WILLIAM M. MCGOVERN

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Standard domino tableaux and asymptotic Hecke algebras

WILLIAM M. McGOVERN*
Department of Mathematics, University of Washington, Seattle, WA 98195

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1. Introduction

This paper is a continuation of [12], whose notation and references we shall preserve. There we showed how to compute certain structure constants (equal to 0 or 1) in the asymptotic Hecke algebra J of a finite classical Weyl group W. Here we use similar methods and a formula of Joseph to compute the remaining structure constants. Unfortunately these methods cannot be extended to exceptional W. Nevertheless, we show that the structure constants of J can in principle be used to produce bases for the irreducible constituents of a left cell $\mathcal{C}$ in terms of the Kazhdan-Lusztig basis for $\mathcal{C}$ itself. This was done for classical W in [12], where we showed that the transition matrices between these bases have all entries equal to ± 1. We also showed how to compute these entries, using Garfinkle's standard domino tableaux (but not using the Kazhdan-Lusztig polynomials). The situation is less satisfactory in the exceptional case. Here we use Lusztig's character tables in [11] to write down nonzero representatives of every isotypic component of a left cell, and then observe that these representatives may be extended to bases for their components by using the multiplication table of J. Unfortunately, just as Lusztig cannot say which columns of his character tables correspond to which Weyl group elements, so we cannot say which coefficients go with which Kazhdan-Lusztig basis vectors to produce our representatives. This ambiguity exists for classical Weyl groups as well, but we will see that it can be circumvented in that case.

The main applications of these results are the same as in [12]. One can now compute the socle of the bimodule of Ad $\mathfrak{g}$-finite maps between two simple highest weight modules whenever the semisimple Lie algebra $\mathfrak{g}$ (with Weyl group W) is classical. One also has severe and explicit constraints on the behavior of the Jacquet functor between Harish-Chandra and category $\mathcal{O}$ modules, since this functor may be viewed as a Hecke module map. (The applications to tensor products of special unipotent representations in [12] depend only on the results in that paper.)

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2. Intertwining operators on Hecke algebras

Let α and β be simple roots of $\mathfrak{g}$ with respect to a fixed Certain subalgebra $\mathfrak{h}$ and assume that they span a subsystem S of type A_2 or B_2. If S is of type A_2, then there is a wall-crossing operator $T_{\alpha\beta}$ introduced by Vogan in [13]. It can be defined on Weyl group elements or left cells. Following [12], we define it on elements w whose τ-invariant contains exactly one of α and β via $T_{\alpha\beta}(w) := u$, where u is uniquely specified by three properties. First, $u \in wW'$, where W' is the parabolic subgroup of W generated by the reflections through α and β; second, u and w differ in length by one; and third, the τ-invariants of u and w meet $\{\alpha, \beta\}$ in disjoint singletons. Then u and w always lie in the same right cell in W, but different left cells. As mentioned above, the map $T_{\alpha\beta}$ is also well defined on left cells and in fact induces an injective H_F-equivariant map from certain left cell representations $[\mathcal{C}]$ in H_F to left cell representations. This fact (first observed by Kazhdan and Lusztig) plays a crucial role in [12].

Now suppose instead that S is of type B_2. As noted in [13], one can still define a map $T_{\alpha\beta}$ on Weyl group elements or left cells via the same three properties as above, but this time $T_{\alpha\beta}$ is not single-valued; indeed, a typical $T_{\alpha\beta}(w)$ has either one or two elements. Nevertheless, $T_{\alpha\beta}$ does induce a single-valued equivariant map $T'_{\alpha\beta}$ on H_F sending a typical $[\mathcal{C}]$ on which it is defined either to some $[\mathcal{C}']$ or to a sum $[\mathcal{C}'] + [\mathcal{C}'']$. More precisely, if $T_{\alpha\beta}(w) = u$, then $T'_{\alpha\beta}(C_w) = C_u$, while if $T_{\alpha\beta}(w) = \{u, v\}$, then $T'_{\alpha\beta}(C_w) = C_u + C_v$. (That the induced map really is H_F-equivariant is proved in the same way as Theorem 4.1 of [12].) To avoid ambiguity, we henceforth use the notation $T_{\alpha\beta}$ only when the subsystem S is of type A_2 and $T_{\alpha\beta}, T'_{\alpha\beta}$ will always denote the induced maps on the Hecke algebra. The key difference between the maps $T_{\alpha\beta}$ and $T'_{\alpha\beta}$ is that the former is injective (and thus preserves the module structure of a cell) while the latter is not. If the subsystem S is of type B_2, then we will also need the map $S_{\alpha\beta}$ defined in [14] and used extensively in [12]. It too induces (and henceforth denotes) an H_F-equivariant map between left cells, which unlike $T'_{\alpha\beta}$ is injective and sends basis vectors to basis vectors.

