P}$. □

3.4.1. REMARK. Let $A\to A[X_1,\ldots,X_d]^\wedge_{\mathfrak{p}}$ be in $\mathfrak{C}_{rf}$. Corollary (7.7) of [10] says that $\text{res}_{X_1,\ldots,X_d;S/M}$ of *ibid.* (7.1) does not depend on the order of the $X_i$. The discussion after the proof of *ibid.* (7.7) shows that the subscript $X_1,\ldots,X_d$ (ordered or unordered) is unnecessary.

3.5. DEFINITION. Let $f\colon A\to S$ be a morphism in $\mathfrak{C}_{fs}\cap\mathfrak{C}_{rf}$ of relative dimension $d$, and let $x_1,\ldots,x_d\in S$ be a system of parameters for $S/A$. Let $J = (x_1,\ldots,x_d)S$. Note that for any $M\in A_\#$, there is a canonical identification $H^d_{\mathfrak{m}_S}(\widetilde{\Omega}^d_{S/A}\otimes M) = H^d_J(\widetilde{\Omega}^d_{S/A}\otimes M)$. For $i\geqslant 0$, let $M_i = A/\mathfrak{m}^i_A$. The maps $\text{Res}_{S/P,M_i}$ give us a map – the *residue map of $S/A$ along $J$*

$$\widehat{\text{res}}_{S/A,J}\colon H^d_J(\widetilde{\Omega}^d_{S/A})\to A \tag{3.5.1}$$

via the isomorphism 3.1.1.

PROPOSITION 3.6. *Let $A\to S$ be a morphism in $\mathfrak{C}_{fs}\cap\mathfrak{C}_{rf}$, and let $J = (x_1,\ldots,x_d)S$ be as above. Then*

(a) *For any $M\in A_\#$,*

$$\text{res}_{S/A,M} = \widehat{\text{res}}_{S/A,J}\otimes\text{id}_M.$$

(b) *If $(P,\eta,\pi)$ is a factorization of $A\to S$, and if $x'_1,\ldots,x'_d,t_1,\ldots,t_n\in P$ are a system of parameters of $P/A$ with $\pi(x'_i) = x_i$, $(i = 1,\ldots,d)$, and $(t_1,\ldots,t_n)P = \ker(\pi)$, then with $J' = (x'_1,\ldots,x'_d,t_1,\ldots,t_n)P$ we have*

$$\widehat{\text{res}}_{S/A,J} = \widehat{\text{res}}_{P/A,J'}\circ\widehat{\text{Res}}_{S/P,J,J'}.$$

*Proof.* Assertion (a) can be proved from definitions. Part (b) follows from (a) and part (c) of 3.4. □

### 3.7. SURJECTIVE BASE CHANGE

Let $P$ be the push-out of $f\colon A\to B$ and $g\colon A\to S$, where $f,g\in\mathfrak{C}_{rf}$. Assume $A\to S$ is surjective and $A\to B$ (and hence $S\xrightarrow{f'}P$) is formally smooth of relative dimension $m$. Let $M\in A_\#$ and set $N := \text{Hom}_A(S,M)\in S_\#$. If # is the pseudo-functor on $\mathfrak{C}_{rf}$ in [10] Section 6, then $N\xrightarrow{\sim}g_\# M$. Using the fact that $f'_\# g_\# M\xrightarrow{\sim}g'_\# f_\# M$ we get an isomorphism of $P$-modules:

$$H^m_{\mathfrak{m}_P}(\widetilde{\Omega}^m_{P/S}\otimes_S N)\xrightarrow{\sim}\text{Hom}_B(P,H^m_{\mathfrak{m}_B}(\widetilde{\Omega}^m_{B/A}\otimes M)). \tag{3.7.1}$$

If $\phi\colon H^m_{\mathfrak{m}_P}(\widetilde{\Omega}^m_{P/S}\otimes N)\to H^n_{\mathfrak{m}_B}(\widetilde{\Omega}^m_{B/A}\otimes M)$ is the $B$-linear map gotten by following the isomorphism above by 'evaluation at 1', then chasing various definitions one sees that

$$\phi\begin{bmatrix} \nu\otimes n \\ t_1^{a_1},\dots,t_m^{a_n}\end{bmatrix} = \begin{bmatrix} \nu'\otimes\psi(n) \\ t_1'^{a_1},\dots,t_m'^{a_n}\end{bmatrix}, \tag{3.7.2}$$

where $t_1,\dots,t_m\in P$ are a regular system of parameters for $P/S$, $t'_1,\dots,t'_m\in B$ pre-images of $t_1,\dots,t_m$, $\nu'\in\widetilde{\Omega}^m_{B/A}$ a pre-image of $\nu$. The right-side does not depend on the choice of $\nu', t'_1,\dots,t'_m$ since $\psi(n)$ is annihilated by $I=\ker(g)$. The map $\phi$ does not depend on the sequence $t_1,\dots,t_m$ either. One checks easily from definitions that

$$\psi\circ\mathrm{res}_{P/N,S}=\mathrm{res}_{B/A,N}\circ\phi$$

where $\psi$ is 'evaluation at 1'. One immediate consequence of the above is:

PROPOSITION 3.8. [Transitivity of Residues]. *Let $R\xrightarrow{g_1}S$ and $S\xrightarrow{g_2}T$ be morphisms in $\mathfrak{C}_{fs}\cap\mathfrak{C}_{rf}$ of relative dimensions $d$ and $n$ respectively. Consider $T$ as an $R$-algebra by comparing the above maps. Then*

(a) *For $M\in R_\#$, identifying $H^{n+d}_{\mathfrak{m}_T}(\widetilde{\Omega}^{n+d}_{T/R}\otimes M)$ with $H^n_{\mathfrak{m}_T}(\widetilde{\Omega}^n_{T/S}\otimes H^d_{\mathfrak{m}_S}(\widetilde{\Omega}^d_{S/R}\otimes M))$ (cf. [10], p. 19, (4.5)(iii)) we have*

$$\mathrm{res}_{T/R,M}=\mathrm{res}_{S/R,M}\circ\mathrm{res}_{T/S,N},$$

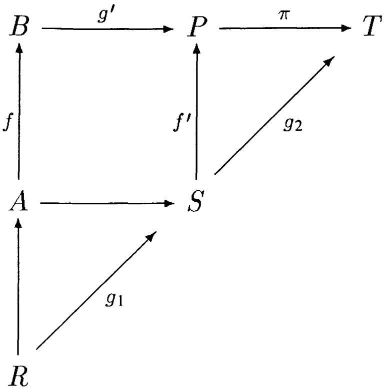
*where $N=H^d_{\mathfrak{m}_S}(\widetilde{\Omega}^d_{S/R}\otimes M)$.*

(b) *Let $t_1,\dots,t_d\in S$ be a regular system of parameters for $S/R$, and extend this to a regular system of parameters $t_1,\dots,t_d,y_1,\dots,y_n\in T$ for $T/R$ (so that $y_1,\dots,y_n$ is a regular system of parameters for $T/S$). Let $I=(t_1,\dots,t_d)S$, $I'=(y_1,\dots,y_n)T$, $J=IT+I'$. Identifying $H^{n+d}_J(\widetilde{\Omega}^{n+d}_{T/R})$ with $H^n_{I'}(\widetilde{\Omega}^n_{T/S}\otimes_S H^d_I(\widetilde{\Omega}^d_{S/R}))$, we have*

$$\widehat{\mathrm{res}}_{T/R,J}=\widehat{\mathrm{res}}_{S/R,J}\circ(\widehat{\mathrm{res}}_{T/S,I'}\otimes\mathrm{id}_N),$$

*where $N=H^d_I(\widetilde{\Omega}^d_{S/R})$.*

*Proof.* Clearly it is enough to prove part (a). If $S = R[X_1, \dots, X_d]^{\wedge}_{\mathfrak{p}}$ and $T = S[Y_1, \dots, Y_n]^{\wedge}_{\mathfrak{q}}$ then it is not difficult to prove the assertion (cf. [10] (7.2)(1)). In general we have a commutative diagram




where $A = R[X_1, \dots, X_{d+p}]^{\wedge}_{\mathfrak{p}}$, $B = A[Y_1, \dots, Y_{n+q}]^{\wedge}_{\mathfrak{q}}$, the rectangle on the top left corner is as in the beginning of 3.7. Existence of such diagrams can be seen, for e.g., by picking a factorization $R \to A \to S$ in $F_{g_1}$, and one $A \to B \to T$ in $F_{g_2 \circ g}$, and setting $P = B \otimes_A S$. For an $R$-algebra $R \to U$ in $\mathfrak{C}_{fs} \cap \mathfrak{C}_{rf}$, let $G(U) := H^t_{\mathfrak{m}_U}(\widetilde{\Omega}^t_{U/R} \otimes M)$, where $t$ is the relative dimension of $U/R$. For $U$ as above, and a $U$-algebra $U \to V$ in $\mathfrak{C}_{fs} \cap \mathfrak{C}_{rf}$ of relative dimension $s$, let $H(V) := H^s_{\mathfrak{m}_V}(\widetilde{\Omega}^s_{V/U} \otimes G(U))$. Then we can identify $G(V)$ with $H(V)$. Under this identification, the map $\operatorname{Res}_{P/B}\colon G(P) \to G(B)$ can be identified with $\operatorname{Res}_{S/A}\colon H(P) \to H(B)$. Now replacing all the rings $U$ in the diagram above by $G(U)$, and replacing all horizontal rows by Res (but going from right to left) and vertical arrows by res (but now pointing downwards), we get a diagram that continues to commute (use the results of 3.7 and 3.4). By previous comments $\operatorname{res}_{A/R} \circ \operatorname{res}_{B/A} = \operatorname{res}_{B/R}$, and by 3.2.5(a), $\operatorname{Res}_{P/B} \circ \operatorname{Res}_{T/P} = \operatorname{Res}_{T/B}$, and hence

$$\operatorname{res}_{S/R} \circ \operatorname{res}_{T/S} = \operatorname{res}_{B/R} \circ \operatorname{Res}_{T/B} = \operatorname{res}_{T/R}$$

as required. □

3.9. Let $A$ be a complete local ring. Let $R \to S$ be a map of $A$-algebras with $R/A$ and $S/A$ in $\mathfrak{C}_{fs}$, and $S/R$ a finite local (and hence global) complete intersection algebra (so that $S$ and $R$ have the same dimension). In this situation, there is, according to [20] Section 16, a degree zero map of graded $R$-modules

$$\sigma^{\bullet}_{S/R}\colon \widetilde{\Omega}^{\bullet}_{S/A} \to \widetilde{\Omega}^{\bullet}_{R/A} \tag{3.9.1}$$

such that for each $p \in \mathbb{N}$, the map

$$\widetilde{\Omega}^p_{S/A} \to \operatorname{Hom}_R(\widetilde{\Omega}^{r-p}_{S/A}, \widetilde{\Omega}^r_{R/A})$$
$$\eta \mapsto (\omega \mapsto \sigma^p_{S/A}(\omega\eta))$$

is bijective for each $r \in \mathbb{N}$.

Next let $(P, \eta, \pi)$ be a factorization of $R \to S$, and $I = \ker(\pi)$. Let $\operatorname{rel\,dim}(P/R) = n$; that of $R/A$ be $d$; and let $q := \operatorname{tr\,deg}_{k_A} k_R$. Now $I$ is generated by a $P$-regular sequence $\boldsymbol{t} = (t_1, \ldots, t_{n-d})$. Note that $\boldsymbol{t}$ forms a system of parameters $P/R$ and hence $\widehat{\operatorname{res}}_{P/R,I}\colon H^n_I(\widetilde{\Omega}^n_{P/R}) \to R$ is defined. Set $\sigma_{S/R} = \sigma^{d+q}_{S/R}$ (see 3.9.1 above).

THEOREM 3.10. *In the above situation*

(a) *Let* $M \in A_\#$ *and set* $N := H^d_{\mathfrak{m}_R}(\widetilde{\Omega}^{d+q}_{R/A} \otimes M) \in R_\#$. *Identifying* $H^{n+d}_{\mathfrak{m}_P}(\widetilde{\Omega}^{n+d+q}_{P/A} \otimes M)$ *with* $H^n_{\mathfrak{m}_P}(\widetilde{\Omega}^n_{P/R} \otimes N)$ *we have:*

$$\operatorname{res}_{P/R,N} \circ \operatorname{Res}_{S/P,M} = H^d_{\mathfrak{m}_R}(\sigma_{S/R} \circ \operatorname{id}_M).$$

(b) *Let* $\boldsymbol{y} = (y_1, \ldots, y_d)$ *be a system of parameters for* $R/A$ *(so that it is one for* $S/A$ *also). Then identifying* $H^{n+d}_{(\boldsymbol{t},\boldsymbol{y})P}(\widetilde{\Omega}^{n+d+q}_{P/A})$ *with* $H^n_I(\widetilde{\Omega}^n_{P/R} \otimes_R H^d_{\boldsymbol{y}R}(\widetilde{\Omega}^{d+q}_{R/A}))$ *we have*

$$(\widehat{\operatorname{res}}_{P/R,I} \otimes \operatorname{id}) \circ \widehat{\operatorname{Res}}_{S/P,\boldsymbol{y}S} = H^d_{\boldsymbol{y}R}(\sigma_{S/R}).$$

*Proof.* Clearly it is enough to prove (a). By 3.4(c) and comments in 3.3.2 (esp. 3.3.3) we conclude that the composition $\operatorname{res}_{P/R,N} \circ \operatorname{Res}_{S/P,M}$ is independent of the factorization $(P, \eta, \pi)$. Since $S/R$ is a global complete intersection there is a presentation $R[X_1, \ldots, X_n]/(t_1, \ldots, t_n) \xrightarrow{\sim} S$. Let $Q = R[X_1, \ldots, X_n]$ and $P = R[X_1, \ldots, X_n]^\wedge_{\mathfrak{p}}$ where $\mathfrak{p}$ is the inverse image of $\mathfrak{m}_S$ in $Q$.

