

COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 101, n° 2 (1996), p. 115-131

http://www.numdam.org/item?id=CM_1996__101_2_115_0

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Dimensions of Demazure modules for rank two affine Lie algebras

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Received 2 August 1994; accepted in final form 6 February 1995

Abstract. Using the path model for highest weight representations of Kac-Moody algebras, we calculate the dimension of all Demazure modules of $A_1^{(1)}$ and $A_2^{(2)}$.

1. Introduction

Let \mathfrak{g} be a symmetrizable Kac-Moody algebra. Recall that for every dominant weight λ , there exists a unique (up to isomorphism) irreducible highest weight module $V = V(\lambda)$ of highest weight λ . In order to gain some insight into the structures of these modules, one can study their characters. Recall that these are formal sums $\chi(V) = \sum (\dim V_\mu) e^\mu$ over all weights μ and where V_μ is the weight space of V of weight μ . Another reason to study character formulas is because of their connection with combinatorial identities. For example, the Rogers-Ramanujan and Euler identities have been derived from specializing the characters of certain level 2 and level 3 highest weight modules for $A_1^{(1)}$ (see [LW]).

For symmetrizable Kac-Moody algebras, character formulas for all highest weight modules exist, the most well-known being Weyl's formula, due to Kac [Ka]. Unfortunately, none of these expressions are combinatorial (they contain many + and – signs) making it difficult to obtain from them a general formula for $\dim V_\mu$ for an arbitrary weight μ .

There is an analogous problem for Demazure modules. Recall that for every element w of the Weyl group, the weight space $V_{w(\lambda)}$ is of dimension one. Let $E_w(\lambda)$ denote the \mathfrak{b} -module generated by $V_{w(\lambda)}$, where \mathfrak{b} is the Borel subalgebra of \mathfrak{g} . The finite-dimensional vector spaces $E_w(\lambda)$ are called Demazure modules and form a filtration of V . These modules often occur in inductive proofs and are connected with the Schubert varieties of the Kac-Moody algebra. Character formulas also exist for Demazure modules [D], [Ku], [M], but they are not combinatorial. So far, very little is known about them. In particular, their dimensions have not been determined except in isolated cases.

In this paper, we determine explicit closed form expressions for the dimensions of all Demazure modules for $A_1^{(1)}$ and $A_2^{(2)}$. Let α_0 and α_1 (respectively $\alpha_0^\vee, \alpha_1^\vee$)

be the two simple roots (resp. coroots). Let Λ_0 and Λ_1 be the two fundamental weights defined by $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$. In the case $\mathfrak{g} = A_2^{(2)}$, we choose the α_i such that $\langle \alpha_0, \alpha_1^\vee \rangle = -1$ and $\langle \alpha_1, \alpha_0^\vee \rangle = -4$. Let r_i be the reflection with respect to α_i . The Weyl group W is generated by the r_i . For every $n > 0$, W contains two elements of length n

$$w_n^+ := r_{i_n} \cdots r_{i_2} r_{i_1} \text{ where } i_j = j + 1 \pmod 2$$

$$w_n^- := r_{i_n} \cdots r_{i_2} r_{i_1} \text{ where } i_j = j \pmod 2$$

THEOREM 1. *Let $\lambda = s\Lambda_0 + t\Lambda_1$.*

(1.1) *If $\mathfrak{g} = A_1^{(1)}$, then:*

$$\dim E_w(\lambda) = \begin{cases} (s + 1)(s + t + 1)^{n-1} & \text{if } w = w_n^+ \text{ for some } n \geq 1; \\ (t + 1)(s + t + 1)^{n-1} & \text{if } w = w_n^- \text{ for some } n \geq 1. \end{cases}$$

(1.2) *If $\mathfrak{g} = A_2^{(2)}$, then:*

$\dim E_w(\lambda) =$

$$\begin{cases} 2^{-n}(s + 1)(2t + s + 1)^n(2t + s + 2)^n & \text{if } w = w_{2n+1}^+ \text{ for some } n \geq 0; \\ 2^{-n}(s + 1)(2t + s + 1)^{n-1}(2t + s + 2)^n & \text{if } w = w_{2n}^+ \text{ for some } n \geq 1; \\ 2^{-n}(t + 1)(2t + s + 1)^n(2t + s + 2)^n & \text{if } w = w_{2n+1}^- \text{ for some } n \geq 0; \\ 2^{-n+1}(t + 1)(2t + s + 1)^n(2t + s + 2)^{n-1} & \text{if } w = w_{2n}^- \text{ for some } n \geq 1. \end{cases}$$

This result was known previously for the case $\mathfrak{g} = A_1^{(1)}$ when $\lambda = \Lambda_0$ or Λ_1 , [LS]. The main tool used in establishing these formulas is Littelmann’s path model for highest weight representations [L1]. The path model is a recently discovered combinatorial parametrization for a base for integrable modules. In Littelmann’s work, the base is parametrized by certain piecewise linear paths whose images lie in \mathfrak{h}^* , where \mathfrak{h} is the Cartan subalgebra of our Kac-Moody algebra. A brief summary of this theory is given in the Section 3. In this paper, we start off with a specific type of path, called Lakshmibai-Seshadri (or L-S) paths, defined in [L1]. By working from the definition of L-S paths, we obtain an explicit description of all such paths for the basic modules. This approach, however, is limited to the basic modules. When λ is other than a fundamental weight, the calculations needed to explicitly describe L-S paths quickly become unwieldy, if not impossible. The idea that we use to bypass this problem is the following: for any highest weight $\lambda = s\Lambda_0 + t\Lambda_1$, start off with the path

$$\pi = \underbrace{\pi_{\Lambda_0} * \cdots * \pi_{\Lambda_0}}_s * \underbrace{\pi_{\Lambda_1} * \cdots * \pi_{\Lambda_1}}_t = \pi_{s\Lambda_0} * \pi_{t\Lambda_1}.$$

The path π is a concatenation of L-S paths representing the fundamental weights. We then obtain an explicit description of the set of paths in the associated path model. These paths will also be concatenations of L-S paths. This description will then allow us to describe $\dim E_w(\lambda)$ as an induction on w and λ . In the case of $A_1^{(1)}$, the parametrization obtained with this method coincides with that given by standard monomial theory [LS]. Although the proofs for these explicit descriptions are only given for the rank two affine cases, they are not dependent on the algebras being affine and it is easy to see that they would generalize to the case of higher ranks.