In types A, B, and C, the maps $T_{\alpha\beta}$ and $T'_{\alpha\beta}$ suffice to generate the right cells, but this is not so in type D. We therefore need two further maps (denoted S_D and T_D), which are attached to quadruples $(\alpha, \beta, \gamma, \delta)$ of simple roots spanning a subsystem S of type D_4. As in [12], we assume that α is the inner root in the Dynkin diagram of S, but it does not matter how we label the outer roots β, γ, δ. Furthermore, since the choice of $(\alpha, \beta, \gamma, \delta)$ is unique if $\mathfrak{g}$ is simple, we omit it from the notation. The map S_D was defined in [12] and is entirely analogous to the map $S_{\alpha\beta}$; we regard it as an injective equivariant map between certain pairs of left cells sending basis vectors to basis vectors. Like $S_{\alpha\beta}$, but unlike $T_{\alpha\beta}$, it has fixed points. The map T_D is the analogue of $T'_{\alpha\beta}$. More precisely, when regarded as a map on Weyl group elements, its domain consists of all $w \in W$ satisfying either Hypothesis $\mathcal{D}$ or Hypothesis $\mathcal{A}_{\mathcal{B}}$ of [5]. The image u or $\{u, v\}$ of w is defined by the same three properties as for S_D in [12], except that u or u, v should satisfy Hypothesis $\mathcal{A}_{\mathcal{B}}$ (resp. $\mathcal{D}$) if w satisfies Hypothesis $\mathcal{D}$ (resp. $\mathcal{A}_{\mathcal{B}}$). As with the other maps, we regard T_D as a single-valued H_F-equivariant map sending a typical C_w on which it is defined either to some C_u or to a sum $C_u + C_v$. It is sometimes convenient to extend the domains of the maps π defined above to all of H_F by declaring that $\pi(C_w) := 0$ if it has not already been defined above.

The above paragraphs apply to any simple Lie algebra $\mathfrak{g}$ and its Weyl group W. For the remainder of this section we assume that $\mathfrak{g}$ and W are classical. A central result in the program of [3] and [4], appearing in [3] as Theorem 3.2.2, asserts that one can get from any C_x to some sum of basis vectors involving C_y via some sequence of maps $T_{\alpha\beta}, T'_{\alpha\beta}, T_D$ whenever x lies in the same right cell as y. In [12], we proved the analogue of this result for $T_{\alpha\beta}, S_{\alpha\beta}, S_D$. The conclusion is sharper (the sum involving C_y may be replaced by C_y itself), but the hypothesis is stronger (x and y must also lie in left cells $\mathcal{C}_1$ and $\mathcal{C}_2$ with $[\mathcal{C}_1]$ isomorphic to $[\mathcal{C}_2]$). We now return to Garfinkle's original setting and make her result more precise. To do this we need some terminology. If $[\mathcal{C}_1] \cong [\mathcal{C}_2]$, then we call the left cells $\mathcal{C}_1$ and $\mathcal{C}_2$ isomorphic. If instead exactly half of the representations in $[\mathcal{C}_1]$ appear in $[\mathcal{C}_2]$, then we say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are adjacent. We extend the notions of isomorphism and adjacency to pairs of left or right tableaux in the obvious way, by passing to Weyl group elements having these tableaux and looking at their left or right cells. Then we have

THEOREM 2.1. *Let $w_1, w_2 \in W$ belong to the same right cell $\mathcal{R}$ and adjacent left cells $\mathcal{C}_1, \mathcal{C}_2$. Then there is a sequence of maps $T_{\alpha\beta}, S_{\alpha\beta}, S_D$ followed by a single map $T'_{\alpha\beta}$ or T_D that either sends C_{w_1} to C_{w_2} or C_{w_2} to C_{w_1}.*