For any system of parameters $\boldsymbol{g} = (g_1, \ldots, g_n)$ of $P/R$, define

$$\tau^X_{\boldsymbol{g}}\colon P \to R$$

as follows: Set $S_{\boldsymbol{g}} := P/\boldsymbol{g}P$. By [20], p. 370, F. 20 and F. 21 there is a trace map $\tau^x_{\boldsymbol{g}}\colon S_{\boldsymbol{g}} \to R$. Define $\tau^X_{\boldsymbol{g}}$ by following the surjection $P \to S_{\boldsymbol{g}}$ by $\tau^x_{\boldsymbol{g}}$. By [20], p. 375, Prop. F. 26, if $\boldsymbol{g}' = (g'_1, \ldots, g'_n)$ is another such sequence in $P$ with

$$g_i = \sum a_{ij} g'_j a_{ij} \in P$$

then

$$\tau^X_{\boldsymbol{g}} = \tau^X_{\boldsymbol{g}} \circ \Delta^{\boldsymbol{g}}_{\boldsymbol{g}'}, \tag{3.10.1}$$

where $\Delta^{\boldsymbol{g}}_{\boldsymbol{g}'}$ is multiplication by $\det(a_{ij})$. For $a_1, \ldots, a_n > 0$, set $\boldsymbol{g}^a := (g_1^{a_1}, \ldots, g_n^{a_n})$. Huang in [10] Section 7 (cf. especially (7.1) and (7.2) (b) of *loc.cit.*) finds $P$-regular sequence $\boldsymbol{f} = (f_1, \ldots, f_n)$ in $P$ such that for $M \in R_\#$, and any $m \in M$, $p \in P$

$$\operatorname{res}_{P/R}\begin{bmatrix} p\,\mathrm{d}X_1 \ldots \mathrm{d}X_n \otimes m \\ f_1^{a_1}, \ldots, f_n^{a_n} \end{bmatrix} = \tau^X_{\boldsymbol{f}^a}(p)m \tag{3.10.2}$$

This is seen by modifying slightly the arguments in [20], F. 22(b).

Let $r \in \mathbb{N}$ be such that $\boldsymbol{f}^r P := (f_1^r, \dots, f_n^r)P$ is contained in $(t_1, \dots, t_n)P + \mathfrak{m}_R P$ (such an $r$ exists since the radical of $\boldsymbol{t}P + \mathfrak{m}_R P$ is $\boldsymbol{f}P + \mathfrak{m}_R P$). Let

$$f_i^r = \sum_{j=1}^{n} a_{ij} t_j + m_i \qquad (a_{ij} \in P, m_i \in \mathfrak{m}_R P).$$

One checks (by applying [20], p. 376, Prop. F. 27 to the base change $R \to R/\mathfrak{m}_R$) that $\tau_{\boldsymbol{t}}^X(P)$ differs from $\tau_{\boldsymbol{f}^r}^X(\det(a_{ij})p)$ by an element in $\mathfrak{m}_R$.

Let $\omega = s\,\mathrm{d}z_1 \dots \mathrm{d}z_d\,\mathrm{d}\xi_1 \dots \mathrm{d}\xi_q$ and let $s', \boldsymbol{z}', \boldsymbol{\xi}'$ be pre-images of $s, \boldsymbol{z}, \boldsymbol{\xi}$ in Q. Set $\omega' = s'\,\mathrm{d}t_1 \dots \mathrm{d}t_n\,\mathrm{d}z_1' \dots \mathrm{d}z_d'\,\mathrm{d}\xi_1' \dots \mathrm{d}\xi_q' \in \widetilde{\Omega}_{Q/A}^{n+d+q}$. According to [20], p. 254, 16.4 if $\omega' = p\,\mathrm{d}X_1 \dots \mathrm{d}X_n \otimes \nu$ $(p \in Q, \nu \in \widetilde{\Omega}_{Q/A}^{d+q})$ then

$$\sigma_{S/R}(\omega) = \tau_{\boldsymbol{t}}^X(p).$$

Now for any system of parameters $\boldsymbol{y} = (y_1, \dots, y_d)$ of $R/A$, we have

$$\begin{aligned}\mathrm{Res}_{S/P}\begin{bmatrix}\omega \otimes \alpha \\ \mathbf{y}\end{bmatrix} &= \begin{bmatrix} p\,\mathrm{d}\mathbf{X} \otimes \nu \otimes \alpha \\ \mathbf{t}, \mathbf{y}\end{bmatrix}, \\ &= \begin{bmatrix} p\,\mathrm{d}\mathbf{X} \otimes \begin{bmatrix}\nu\otimes\alpha \\ \mathbf{y}\end{bmatrix} \\ \mathbf{t}\end{bmatrix}, \\ &= \begin{bmatrix} \det(a_{ij}) p\,\mathrm{d}\mathbf{X} \begin{bmatrix}\nu\otimes\alpha \\ \mathbf{y}\end{bmatrix} \\ \mathbf{f}^r\end{bmatrix}.\end{aligned}$$

Whence

$$\begin{aligned}\mathrm{res}_{P/A} \circ \mathrm{Res}_{S/P}\begin{bmatrix}\omega \otimes \alpha \\ \mathbf{y}\end{bmatrix} &= \tau_{\mathbf{f}^r}^X(\det(a_{ij}) \cdot p)\begin{bmatrix}\nu \otimes \alpha \\ \mathbf{y}\end{bmatrix}, \\ &= \tau_t^X(p)\begin{bmatrix}\nu \otimes \alpha \\ \mathbf{y}\end{bmatrix}, \\ &= \begin{bmatrix}\sigma_{S/R}(\omega) \\ \mathbf{y}\end{bmatrix},\end{aligned}$$

as required.

3.11. REMARK. Let $A \to B$ be a smooth algebra of finite type, equidimensional of dimension $d$; $A$ excellent; $\mathrm{Ass}(A) = \mathrm{Min}(A)$. Let $\mathfrak{p} \in \mathrm{Max}(B)$ and $\mathfrak{q} \in \mathfrak{p} \cap A$, and let $\boldsymbol{t} = (t_1, \dots, t_d)$ be a system of parameters for $B_\mathfrak{p}/A_\mathfrak{q}$. Set $R := \hat{A}_\mathfrak{q}$, $S := \hat{B}_\mathfrak{p}$. In this situation Hübl and Kunz define a residue map

$$\mathrm{Res}_{\,\mathfrak{p}S, \boldsymbol{t}S}\colon H_{\boldsymbol{t}S}^d(\widetilde{\Omega}_{S/R}^d) \to R.$$

(Cf. [15] Sect. 3, p. 76. Cf. also *ibid.*, p. 64, (2.1)). Clearly, in view of 3.8(b) and 3.10(b) and the definition of $\operatorname{Res}_{\mathfrak{p}S,tS}$, we have

$$\operatorname{Res}_{\mathfrak{p}S,tS} = \widehat{\operatorname{res}}_{S/A,tS}.$$

THEOREM 3.12. [Local Duality]. *Let $f: A \to S$ be a morphism in $\mathfrak{C}_{rf} \cap \mathfrak{C}_{fs}$ of relative dimension $d$.*

(a) *For each $M \in A_{\#}$, the residue map $\operatorname{res}_{S/A,M}$ induces an isomorphism of $S$-modules*

$$H^n_{\mathfrak{m}_S}(\tilde{\Omega}^d_{S/A} \otimes M) \xrightarrow{\sim} \operatorname{Hom}^c_A(S, M),$$

*whence we have an isomorphism of $S$-modules*

$$\begin{aligned} \operatorname{Hom}_S(N, H^d_{\mathfrak{m}_S}(\tilde{\Omega}^d_{S/A} \otimes M)) &\xrightarrow{\sim} \operatorname{Hom}_A(N, M) \\ \phi &\mapsto \operatorname{res}_{S/A} \circ \phi \end{aligned}$$

*for all $N \in S_{\#}$.*

(b) *If $\boldsymbol{x} = (x_1, \ldots, x_d)$ is a system of parameters of $S/A$, then for every finitely generated $S$-module $N$, the $S$-homomorphism*

$$\begin{aligned} \operatorname{Hom}_S(N, \tilde{\Omega}^d_{S/A}) &\to \operatorname{Hom}_A(H^d_{\boldsymbol{x}S}(N), A) \\ \phi &\mapsto \widehat{\operatorname{res}}_{S/A,\boldsymbol{x}S} \circ H^d_{\boldsymbol{x}S}(\phi) \end{aligned}$$

*is an isomorphism of $S$-modules.*

*Proof.* For part (a), let $(P, \eta, \pi)$ be a factorization of $A \to S$. From 3.2.6 and [10] (7.3), the map $H^d_{\mathfrak{m}_S}(\tilde{\Omega}^{\otimes}_{S/A} M) \to \operatorname{Hom}^c_A(S, M)$ given by $\operatorname{res}_{S/A}$ is seen to be an isomorphism. The remaining part of (a) is a trivial consequence of the natural isomorphism $\operatorname{Hom}_S(N, \operatorname{Hom}^c_A(S, M)) \xrightarrow{\sim} \operatorname{Hom}_A(N, M)$ for $N \in S_{\#}$.

Note that (b) is well-known if $S$ is a power series ring over $A$. We first prove (b) under the assumption that $A$ is an Artin local ring. Let $x_1, \ldots, x_d \in S$ be a regular system of parameters for $S/A$. Set $R := A[|X_1, \ldots, X_d|]$ ($X_1, \ldots, X_d$ analytically independent). Define $R \to S$ by sending $X_i$ to $x_i$. Clearly $R \to S$ is finite. It is well-known that it is flat (cf. for example [20], p. 310, B. 27), and using [20], p. 314, C. 4, it is a global complete intersection. Let, as usual, $\sigma_{S/R}: \tilde{\Omega}^d_{S/A} \to \tilde{\Omega}^d_{R/A}$ be the $d$th component of the trace map 3.9.1. By 3.10, $\widehat{\operatorname{res}}_{S/A} = \widehat{\operatorname{res}}_{R/A} \circ H^d_{\mathfrak{m}_R}(\sigma_{S/R})$. Using the fact that $\tilde{\Omega}^d_{S/A}$ is isomorphic to $\operatorname{Hom}_R(S, \tilde{\Omega}^d_{R/A})$ via $\sigma_{S/R}$ one sees that the result is true in this case (cf. [21], p. 69, (7.5)).

Now we drop the assumption that $A$ is Artinian. Let $N$ be a finitely generated $S$-module and $\phi: H^d_{\boldsymbol{x}S}(N) \to A$ an $A$-linear map. For $p \in \mathbb{N}$, let $A(p) = A/\mathfrak{m}_A{}^p$, $S(p) = S/\mathfrak{m}_A{}^p S$, $N(p) = N/\mathfrak{m}_A{}^p N$ and $\phi(p) = \phi \otimes_A A/\mathfrak{m}_A{}^p: H^d_{\boldsymbol{x}S(p)}(N(p)) \to A(p)$. By 3.7.1 and by local duality for $S(p)/A(p)$ ($A(p)$ being Artin) we have a unique $S(p)$-linear map

$$\Psi(p): N(p) \to \tilde{\Omega}^d_{S(p)/A(p)} = \tilde{\Omega}^d_{S/A} \otimes_A A(p)$$

such that $\phi(p) = \widehat{\text{res}}_{S(p)/A(p)} \circ H^d_{xS(p)}(\Psi(p))$. One checks by the universal property of $(\widetilde{\Omega}^d_{S(p)/A(p)}, \text{res}\, S(p)/A(p))$ that the diagram

$$\begin{array}{ccc} N(p) & \xrightarrow{\psi(p)} & \widetilde{\Omega}^d_{S/A} \otimes A(p) \\ \uparrow & & \uparrow \\ N(p+1) & \xrightarrow[\psi(p+1)]{} & \widetilde{\Omega}_{S/A} \otimes A(p+1) \end{array}$$

commutes, where the vertical arrows are natural. Define $\Psi$ using 3.1.1 via $\{\Psi(p)\}$. Clearly (by definition of $\widehat{\text{res}}_{S/A}$), $\phi = \widehat{\text{res}}_{S/A} \circ H^d_{xS}(\Psi)$. Uniqueness of the $S$-map satisfying the above formula follows from the uniqueness of $\Psi(p)$ for each $p \in \mathbb{N}$. □

3.12.1. REMARK. Let $A$ be a complete local ring and let $A \to R$, $R \to P$ be in $\mathfrak{C}_{fs}$. Let $S$ be a homomorphic image of $P$, with $S/A$ formally smooth, and the induced map $R \to S$ surjective. Then by [10], p. 21, Section 5, $P$ is a power series ring over $R$. Using this it is not hard to show that

$$\text{res}_{P/R} \circ \text{Res}_{S/P} = \text{Res}_{S/R},$$

and

$$\widehat{\text{res}}_{P/R} \circ \widehat{\text{Res}}_{S/P} = \widehat{\text{Res}}_{S/R}.$$

## 4. Pseudo-coefficient fields and residues

For the rest of the paper we will work over a fixed perfect field $k$.