2. Preliminaries

We recall a few basic facts about Kac-Moody algebras. We follow the notations in [Ka]. Let \mathfrak{g} be a complex symmetrizable Kac-Moody algebra of rank n . As an algebra, \mathfrak{g} is generated by e_i, \mathfrak{h} , and f_i for $0 \leq i \leq n - 1$ where \mathfrak{h} is the Cartan subalgebra of \mathfrak{g} . Let \mathfrak{b} be the Borel subalgebra of \mathfrak{g} . We have that $\mathfrak{h} \supseteq \bigoplus_{i=0}^{n-1} \mathbb{C}\alpha_i^\vee$ and that \mathfrak{b} is the subalgebra generated by \mathfrak{h} and the e_i . For $0 \leq i \leq n - 1$, we denote the simple roots by α_i . The α_i are the elements of \mathfrak{h}^* defined by the relation $[h, e_i] = \langle \alpha_i, h \rangle e_i$ for all $h \in \mathfrak{h}$. In addition, for $0 \leq i \leq n - 1$, we denote the fundamental weights by Λ_i . These are elements of \mathfrak{h}^* defined by the relations $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$. For all i , let r_i be the reflection of \mathfrak{h}^* with respect to α_i . By definition, the Weyl group W is generated by the r_i .

Let $P_+ = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}\}$ denote the set of all dominant weights. For every $\lambda \in P_+$, there exists a unique (up to isomorphism) highest weight module $V = V(\lambda)$ of highest weight λ . By definition, $\mu \in \mathfrak{h}^*$ is called a weight if the corresponding weight space $V_\mu = \{v \in V \mid h \cdot v = \langle \mu, h \rangle v \ \forall h \in \mathfrak{h}\}$ is non-zero. We have $V = \bigoplus V_\mu$ where the sum is over all possible weights. For every $w \in W$, the weight space $V_{w(\lambda)}$ is of dimension 1.

Let $E_w(\lambda)$ be the \mathfrak{b} -module generated by $V_{w(\lambda)}$. The $E_w(\lambda)$ are called Demazure modules. They are finite-dimensional subspaces of V that satisfy the following property: for any $w, w' \in W$ such that $w \leq w'$ (where \leq is the Bruhat order on W), we have $E_w(\lambda) \subseteq E_{w'}(\lambda)$. In addition, $\bigcup_{w \in W} E_w(\lambda) = V$.

3. The path model

The path model, introduced in [L1], [L2], gives a combinatorial parametrization of the basis vectors of a highest weight module for a Kac-Moody algebra. This parametrization is in the form of certain piecewise linear paths $\pi: [0, 1] \rightarrow \mathfrak{h}^*$. We now give a brief synopsis of those parts of the theory necessary for this paper. (See [L1], [L2] for more details).

3.1. GENERAL THEORY

Let Π denote the set of all continuous piecewise linear paths $\pi : [0, 1] \rightarrow \mathfrak{h}^*$ with $\pi(0) = 0$ and $\pi(1)$ a weight. We identify any two such paths if their images coincide. To every simple root α_i , we associate linear operators $e_i, f_i : \Pi \rightarrow \Pi$. The e_i, f_i , which we call root operators, act in the following way. Let $\tau \in \Pi$ be a path. According to the behavior of the function $t \mapsto \langle \tau(t), \alpha_i^\vee \rangle$, either $f_i\tau$ is undefined or $f_i\tau$ is a new path in Π such that $f_i\tau(1) = \tau(1) - \alpha_i$. Likewise, either $e_i\tau$ is undefined or $e_i\tau$ is in Π with $e_i\tau(1) = \tau(1) + \alpha_i$. Before we give a more detailed description of their action, we will need the following definitions.

For any two paths $\tau_1, \tau_2 \in \Pi$, we denote by $\tau_1 * \tau_2$ their concatenation,

$$(\tau_1 * \tau_2)(t) := \begin{cases} \tau_1(2t) & \text{for } t \in [0, 1/2] \\ \tau_1(1) + \tau_2(2t - 1) & \text{for } t \in [1/2, 1] \end{cases}.$$

For any path $\tau \in \Pi$ and any $r_i \in W$, we define the path $(r_i\tau)(t) := r_i(\tau(t))$. Let $m_i(\tau) := \min_{t \in [0, 1]} \langle \tau(t), \alpha_i^\vee \rangle$.

We can now describe the action of f_i . Let $p \in [0, 1]$ be maximal such that $\langle \tau(p), \alpha_i^\vee \rangle = m_i(\tau)$. If $\langle \tau(1), \alpha_i^\vee \rangle - m_i(\tau) < 1$, then $f_i\tau$ is undefined. If not, we do the following. Let $x \in [p, 1]$ be minimal such that $\langle \tau(x), \alpha_i^\vee \rangle = m_i(\tau) + 1$. We now ‘cut’ τ into three parts τ_1, τ_2 and τ_3 . Each of these parts is a path in Π and they are defined by:

$$\begin{aligned} \tau_1(t) &:= \tau(tp); & \tau_2(t) &:= \tau(p + t(x - p)) - \tau(p); \\ \tau_3(t) &:= \tau(x + t(1 - x)) - \tau(x). \end{aligned}$$

Notice that $\tau = \tau_1 * \tau_2 * \tau_3$. We define $f_i\tau := \tau_1 * r_i\tau_2 * \tau_3$.

We define e_i similarly. Let $q \in [0, 1]$ be minimal such that $\langle \tau(q), \alpha_i^\vee \rangle = m_i(\tau)$. If $-m_i(\tau) < 1$, then $e_i\tau$ is undefined. If not, we do the following. Let $y \in [0, q]$ be maximal such that $\langle \tau(y), \alpha_i^\vee \rangle = m_i(\tau) + 1$. We now ‘cut’ τ into three parts τ_1, τ_2 and τ_3 . Each of these parts is a path in Π and they are defined by:

$$\begin{aligned} \tau_1(t) &:= \tau(ty); & \tau_2(t) &:= \tau(y + t(q - y)) - \tau(y); \\ \tau_3(t) &:= \tau(q + t(1 - q)) - \tau(q). \end{aligned}$$

Notice that $\tau = \tau_1 * \tau_2 * \tau_3$. We define $e_i\tau := \tau_1 * r_i\tau_2 * \tau_3$.

These definitions of the root operators are those found in [L1]. A second, more general and slightly more complicated, definition is given in [L2]. This second definition allows the path model to include a wider class of paths. However, for those paths that we will be using (concatenations of Lakshmibai-Seshadri paths, defined below) these two definitions coincide. Therefore, we only give the simpler version.

With these two root operators we can define an operator D_i on $\mathbb{Z}[\Pi]$, the free \mathbb{Z} -module with basis Π

$$D_i \tau = \begin{cases} \tau + f_i \tau + \dots + f_i^n \tau & \text{if } n := \langle \tau(1), \alpha_i^\vee \rangle \geq 0 \\ \text{undefined} & \text{if } \langle \tau(1), \alpha_i^\vee \rangle = -1 \\ -e_i \tau - \dots - e_i^{-n-1} \tau & \text{if } n := \langle \tau(1), \alpha_i^\vee \rangle \leq -2 \end{cases}$$

Choose a path $\pi \in \Pi$ whose image lies in the fundamental chamber and whose endpoint $\pi(1)$ is a dominant weight λ . Such a path, π , is called a dominant path. Let $B\pi = \{f_{i_1} \dots f_{i_k} \pi \mid i_j \in [0, \dots, n-1], k \geq 0\}$ be the set of all possible paths that one obtains by applying the root operators to π . We call $B\pi$ the path model associated to π . We then have:

THEOREM 2. [L1]. $\chi(V) = \sum_{\tau \in B\pi} e^{\tau(1)}$.