Proof. As in the proof of Theorem 4.2 of [12], we suppose first that W is of type B or C and then imitate the proof of Theorem 3.2.2 of [3], proceeding by induction on the rank of W. Once again one must strengthen both the hypothesis and the conclusion of Lemma 3.2.9 of [3]. The new hypothesis states that we are given a tableau T_1 and an extremal position P' in it such that there is another tableau $\tilde{T}_1$ either equivalent or adjacent to T_1 having its largest label in position P'. The new conclusion replaces the sequence of maps in the old conclusion with a sequence of maps $T_{\alpha\beta}, S_{\alpha\beta}$ followed by a single map $T'_{\alpha\beta}$. The proof follows the lines of the earlier proof but also uses the following fact: whenever two tableaux T_1, T_2 are adjacent, then the cycle structure of one of them (in the sense of [2]) is obtained from that of the other by interchanging the holes attached to exactly two corners. This fact follows easily from the definition of adjacency and Theorems 2.12 and 3.2 in [12]. One also strengthens Lemma 3.2.8 exactly as in the proof of Theorem 4.2 of [12]. As in that proof, the new versions of Theorem 3.2.2 and Proposition 3.2.4 are easily verified if $r = 2$ (in the notation of [3]). In general, the arguments of [3] now carry over to our situation and show that there are sequences of maps as in our conclusion respectively sending C_{w_1} to a sum involving C_{w_2} and C_{w_2} to a sum involving C_{w_1}. Using Theorem 2.12 of [12] again, one checks that one of the intersections $\mathcal{C}_1 \cap \mathcal{R}, \mathcal{C}_2 \cap \mathcal{R}$, say the first one, has cardinality exactly twice that of the other. It follows that the first sequence of maps, when restricted to the F-span of the C_w for $w \in \mathcal{C}_1 \cap \mathcal{R}$, sends basis vectors to basis vectors in a two-to-one fashion. The result follows.

As in [12], a similar strategy, using [4], takes care of the case when W is of type D. There the base case is $r = 4$ and one replaces the maps $S_{\alpha\beta}, T'_{\alpha\beta}$ by S_D, T_D, respectively. □

We conclude this section with an extension of Theorem 2.1 that is the crucial step in computing the multiplication table of the asymptotic Hecke algebra J. Before stating it, we introduce a useful bit of machinery. Given a double cell $\mathcal{D}$ of W, let L, R be the perfectly paired finite-dimensional vector spaces over the two-element field $\mathbb{F}_2$ attached to $\mathcal{D}$ by Theorem 2.12 of [12]. Then isomorphism types of left cells $\mathcal{C}$ in $\mathcal{D}$ correspond bijectively to supersmooth subspaces of L, in the language of [12]. We introduce a metric ρ on the set of such subspaces via

$$\rho(S_1, S_2) = \dim S_1 + \dim S_2 - 2\dim(S_1 \cap S_2),$$

where all dimensions are over $\mathbb{F}_2$. One easily checks that ρ is indeed a metric and that $\rho(S_1, S_2) = 1$ if and only if any pair of left cells corresponding to S_1, S_2 are adjacent. Now we have

THEOREM 2.2. *Let $x, w \in W$ be an involution and another element in its right cell $\mathcal{R}$, respectively. Then there is a sequence of maps $T_{\alpha\beta}, S_{\alpha\beta}, S_D, T'_{\alpha\beta}, T_D$ sending C_x to C_w.*

Proof. We know that x lies in the left cell $\mathcal{C} := \mathcal{R}^{-1}$; let $\mathcal{C}'$ be the left cell of w. By Theorem 2.1 and Theorem 4.2 of [12], it suffices to construct a sequence $[\mathcal{C}_1], \ldots, [\mathcal{C}_n]$ of left cells such that $[\mathcal{C}_1] = [\mathcal{C}], [\mathcal{C}_n] \cong [\mathcal{C}']$, each $\mathcal{C}_i$ is adjacent to $\mathcal{C}_{i+1}$, and each $\mathcal{C}_i$ meets $\mathcal{R}$ in exactly twice as many elements as $\mathcal{C}_{i+1}$. Translating the above properties into the language of supersmooth subspaces, we see that we must show how to produce a chain of supersmooth subspaces connecting any two given ones such that each subspace in the chain is at ρ-distance one from the next and the length of the chain equals the ρ-distance between the given subspaces. To do this it suffices by induction to assume that T, U are supersmooth subspaces with $\rho(T, U) = i$ and to produce another supersmooth subspace V at distance 1 from one of T, U and $i - 1$ from the other.