Given a map $k \to S$ in $\mathfrak{C}_{rfg}$, and a field $L$ in $S$ containing the image of $k$, such that $k_S$ is finite over $L$ – there is a way of constructing an injective hull of the $S$-module $k_S$, described for example by Hartshorne in [8], p. 63, Example 1. Given two such fields, we describe in 4.6, a *canonical* isomorphism between the two constructions. The isomorphism depends heavily on the theory of residues developed below and in the last section. Moreover, this isomorphism agrees with the one in [31] wherever both make sense (cf. 4.7).

### 4.1. DEFINITIONS, NOTATIONS AND REMARKS

(a) If $A$ is a $k$-algebra such that the structural map $k \to A$ is in $\mathfrak{C}_{rfg}$, then we write $A \in \mathfrak{C}_k$ (or, more often than not, simply $A \in \mathfrak{C}$). $\mathfrak{C}$ with morphisms being members of $\mathfrak{C}_{rfg}$, is a category.

(b) If $A$ is in $\mathfrak{C}$, and $A$ is formally smooth over $k$, then we write $A \in \boldsymbol{k}_{fs}$. In this case we set $H(A) = H^m_{\mathfrak{m}_A}(\widetilde{\Omega}^{m+q}_{A/k})$ where $m = \dim A$, and $q = \text{tr.deg}_k k_A$.
(c) Since $k$ is perfect, if $L \in \mathfrak{C}$ is a field, then $L \in \boldsymbol{k}_{fs}$. In this case we define

$$\omega_L := H(L) = \bigwedge_L^q \Omega^1_{L/k}$$

where $q = \text{tr.deg}_k L$.
(d) Let $\sigma$: $L \to A$ be a morphism in $\mathfrak{C}$, with $L$ being a field. We say $\sigma$, is a *coefficient field* (resp. *pseudo-coefficient field*, resp. *quasi-coefficient field*) if $\sigma$, followed by the natural surjection $A \to k_A$ is an isomorphism (resp. a finite extension of fields, resp. an étale extension of fields).
(e) For $A \in \mathfrak{C}$, set

$$C_A = \{\sigma: L \to A | \sigma \text{ is a pseudo-coefficient field}\}$$

If the domain of $\sigma \in C_A$ is not specified, then we write $L_\sigma$ for the domain of $\sigma$, and $\omega_\sigma$ for $\omega_{L_\sigma}$.
(f) For $A \in \mathfrak{C}$, and $\sigma \in C_A$, we define

$$\mathcal{K}(\sigma) := \text{Hom}^c_\sigma(A, \omega_\sigma)$$

Note that $\mathcal{K}(\sigma)$ is an injective hull of $k_A$ over $A$ (cf. [8], p. 63, Example 1). Define

$$\boldsymbol{e}_\sigma: \mathcal{K}(\sigma) \to \omega_\sigma$$

to be 'evaluation at 1'.
(g) Let $A \in \mathfrak{C}$, and $\sigma \in C_A$. Define a functor $G_\sigma: A_\sharp \to \text{Mod}_A$ by setting $G_\sigma := \text{Hom}_\sigma(-, \omega_\sigma)$. Then by 1.7.1, the pair $(\mathcal{K}(\sigma), \boldsymbol{e}_\sigma)$ represents $G_\sigma$.
(h) Let $S \xrightarrow{f} R$ be a finite map in $\mathfrak{C}$. Let $\tau$: $L \to S$ be a pseudo-coefficient field and set $\sigma = f \circ \tau \in C_R$. Since $(\mathcal{K}(\tau), \boldsymbol{e}_\tau)$ represents $G_\tau$, therefore there is a unique $S$-map

$$\gamma^\tau_\sigma: \mathcal{K}(\sigma) \to \mathcal{K}(\tau)$$

such that

$$\boldsymbol{e}_\tau \circ \gamma^\tau_\sigma = \boldsymbol{e}_\sigma.$$

We claim that the map of $R$-modules

$$\Gamma^\tau_\sigma: \mathcal{K}(\sigma) \to \text{Hom}_S(R, \mathcal{K}(\tau))$$

induced by $\gamma^\tau_\sigma$ is an *isomorphism*. This is seen by noting that the natural map

$$\text{Hom}_S(R, \text{Hom}^c_\tau(S, \omega_L)) \to \text{Hom}^c_{f\circ\tau}(R, \omega_L)$$
$$\phi \mapsto (r \mapsto \phi(r)(1))$$

is an isomorphism.

(i) In particular, if $S \to R$ above is *surjective*, then $\gamma_\sigma^\tau$ is injective, since $\Gamma_\sigma^\tau$ identifies $\mathcal{K}(\sigma)$ with the $S$-submodule of $\mathcal{K}(\tau)$ annihilated by ker $f$.

## 4.2. RESIDUES FOR PSEUDO-COEFFICIENT FIELDS

Let $S \in \boldsymbol{k}_{fs}$. Let $\sigma: L \to S$ be a pseudo-coefficient field. Let $(P, \eta, \pi)$ be a factorization of $\sigma$. As in the proof of 3.4(c), one checks that $\mathrm{res}_{P/L,\omega_L} \circ \mathrm{Res} S/P, k: H(S) \to \omega_L$ is independent of the factorization $(P, \eta, \pi)$. Denote this map:

$$\mathrm{res}_\sigma: H(S) \to \omega_\sigma. \tag{4.2.1}$$

If $\boldsymbol{x} = (x_1, \ldots, x_d)$ is a regular system of parameters for $S$, then $R = L[|x_1, \ldots, x_d|](\subset S)$ is a power series ring over $R$ and the algebra $S/R$ is finite, flat, and a global complete intersection. Let $\tau: L \to R$ be the natural map. Let $q = \mathrm{tr.deg}_k k_S$ and let $p = d + q$. Then by 3.10 $\mathrm{res}_\sigma = \mathrm{res}_\tau \circ H^d_{\mathfrak{m}_R}(\sigma_{S/R})$ where $\sigma_{S/R}$ is the $p$th component of the map 3.9.1. Consequently $\mathrm{res}_\sigma$ is a special case of the residue defined in [14] (1.1). Using this, one checks the following:

PROPOSITION 4.2.2. *The pair* $(\widetilde{\Omega}^p_{S/k}, \mathrm{res}_\sigma)$ *represents the functor* $\mathrm{Hom}_\sigma(H^d_{\mathfrak{m}_S}(M), \omega_L)$ *of finitely generated $S$-modules $M$.*

4.2.3. REMARK. Let $f: R \to S$ be in $\mathfrak{C}_{fs}$ with $R \in \boldsymbol{k}_{fs}$. Then identifying $H(S)$ with $H^n_{\mathfrak{m}_S}(\widetilde{\Omega}^n_{S/R} \otimes H(R))$ $(n = \mathrm{rel.dim}\ S/R)$ via [10] p.19, 4.5 (ii), we can show, as in 3.8 that for $\sigma \in C_R$, $\mathrm{res}_{f\circ\sigma} = \mathrm{res}_\sigma \circ \mathrm{res}_{S/R,H(R)}$ (cf. also [14] Prop. 1.3 and Cor. 1.4)).

PROPOSITION 4.2.4. *Let $S \in \boldsymbol{k}_{fs}$. The unique $S$-linear map*

$$\delta_\sigma: H(S) \to \mathcal{K}(\sigma)$$

*satisfying* $\boldsymbol{e}_\sigma \circ \delta_\sigma = \mathrm{res}_\sigma$ *is an isomorphism. In other words, the pair* $(H(S), \mathrm{res}_\sigma)$ *represents the functor* $G_\sigma$ *of* 4.1(g).

*Proof.* It is well-known that $H(S)$ is an injective hull of $k_S$ with respect to $S$, and so is $\mathcal{K}(\sigma)$. Hence there is a (non-canonical) isomorphism

$$\psi_\sigma: \mathcal{K}_\sigma \xrightarrow{\sim} H(S).$$

Let $\theta_\sigma: H(S) \to \omega_L$ be the map corresponding to $\boldsymbol{e}_\sigma$ under $\psi_\sigma$. Then by 4.2.2 there is a map $h: \widetilde{\Omega}^p_{S/k} \to \widetilde{\Omega}^p_{S/k}$ such that $\mathrm{res}_\sigma \circ H^d_{\mathfrak{m}_S}(h) = \theta_\sigma$. Applying 1.7.2 to $J = H(S)$, $q = \mathrm{res}_\sigma$, $\gamma = H^d_{\mathfrak{m}_S}(h) \circ \psi_\sigma$ we get the proposition. □

4.3. DEFINITION. Let $S \xrightarrow{f} R$ be a surjective ring homomorphism in $\mathfrak{C}$, and let $\sigma \in C_R$. We say $\tau \in C_S$ is a *lift* of $\sigma$, (or $\tau$ lifts $\sigma$ to $S$) if $f \circ \tau = \sigma$.

We need the following well known (and easy) fact:

PROPOSITION 4.4. *Let $S \to R$ be a surjective ring homomorphism in $\mathfrak{C}$. Then every $\sigma \in C_R$ has a lift in $C_S$.*

4.4.1. REMARKS. Let $S \xrightarrow{f} R$, $\sigma \in C_R$, $\tau \in C_S$ be as in the Proposition, and let $S \in \boldsymbol{k}_{fs}$.

(a) By 4.2.2, we have a unique $S$-map

$$\phi_\sigma^\tau : \mathcal{K}(\sigma) \to H(S),$$

such that $\text{res}_\tau \circ \phi_\sigma^\tau = \boldsymbol{e}_\sigma$. If $\delta_\tau : H(S) \to \mathcal{K}(\tau)$ is the isomorphism in 4.2.2 then clearly

$$\delta \circ \phi_\sigma^\tau = \gamma_\sigma^\tau,$$

where $\gamma_\sigma^\tau$ is as in 4.1(h).

(b) Let $I = \ker(f)$ and define an $S$-submodule of $H(S)$ thus:

$$H(S)(f) = \{\nu \in H(S) | I\nu = 0\}.$$

In view of (a) above, and 4.1(i), $\phi_\sigma^\tau : \mathcal{K}(\sigma) \to H(S)$ is injective and takes values in $H(S)(f)$. Let

$$\Phi_\sigma^\tau : \mathcal{K}(\sigma) \xrightarrow{\sim} H(S)(f),$$

denote the resulting isomorphism.

(c) Let $S' \xrightarrow{f'} S$ be another surjective ring homomorphism in $\mathfrak{C}$, with $S' \in \boldsymbol{k}_{fs}$. Let $\tau' \in C_{S'}$ be a lift of $\tau \in C_S$. Then,

$$\phi_\sigma^{\tau'} = \text{Res}_{S/S'} \circ \phi_\sigma^\tau.$$

THEOREM 4.5. *Let $S \xrightarrow{f} R$ be a surjective ring homomorphism in $\mathfrak{C}$, with $S \in \boldsymbol{k}_{fs}$. Let $\tau_1, \tau_2 \in C_S$ be two lifts of a pseudo-coefficient field $\sigma : L \to R$. Then*

$$\phi_\sigma^{\tau_1} = \phi_\sigma^{\tau_2}.$$

*Proof.* Let $\tau$ be any lift of $\sigma$. We have to show that $\phi_\sigma^\tau$ is independent of $\tau$. Let $i, j$ be the canonical inclusions of $k$ in $S$ and $L$ respectively. Then $i, j$ are formally smooth, $\tau, \sigma$ are residually finite and $\tau j = i$ and $f\tau = \sigma$.

Consider the following three pseudo-functors defined in [10] (our notations differ from *ibid.*):

(i) $(\_)_\flat$ on $\mathfrak{C}_{rf}$ defined in (4.4) of *ibid.* Here, for $T \in \mathfrak{C}_{rf}$, $T_\flat$ is the category of $T$-modules with 0-dimensional support, and for a morphism $T \xrightarrow{g} T'$ in $\mathfrak{C}_{rf}$, $g_\flat = \text{Hom}_T^c(T', -)$.