We also have an analogue for Demazure modules. Let $w = r_{j_1} \dots r_{j_k}$ be a reduced decomposition of w . Let $P_w \pi = D_{j_1} \dots D_{j_k} \pi = \{f_{j_1}^{i_1} \dots f_{j_k}^{i_k} \pi \mid i_1, \dots, i_k \in \mathbb{Z}_{\geq 0}\}$. Then

THEOREM 3. [L1]. $\chi(E_w(\lambda)) = \sum_{\tau \in P_w \pi} e^{\tau(1)}$.

From this description, we see that, once we have chosen a dominant path π , the problem is to characterize all of the paths contained in $B\pi$ (or $P_w \pi$). In general, this is not an easy task. However, for one particular type of dominant path, this has already been done for us [L1]. The dominant path in this case is the straight path $\pi_\lambda(t) = t\lambda$ from 0 to λ . The paths in the associated path model are called Lakshmibai-Seshadri paths.

3.2. LAKSHMIBAI-SESHADRI PATHS

For a given $\lambda \in P^+$, we choose as dominant path $\pi_\lambda(t) = t\lambda$. The elements in $B\pi_\lambda$ are called Lakshmibai-Seshadri (or L-S) paths of shape λ . These paths can be characterized as follows. There is a bijection between paths $\tau \in B\pi_\lambda$ and pairs of sequences $(\underline{\sigma}, \underline{a})$ that satisfy the following conditions:

$$\begin{aligned} \underline{\sigma} : \sigma_1 > \dots > \sigma_n \quad \sigma_i \in W/W_\lambda \quad \forall i, \\ \underline{a} : 0 < a_1 < \dots < a_n = 1 \quad a_i \in \mathbb{Q} \quad \forall i, \end{aligned}$$

where W_λ is the stabilizer of λ in W and where $>$ is the relative Bruhat order. In addition, we require that for every $i \in [1, \dots, n-1]$, there exists an a_i -chain for the pair (σ_i, σ_{i+1}) . Recall that, by definition [L1], the existence of an a -chain for a pair (σ, σ') of cosets in W/W_λ means that there exists a sequence of cosets in W/W_λ

$$\kappa_0 := \sigma > \kappa_1 := r_{\beta_1} \sigma > \kappa_2 := r_{\beta_2} r_{\beta_1} \sigma > \dots > \kappa_s := r_{\beta_s} \dots r_{\beta_1} \sigma = \sigma'$$

where the r_{β_i} are the reflections with respect to the positive real root β_i , such that for all $i \in [1, \dots, s]$

$$l(\kappa_i) = l(\kappa_{i-1}) - 1 \quad \text{and} \quad a\langle \kappa_i(\lambda), \beta_i^\vee \rangle \in \mathbb{Z}.$$

We shall also refer to such a pair $(\underline{\sigma}, \underline{a})$ as Lakshmibai-Seshadri (or L-S). For any such pair, the corresponding path is

$$\sigma(t) := \sum_{i=1}^{j-1} (a_i - a_{i-1})\sigma_i(\lambda) + (t - a_{j-1})\sigma_j(\lambda) \quad \text{for } a_{j-1} \leq t \leq a_j.$$

Finally, we will need

THEOREM 4. [L1]. $P_w \pi_\lambda = \{(\underline{\sigma}, \underline{a}) \text{ L-S} \mid \sigma_1 \leq w\}$.

4. Description of L-S paths for the basic modules

From here on, we assume that \mathfrak{g} is isomorphic to $A_1^{(1)}$ or to $A_2^{(2)}$. In this section, we will work directly from the definition of L-S pairs in order to obtain an explicit characterization of all such pairs. Note that

$$\begin{aligned} W/W_{\Lambda_0} &= \{w_n^+ := r_{i_n} \cdots r_{i_2} r_{i_1} \mid n \in \mathbb{N} \text{ and } i_j = j + 1 \pmod{2}\}, \\ W/W_{\Lambda_1} &= \{w_n^- := r_{i_n} \cdots r_{i_2} r_{i_1} \mid n \in \mathbb{N} \text{ and } i_j = j \pmod{2}\}. \end{aligned}$$

On each of these two coset spaces, the Bruhat order is a total ordering. For $\varepsilon \in \{-, +\}$, $w_n^\varepsilon > w_m^\varepsilon \Leftrightarrow n > m$. For $n \geq 0$, set:

$$\begin{aligned} d_{2n}^+ &:= \langle w_{2n}^+(\Lambda_0), \alpha_1^\vee \rangle = \langle \alpha_0, \alpha_1^\vee \rangle n, \\ d_{2n+1}^+ &:= \langle w_{2n+1}^+(\Lambda_0), \alpha_0^\vee \rangle = -(2n + 1), \\ d_{2n}^- &:= \langle w_{2n}^-(\Lambda_1), \alpha_0^\vee \rangle = \langle \alpha_1, \alpha_0^\vee \rangle n, \\ d_{2n+1}^- &:= \langle w_{2n+1}^-(\Lambda_1), \alpha_1^\vee \rangle = -(2n + 1). \end{aligned}$$

In the case of $A_1^{(1)}$, $\langle \alpha_0, \alpha_1^\vee \rangle = \langle \alpha_1, \alpha_0^\vee \rangle = -2$. In the case of $A_2^{(2)}$, to fix notation, we choose the α_i such that $\langle \alpha_0, \alpha_1^\vee \rangle = -1$ and $\langle \alpha_1, \alpha_0^\vee \rangle = -4$.

LEMMA 1. *Let Λ be a fundamental weight. The L-S paths π of shape Λ are those paths $\pi = (\underline{\sigma}, \underline{a})$ such that*

$$\underline{\sigma} : w_{n+k}^\varepsilon > w_{n+k-1}^\varepsilon > w_{n+k-2}^\varepsilon > \cdots > w_n^\varepsilon \quad n, k \geq 0,$$

$$\underline{a} : 0 < a_{n+k}^\varepsilon < a_{n+k-1}^\varepsilon < \cdots < a_{n+1}^\varepsilon < 1,$$

where $\varepsilon = +$ if $\Lambda = \Lambda_0$, $\varepsilon = -$ if $\Lambda = \Lambda_1$ and where $a_j^\varepsilon \cdot d_j^\varepsilon \in \mathbb{Z}$.