Recall now the standard basis $\ell_1, \ldots, \ell_n$ of the space L used in [12] to define the notion of supersmooth subspace. Call a sum $v = \ell_a + \cdots + \ell_b$ of consecutive basis vectors with $v \in T$ an *atom* if no proper subsum $\ell_a + \cdots + \ell_c$ of consecutive basis vectors with $a \leq c < b$ also lies in T. Then the atoms in T form a basis of the latter satisfying the nonoverlapping condition of [12, Lemma 2.14], and similarly for U. We now attach two finite sequences of indices $\{i_j\}, \{i'_j\}$ to T, U, respectively, as follows. Let i_1 be the least index appearing in any atom of T. Assume inductively that the indices $i_1, \ldots, i_k$ have been defined and correspond to the atoms $a_1, \ldots, a_k$

of T. If there is an atom of T that is a proper subsum of a_k, then let i_{k+1} be the least index appearing in any such atom. Otherwise i_{k+1} and all subsequent i_j are undefined. Define the sequence $\{i'_j\}$ in the same way, replacing T by U. We now consider several cases.

Suppose first that there is some k such that i_j, i'_j are defined for $j \leq k$, that $i_j = i'_j$ if $j < k$, and that $i_k < i'_k$. We claim that no sum of atoms in T involving a_k can lie in U. This follows because the coefficients of $\ell_{i_k}, \ell_{i_k - 1}$ in any such sum must differ, but no sum of atoms in U has this property. If we now take T' to be the span of all atoms in T except a_k, then one checks that $T' \cap U = T \cap U$ but $\dim T' = \dim T - 1$. Thus T' has the desired property. Of course the same argument works if instead $i'_k < i_k$.

Now suppose that there is some k such that i_j, i'_j are defined and equal if $j < k$, that i_k is defined, and that i'_k is not. Look at the *largest* index i in the atom $a'_{k-1} \in U$ corresponding to i'_{k-1}. If $i \geq i_k$, then we can take T' to be the span of all the atoms in T other than a_k and the argument in the last paragraph shows that T' has the desired property. If $i < i_k$, then the definition of atom shows that $a'_{k-1} \notin T$. Let T' be the span of the atoms in T and a'_{k-1}. Then T' is supersmooth and $\dim T' = \dim T + 1$ while $\dim(T' \cap U) = \dim(T \cap U) + 1$. Thus T' again has the desired property. Again the same argument works if instead i'_k is defined and i_k is not.

If neither of the hypotheses of the above paragraphs holds and the sequence of defined i_j is $i_1, \ldots, i_m$, then the sequence of defined i'_j must also be $i_1, \ldots, i_m$. One of the atoms a_m, a'_m, say a'_m, must be a subsum (not necessarily proper) of the other. If $a_m = a'_m$, then let T' be the span of the atoms in T other than a_m. Otherwise $a'_m \notin T$; let T' be the span of the atoms in T and a'_m. In either case T' is supersmooth and we can argue as in the last paragraph to show that T' has the desired property. Thus in all cases we can find a suitable T', as desired. □

3. Structure constants in the classical cases

We continue to assume that W is classical. We begin by recalling a special case of the main result of [12] on structure constants. Recall that we denote the standard basis of the asymptotic Hecke algebra J by $\{t_w : w \in W\}$.

LEMMA 3.1. *Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be three left cells in the same double cell $\mathcal{D}$. The set $J_1 := \{t_x : x \in \mathcal{C}_1^{-1} \cap \mathcal{C}_1\}$ has a natural structure of elementary abelian 2-group acting on the sets $J_i := \{t_y : y \in \mathcal{C}_1^{-1} \cap \mathcal{C}_i\}$ for $i = 2, 3$. Any two elements in J_2 have the same stabilizer in J_1. Any two orbits of this stabilizer in J_3 are isomorphic as homogeneous spaces. Both this stabilizer and its orbits in J_3 may be explicitly computed in terms of standard domino tableaux.*

Proof. The first three assertions are proved in [11, 2.12, 3.13]. More precisely, suppose that $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ correspond to the supersmooth subspaces S_1, S_2, S_3 of the