(ii) $(\_)_\sharp$ on $\mathfrak{C}_{fs}$ defined in (4.5) of *ibid.* For $T \in \mathfrak{C}_{fs}$, $T_\sharp$ is the category of $T$-modules with 0-dimensional support. For a morphism $T \xrightarrow{g} T'$ in $\mathfrak{C}_{fs}$, $g_\sharp = H^d_{\mathfrak{m}_{T'}}(\widetilde{\Omega}^t_{T'/T} \otimes_T -)$, where $d$ is the relative dimension of $T'/T$ and $t$ is defined by $t - d = \text{tr.deg}_{k_T} k_{T'}$.

(iii) $(\_)_*$ on $\mathfrak{C}_{rfg}$ defined in Section 6 of *ibid.*

On $\mathfrak{C}_{rf}$, $(\_)_\flat$ and $(\_)_*$ are isomorphic by Section 7 of [10]. On $\mathfrak{C}_{fs}$, $(\_)_\sharp$ and $(\_)_*$ are isomorphic by construction. Identifying $H(S)(f)$ with $\operatorname{Hom}_S(R, H(S))$, we see that $(\phi_\sigma^\tau)^{-1}$ is the composition (with all arrows arising from the natural isomorphisms between the various pseudo-functors)

$$f_\flat i_\sharp k \to f_\flat i_* k \to f_\flat \tau_* j_* k \to f_\flat \tau_* j_\sharp k \to f_\flat \tau_\flat j_\sharp k \to \sigma_\flat j_\sharp k.$$

To show that this composition is independent of $\tau$ it is enough to show that the following diagram commutes (for the composition along the western border followed by the composition along the southern border does not involve $\tau$).

$$\begin{array}{ccccccc}
f_\flat i_\sharp & \longrightarrow & f_\flat i_* & & & & \\
\downarrow & & \downarrow & & & & \\
f_* i_\sharp & & f_\flat \tau_* j_* & \longrightarrow & f_\flat \tau_* j_\sharp & \longrightarrow & f_\flat \tau_\flat j_\sharp \\
\downarrow & & \downarrow & & & & \downarrow \\
f_* i_* & \longrightarrow & f_* \tau_* j_* & \longrightarrow & f_* \tau_\flat j_* & \longrightarrow & f_\flat \tau_\flat j_* \\
\downarrow & & \Big| & & \downarrow & & \downarrow \\
(fi)_* & & \Big| & & f_\flat \tau_\flat j_* & \longrightarrow & f_\flat \tau_\flat j_\sharp \\
\| & & \downarrow & & \downarrow & & \downarrow \\
(\sigma j)_* & \longrightarrow & \sigma_* j_* & \longrightarrow & \sigma_\flat j_* & \longrightarrow & \sigma_\flat j_\sharp
\end{array}$$

All arrows here are isomorphisms, and one checks easily that each subrectangle commutes. □

4.5.1. REMARK. Let $f\colon S \to R$ be a map in $\mathfrak{C}$ which is surjective, with $S \in \boldsymbol{k}_{fs}$. Then for $\sigma \in C_R$, the above theorem tells us that we have a unique map

$$\phi_\sigma^f\colon \mathcal{K}(\sigma) \to H(S),$$

such that for every $\tau \in C_S$ which lifts $\sigma$, $\operatorname{res}_\tau \circ \phi_\sigma$ equals $\boldsymbol{e}_\sigma$. $\phi_\sigma^f$ maps $\mathcal{K}(\sigma)$ isomorphically onto $H(S)(f)$, and hence we have an isomorphism

$$\Phi_\sigma^f\colon \mathcal{K}(\sigma) \xrightarrow{\sim} H(S)(f),$$

induced by $\phi_\sigma^f$.

4.6. DEFINITION. Let $R \in \mathfrak{C}$, and $\sigma, \sigma' \in C_R$. Define an isomorphism $\Psi_\sigma^{\sigma'}$: $\mathcal{K}(\sigma) \xrightarrow{\sim} \mathcal{K}(\sigma')$ as follows: Pick a surjective map $f: S \to R$ in $\mathfrak{C}$ with $S \in \boldsymbol{k}_{fs}$ (such a map always exists). By 4.4 we can find lifts $\tau, \tau' \in C_S$ of $\sigma$ and $\sigma'$ respectively. With $\Phi_\sigma^f$ and $\Phi_{\sigma'}^f$ as in 4.5.1 we define:

$$\Psi_\sigma^{\sigma'} := (\Phi_{\sigma'}^f)^{-1} \circ \Phi_\sigma^f : \mathcal{K}(\sigma) \xrightarrow{\sim} \mathcal{K}(\sigma'). \tag{4.6.1}$$

*A priori* the isomorphism depends on the surjective map $f: S \to R$, but we will show that it is independent of this map. Let $f': S' \to R$ be another map in $\mathfrak{C}$ which is surjective, with $S' \in \boldsymbol{k}_{fs}$. By 3.3.2 and 3.3.3 we may assume that there is a surjective $k$-algebra homomorphism $p: S' \to S$ such that $f' = f \circ p$. Let $\epsilon, \epsilon' \in C_{S'}$ be lifts of $\tau$ and $\tau'$ respectively. Now $\mathrm{Res}_{S/S'}: H(S) \to H(S')$ maps $H(S)$ isomorphically onto the $S'$-submodule $H(S')(p)$, and hence maps $H(S)(f)$ isomorphically onto $H(S')(f')$. Let $\gamma: H(S)(f) \xrightarrow{\sim} H(S')(f')$ be the induced isomorphism. By 4.4.1(c), $\Phi_\sigma^{f'} = \gamma \circ \Phi_\sigma^f$ and $\Phi_{\sigma'}^{f'} = \gamma \circ \Phi_{\sigma'}^f$. It follows that $\Psi_\sigma^{\sigma'} = (\Phi_{\sigma'}^{f'})^{-1} \circ \Phi_\sigma^{f'}$. Thus the isomorphism $\Psi_\sigma^{\sigma'}$ is independent of all choices.

From the definitions it is immediate that for three pseudo-coefficient fields, $\sigma, \sigma', \sigma'' \in C_R$, we have

$$\Psi_\sigma^{\sigma''} = \Psi_{\sigma'}^{\sigma''} \circ \Psi_\sigma^{\sigma'}. \tag{4.6.2}$$

Note that if $R' \in \mathfrak{C}$, and if we have a surjective map $g: R' \to R$ in $\mathfrak{C}$, and $\tau, \tau' \in C_{R'}$ are lifts of $\sigma, \sigma' \in C_R$, then

$$\Psi_\tau^{\tau'} \circ \gamma_\sigma^\tau = \gamma_{\sigma'}^{\tau'} \circ \Psi_\sigma^{\sigma'}, \tag{4.6.3}$$

where $\gamma_\sigma^\tau$ and $\gamma_{\sigma'}^{\tau'}$ are as in 4.1(h).

THEOREM 4.7. *Let $X$ be reduced algebraic $k$-scheme and $x \in X$ a point. Let $R = \mathcal{O}_{X,x}$ and let $\sigma, \sigma' \in C_R$. Let $\Phi_{\sigma,\sigma'}: \mathcal{K}(\sigma) \to \mathcal{K}(\sigma')$ be the $R$-isomorphism in* [31], p. 96, 4.3.13. *Then*

$$\Phi_{\sigma,\sigma'} = \Psi_\sigma^{\sigma'}.$$

*Proof.* If $R \in \boldsymbol{k}_{fs}$ then this is a consequence of [29] 0.2.11 and [14] Theorem 2.2. Otherwise, by replacing $X$ by a smaller open neighbourhood of $x$ if necessary, we may assume that there is a closed immersion $i: X \hookrightarrow Y$, with $Y$ a smooth variety. Let $y = i(x)$, and $R' = \mathcal{O}_{Y,\hat{y}}$. We have a natural surjection $g: R' \to R$ (corresponding to $i$). Let $\tau, \tau' \in C_{R'}$ be lifts of $\sigma$ and $\sigma'$ respectively. Lemma 4.4.8 of [31] and the definition of $\Phi_{\sigma,\sigma'}$ gives:

$$\Phi_{\tau,\tau'} \circ \gamma_\sigma^\tau = \gamma_{\sigma'}^{\tau'} \circ \Phi_{\sigma,\sigma'}. \tag{4.7.1}$$

Since $\Phi_{\tau,\tau'} = \Psi_\tau^{\tau'}$ ($R'$ being in $\boldsymbol{k}_{fs}$) and since $\gamma_\sigma^\tau, \gamma_{\sigma'}^{\tau'}$ are injective maps, therefore the proposition follows by comparing the above equation with 4.6.3. □

We end this section with a change of rings theorem for 4.6.1 which contains as a special case 4.6.3.

THEOREM 4.8. *Let $f\colon S \to R$ be a finite map in $\mathfrak{C}$. Let $\sigma, \sigma' \in C_R$ and let $\tau, \tau' \in C_S$ be lifts of $\sigma$ and $\sigma'$. Then*

$$\Psi_{\tau}^{\tau'} \circ \gamma_{\sigma}^{\tau} = \gamma_{\sigma'}^{\tau'} \circ \Psi_{\sigma}^{\sigma'}.$$

*Proof.* Pick a surjective ring homomorphism $S' \xrightarrow{g} S$ in $\mathfrak{C}$ with $S' \in \boldsymbol{k}_{fs}$. Let $(f', R', g')$ be a factorization of $f \circ g\colon S' \to R$. One checks (using 4.2.3) that $\phi_{\tau}^{g} \circ \gamma_{\sigma}^{\tau} = \mathrm{Res}_{R'/S'} \circ \phi_{\sigma}^{g'}$ and $\phi_{\tau'}^{g} \circ \gamma_{\sigma'}^{\tau'} = \mathrm{Res}_{R'/S'} \circ \phi_{\sigma'}^{g'}$. It follows that we have

$$\Phi_{\tau}^{g} \circ \gamma_{\sigma}^{\tau} \circ (\Phi_{\sigma}^{g'})^{-1} = \Phi_{\tau'}^{g} \circ \gamma_{\sigma'}^{\tau'} \circ (\Phi_{\sigma'}^{g'})^{-1}.$$

The theorem follows. □

## 5. Trace structures

5.1. DEFINITION A *trace-structure* on $\mathfrak{C}$ consists of:

(i) An injective hull $\mathcal{K}(R)$ of the $R$-module $R/\mathfrak{m}_R$ for each $R \in \mathfrak{C}$.

(ii) For each $R \in \mathfrak{C}$ and each $\sigma \in C_R$, a $\sigma$-linear map $t_\sigma\colon \mathcal{K}(R) \to \omega(\sigma)$ such that for $R \in \mathfrak{C}$:

  (a) The pair $(\mathcal{K}(R), t_\sigma)$ represents the functor $G_\sigma$ for each $\sigma \in C_R$.

  (b) If $R$ is formally smooth over $k$, there exists an $R$-map (unique by (a) above) $\phi_R\colon H(R) \to \mathcal{K}(R)$ such that $t_\sigma \circ \phi_R = \mathrm{res}_\sigma$ (cf. 4.2.1) for every $\sigma \in C_R$. (*Note*: Such a $\phi_R$ is necessarily an isomorphism by 4.2.4).

  (c) If $f\colon R \to S$ is a morphism in $\mathfrak{C}$ such that the residue field extension is finite, there exists an $R$-map (unique by (a) above): $\tau_f\colon \mathcal{K}(S) \to \mathcal{K}(R)$ such that $t_\sigma \circ \tau_f = t_{f\circ\sigma}$ for every $\sigma \in C_R$.

5.2. REMARKS.

(i) We stress that $\phi_R$ and $\tau_f$ in (b) and (c) above are required to be independent of $\sigma \in C_R$.

(ii) If $f\colon R \to S$ is as in (c) above, then one checks, rather easily, that the map $\tau_f$ induces an $S$-isomorphism $T_f\colon \mathcal{K}(S) \xrightarrow{\sim} \mathrm{Hom}_R^{\mathrm{c}}(S, \mathcal{K}(R))$, where the right side is the $S$-module of continuous $R$-maps from $S$ to $\mathcal{K}(R)$), such that $T_f$ followed by 'evaluation at 1' gives $\tau_f$. Note that if $f$ is finite then $\mathrm{Hom}_R^{\mathrm{c}}(S, \mathcal{K}(R)) = \mathrm{Hom}_R(S, \mathcal{K}(R))$.

(iii) Let $f\colon R \to S$ be a morphism in $\mathfrak{C}_{rf} \cap \mathfrak{C}_{fs}$ of relative dimension $d$, and assume $R$ (and hence $S$) is formally smooth over $k$. Using the transitivity of residues (cf. 4.2.3) and the identification $H^d_{\mathfrak{m}_S}(\tilde{\Omega}^d_{S/R} \otimes H(R))$ with $H(S)$, we see that the diagram:

$$\begin{array}{ccccc} H^d_{\mathfrak{m}_S}(\tilde{\Omega}^d_{S/R} \otimes_R H(R)) & \xrightarrow[(=)]{\sim} & H(S) & \xrightarrow[\phi_S]{\sim} & \mathcal{K}(S) \\ \downarrow \mathrm{res}_{S/R} & & & & \downarrow \tau_f \\ H(R) & & \xrightarrow[\phi_R]{\sim} & & \mathcal{K}(R) \end{array}$$

commutes. Proposition 5.5 below generalizes this to the case where $R$ is not necessarily formally smooth over $k$, and $f: R \to S$ induces a finite field extension of residue fields.