Proof. Clearly, π is a path of shape Λ . We need to show that the chain condition is satisfied by these paths only. Notice that d_j^ε and d_{j+1}^ε are relatively prime. Therefore, if $a \in \mathbb{Q}$, $0 < a < 1$ is such that $a \cdot d_j^\varepsilon \in \mathbb{Z}$, then $a \cdot d_{j+1}^\varepsilon \notin \mathbb{Z}$. Therefore, π cannot be L-S unless $\underline{\sigma}$ is of the form above. In other words, there can be no ‘skips’ in the sequence of Weyl group elements. The chain condition demands that a_j^ε be as stated above. \square

Lemma 1 shows us that for any given $\underline{\sigma}$, there exists possibly more than one \underline{a} such that $\pi = (\underline{\sigma}, \underline{a})$ is L-S. For any L-S path $\pi = (\underline{\sigma}, \underline{a})$ where $\underline{\sigma} : \sigma_1 > \cdots > \sigma_r$, define $\text{beg}(\pi) := \sigma_1$ and $\text{end}(\pi) := \sigma_r$. For Λ one of the fundamental weights, let

$$p_\Lambda(m, n) := \{ \pi = (\underline{\sigma}, \underline{a}) \text{ L-S of shape } \Lambda \mid \text{beg}(\pi) = w_m^\varepsilon \text{ and } \text{end}(\pi) = w_n^\varepsilon \}.$$

For ease of notation and when there can be no ambiguity concerning which fundamental weight is Λ , we will write $p(m, n)$ instead of $p_\Lambda(m, n)$. In addition we will write $P_n(\pi_{s\Lambda_0})$ for $P_{w_n^+}(\pi_{s\Lambda_0})$ and $P_n(\pi_{s\Lambda_1})$ for $P_{w_n^-}(\pi_{s\Lambda_1})$.

In order to determine the dimensions of any given Demazure module of \mathfrak{g} , we will first calculate $|p_{\Lambda_0}(m, n)|$ and $|p_{\Lambda_1}(m, n)|$ for $m \geq n \geq 0$. This we do for $A_1^{(1)}$ and $A_2^{(2)}$ in Sections 4.1–4.2. Notice that these results will immediately give the dimension of any Demazure module for Λ_0 and Λ_1 because

$$\dim E_{w_n^+}(\Lambda_0) = |P_n(\pi_{\Lambda_0})| = \sum_{0 \leq j \leq i \leq n} |p_{\Lambda_0}(i, j)|$$

and similarly for $\dim E_{w_n^-}(\Lambda_1)$.

4.1. CHARACTERIZATION OF L-S PATHS FOR THE BASIC MODULES OF $A_1^{(1)}$

Because of the symmetric roles of α_0 and α_1 , we need only consider L-S paths of shape Λ_0 . The lemma above states that $\pi = (\underline{\sigma}, \underline{a})$ is in $p_{\Lambda_0}(m, n)$ if and only if

$$\underline{\sigma} : w_m^+ > w_{m-1}^+ > w_{m-2}^+ > \cdots > w_n^+ \quad m \geq n \geq 0,$$

$$\underline{a} : 0 < \frac{i_m}{m} < \frac{i_{m-1}}{m-1} < \cdots < \frac{i_{n+1}}{n+1} < 1,$$

where $i_j \in \{1, \dots, j-1\}$ for $n+1 \leq j \leq m$. A $(m-n)$ -tuple $(i_m, i_{m-1}, \dots, i_{n+1}) \in \mathbf{N}^k$ satisfies these inequalities if and only if $1 \leq i_m \leq i_{m-1} \leq \dots \leq i_{n+1} \leq n$. There are $\binom{m-1}{n-1}$ such sequences. We conclude that for all $m \geq n > 0$, $|p(m, n)| = \binom{m-1}{n-1}$. Moreover $|p(0, 0)| = 1$ and $|p(m, 0)| = 0$ for $m > 0$. Thus it will be convenient to set $\binom{-1}{-1} = 1$ and $\binom{m-1}{-1} = 0$ for $m > 0$.

PROPOSITION 5. *Let $\mathfrak{g} = A_1^{(1)}$ and Λ be a fundamental weight. Then $\dim E_w(\Lambda) = 2^n$ where n is the length of $w \in W/W_\Lambda$.*

Proof. Without loss of generality, we can assume that $\Lambda = \Lambda_0$. Then $P_n(\pi_{\Lambda_0}) = \sum_{0 \leq i, j \leq n} p(i, j)$. Therefore, $\dim E_w(\Lambda_0) = |P_n(\pi_{\Lambda_0})| = \sum_{0 \leq j \leq i \leq n} |p(i, j)| = \sum_{j=0}^n \sum_{i=0}^n \binom{i-1}{j-1} = \sum_{j=0}^n \binom{n}{j} = 2^n$. \square

This result was obtained previously by V. Lakshmibai and C.S. Seshadri, (see [LS], (2) p. 194).

4.2. CHARACTERIZATION OF L-S PATHS FOR THE BASIC MODULES OF $A_2^{(2)}$

We first consider L-S paths of shape Λ_0 . Set $a(m, n) := |p_{\Lambda_0}(m, n)|$.

LEMMA 2. *The $a(m, n)$ satisfy the following recursion relations:*

$$\mathbf{R1.} \quad a(m, 2j) = a(m-2, 2j) + 2a(m-2, 2j-1) + a(m-2, 2j-2);$$

$$\mathbf{R2.} \quad a(m, 2j+1) = a(m-2, 2j) + a(m-2, 2j-1).$$

Proof. Clearly, $a(m, m) = 1$. For $m > n \geq 0$, let $(\underline{\sigma}, \underline{a}) \in p_{\Lambda_0}(m, n)$. By Lemma 1, \underline{a} is of the form

$$\underline{a} : 0 < \frac{i_m}{m} < \frac{i_{m-1}}{m-1} < \dots < \frac{i_{n+1}}{n+1} < 1,$$

where i_j is even whenever j is even. Then $(i_m, \dots, i_{n+1}) \in \mathbf{N}^{m-n}$ satisfies these inequalities if and only if

$$1 \leq i_m \leq i_{m-1} \leq \dots \leq i_{n+1} \leq n.$$

Setting $k_r := i_r + m - r$, we see that this is equivalent to

$$1 \leq k_m < \dots < k_{n+2} < k_{n+1} \leq m-1,$$

where, for j even, $k_j = m \bmod 2$.