$\mathbb{F}_2$-vector space L in the parametrization of [12]. Then J_1 is naturally isomorphic to the product $(L/S_1)\times S_1^*$ and we may identify J_i with $(L/S_1S_i)\times(S_1\cap S_i)^*$, ignoring the group structure on this last set. Then L/S_1 acts on L/S_1S_i by left translation and S_1^* acts on $(S_1\cap S_i)^*$ by restriction and translation. The transitive action of J_1 on J_2 is then the direct product of these two actions; the common stabilizer of any element identifies with $(S_1S_2/S_1)\times(S_1/(S_1\cap S_2))^*$. The orbits of this group on J_3 may all be identified with $(S_1S_2/(S_1(S_2\cap S_3)))\times(S_1/(S_1\cap S_2)(S_1\cap S_3))^*$ as homogeneous spaces, where again we ignore the group structure on this set. Recipes for multiplication in J_1 and the J_1 action on J_2, J_3 may be found in Theorem 5.1 of [12], where they are expressed in terms of extended open cycles of one standard domino tableau relative to another (and the bijection between Weyl group elements and pairs of standard domino tableaux of the same shape). □

Recall from [2] and [12] the notion of extended open cycles of one tableau relative to another. Our main result is

THEOREM 3.2. *Retain the notation of Lemma 3.1 and its proof. For* $i,j,k\in\{1,2,3\}$, *let* ℓ_{ij} (resp. ℓ_{ijk}) *denote the number of* W*-representations common to* $[\mathcal{C}_i]$ *and* $[\mathcal{C}_j]$ (resp. *to* $[\mathcal{C}_i],[\mathcal{C}_j]$, *and* $[\mathcal{C}_k]$). *Suppose that* $x\in\mathcal{C}_1^{-1}\cap\mathcal{C}_2, y\in\mathcal{C}_2^{-1}\cap\mathcal{C}_3$. *Then the product* t_xt_y *equals* s *times the sum of the* t_z *as* z *runs over one orbit in* J_3 *of the stabilizer of* t_x *in* J_1 *and* $s=\ell_{123}\sqrt{\#L/\ell_{12}\ell_{13}\ell_{23}}$. *The left and right tableaux* $T_L(z),T_R(z)$ *of one index* z *appearing in this sum may be obtained as follows. Let* d *be the Duflo involution in* $\mathcal{C}_2$ *and let* $T_L(x)$ (resp. $T_R(y)$) *be obtained from* $T_L(d)$ (*resp.* $T_R(d)=T_L(d)$) *by moving through the open cycles* $c_1,\ldots,c_k$ (resp. $c'_1,\ldots,c'_m$). *Denote by* $e_1,\ldots,e_k$ (resp. $e'_1,\ldots,e'_m$) *the extended open cycles containing* $c_1,\ldots,c_k$ (resp. $c'_1,\ldots,c'_m$) *relative to* $T_L(y)$ (resp. $T_R(x)$). *Denote by* U (resp. U') *the union of the extended open cycles appearing an odd number of times in the list* $e_1,\ldots,e_k$ (resp. $e'_1,\ldots,e'_m$). *Then* $T_L(z)$ *is the right tableau of* $\mathbf{E}((T_L(y),T_L(x));U,L)$; *similarly* $T_R(z)$ *is the right tableau of* $\mathbf{E}((T_R(x),T_R(y));U',L)$.

Proof. All of these assertions except the recipes for $T_L(z),T_R(z)$ follow from Proposition 4.2 of [7]. To get these recipes, we argue as in the proof of Theorem 5.1 of [12]. If $y=d$, then the sum has only one term, namely t_x [11], and the recipes follow at once. In general, we can get from C_d to C_y via a sequence of maps $T_{\alpha\beta},S_{\alpha\beta},S_D,T'_{\alpha\beta},T_D$ as in Theorem 2.2. As each of these maps induces a left J-module map on J in the obvious way (cf. [12]), we obtain the product t_xt_y by applying the same sequence of maps to C_x and then replacing C by t. Garfinkle has computed these maps on the level of domino tableaux in [2] and [4]. Her recipes reduce in this situation to the ones in the theorem. □

Of course by combining Lemma 3.1 and Theorem 3.2 (or rather Theorem 5.1 in [12] and Theorem 3.2), we get a formula for the product t_xt_y whenever x,y satisfy the hypotheses of Theorem 3.2 for some left cells $\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3$. If they do not satisfy these hypotheses for any choice of left cells C_i, then $t_x t_y$ is zero [11]. So we now know the complete multiplication table of J in the classical cases (as promised in §5 of [12]). As a consequence, it is not difficult in the classical cases to verify the general version of Conjecture 3.15 in [11] (that is, the one determining all the structure constants of J, not just those of certain of its subrings). In the exceptional cases, Joseph has computed a large number of structure constants in [8].