(iv) Let $R \in \mathfrak{C}$ be reduced and equidimensional. Let $\omega_R$ be the module of regular differentials over $k$ and $H(R) = H^{\dim R}_{\mathfrak{m}_R}(\omega_R)$. Then as in [14], (1.1), we have maps $\mathrm{res}_\sigma: H(R) \to \omega(\sigma)$ for each $\sigma \in C_R$, such that $(\omega_R, \mathrm{res}_\sigma)$ represents the functor $\mathrm{Hom}_\sigma(H^{\dim R}_{\mathfrak{m}_R}(M), \omega(\sigma))$ of finitely generated $R$-modules $M$, and hence a map $\phi^\sigma_R: H(R) \to \mathcal{K}(R)$ such that $t_\sigma \circ \phi^\sigma_R = \mathrm{res}_\sigma$. It turns out that $\phi^\sigma_R = \phi^\tau_R$ for $\sigma, \tau \in C_R$. However, we will only prove this under the assumption that $R$ is the completion of a local ring at a point of a reduced equidimensional algebraic $k$-scheme $X$ (cf. 9.3). Note that if $R$ is formally smooth over $k$, then this is part of the definition of a trace structure.

(v) Let $R \xrightarrow{f} S \xrightarrow{g} T$ be a sequence of morphisms in $\mathfrak{C}$, such that the induced map on residue fields is finite for both $f$ and $g$ (and hence $g \circ f$). Then

$$\tau_{g \circ f} = \tau_f \circ \tau_g.$$

(vi) Let $f: R \to S$ be a morphism in $\mathfrak{C}$ such that the residue field extension is finite. Then for 0-dimensional $S$-modules $M$, the natural $S$-map

$$\mathrm{Hom}_S(M, \mathcal{K}(S)) \to \mathrm{Hom}_R(M, \mathcal{K}(R))$$

(arising from $\tau_f: \mathcal{K}(S) \to \mathcal{K}(R)$) is an isomorphism, i.e., $(\mathcal{K}(R), \tau_f)$ represents the functor $\mathrm{Hom}_R(M, \mathcal{K}(R))$ of 0-dimensional $S$-modules $M$.

PROPOSITION 5.3. *Trace structures exist and are unique up to unique isomorphism, i.e., if* $\{(\mathcal{K}(R), \{t_\sigma\})\}$ *and* $\{(\mathcal{K}'(R), \{t'_\sigma\})\}$ *are two trace structures, then there exists a unique family of isomorphisms*

$$\mu_R: \mathcal{K}(R) \xrightarrow{\sim} \mathcal{K}'(R), \quad R \in \mathfrak{C}$$

*such that* $t'_\sigma \circ \mu_R = t_\sigma$ *for every* $\sigma \in C_R$.

*Proof.* We first prove uniqueness. Fix $R \in \mathfrak{C}$. For each $\sigma \in C_R$, clearly there is an unique isomorphism

$$\mu^\sigma_R: \mathcal{K}(R) \xrightarrow{\sim} \mathcal{K}'(R)$$

such that $t'_\sigma \circ \mu^\sigma_R = t_\sigma$. We are done if we show that $\mu^\tau_R = \mu^\sigma_R$ for $\sigma, \tau \in C_R$. First assume $R$ is formally smooth over $k$, and $\phi_R\colon H(R) \xrightarrow{\sim} \mathcal{K}(R)$ and $\phi'_R\colon H(R) \xrightarrow{\sim} \mathcal{K}'(R)$ the isomorphism arising from 5.1(b). Then for every $\sigma \in C_R$, it is easy to check that $\mu^\sigma_R = \phi'_R \circ \phi_R^{-1}$. This proves the assertion in case $R$ is formally smooth over $k$. For general $R \in \mathfrak{C}$, let $f\colon S \to R$ be a surjective map in $\mathfrak{C}$ with $S$ formally smooth over $k$. Let $\sigma, \tau \in C_R$ and let $\sigma^*, \tau^* \in C_S$ be some lifts of $\sigma$ and $\tau$ respectively. Let $T_f\colon \mathcal{K}(R) \xrightarrow{\sim} \mathrm{Hom}_S(R, \mathcal{K}(S))$ and $T'_f\colon \mathcal{K}'(R) \xrightarrow{\sim} \mathrm{Hom}_S(R, \mathcal{K}'(S))$ be the isomorphisms in 5.2(ii). Let $\mu_R\colon \mathcal{K}(R) \xrightarrow{\sim} \mathcal{K}(R')$ be given by $T'^{-1}_f \circ \mu_S \circ T_f$. Clearly $t'_\sigma \circ \mu_R = t_\sigma$ and $t'_\tau \circ \mu_R = t_\tau$, and hence $\mu^\sigma_R = \mu_R = \mu^\tau_R$ as required. This proves the uniqueness of trace structures.

For $R \in \mathfrak{C}$, consider the collection $\{\mathcal{K}(\sigma)\}_{\sigma \in C_R}$. The maps $\Psi^{\sigma'}_\sigma$ of 4.6.1 make $\{\mathcal{K}(\sigma)\}$ into a direct system. Set

$$\mathcal{K}(R) := \varinjlim_{\sigma} \mathcal{K}(\sigma).$$

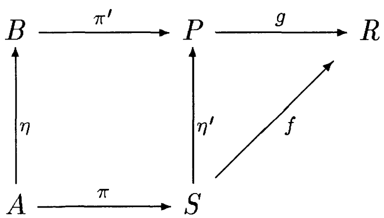
Then we have unique isomorphisms $\mu_\sigma\colon \mathcal{K}(\sigma) \xrightarrow{\sim} \mathcal{K}(R)$, one for each $\sigma \in C_R$, such that $\Psi^{\sigma'}_\sigma = \mu^{-1}_{\sigma'} \circ \mu_\sigma$. Set $t_\sigma = e_\sigma \circ \mu_\sigma^{-1}$ (where, as usual, $e_\sigma$ denotes 'evaluation at 1').

For $f\colon S \to R$ a finite map, $\sigma \in C_R$, and $\tau \in C_S$ a lift of $\sigma$, define $\tau_f\colon \mathcal{K}(R) \to \mathcal{K}(S)$ by the formula

$$\tau_f = \mu_\tau \circ \gamma^\tau_\sigma \circ \mu_\sigma^{-1}.$$

(cf. 4.1(h) for definition of $\gamma^\tau_\sigma$). Theorem 4.8 says that $\tau_f$ is well-defined. Clearly $t_\tau \circ \tau_f = t_\sigma$. This proves the existence of a trace when $f$ is finite.

Now suppose $f\colon S \to R$ is a morphism in in $\mathfrak{C}_{rf}$. Let $\pi\colon A \to S$ be a surjective map in $\mathfrak{C}$ with $A$ formally smooth over $k$. The map $f \circ \pi$ has a factorization $(B, \eta, \psi)$, and hence we have a commutative diagram (with $P = B \otimes_A S$ and $g \circ \pi' = \psi$).




Since all horizontal arrows are surjective, traces for them are defined. Further $\mathcal{K}(A)$ can be identified with $H^m_{\mathfrak{m}_A}(\widetilde{\Omega}^{n+q}_{A/k})$ (where $n = \dim A$ and $q = \mathrm{tr.deg}_k(A/\mathfrak{m}_A)$ and $\mathcal{K}(B)$ can be identified with $H^{n+d}_{\mathfrak{m}_B}(\widetilde{\Omega}^{n+d+q}_{B/k}) = H^d_{\mathfrak{m}_B}(\widetilde{\Omega}^d_{B/A} \otimes \mathcal{K}(A))$. Define an $A$-linear map $\tau_{f\circ\pi}\colon \mathcal{K}(R) \to \mathcal{K}(A)$ by the formula $\tau_{f\circ\pi} = \mathrm{res}_{B/A} \circ \tau_{\pi'} \circ \tau_g$. For $\sigma \in C_A$, the 'transitivity of residues' formula 4.2.3 gives the relation $t_\sigma \circ \tau_{f\circ\pi} = t_{f\circ\pi\circ\sigma}$, whence $\tau_{f\circ\pi}$ depends only on the algebra structure of $A \to R$. Using 3.7, 3.7.1 and 3.7.2 we see that $\tau_{f\circ\pi}$ takes values in the image of $\tau_\pi$. Since $\tau_\pi$ is injective, this gives an $S$-linear map $\tau_f\colon \mathcal{K}(R) \to \mathcal{K}(S)$. From the construction of $\tau_f$, the relation $t_\sigma \circ \tau_f = t_{f\circ\sigma}$ is immediate for $\sigma \in C_S$. It follows that $\tau_f$ is independent of the surjective map $\pi\colon A \to S$. Thus trace structures exist. □

5.4. REMARKS

(a) Observe that if $\mu_R$ is as in the proposition, and $R$ is formally smooth over $k$, then $\mu_R \circ \phi_R = \phi'_R$ where $\phi_R\colon H(R) \xrightarrow{\sim} \mathcal{K}(R)$ and $\phi'_R\colon H(R) \to \mathcal{K}'(R)$ are as in 5.1(b).
(b) If $f\colon S \to R$ is in $\mathfrak{C}_{rf}$, then $\mu_S \circ \tau_f = \tau'_f \circ \mu_R$, where $\tau_f$ and $\tau'_f$ are as in 5.1(c).

PROPOSITION 5.5. *Let $f\colon R \to S$ be a formally smooth morphism of relative dimension $n$ in $\mathfrak{C}_{rf}$. Then there exists a unique isomorphism*

$$\psi_f\colon H^n_{\mathfrak{m}_S}(\tilde{\Omega}^n_{S/R} \otimes \mathcal{K}(R)) \xrightarrow{\sim} \mathcal{K}(S)$$

*such that* $\tau_f \circ \psi_f = \operatorname{res}_{S/R}$.

*Proof.* Follows from 5.2(vi) and 3.12(a). □

## 6. Construction of the residue complex

Fix a trace structure $\{(\mathcal{K}(R), \{t_\sigma\})\}$.

6.1. CONVENTIONS

(a) For $X$ an algebraic $k$-scheme, $x$ a point in $X$, and $R = \widehat{\mathcal{O}}_{X,x}$, we write $\mathcal{K}(x)$ for $\mathcal{K}(R)$, $C_x$ for $C_R$, $\phi_x$ for $\phi_R$ etc. If $x$ is a smooth point, we write $H(x)$ for $H(R)$.
(b) If $f\colon U \to V$ is a map of algebraic $k$-schemes, then for $u \in U$ which is closed in the fibre $f^{-1}f(u)$, the map $\mathcal{K}(u) \to \mathcal{K}(f(u))$ of 5.1(c) will be denoted $\theta_{f,u}$.

6.2. CONSTRUCTION

Let $X$ be a connected algebraic $k$-scheme of dimension $n$ and let $\Delta_X = \Delta\colon X \to \mathbb{Z}$ be the codimension function $\Delta(x) = -\dim\{x\}^-$ for $x \in X$. Let $\mathcal{K}^p_X$ be the injective $\mathcal{O}_X$-module given by

$$\mathcal{K}^p_X = \bigoplus_{\Delta(x)=p} \bar{\mathcal{K}}(x).$$

We intend to define a coboundary map $\delta$ on the graded module $\bigoplus_{p\in\mathbb{Z}} \mathcal{K}^p_X$ so that $(\mathcal{K}^\bullet_X, \delta)$ becomes a residual complex.

First assume $X$ is smooth. Then by property (b) of a trace structure, we have a canonical isomorphism $\phi_x\colon H(x) \xrightarrow{\sim} \mathcal{K}(x)$ for each $x \in X$. Let $E^\bullet$ be the Cousin complex $E^\bullet(\Omega^n_{X/k})[n]$. Then $E^\bullet$ is a residual complex, and for $x \in X$, $E(x) = H(x)$. Further, the codimension function associated to $E^\bullet$ is precisely $\Delta_X$. Thus for $p \in \mathbb{Z}$, we have an isomorphism

$$\bar{\phi}^p = \sum_{\Delta(x)=p} \bar{\phi}_x \colon E^p \xrightarrow{\sim} \mathcal{K}^p_X.$$

Define $\delta^p \colon \mathcal{K}^p_X \xrightarrow{\sim} \mathcal{K}^{p+1}_X$ in the obvious way, viz., $\delta^p = \bar{\phi}^{p+1} \circ \delta^p_{E^\bullet} \circ (\bar{\phi}^p)^{-1}$, where $\delta^p_{E^\bullet}$ is the coboundary on $E^\bullet$. We thus get a residual complex $(\mathcal{K}^\bullet_X, \delta)$ isomorphic to $E^\bullet$, when $X$ is smooth.