If n is even, we claim

$$a(m, n) = \sum_{i=1}^{n/2} (2i-1)a(m-2i, n-2i+2).$$

To obtain this identity, for each $i \in \{1, \dots, n/2\}$, fix $k_{n+2} = m - 2i$. Then the number of possible subsequences (k_m, \dots, k_{n+3}) that satisfy $1 \leq k_m < \dots < k_{n+4} < k_{n+3} \leq k_{n+2} - 1$ (with the same parity requirements) equals $a(m - 2i, n - 2i + 2)$. There are still $2i - 1$ possibilities for k_{n+1} . Now sum over i . If n is odd,

$$a(m, n) = \sum_{i=1}^{(n-1)/2} a(m - 2i, n - 2i + 2).$$

To see this identity, for each $i \in \{1, \dots, (n - 1)/2\}$, fix $k_{n+1} = m - 2i$. Then the number of possible subsequences (k_m, \dots, k_{n+2}) that satisfy $1 \leq k_m < \dots < k_{n+3} < k_{n+2} \leq k_{n+1} - 1$ (with the same parity requirements) equals $a(m - 2i, n - 2i + 2)$. Now sum over i . \square

REMARK. The $a(m, n)$ can be calculated explicitly. For k and j such that $m > n \geq 0$ below,

$$a(2k, 2j) = \sum_{i=0}^{k-j} 2^i \binom{k-j}{i} \binom{k-1}{k-j+i},$$

$$a(2k, 2j+1) = \sum_{i=0}^{k-j-1} 2^i \binom{k-j-1}{i} \binom{k-1}{k-j+i},$$

$$a(2k+1, 2j) = 2 \sum_{i=0}^{k-j} 2^i \binom{k-j}{i} \binom{k}{k-j+i+1},$$

$$a(2k+1, 2j+1) = 2 \sum_{i=0}^{k-j-1} 2^i \binom{k-j-1}{i} \binom{k}{k-j+i+1},$$

where $\binom{x}{y} = 0$ whenever $x < y$ or $y < 0$.

We now consider L-S paths of shape Λ_1 . Set $b(m, n) := |p_{\Lambda_1}(m, n)|$.

LEMMA 3. *The $b(m, n)$ satisfy the following recursion relations:*

R3. $b(m, 2k) = b(m - 2, 2k) + 2b(m - 2, 2k - 1) + b(m - 2, 2k - 2)$

R4. $b(m, 2k + 1) = 3b(m - 2, 2k + 1) + 4b(m - 2, 2k) + b(m - 2, 2k - 1)$

Proof. By Lemma 1, $b(m, n)$ is the cardinality of the set of $(m - n)$ -tuples $(i_m, \dots, i_{n+1}) \in \mathbf{N}^{m-n}$ such that

$$0 < \frac{i_m}{m} < \frac{i_{m-1}}{m-1} < \dots < \frac{i_{n+1}}{n+1} < 2$$

and where i_j is even if j is odd and $n + 1 \leq j \leq m$.

To see what conditions the (i_m, \dots, i_{n+1}) must satisfy we do the following. For each $j \in \{n, \dots, m\}$ look at the subset of these $(m - n)$ -tuples that satisfy

$$0 < \frac{i_m}{m} < \dots < \frac{i_{j+1}}{j+1} \leq 1 < \frac{i_j}{j} < \dots < \frac{i_{n+1}}{n+1} < 2.$$

This is true if and only if

$$\begin{aligned} 1 &\leq i_m \leq i_{m-1} \leq \dots \leq i_{j+1} \leq j+1 \\ &\leq i_j \leq i_{j-1} + 1 \leq i_{j-2} + 2 \leq \dots \leq i_{n+1} + j - n - 1 \leq n + j. \end{aligned}$$

Setting $k_r := i_r + m - r$ for $j + 1 \leq r \leq m$ and $k_r := i_r + m + j - 2r$ for $j \leq r \leq n + 1$, we see that this is equivalent to

$$1 \leq k_m < k_{m-1} < \dots < k_j \leq m < k_{j-1} < \dots < k_{n+1} \leq m + j - 1,$$

where for r odd, $r \geq j + 1$, $k_r = m + 1 \pmod{2}$ and for r odd, $r \leq j$, $k_r = j + m \pmod{2}$. Note that $\sum_{j=n}^m b_j(m, n) = b(m, n)$. We now determine $b_j(m, n)$.

The number of $(m - j)$ -tuples that satisfy

$$1 \leq k_m < k_{m-1} < \dots < k_{j+1} \leq m$$

(with the appropriate parity conditions on k_r for r odd) is $a(m + 1, j + 1)$. The number of $(j - n)$ -tuples that satisfy

$$m < k_j < \dots < k_{n+1} \leq m + j - 1$$

(with the appropriate parity conditions on k_r for r odd) equals $a(j + 1, n + 1)$ if j is odd. If j is even, then it equals

$$\begin{cases} a(j, n) & \text{for } m, n \text{ both even;} \\ a(j, n + 1) + 2a(j, n) & \text{for } m \text{ even, } n \text{ odd;} \\ a(j + 1, n) & \text{for } m \text{ odd, } n \text{ even;} \\ a(j + 1, n + 1) & \text{for } m, n \text{ both odd.} \end{cases}$$

Then $b_j(m, n)$ is just the product of these two cardinalities. The recursion relations are obtained by applying R1–R2 to each summand of $b(m, n) = \sum_{j=n}^m b_j(m, n)$. \square

5. Description of paths for all other modules

When our dominant weight λ is not one of the fundamental weights, it is difficult to calculate the dimensions of the Demazure modules directly by counting the

number of L-S paths of shape λ . Such a path $\pi = (\underline{\sigma}, \underline{a})$ can be one where $\underline{\sigma}$ has ‘skips’ in its sequence of Weyl group elements. For example, $\underline{\sigma}$ could resemble $w_m^\varepsilon > w_{m-2}^\varepsilon > w_{m-3}^\varepsilon > \dots > w_n^\varepsilon$. In addition, if λ is regular, ε is variable. The difficulty with keeping track of such $\underline{\sigma}$ makes counting such L-S paths a real pain.

In order to bypass this problem, we choose a dominant path that is a concatenation of L-S paths for the basic modules. Suppose that $\lambda = s\Lambda_0 + t\Lambda_1$. Set

$$\pi = \underbrace{\pi_{\Lambda_0} * \dots * \pi_{\Lambda_0}}_s * \underbrace{\pi_{\Lambda_1} * \dots * \pi_{\Lambda_1}}_t = \pi_{s\Lambda_0} * \pi_{t\Lambda_1}$$

This path traces a straight path from $\pi(0) = 0$ to $s\Lambda_0$ and then one from $s\Lambda_0$ to $\pi(1) = \lambda$. Note that this is an L-S path if and only if $st = 0$. For two sets S_1, S_2 of paths, let $S_1 * S_2 := \{\pi_1 * \pi_2 \mid \pi_1 \in S_1, \pi_2 \in S_2\}$. There is a natural injection

$$B\pi \hookrightarrow \underbrace{B\pi_{\Lambda_0} * \dots * B\pi_{\Lambda_0}}_s * \underbrace{B\pi_{\Lambda_1} * \dots * B\pi_{\Lambda_1}}_t.$$

The problem then is to determine exactly which paths lie in the image of $B\pi$.