For the last result of this section we adopt the notation of Corollary 5.2 of [12].

COROLLARY 3.3. *Let $x, y \in W$ be arbitrary. Then one can compute the socle of the bimodule $L(L(x \cdot \lambda), L(y \cdot \lambda))$ of* Ad $\mathfrak{g}$*-finite maps from $L(x \cdot \lambda)$ to $L(y \cdot \lambda)$, given a knowledge of the Duflo involution in the right cell of x or y. All constituents of this socle have the same multiplicity, which is a power of 2. The socle is nonzero if and only if x and y lie in the same right cell.*

Proof. This follows at once from Joseph's formula for the socle of $L(L(x \cdot \lambda), L(y \cdot \lambda))$ in terms of his modified versions $c^*_{x^{-1},y,z^{-1}}$ of the structure constants $c_{x^{-1},y,z^{-1}}$ of J [6, 4.8], together with his later observation that, in fact, $c^*_{x,y,z} = c_{x,y,z}$ for any x, y, and z [9]. □

4. Kazhdan-Lusztig bases of irreducible constituents

We now drop the assumption that W is classical. As noted above, the maps $T_{\alpha\beta}, S_{\alpha\beta}, S_D, T'_{\alpha\beta}, T_D$ are still defined (under the same hypotheses as above) but are not as well behaved as in the classical case. In particular, there are left cells $\mathcal{C}$ in every exceptional Weyl group such that self-intertwining operators on $[\mathcal{C}]$ sending basis vectors to basis vectors do not act transitively on $\mathcal{C}^{-1} \cap \mathcal{C}$. Thus one cannot use such operators to decompose $[\mathcal{C}]$ explicitly into irreducible constituents, as was done for all classical left cells $[\mathcal{C}]$ in Section 4 of [12]. It turns out, however, that the algebra J always furnishes enough intertwining operators to do this. If $\mathcal{C}$ is a left cell, then we define the subrings $J_{\mathcal{C}^{-1}\cap\mathcal{C}}, J_{\mathcal{C}}$ as in Section 5 of [12].

THEOREM 4.1. *Retain the above notation. Right multiplication ρ_x of $J_{\mathcal{C}}$ by any $x \in J_{\mathcal{C}^{-1}\cap\mathcal{C}}$ induces a left H_F-equivariant map $\bar{\rho}_x$ on $[\mathcal{C}]$ via the vector space isomorphism $\pi : J_{\mathcal{C}} \to [\mathcal{C}]$ sending t_w to C_w. Moreover, there is a natural bijection $E_J \leftrightarrow E_H$ between simple $J_{\mathcal{C}^{-1}\cap\mathcal{C}} \otimes \mathbb{Q}$-modules and simple H_F-submodules of $[\mathcal{C}]$ and an isomorphism i (compatible with this bijection) between the endomorphism rings of $J_{\mathcal{C}}$ and $[\mathcal{C}]$ such that ρ_x and $\bar{\rho}_x$ correspond under i.*

Proof. Lusztig has written down a homomorphism $H \to J \otimes A$ which is injective because it becomes an isomorphism after extending scalars [10, 2.4, 2.8] . Thus it also becomes an isomorphism after extending scalars to F. The formula for this homomorphism shows that any nonzero left H_F-equivariant map on $[\mathcal{C}]$ induces a nonzero $J_{\mathcal{C}^{-1}\cap\mathcal{C}} \otimes F$-equivariant one on $J_{\mathcal{C}} \otimes F$, which must be given by right multiplication by some element of $J_{\mathcal{C}^{-1}\cap\mathcal{C}} \otimes F$. On the other hand, the space of H_F-equivariant maps on $[\mathcal{C}]$ has dimension equal to dim $\mathrm{Hom}_{\mathbb{Q}}(\langle\mathcal{C}\rangle, \langle\mathcal{C}\rangle) = \#(\mathcal{C}^{-1} \cap \mathcal{C}) = \dim_F(J_{\mathcal{C}^{-1}\cap\mathcal{C}} \otimes \mathcal{F})$. The first assertion follows; to get the second one, we use Lusztig's homomorphism again and [11, Proposition 3.3]. □