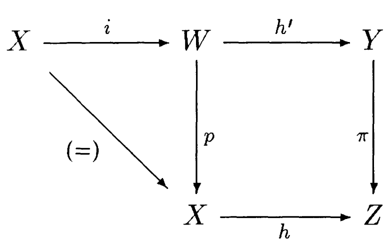
We now construct the coboundary map $\delta$ on $\bigoplus_{p\in\mathbb{Z}} \mathcal{K}^p_X$ for general $X$.

Let $x_1 \mapsto x_2$ be an immediate specialization in $X$. We define an $\mathcal{O}_{X,x_2}$-homomorphism $\delta(x_1, x_2) \colon \mathcal{K}(x_1) \to \mathcal{K}(x_2)$ as follows:

**Case 1.** Suppose we have a closed immersion $g \colon X \to Y$ with $Y$ a smooth $k$-variety. Let $K^\bullet_Y$ be the complex just constructed. Let $J^\bullet = g^\flat \mathcal{K}^\bullet_Y$, where, as usual, if $\mathcal{F}^\bullet$ is an $\mathcal{O}_Y$-injective complex, then $g^\flat \mathcal{F}^\bullet$ is the $\mathcal{O}_X$-injective complex corresponding to the complex of $g_*\mathcal{O}_X$-modules $\mathcal{H}om_Y(g_*\mathcal{O}_X, \mathcal{K}^\bullet_Y)$. $J^\bullet$ is clearly a residual complex.

For $x \in X$, set $y = g(x)$, $R = \widehat{\mathcal{O}}_{X,x}$ and $S = \widehat{\mathcal{O}}_{Y,y}$. From the definition of $J^\bullet$, one has a canonical identification of $J(x)$ with $\mathrm{Hom}_S(R, \mathcal{K}(y))$. On the other hand, $\theta_{y,x} \colon \mathcal{K}(x) \to \mathcal{K}(y)$ induces an isomorphism $\Theta_{g,x} \colon \mathcal{K}(\tau) \xrightarrow{\sim} \mathrm{Hom}_S(R, \mathcal{K}(y)) = J(y)$ (cf. 5.2(ii)). The differential on $J^\bullet$ gives us a map $d(x_1, x_2) \colon J(x_1) \to J(x_2)$, whence (via $\Theta_{g,x_1}$ and $\Theta_{g,x_2}$) a map $\delta^g(x_1, x_2) \colon \mathcal{K}(x_1) \to \mathcal{K}(x_2)$.

Next we prove that if $h \colon X \to Z$ is another closed immersion with $Z$ smooth, then $\delta^g(x_1, x_2) = \delta^h(x_1, x_2)$. We may assume (by replacing $Z$ by $Y \times_k Z$ if necessary) that there is a smooth map $\pi \colon Y \to Z$ such that $\pi \circ g = h$. If $W = X \times_Z Y$ and $p \colon W \to X$, $h' \colon W \to Y$ the resulting projection maps, we have a commutative diagram (with the square being cartesian)




(6.2.1)

with $h' \circ i = g$ and $p \circ i =$ identity. It is immediate that $i \colon X \to W$ is a closed immersion, and that the ideal sheaf of $X$ is generated (locally) by a regular sequence.

For the immediate specialization $x_1 \mapsto x_2$, let $w_1 \mapsto w_2$, $y_1 \mapsto y_2$ and $z_1 \mapsto z_2$ be the corresponding immediate specializations in $W, Y$ and $Z$ respectively. Let $s_1, \ldots, s_d \in \mathcal{O}_{W,w_2}$ generate that kernel of the surjection $\mathcal{O}_{W,w_2} \to \mathcal{O}_{X,x_2}$. We may assume that $s_1, \ldots, s_d$ is a regular system of parameters for the algebra

$\widehat{\mathcal{O}}_{W,w_2}/\widehat{\mathcal{O}}_{X,x_2}$. Lifting $s_1, \ldots, s_d$ to $\mathcal{O}_{Y,y_2}$ (and denoting the lifted elements also by $s_1, \ldots, s_d$) we note that $s_1, \ldots, s_d$ are a system of parameters for $\widehat{\mathcal{O}}_{Y,y_2}/\widehat{\mathcal{O}}_{Z,z_2}$. Fix $j \in \{1,2\}$, and let $Q_j = \widehat{\mathcal{O}}_{W,w_j}$, $R_j = \widehat{\mathcal{O}}_{X,x_j}$, $S_j = \widehat{\mathcal{O}}_{Y,y_j}$ and $T_j = \widehat{\mathcal{O}}_{Z,z_j}$. One sees that $Q_j = R_j[|s_1, \ldots, s_d|]$ and $S_j = T_j[|s_1, \ldots, s_d|]$, (with s being analytically independent in both cases) and the maps $R_j \to Q_j$, $T_j \to S_j$ (induced by $p\colon W \to X$ and $\pi\colon Y \to Z$), are the natural inclusions of coefficient rings in power series rings.

Let $m = \dim Z$, so that $\dim Y = m + d$. Let $\psi\colon \mathcal{K}(x_1) \to H(z_1)$ be the map $\phi_{z_1}^{-1} \circ \theta_{h,x_1}$. Let $\delta(y_1, y_2)\colon H(y_1) \to H(y_2)$ and $\delta(z_1, z_2)\colon H(z_1) \to H(z_2)$ be the maps arising from the Cousin complexes $E(\Omega_{Y/k}^{m+d}[m+d])$ and $E(\Omega_{Z/k}^{m}[m])$. Let $\mu_j\colon H(z_j) \to H(y_j)$ be the map

$$\begin{bmatrix} \omega \\ \mathbf{t} \end{bmatrix} \mapsto \begin{bmatrix} d\mathbf{s} \otimes \omega \\ \mathbf{s}, \mathbf{t} \end{bmatrix}$$

where $\mathbf{t}$ is a system of parameters for $\mathcal{O}_{Z,z_j}$. Using the fact that

$$\operatorname{res}_{Q_j/R_j} \begin{bmatrix} d\mathbf{s} \otimes \begin{bmatrix} \omega \\ \mathbf{t} \end{bmatrix} \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} \omega \\ \mathbf{t} \end{bmatrix},$$

5.2 (iii) and (v), we are reduced to showing the following:

For $\xi \in \mathcal{K}(x_1)$;

$$\delta(y_1, y_2) \circ \mu_1 \circ \psi(\xi) = \mu_2 \circ \delta(z_1, z_2) \circ \psi(\xi).$$

Let $\psi(\xi) = \begin{bmatrix} \omega/t_0 \\ t_1, \ldots, t_r \end{bmatrix}$ where $\omega \in \Omega_{T_2/k}^{m}$ – we are using the notations in 1.1. Then, using 1.1.1, we have

$$\begin{aligned} \delta(y_1, y_2) \circ \mu_1 \begin{bmatrix} \omega/t_0 \\ t_1, \ldots, t_r \end{bmatrix} &= \delta(y_1, y_2) \begin{bmatrix} ds_1 \wedge \cdots \wedge ds_d \otimes \omega/t_0 \\ s_1, \ldots, s_d, t_1, \ldots, t_r \end{bmatrix} \\ &= (-1)^{n+d} \begin{bmatrix} ds_1 \wedge \cdots \wedge ds_d \otimes \omega/t_0 \\ t_0, s_1, \ldots, s_d, t_1, \ldots, t_r \end{bmatrix} \\ &= (-1)^{n} \begin{bmatrix} ds_1 \wedge \cdots \wedge ds_d \otimes \omega \\ s_1, \ldots, s_d, t_0, t_1, \ldots, t_r \end{bmatrix} \qquad (6.2.2) \\ &= (-1)^{n} \mu_2 \begin{bmatrix} \omega \\ t_0, t_1, \ldots, t_r \end{bmatrix} \\ &= \mu_2 \circ \delta(z_1, z_2) \begin{bmatrix} \omega \\ t_1, \ldots, t_r \end{bmatrix} \end{aligned}$$

as required. Thus $\delta^g(x_1,x_2) = \delta^h(x_1,x_2)$. Call the common value $\delta(x_1,x_2)$. It is now immediate that if $\delta^p = \Sigma_{\substack{x_1\mapsto x_2\\ \Delta(x_1)=p}} \bar{\delta}(x_1,x_2)\colon \mathcal{K}_X^p \to \mathcal{K}_X^{p+1}$ then $\delta^{p+1}\circ\delta^p = 0$ so that we have a differential $\delta$ making $\mathcal{K}_X^\bullet$ a residual complex.

REMARK. One checks, easily, that for $U$ open in $X$, $K_U^\bullet = \mathcal{K}_X^\bullet \mid U$. In fact, if $g\colon X\to Y$ is a closed immersion with $Y$ smooth, then there exists an open subscheme $V$ of $Y$ such that $g^{-1}(V) = U$, and hence there is a closed immersion $g_U\colon U\to V$ induced by $g$. Since $E^\bullet(\Omega^r_{Y/k}[r]) \mid V = E^\bullet(\Omega^r_{V/k}[r])$ where $r = \dim Y$, the assertion follows.

**Case 2.** Now let $X$ be an arbitrary algebraic $k$-scheme. Pick an open neighbourhood $U$ of $x_2$ in $X$ which admits a closed immersion into a smooth variety, so that the complex $\mathcal{K}_U^\bullet$ can be defined. Define $\delta(x_1,x_2)\colon \mathcal{K}(x_1)\to\mathcal{K}(x_2)$ as the map induced by the differential in $K_U^\bullet$. By the remark above, $\delta(x_1,x_2)$ does not depend on the open neighbourhood $U$ of $x_2$. One checks that if $\delta^p = \Sigma_{\substack{x_1\mapsto x_2\\ \Delta(x_1)=p}} \bar{\delta}(x_1,x_2)\colon \mathcal{K}_X^p\to\mathcal{K}_X^{p+1}$, then $\delta^{p+1}\circ\delta^p = 0$ for all $p\in\mathbb{Z}$. The resulting complex is clearly residual (since it is so locally).

6.3. REMARK

(i) As before, for $U$ an open subscheme of $X$, $\mathcal{K}_X^\bullet \mid U = \mathcal{K}_U^\bullet$. In fact, the collection $\mathcal{K}^\bullet = \{\mathcal{K}_X^\bullet \mid X \text{ is an algebraic } k\text{-scheme}\}$ forms a Zariski sheaf of complexes, i.e., $\mathcal{K}^\bullet$ is a residual $\mathcal{O}$-complex in the sense of [29]. For an open immersion $g\colon U\to V$, the restriction isomorphism $\beta_g\colon g^*\mathcal{K}_V^\bullet \xrightarrow{\sim} \mathcal{K}_U^\bullet$ is given by $\beta_g(u) = \theta_{g,u}^{-1}$ for $u\in U$.

(ii) Let $f\colon X\to Y$ be a closed immersion. Then the maps $\theta_{f,x}\colon \mathcal{K}(x)\to\mathcal{K}(f(x))$ (for $x\in X$) give a map (of complexes) $\theta_f\colon f_*\mathcal{K}_X^\bullet\to\mathcal{K}_Y^\bullet$ such that the resulting map $\Theta_f\colon \mathcal{K}_X^\bullet\to f^\flat\mathcal{K}_Y^\bullet$ is an isomorphism of complexes. To see this we reduce to the case where $Y$ admits a closed immersion $g\colon Y\to Z$ where $Z$ is smooth, and use the identity $\theta_{g,f(x)}\circ\theta_{f,x} = \theta_{gf,x}$ for each $x\in X$ (cf. 5.2 (v)). Again details are left to the reader.

## 7. Trace in the absolute case

Let $x\in X$ be a closed point, and $\sigma_x\colon k\to\widehat{\mathcal{O}}_{X,x}$, the natural map. Set $\theta_x = t_{\sigma_x}\colon \mathcal{K}(x)\to k$.

PROPOSITION 7.1. *For X proper, the map*

$$\sum_{x \text{ closed}} \theta_x\circ\delta^{(-1)}\colon \Gamma(X,\mathcal{K}_X^{-1})\to k$$

*is the zero map, and hence*

$$\theta_X = \sum_{x \text{ closed}} \theta_x\colon \Gamma(X,\mathcal{K}_X^\bullet)\to k$$

*is a map of complexes.*

*Proof.* First assume $X$ is smooth of dimension $n$. Identifying $\mathcal{K}_X^\bullet$ with $E^\bullet(\Omega^n_{X/k}[n])$, the proposition follows from 2.2.1.

Next assume there is a closed immersion $i: X \to Y$ where $Y$ is smooth and proper. As in 6.3 (ii) we have a map of complexes $\theta_i: i_*\mathcal{K}_X^\bullet \to \mathcal{K}_Y^\bullet$, whence a map of complexes $\theta_y \circ \Gamma(Y, \theta_i): \Gamma(X, \mathcal{K}_X^\bullet) \to k$. Since $\theta_{g(x)} \circ \theta_{i,x} = \theta_x$, the proposition follows in this case also.