THEOREM 5. *Let $\lambda = s\Lambda_0 + t\Lambda_1 \neq 0$. Let $\pi_1, \dots, \pi_s \in B\pi_{\Lambda_0}$ and let $\zeta_1, \dots, \zeta_t \in B\pi_{\Lambda_1}$. Then $\pi_1 * \dots * \pi_s * \zeta_1 * \dots * \zeta_t \in \mathcal{P}_w(\pi_{s\Lambda_0} * \pi_{t\Lambda_1})$ if and only if*

- C1.** $\text{end}(\pi_i) \geq \text{beg}(\pi_{i+1})$ for all i and $\text{end}(\zeta_j) \geq \text{beg}(\zeta_{j+1})$ for all j .
- C2.** $\text{end}(\pi_s) \cdot r_1 \geq \text{beg}(\zeta_1)$
- C3.** If $s = 0$, then $\text{beg}(\zeta_1) \leq w$. If $s > 0$, then $\text{beg}(\pi_1) \leq w$. In addition, if $\text{end}(\pi_s) \not\geq \text{beg}(\zeta_1)$, then $\text{beg}(\pi_1) \cdot r_1 \leq w$.

Proof. The proof is as follows:

- (1) We first show that for $\pi_1, \dots, \pi_s \in B\pi_{\Lambda_0}$, we have $\pi_1 * \dots * \pi_s \in B\pi_{s\Lambda_0}$ if and only if the π_i satisfy C1. (In fact, the same proof shows that for $\zeta_1, \dots, \zeta_t \in B\pi_{\Lambda_1}$, we have $\zeta_1 * \dots * \zeta_t \in B\pi_{t\Lambda_1}$ if and only if the ζ_i satisfy C1.)
- (2) We then show that for $\pi \in B\pi_{s\Lambda_0}$ and for $\zeta \in B\pi_{t\Lambda_1}$, if $\pi * \zeta \in B(\pi_{s\Lambda_0} * \pi_{t\Lambda_1})$ then $\pi * \zeta$ must satisfy C2.
- (3) Finally, we show that for $\pi \in B\pi_{s\Lambda_0}$ and $\zeta \in B\pi_{t\Lambda_1}$, we have $\pi * \zeta \in P_w(\pi_{s\Lambda_0} * \pi_{t\Lambda_1})$ if and only if $\pi * \zeta$ satisfies C1–C3. This proof is by induction on w and implies the ‘only if’ direction for 2.

Proof of 1.: Any path in $B(\pi_{s\Lambda_0})$ is obviously the concatenation of s paths that satisfy condition C1. We now show that the concatenation of any s paths π_1, \dots, π_s that satisfy C1 is in $B(\pi_{s\Lambda_0})$. We proceed by induction and suppose that we have

shown this for $s - 1$ paths. Let $\pi_1 \in B\pi_{(s-1)\Lambda_0}$ and let $\pi_2 \in B(\pi_{\Lambda_0})$. Being L-S paths, $\pi_1 = (\underline{\sigma}, \underline{a})$ and $\pi_2 = (\underline{\tau}, \underline{b})$ are defined as follows:

$$\begin{aligned} \underline{\sigma} &: \sigma_1 > \cdots > \sigma_r, \\ \underline{a} &: 0 < a_1 < \cdots < a_r = 1, \\ \underline{\tau} &: \tau_1 > \cdots > \tau_q, \\ \underline{b} &: 0 < b_1 < \cdots < b_q = 1. \end{aligned}$$

If $\text{end}(\pi_1) = \text{beg}(\pi_2)$, (in other words, if $\sigma_r = \tau_1$) then define $\pi = (\underline{\kappa}, \underline{c})$ where

$$\begin{aligned} \underline{\kappa} &: \sigma_1 > \cdots > \sigma_r = \tau_1 > \cdots > \tau_q, \\ \underline{c} &: 0 < c_1 < \cdots < c_r < c_{r+1} < \cdots < c_{r+q-1} = 1, \end{aligned}$$

where $a_i = \frac{a_i(s-1)}{s}$ for $1 \leq i < r$ and $a_{i-1} = \frac{b_i+s-1}{s}$ for $r+1 \leq i \leq r+q$.

If $\text{end}(\pi_1) > \text{beg}(\pi_2)$, (in other words, if $\sigma_r > \tau_1$) then define $\pi = (\underline{\kappa}, \underline{c})$ where

$$\begin{aligned} \underline{\kappa} &: \sigma_1 > \cdots > \sigma_r > \tau_1 > \cdots > \tau_{r+q}, \\ \underline{c} &: 0 < c_1 < \cdots < c_r < c_{r+1} < \cdots < c_{r+q} = 1, \end{aligned}$$

where $c_i = \frac{a_i(s-1)}{s}$ for $1 \leq i \leq r$ and $c_{i-1} = \frac{b_i+s-1}{s}$ for $r < i \leq r+q$.

$\pi = (\underline{\kappa}, \underline{c})$ is obviously a path of shape $s\Lambda_0$. Again, to show that it is L-S, we need only check the chain condition. This is immediate for π because if $w \in W/W_{\Lambda_0}$ and α , a positive real root, are such that $a_i \langle w(\Lambda_0), \alpha^\vee \rangle \in \mathbb{Z}$ (resp. $b_i \langle w(\Lambda_0), \alpha^\vee \rangle \in \mathbb{Z}$) then

$$c_i \langle w(s\Lambda_0), \alpha^\vee \rangle = \frac{a_i(s-1)}{s} \langle w(s\Lambda_0), \alpha^\vee \rangle = a_i \langle w((s-1)\Lambda_0), \alpha^\vee \rangle \in \mathbb{Z}$$

(resp. $c_i \langle w(s\Lambda_0), \alpha^\vee \rangle = \frac{b_i+s-1}{s} \langle w(s\Lambda_0), \alpha^\vee \rangle = b_i \langle w(\Lambda_0), \alpha^\vee \rangle + (s-1) \times \langle w(\Lambda_0), \alpha^\vee \rangle \in \mathbb{Z}$.) This proves (1).

Proof of 2.: We now show that if $\pi \in B\pi_{s\Lambda_0}$ and $\zeta \in B\pi_{t\Lambda_1}$ are such that $\pi * \zeta \in B(\pi_{s\Lambda_0} * \pi_{t\Lambda_1})$ then property C2 must hold. In fact, we need only show that the root operators preserve property C2; since the dominant path $\pi_{s\Lambda_0} * \pi_{t\Lambda_1}$ satisfies C2 then so should all paths in $B(\pi_{s\Lambda_0} * \pi_{t\Lambda_1})$.

For any two paths τ, σ recall that

$$e_i(\tau * \sigma) = \begin{cases} \tau * (e_i\sigma) & \text{if } m_i(\tau) > \langle \tau(1), \alpha_i^\vee \rangle + m_i(\sigma), \\ (e_i\tau) * \sigma & \text{if } m_i(\tau) \leq \langle \tau(1), \alpha_i^\vee \rangle + m_i(\sigma). \end{cases}$$

If $e_i(\pi * \zeta) = \pi * e_i \zeta$ then $\text{beg}(e_i \zeta) = \text{beg}(\zeta)$ or $\text{beg}(e_i \zeta) = r_i \cdot \text{beg}(\zeta) < \text{beg}(\zeta)$ so $e_i(\pi * \zeta)$ satisfies C2. Suppose now that $e_i(\pi * \zeta) = e_i \pi * \zeta$. If $\text{end}(e_i \pi) = \text{end}(\pi)$ then $e_i(\pi * \zeta)$ satisfies C2. Suppose then that $\text{end}(e_i \pi) = r_i \cdot \text{end}(\pi) < \text{end}(\pi)$. This would happen when $m_i(\pi * \zeta)$ occurs at the endpoint of π . Then $\langle \text{end}(\pi)(\Lambda_0), \alpha_i^\vee \rangle < 0$ and $\langle \text{beg}(\zeta)(\Lambda_1), \alpha_i^\vee \rangle > 0$ so $r_i \cdot \text{end}(\pi) r_i \geq \text{beg}(\zeta)$ and $e_i(\pi * \zeta)$ satisfies C2. A similar argument shows that the action of the f_i preserves property C2.

Proof of 3.: If $s = 0$, then $\pi_{\Lambda_1} * \dots * \pi_{\Lambda_1} = \pi_{t\Lambda_1}$ is a L-S path of shape $t\Lambda_1$. Therefore, C3 follows immediately from C1 and Theorem 4. Now assume that $s > 0$. Let $w \in W$. We now show that a path $\pi * \zeta$ lies in $P_w(\pi_{s\Lambda_0} * \pi_{t\Lambda_1})$ if and only if $\text{beg}(\pi) \leq w$ and, if $\text{end}(\pi) \not\leq \text{beg}(\zeta)$, then $\text{beg}(\pi) \cdot r_1 \leq w$.

- (i) Set $w = r_i u$ where $u < w$. We will assume that $P_u := P_u(\pi_{s\Lambda_0} * \pi_{t\Lambda_1})$ equals the set of paths that satisfy C1–C3.
- (ii) Set P'_w equal to the set of paths in $B(\pi_{s\Lambda_0}) * B(\pi_{t\Lambda_1})$ that satisfy C1–C3. We will be done once we have shown that $P'_w = \bigcup_{s \geq 0} f_i^s P_u$. This is equivalent to showing

$$(a) \bigcup_{s \geq 0} e_i^s P'_w \subseteq P_u \quad \text{and} \quad (b) P'_w \supseteq \bigcup_{s \geq 0} f_i^s P_u.$$

We show (a). Let $\pi * \zeta \in P'_w \setminus P_u$. Let $v = \text{beg}(\pi)$. If w equals some w_n^+ , then $v = w_n^+$. If w equals some w_n^- , then $v = w_{n-1}^+$. In any case, we have $\langle v(\lambda), \alpha_i^\vee \rangle \leq -1$. So, there exists some power $l > 0$ such that $\text{beg}(e_i^l \pi) = r_i w = u$. Therefore, there exists a power $n \geq l$ such that $e_i^n(\pi * \zeta) = e_i^n(\pi) * e_i^{n-l}(\zeta) \neq 0$. Conditions C1–C2 force $\text{beg}(e_i^l(\pi)) \cdot r_1 \leq u$ if $\text{end}(e_i^l(\pi)) \not\leq \text{beg}(e_i^{n-l}(\zeta))$. Therefore, $e_i^n(\pi * \zeta) \in P_u$. The proof of (b) is similar. \square

In the case of $A_1^{(1)}$, this parametrization of the basis vectors is equivalent to that given by standard monomial theory [LS]. Theorem 5 implies the following relations, which will be used to prove Theorem 1. Here, let Λ denote any of the fundamental weights.

- E1.** $P_n(\pi_{k\Lambda}) = \bigcup_{0 \leq j \leq i \leq n} p_\Lambda(i, j) * P_j(\pi_{(k-1)\Lambda})$.
- E2.** $P_{w_n^-}(\pi_{s\Lambda_0} * \pi_{t\Lambda_1}) = \bigcup_{i=0}^{n-1} \bigcup_{j=0}^i p_{\Lambda_0}(i, j) * P_{w_{j+1}^-}(\pi_{(s-1)\Lambda_0} * \pi_{t\Lambda_1})$.
- E3.** $P_{w_n^+}(\pi_{t\Lambda_1} * \pi_{s\Lambda_0}) = \bigcup_{i=0}^{n-1} \bigcup_{j=0}^i p_{\Lambda_1}(i, j) * P_{w_{j+1}^+}(\pi_{(t-1)\Lambda_1} * \pi_{s\Lambda_0})$.

The last relation comes from using an analogue of Theorem 5 for the path

$$\pi' := \underbrace{\pi_{\Lambda_1} * \dots * \pi_{\Lambda_1}}_t * \underbrace{\pi_{\Lambda_0} * \dots * \pi_{\Lambda_0}}_s = \pi_{t\Lambda_1} * \pi_{s\Lambda_0}.$$

This path traces a straight path from $\pi'(0) = 0$ to $t\Lambda_1$ and then one from $t\Lambda_1$ to $\pi'(1) = \lambda$.

6. Proof of Theorem 1

Proof for $\mathfrak{g} = A_1^{(1)}$. We first prove our formula for the case $\lambda \neq 0$ is singular (i.e. $st = 0$). We have already proved the formula for λ a fundamental weight. By Theorem 3 and relation E1,

$$\begin{aligned} \dim E_{w_n^+}(s\Lambda_0) &= \left| \bigcup_{i=0}^n \bigcup_{j=0}^i p(i, j) * P_j(\pi_{(s-1)\Lambda_0}) \right| \\ &= \sum_{0 \leq j \leq i \leq n} |p(i, j)| \cdot |P_j(\pi_{(s-1)\Lambda_0})| \\ &= \sum_{j=0}^n \left(\sum_{i=0}^n \binom{i-1}{j-1} \right) s^j \text{ by induction on } s \\ &= \sum_{j=0}^n \binom{n}{j} s^j \\ &= (s+1)^n \end{aligned}$$

and similarly for $\dim E_{w_n^-}(t\Lambda_1)$. Now suppose that $\lambda = s\Lambda_0 + t\Lambda_1$ regular. Then, by Theorems 3, 5 and relation E2,

$$\begin{aligned} \dim E_{w_n^-}(\lambda) &= \sum_{i=0}^{n-1} \sum_{j=0}^i |p(i, j) * P_{w_{j+1}^-}(\pi_{(s-1)\Lambda_0} * \pi_{t\Lambda_1})| \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} (t+1)(s+t)^j \text{ by induction} \\ &= (t+1)(s+t+1)^{n-1}. \end{aligned}$$

Use relation E3 to obtain $\dim E_{w_n^+}(\lambda)$. □

Proof for $\mathfrak{g} = A_2^{(2)}$. As in the case above, we first prove the dimension formulas for λ singular and then for λ regular.

Case $\lambda = s\Lambda_0$. We must show that $\dim E_{w_{2n}^+}(s\Lambda_0) = 2^{-n}(s+1)^n(s+2)^n$ and that $\dim E_{w_{2n+1}^+}(s\Lambda_0) = 2^{-n}(s+1)^{n+1}(s+2)^n$.