Finally, let $X$ be proper, and $x \in X$ be such that $\dim\{x\}^- = 1$ (i.e., $\Delta(x) = -1$). We have to show that $\Sigma_{x \mapsto y} \theta_y \circ \delta(x, y) = 0$. Let $\mathcal{I}_x$ be the ideal sheaf giving the reduced structure on $\{x\}^-$, and for $n \in \mathbb{N}$, let $Z_n$ be the 1-dimensional algebraic $k$-scheme defined by $\mathcal{I}_x^n$. Let $g_n: Z_n \to X$ be the natural closed immersion, and $x_n$ the unique pre-image of $x$ under $g_n$. For $y$ an immediate specialization of $X$, let $y_n \in Z_n$ be the closed point corresponding to $y$. By [6] §22, §25, we may assume that $Z_n$ is projective, whence $\Sigma_{y_n} \theta_{y_n} \circ \delta(x_n, y_n) = 0$ by our previous case. The proposition follows by taking direct limits as $n \to \infty$. □

From now on, for $X$ proper, let $\theta_X: \Gamma(X, \mathcal{K}_X^\bullet) \to k$ be the $k$-linear map given by the last proposition. Note that for $x$ a closed point of $X$, $(\mathcal{K}(x), \theta_x)$ represents the functor $\mathrm{Hom}_k(M, k)$ of $\mathcal{O}_{x,x}$-modules of 0-dimensional support; in other words, $(\mathcal{K}_X^\bullet, \theta_X)$ is a *pointwise residue pair* [28], p. 114. By 1.7 we see that $(\mathcal{K}_X^\bullet, \theta_X)$ is a residue pair. More precisely, with $\{p\} = \mathrm{Spec}\, k$, we have:

THEOREM 7.2. *The map $\theta_X: \Gamma(X, \mathcal{K}_X^\bullet) \to k$ induces an isomorphism.*

$$\mathbb{R}\,\mathrm{Hom}_X^\bullet(\mathcal{F}^\bullet, \mathcal{K}_X^\bullet) \xrightarrow{\sim} \mathbb{R}\,\mathrm{Hom}_{\{p\}}(\mathbb{R}\Gamma(X, \mathcal{F}^\bullet), k)$$

*in $D\{p\}$ for every $\mathcal{F}^\bullet \in D^+_{qc}(X)$.*

## 8. Trace for proper morphisms

Let $f: X \to Y$ be a morphism of algebraic $k$-schemes. For each $x \in X$ which is *closed* in the fibre $f^{-1}f(x)$, we have a map $\theta_{f,x}: \mathcal{K}(x) \to \mathcal{K}(f(x))$ as in 6.1(b). We extend the definition of $\theta_{f,x}$ to points $x \in X$ which are *not* closed in their fibre by setting $\theta_{f,x}: \mathcal{K}(x) \to \mathcal{K}(f(x))$ equal to zero in this case. One checks that $\theta_f: f_*\mathcal{K}_X^\bullet \to \mathcal{K}_Y^\bullet$ given by $\theta_f = \Sigma_{x \in X} \overline{\theta}_{f,x}$ is a map of graded $\mathcal{O}_Y$-modules. The main result of this section is:

PROPOSITION 8.1. *If $f$ is proper, then $\theta_f: f_*\mathcal{K}_X^\bullet \to \mathcal{K}_Y^\bullet$ is a map of complexes.*

The proof in carried out in 8.4. We need some preliminary material.

### 8.2. Curves over Artin local rings

Let $S = \mathrm{Spec} A$, $A \in \mathfrak{C}_k$ an Artin local $k$-algebra. Let $\pi: \mathbb{P}^n_S \to S$ be the structural map. Let $I = \mathcal{K}(A)$. Define

$$\mathcal{K}^\bullet_{\mathbb{P}^n_S} := E^\bullet_{\mathbb{P}^n_S}(\Omega^n_\pi \otimes I)[n]$$

Let

$$\theta_{\mathbb{P}^n_S}: \Gamma(\mathbb{P}^n_S, \mathcal{K}^\bullet_{\mathbb{P}^n_S}) \to I \tag{8.2.1}$$

be the map of complexes defined in 2.2.1. For a proper irreducible $S$-scheme $f: C \to S$ of dimension 1, there is a closed immersion of $S$-schemes $i: C \to \mathbb{P} = \mathbb{P}^n_S$ for some $n \in \mathbb{N}$. Indeed an ample invertible sheaf on the closed fibre over $S$ can be lifted to an invertible sheaf on $C$ via [9], p. 224, (4.6), and this sheaf must necessarily be ample on $C$ by *ibid.*, p. 232, (5.7)(d). For $c \in C$ if $\mathcal{R}(c) := \mathcal{K}(\hat{\mathcal{O}}_{C,c})$, $x = i(c)$, then we have a natural map

$$\theta_{i,c}: \mathcal{R}(c) \to \mathcal{K}_{\mathbb{P}}(x) \tag{8.2.2}$$

given by our trace structure. In fact, using the fact that $\mathcal{R}(c) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_{\mathbb{P},x}}(\mathcal{O}_{C,c}, \mathcal{K}_{\mathbb{P}}(x))$, we can string together the $\mathcal{R}(c)$ (as $c \in C$ varies) to get a residual complex $\mathcal{R}^\bullet_C$, such that

$$\theta_i = \sum_{c \in C} \overline{\theta_{i,c}}: i_* \mathcal{R}^\bullet_C \to \mathcal{K}^\bullet_{\mathbb{P}}$$

is a map of complexes.

In this situation we have the following Proposition, whose easy (though tedious) proof we leave to the reader (cf. Sect. 6 for the techniques involved).

PROPOSITION 8.2.3. *With notations as above, suppose $g: X \to Y$ is a proper map of relative dimension one of algebraic $k$-schemes, and suppose $A = \mathcal{O}_{Y,y}$ for some $y \in Y$, $C = S \times_Y X$, and $f: C \to S$ the projection. Let $h: S \to Y$ be the natural map and $h': C \to X$ the projection on to $X$. In this situation*

$$\mathcal{R}^\bullet_C = h'^* \mathcal{K}^\bullet_X\{-d\}$$

*where $d =$* $\dim Y$, *and '$\{-d\}$' denotes translation to the right by $d$ units* without changing the signs of the coboundary maps.

One immediate consequence is the following Lemma, in which all schemes mentioned are algebraic $k$-schemes.

LEMMA 8.3. *Let $g: V \to W$ be proper, surjective, with $V$ irreducible. Let $v \in V$ be the generic point of $V$ and let $w = g(v)$. Suppose $\Delta(v) = \Delta(w) - 1$ then the composition*

$$\mathcal{K}(v) \xrightarrow{\Sigma\delta(v,v')} \bigoplus_{v'} \mathcal{K}(v') \xrightarrow{\Sigma\theta_{g,v'}} \mathcal{K}(w) \tag{8.3.1}$$

*is zero, where the direct sum is over $v' \in g^{-1}(w)$ with $v \mapsto v'$.*

*Proof.* Let $A = \mathcal{O}_{W,w}$, $S = \operatorname{Spec} A$ and $h\colon S \to W$ the natural map. Let $C = S \times_W V$, $f\colon C \to S$ and $h'\colon C \to V$ the two projections. $C$ is irreducible of dimension one. For $f\colon C \to S$ we use the notations in 8.2. By Proposition 8.2.3 the complex $\Gamma(C, \mathcal{R}_C^\bullet)$ is precisely

$$0 \to \mathcal{K}(v) \xrightarrow{\Sigma\delta(v,v')} \bigoplus_{v'} \mathcal{K}(v') \to 0$$

and the map $\Sigma\theta_{g,w'}$ is induced by the map of complexes $\theta_{\mathbb{P}^n_S} \circ \theta_i\colon \Gamma(C, \mathcal{R}_C^\bullet) \to I = \mathcal{K}(w)$. The Lemma follows.

8.4. *Proof of Proposition* 8.1.

**Case 1.** If $f$ is a closed immersion, then we have already seen that the proposition is true [cf. 6.3(ii)].

Let $Y'$ be the scheme theoretic image of $f$, and $X \xrightarrow{f'} Y' \xrightarrow{i} Y$ the resulting factorization of $f$. One checks that $\theta_f = \theta_i \circ i_*\theta_{f'}$. By case 1, $\theta_i$ is a map of complexes, and hence we are reduced to showing that $\theta_{f'}$ is a map of complexes.

So we may assume that $f\colon X \to Y$ as in the proposition is *surjective.*

**Case 2.** Let $Y$ be smooth of dimension $n$, and $f$ smooth of relative dimension $d$. We identify $\mathcal{K}_Y^\bullet$ with $E^\bullet(\Omega^n_{Y/k}[n])$ and $\mathcal{K}_X^\bullet$ with $E^\bullet(\Omega^{n+d}_{X/k}[n+d])$. Then we have a map

$$\int_f : R^d f_* \Omega^{n+d}_{X/k} \to \Omega^n_{Y/k}$$

as in [16], (4.2) (there the map $\int_f$ is denoted $\int_{X/Y}^{(\text{Spec } k)}$). Combining this with Proposition 2.1 we get a morphism of complexes

$$f_* E^\bullet(\Omega^{n+d}_{X/k}[n+d]) \xrightarrow{\rho_f[n]} E^\bullet(R^d f_* \Omega^{n+d}_{X/k}[n]) \xrightarrow{\int_f} E^\bullet(\Omega^n_{Y/k}[n]).$$

The definition of $\rho_f$ along with [23], Proposition (4.2.2) and 5.2 (iv) gives the result in this case.

**Case 3.** Suppose $f\colon X \to Y$ is finite and surjective. First, assume $Y$ is projective, (so that $X$ is also projective). Let $g\colon Y \to \mathbb{P}^n$ and $h\colon X \to \mathbb{P}^m$ be closed immersions. Set $Z = \mathbb{P}^n \times_k \mathbb{P}^m$. Then we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & W & \xrightarrow{g'} & Z \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ & & Y & \xrightarrow{g} & \mathbb{P}^n \end{array}$$

with $p\colon Z \to \mathbb{P}^n$ the canonical projection, the square cartesian, and $g' \circ i = h$. We already know that $\theta_i$, $\theta_{g'}$, $\theta_p$ and $\theta_g$ are morphisms of complexes. Now $\theta_f = \theta_q \circ q_*\theta_i$, and hence it is enough to prove that $\theta_q$ is a map of complexes. Note that $\theta_g \circ g_*\theta_q = \theta_p \circ p_*\theta_{g'}$, and the right side is a map of complexes. Since $\theta_g^j\colon g_*\mathcal{K}_Y^j \to \mathcal{K}_{\mathbb{P}^n}^j$ is an inclusion and $g$ is an affine map, it follows that $\theta_q$ is a map of complexes, and hence so is $\theta_f$ in this case. Now assume $Y$ is quasi-projective instead of projective. Let $j\colon Y \to \bar{Y}$ be a projective compactification of $Y$. By Zariski's Main Theorem we can find a cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\ j'\ } & \bar{X} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \bar{f}} \\ Y & \xrightarrow{\ j\ } & \bar{Y} \end{array}$$

with the horizontal arrows being open immersions, and with $\bar{f}$ finite. It is easy to see that $\theta_f = j^*\theta_{\bar{f}}$, and hence $\theta_f$ is a map of complexes.

Now let $f\colon X \to Y$ be an arbitrary finite surjective map. Since $Y$ can be covered locally by quasi-projectives, and since the question is local, therefore, by the above arguments, $\theta_f$ is a map of complexes.

**Case 4.** We now prove the Proposition for a general proper map $f\colon X \to Y$. Let $Y'$ be the scheme theoretic image of $f$, and $X \xrightarrow{f'} Y' \xrightarrow{i} Y$ the resulting factorization of $f$. One checks that $\theta_f = \theta_i \circ i_*\theta_{f'}$. By case 1, $\theta_i$ is a map of complexes, and hence we are reduced to showing that $\theta_{f'}$ is a map of complexes.

So we will assume, without loss of generality, that $f\colon X \to Y$ as in the proposition is *surjective*. Let $x \in X$, $y = f(x)$, $y' \in Y$, $\Lambda = \{x' \in f^{-1}(y') \mid x \mapsto x'\}$. We have to show that

$$\sum_{x' \in \Lambda} \theta_{f,x'} \circ \sum_{x' \in \Lambda} \delta(x, x') = \delta(y, y') \circ \theta_{f,x} \tag{8.4.1}$$

where $\delta(y, y') = 0$ if $y'$ is not an immediate specialization of $y$. We do this in two stages.