The proof is by induction on n . By using the definition of L-S paths, we can compute the following directly for all s : $\dim E_{w_1^+}(s\Lambda_0) = (s+1)$ and

$\dim E_{w_2^+}(s\Lambda_0) = \frac{1}{2}(s+1)(s+2)$. Set $R_n := P_n(\pi_{s\Lambda_0}) \setminus P_{n-1}(\pi_{s\Lambda_0})$. We need to show that $|R_n| = \frac{1}{2}(s+1)(s+2) |R_{n-2}|$ for $n > 2$. By relation E1, $R_n = \sum_{j \geq 0} p(n, j) * P_j(\pi_{(s-1)\Lambda_0})$. Therefore,

$$\begin{aligned}
 |R_n| &= \sum_{j \geq 0} (a(n, 2j) + sa(n, 2j+1)) \cdot 2^{-j} s^j (s+1)^j \\
 &= \sum_{j \geq 0} ((1+s)a(n-2, 2j) + (2+s)a(n-2, 2j-1) \\
 &\quad + a(n-2, 2j-2)) \cdot 2^{-j} s^j (s+1)^j \tag{1}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2}(s+1)(s+2) \sum_{j \geq 0} (a(n-2, 2j) + sa(n-2, 2j+1)) \\
 &\quad \times 2^{-j} s^j (s+1)^j \tag{2} \\
 &= \frac{1}{2}(s+1)(s+2) |R_{n-2}|.
 \end{aligned}$$

Equation 1 was obtained by using R1 and R2. Equation 2 was obtained by reordering and matching indices.

Case $\lambda = t\Lambda_1$. We must show that $\dim E_{w_{2n}^-}(t\Lambda_1) = (t+1)^n(2t+1)^n$ and that $\dim E_{w_{2n+1}^-}(t\Lambda_1) = (t+1)^{n+1}(2t+1)^n$.

The proof is by induction on n . By using the definition of L-S paths, we can compute the following directly for all t : $\dim E_{w_1^+}(t\Lambda_1) = (t+1)$ and $\dim E_{w_2^+}(t\Lambda_1) = (t+1)(2t+1)$. Set $R_n := P_n(\pi_{t\Lambda_1}) \setminus P_{n-1}(\pi_{t\Lambda_1})$. We need to show that $|R_n| = (t+1)(2t+1) |R_{n-2}|$ for $n > 2$. By relation E1, $R_n = \sum_{j \geq 0} p(n, j) * P_j(\pi_{(t-1)\Lambda_1})$. Therefore,

$$\begin{aligned}
 |R_n| &= \sum_{i \geq 0} |p(n, i)| \cdot |P_{w_i^-}(\pi_{(t-1)\Lambda_1})| \\
 &= \sum_{i \geq 0} (b(n, 2i) + tb(n, 2i+1)) t^i (2t-1)^i \\
 &= \sum_{i \geq 0} (b(n-2, 2i) + 2b(n-2, 2i-1) + b(n-2, 2i-2) \\
 &\quad + 3tb(n-2, 2i+1) + 4tb(n-2, 2i) \\
 &\quad + tb(n-2, 2i-1)) t^i (2t-1)^i \tag{3}
 \end{aligned}$$

$$\begin{aligned}
 &= (t+1)(2t+1) \sum_{i \geq 0} (b(n-2, 2i) \\
 &\quad + tb(n-2, 2i+1)) t^i (2t-1)^i \tag{4} \\
 &= (t+1)(2t+1) |R_{n-2}|
 \end{aligned}$$

Equation 3 was obtained by using R3 and R4. Equation 4 was obtained by reordering and matching indices.

Case $\lambda = s\Lambda_0 + t\Lambda_1$ regular:

The proof is by induction on s and t . For $l(w) \leq 2$ one can verify the dimensions formulas for all s and t by hand using Theorem 5 and relations E2-E3. Let $n > 2$. Set $R_{w_n^-} := P_{w_n^-}(\pi_{s\Lambda_0} * \pi_{t\Lambda_1}) \setminus P_{w_{n-1}^-}(\pi_{s\Lambda_0} * \pi_{t\Lambda_1})$. Then, by Theorem 5 and relation E2,

$$\begin{aligned} |R_{w_n^-}| &= \sum_{j \geq 0} |p(n-1, j) * P_{w_{j+1}^-}(\pi_{(s-1)\Lambda_0} * \pi_{t\Lambda_1})| \\ &= \sum_{j \geq 0} (a(n-1, 2j) + (2t+s)a(n-1, 2j+1)) \\ &\quad \times (t+1)(2t+s)^j (2t+s+1)^j 2^{-j} \\ &= \frac{1}{2}(2t+s+1)(2t+s+2) |R_{w_{n-2}^-}| \end{aligned} \quad (5)$$

Equation 5 was obtained just as in the cases above for λ singular, that is, by applying R1–R2 and then rearranging indices. Therefore, the dimensions hold for all w_n^- .

Now, let $R_{w_n^+} := P_{w_n^+}(\pi_{t\Lambda_1} * \pi_{s\Lambda_0}) \setminus P_{w_{n-1}^+}(\pi_{t\Lambda_1} * \pi_{s\Lambda_0})$. By relation E3,

$$\begin{aligned} |R_{w_n^+}| &= \sum_{j \geq 0} |p(n-1, j) * P_{w_{j+1}^+}(\pi_{(t-1)\Lambda_1} * \pi_{s\Lambda_0})| \\ &= \sum_{j \geq 0} (b(n-1, 2j) + 2^{-1}(2t+s)b(n-1, 2j+1)) \\ &\quad \times (s+1)(2t+s-1)^j (2t+s+1)^j 2^{-j} \\ &= \frac{1}{2}(2t+s+1)(2t+s+2) |R_{w_{n-2}^+}| \end{aligned} \quad (6)$$

Equation 6 was obtained just as in the case above when λ is singular, that is, by applying R3–R4 and then rearranging indices. Therefore, the dimensions hold for all w_n^+ . \square

As we can see from Theorem 1, for fixed $w \in W$, $\dim E_w(s\Lambda_0 + t\Lambda_1)$ are polynomials in the variables s and t . This suggests that each basis vector can be parametrized by an integral point in a certain lattice polytope of dimension equal to the length of w . In the case of the basic modules, this is clearly the case. If n equals the length of w , then each path is represented by a lattice point in the simplex with vertices v_0, v_1, \dots, v_n where $v_i = (d_1^e, d_2^e, \dots, d_i^e, 0, \dots, 0)$. Then $\dim E_w(s\Lambda_0)$ (respectively $\dim E_w(t\Lambda_1)$) equals the number of lattice points in that simplex dilated by a factor of s , (resp. t). Raika Dehy (Rutgers University,

personal communication) has established the existence of a corresponding polytope for the regular highest weights.

Acknowledgements

The author would like to thank F. Knop, P. Littelmann and O. Mathieu for useful conversations.

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