First assume $x \in X$ is closed in its fibre. If $y'$ is not an immediate specialization of $y$, then by reasons of codimension (cf. [7], p. 333, Proposition (3.4)), none of the $x' \in \Lambda$ are closed in their fibres and so 8.4.1 trivially holds. So assume $y \mapsto y'$. Then by loc. cit., all $x' \in \Lambda$ are closed in their fibre. Let $\mathfrak{p} \subset \mathcal{O}_X$ be the ideal sheaf of $\{x\}^-$. Let $Z_n$ be the closed subscheme of $X$ given by $\mathfrak{p}^n$ and $i_n\colon Z_n \to X$ the inclusion. Let $g_n\colon Z_n \to Y$ be the composition $f \circ i_n$. By [5], 4.4.11, there is an open neighbourhood $U$ of $y'$ such that the restriction of $f$ to $g_n^{-1}(U)$ is a finite morphism into $U$. A little thought shows that $U$ is independent of $n$. By replacing $Y$ by $U$ if necessary, we may assume that $g_n\colon Z_n \to Y$ is finite. Let $x_n \in Z_n$ be the point corresponding to $x \in X$, and $\Lambda_n = \{x'_n \in g_n^{-1}(y') | \, x_n \mapsto x_n\}'$. Since $g_n \colon Z_n \to Y$ is finite, the previous cases give

$$\sum_{x_n \in \Lambda_n} \theta_{g_n, x'_n} \circ \sum_{x'_n \in \Lambda_n} \delta(x_n, x'_n) = \delta(y, y') \circ \theta_{g_n, x_n}.$$

8.4.1 follows by passing to the direct limit.

Now suppose $x \in X$ is not closed in its fibre. We are reduced to showing:

$$\sum_{x' \in \Lambda} \theta_{f, x'} \circ \sum_{x' \in \Lambda} \delta(x, x') = 0. \tag{8.4.2}$$

If $y' \neq y$, then no $x' \in \Lambda$ is closed in $f^{-1}(y')$ (apply [7], p. 333, (3.4)), and hence 8.4.2 holds. So assume $y' = y$. If $\Delta(x) \neq \Delta(y) - 1$, then again by loc.cit. – no $x' \in \Lambda$ is closed in its fibre, and hence 8.4.2 holds. So we are reduced to the case where $y' = y$ and $\Delta(x) = \Delta(y) - 1$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of $\{x\}^-$ (with reduced structure). Let $V_n$ be the closed subscheme of $X$ defined by $\mathcal{I}^n$. We have a commutative diagram

$$\begin{array}{ccc} V_n & \xrightarrow{\;i_n\;} & X \\ {\scriptstyle g_n}\downarrow & & \downarrow{\scriptstyle f} \\ W_n & \xrightarrow{\;j_n\;} & Y \end{array}$$

where $W_n$ is the scheme theoretic image of $V_n$ in $Y$, and $i_n, j_n, g_n$ are the induced maps. Since $\Delta(x) = \Delta(y) - 1$, the map $g_n \colon V_n \to W_n$ satisfies the hypothesis of Lemma 8.3. If $v_n \in V_n$ is its generic point, $w_n = g_n(v_n)$, then 8.3 gives

$$\sum \theta_{g_n, v'} \circ \sum \delta(v_n, v') = 0$$

where the sum is over $v' \in g_n^{-1}(w_n)$, with $v_n \mapsto v'$. Passing to the direct limit (as $n \to \infty$), we get 8.4.2. □

8.5. REMARKS

1. If $Y$ is Spec $k$ in the Proposition, then identifying $\mathcal{K}(k)$ with $k$ in the obvious way (i.e., via $t_\sigma$ where $\sigma \colon k \to k$ is the identity map), we have $\theta_f = \theta_X$.
2. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a pair of proper morphisms. Then $\theta_g \circ g_* \theta_f = \theta_{gf}$. In particular, if $Y$ is proper over $k$, then $\theta_Y \circ \Gamma(Y, \theta_f) = \theta_X$.

For $f \colon X \to Y$ a proper map of algebraic schemes, let $(f^!, \int_f)$ be as in [30], Theorem 1. This pair is unique up to unique isomorphism. Clearly if $Y = \text{Spec}\, k$, then we can identify $(f^! k, \int_f k)$ with the pair $(Q\mathcal{K}_X^\bullet, Q\theta_X)$ where $Q$ denotes both localization functors $K(X) \to D(X)$ and $K(Y) \to D(Y)$. This is the content of

Theorem 7.2. This has the following generalization (we are identifying $Q\mathcal{K}^\bullet_X$ with $\mathcal{K}^\bullet_X$, $Q\mathcal{K}^\bullet_Y$ with $\mathcal{K}^\bullet_Y$ $\mathbb{R}f_*\mathcal{K}^\bullet_X$ with $f_*\mathcal{K}^\bullet_X$, and $Q\theta_f$ with $\theta_f$).

THEOREM 8.6. *Let $f\colon X\to Y$ be a proper morphism of algebraic $k$-schemes. The pair $(f^!\mathcal{K}^\bullet_Y, \int_f \mathcal{K}^\bullet_Y)$ is isomorphic to the pair $(\mathcal{K}^\bullet_X, \theta_f)$.*

*Proof.* If $Y$ is proper, the theorem follows from the identity $\theta_Y \circ \Gamma(Y, \theta_f) = \theta_X$ and the universal properties of $\theta_X$ and $\theta_Y$. In general, one can *compactify* the map $f\colon X\to Y$, i.e. get a cartesian square:

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & \bar{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ Y & \xrightarrow{\ j\ } & \bar{Y} \end{array}$$

with $\bar{X}, \bar{Y}$ proper over $k$ and $i, j$ open immersions (cf. [21], p. 50). Since $\theta_f = j^*\theta_{\bar{f}}$, the theorem follows from [30], Theorem 2. □

8.7. REMARK. From the construction of $\theta_f$ for a proper map $f\colon V\to W$ of algebraic $k$-schemes, it is clear that $(\mathcal{K}^\bullet, \{\theta_V\})$ gives a dualizing structure on the $\mathcal{O}$-module $\mathcal{K}^\bullet$ (defined on the Zariski site on the category of algebraic $k$-schemes) (cf. 0.1 in the Introduction).

## 9. Connections with Yekutieli's complex

In this section we show that for $X$ equidimensional and reduced, $\mathcal{K}^\bullet_X$ is essentially the complex constructed by Yekutieli in [31] (cf. 9.2 below). We will work over $\mathcal{V}^{\text{red}}$ – the category of reduced, equidimensional, algebraic $k$-schemes and we point out that the results of [29] are valid in this category (cf. Remark 0.2.12 of *ibid.*)[5]. We deduce a relationship between *regular differential* forms of the top degree and $\mathcal{K}^\bullet_X$, analogous to Theorem 0.2.2 of [29] (Theorem 9.3 below). We assume familiarity with the language of $\mathcal{O}$-modules as laid out in [21], pp. 28–30.

By Remark 6.3, we see that $\{\mathcal{K}^\bullet_X\colon X \in \mathcal{V}^{\text{red}}\}$ is a Zariski sheaf on $\mathcal{V}^{\text{red}}$. We denote this Zariski sheaf $\mathcal{K}^\bullet$. Moreover, one checks that the data $(\mathcal{K}^\bullet, \{\theta_X\}, \{\gamma_X\}, \{\theta_f\})$ gives a *residue complex* on $\mathcal{V}^{\text{red}}$ (in the sense of [29] (1.4)), where $\gamma_X$, for $X$ smooth, is defined as follows: If $n = \dim X$, we identify $\mathcal{K}^\bullet_X$ with $E^\bullet(\Omega^n_{X/k}[n])$ (via $\phi_x$, $x \in X$), whence we have a natural quasi-isomorphism $\Omega^n_{X/k}[n]\to\mathcal{K}^\bullet_X$. We denote this quasi-isomorphism by $\gamma_X$.

9.1. REMARK. If $(\mathcal{C}^\bullet, \{(-1)^{\dim X}\mathrm{Tr}_X\}, \{C_X\}, \{\mathrm{Tr}_f\})$ is the *Yekutieli residue complex* on $\mathcal{V}^{\text{red}}$, i.e. the residue complex of [29], Theorem 0.2.2 , then for $X \in \mathcal{V}^{\text{red}}$,

[5] However in the definition of a canonical structure in *ibid.*,1.3, care must be taken to identify $\theta_f$ for all finite dominant maps (not just generically étale ones). This is done by using the trace in [20]§16. The generalization is routine and straightforward.

and $x \in X$, we can identify $\mathcal{K}(x)$ with $\mathcal{C}(x)$. This is an obvious consequence of 4.7 and the definition of $\mathcal{K}(x)$. Thus $\mathcal{K}_X^p = \mathcal{C}_X^p$ for $p \in \mathbb{Z}$.

By [29], (1.2.6), (1.3.4) and (1.4.3) we have a unique isomorphism of complexes of $\mathcal{O}$-modules

$$\lambda: \mathcal{K}^\bullet \xrightarrow{\sim} \mathcal{C}^\bullet$$

which preserves the dualizing and canonical structures of $\mathcal{K}^\bullet$ and $\mathcal{C}^\bullet$. The main theorem of this section is

THEOREM 9.2. *For $X \in \mathcal{V}^{\text{red}}$, the isomorphism*

$$\lambda_X: \mathcal{K}_X^\bullet \to \mathcal{C}_X^\bullet$$

*satisfies*

$$\lambda_X^p = (-1)^{p(\dim X)}\, 1_{\mathcal{K}_X^p}$$

*for each $p \in \mathbb{Z}$, where $1_{\mathcal{K}_X^p}$ is the identity map on $\mathcal{K}_X^p = \mathcal{C}_X^p$.*

*Proof.* If $X = \mathbb{P}^d$, the projective space of dimension $d$ over $k$, the theorem follows from [29], (0.2.11).

It is enough to prove the proposition for $X$ projective, for then it would be true for $X$ quasi-projective, and hence for all $X$ (by finding an open cover by quasi-projectives). Let $f: X \to \mathbb{P}^d$ be a noether normalization ($d = \dim X$) comparing the definitions of $\theta_f$ in the beginning of Section 8 with the definitions of $\text{Tr}_f$ in [31], 4.4.11(a), we see that for $p \in \mathbb{Z}$, $\theta_f^p = \text{Tr}_f^p$. On the other hand the diagram

$$\begin{array}{ccc} f_*\mathcal{K}_X^\bullet & \xrightarrow{f_*\lambda_X} & f_*\mathcal{C}_X^\bullet \\ \Big\downarrow{\scriptstyle \theta_f} & & \Big\downarrow{\scriptstyle \text{Tr}_f} \\ \mathcal{K}_{\mathbb{P}^d}^\bullet & \xrightarrow{\lambda_{\mathbb{P}^d}} & \mathcal{C}_{\mathbb{P}^d}^\bullet \end{array}$$

commutes. This implies that for $x \in X$, with $\dim\{\bar{x}\} = -p$, and $y = f(x)$, the diagram of $R = \widehat{\mathcal{O}}_{\mathbb{P}^d,y}$-modules

$$\begin{array}{ccc} \mathcal{K}(x) & \xrightarrow{\lambda(x)} & \mathcal{K}(x) \\ \Big\downarrow{\scriptstyle \theta_{f,x}} & & \Big\downarrow{\scriptstyle \theta_{f,x}} \\ \mathcal{K}(y) & \xrightarrow{(-1)^{pd}} & \mathcal{K}(y) \end{array}$$

commutes. The assertion follows from the fact that, with $S = \widehat{\mathcal{O}}_{X,x}$, the natural map

$$\mathrm{Hom}_S(\mathcal{K}(x), \mathcal{K}(x)) \to \mathrm{Hom}_R(\mathcal{K}(x), \mathcal{K}(y))$$

given by $\theta_{f,y}$, is an isomorphism.

Since $\mathcal{K}^\bullet$ has a canonical structure, the $\mathcal{O}$-module $\omega = H^{-\dim}(\mathcal{K}^\bullet)$ is a canonical $\mathcal{O}$-module in the sense of [21] pp. 32–33 (where dim is the 'dimension sheaf' on $\mathcal{V}^{\mathrm{red}}$). For $X \in \mathcal{V}^{\mathrm{red}}$ if we identify $\omega_X$ with a subsheaf of $\Omega^{\dim X}_{k(X)/k}$ via the map $\gamma_{X^{\mathrm{sm}}}: \Omega^{\dim X}_{X^{\mathrm{sm}}} \to \mathcal{K}^\bullet_X$ (where $X^{\mathrm{sm}}$ is the smooth locus of $X$), then clearly $\omega_X = \tilde{\omega}_X$ – the sheaf of regular differential forms of highest degree on $X$. The map $\gamma_{X^{\mathrm{sm}}}$ extends to give a map $\gamma_X: \tilde{\omega}_X[\dim X] \to \mathcal{K}^\bullet_X$. In fact we get a map of complexes of $\mathcal{O}$-modules

$$\gamma: \tilde{\omega}[\dim] \to \mathcal{K}^\bullet.$$

For $x \in X$, $\sigma \in C_x$, let $\mathrm{res}_\sigma: H^d_x(\tilde{\omega}_X) \to \omega(\sigma)$ be the residue map in [14] (1.1), where $d = \dim \widetilde{\mathcal{O}}_{X,x}$. Theorem 9.2 and [29] 0.2.11 gives the following result (compare with 5.2 (iv)):

THEOREM 9.3.

$$t_\sigma \circ H^d_x(\gamma_X) = \mathrm{res}_\sigma.$$

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