Given a left cell $\mathcal{C}$, let (G, H) be the ordered pair of finite groups corresponding to $[\mathcal{C}]$ in Lusztig's classification [11]. If G is abelian (so in particular if W is classical), then this result may be sharpened: the t_w for $w \in \mathcal{C}^{-1} \cap \mathcal{C}$ form an elementary abelian 2-group under multiplication (isomorphic to G) and this group acts on the set $S := \{t_v : v \in \mathcal{C}\}$ on the right. Moreover, if we identify S with the set of vertices of the W-graph of $\mathcal{C}$ in the obvious way, then this group acts by graph automorphisms. In general, the action of $J_{\mathcal{C}^{-1}\cap\mathcal{C}}$ on $J_{\mathcal{C}}$ is too complicated to correspond to graph automorphisms.

Asymptotic Hecke algebras J and their subrings $J_{\mathcal{C}^{-1}\cap\mathcal{C}}$ are examples of 'based rings' in the terminology of [11]. The general theory of such rings is developed in Section 1 of [11] (and in somewhat different language in [7]) and is quite analogous to the classical theory of representations of finite groups. Lusztig uses this theory to construct the character table of $J_{\mathcal{C}^{-1}\cap\mathcal{C}}$ for every left cell $\mathcal{C}$ in [11, Appendix]; it depends only on (G, H). Now one has

COROLLARY 4.2. *Let $\mathcal{C}$ and (G, H) be as above. Then there is a bijection ι between the set $\{\gamma_i\}$ of columns of Lusztig's character table for $J_{\mathcal{C}^{-1}\cap\mathcal{C}}$ and $\mathcal{C}^{-1} \cap \mathcal{C}$ such that if the ith row $(r_1, r_2, \ldots)$ of this table corresponds to the H_F-representation ρ_i, then one element of the ρ_i-isotypic component of $[\mathcal{C}]$ is $\sum_i r_i C_{\iota(\gamma_i)}$.*

Proof. The map ι of course comes from the definition of character table; as noted in the introduction, one does not know how to compute it (although Lusztig sets up his tables so that $\iota(\gamma_1)$ is always the Duflo involution d in $\mathcal{C}$). By the general theory of based rings, the element $t_i := \sum_i \bar{r}_i t_{\iota(i)}$ acts by zero on all simple representations of $J_{\mathcal{C}^{-1}\cap\mathcal{C}}$ other than the one (say ρ) corresponding to ρ_i, and by a nonzero scalar on ρ [11, 1.3]. Now the desired result follows by Theorem 4.1, the cyclicity of C_d in $[\mathcal{C}]$ [6], and the empirical fact that $\bar{r}_i = r_i \in \mathbb{Z}$ for all i. □

In case some ρ_i appears with multiplicity greater than one in $[\mathcal{C}]$ (as can happen for exceptional W), then one can obtain further representatives of ρ_i in the span of the C_w for $w \in \mathcal{C}^{-1} \cap \mathcal{C}$ by postmultiplying the element t_i in the proof of Theorem 4.2 by various basis vectors in $J_{\mathcal{C}^{-1}\cap\mathcal{C}}$ and applying Theorem 4.1. For any ρ_i, if $\mathcal{R}$ is any right cell with ρ_i appearing in $[\mathcal{R}]$, then the proof of Theorem 4.2 shows that one can obtain every representative of ρ_i lying in the span of the C_w for $w \in \mathcal{C} \cap \mathcal{R}$ by premultiplying t_i by suitable elements of $J_{\mathcal{R}\cap\mathcal{C}}$ and again applying Theorem 4.1. Repeating this procedure for every right cell $\mathcal{R}$ meeting $\mathcal{C}$, one obtains a basis for the ρ_i-isotypic component of $[\mathcal{C}]$ in which every element is a $\mathbb{Z}$-linear combination of C_w. Moreover (as promised above) this basis can be computed using only the multiplication table of J; indeed, computing it amounts to decomposing J explicitly as a left module over itself. (It cannot be constructed from Lusztig's character tables alone, for it is well known that the character table of a based ring does not determine its multiplication table. On the other hand, the multiplication table of J *is* completely determined by the general version of Conjecture 3.15 of [11] mentioned above.)

In particular, if W is classical, then we know this multiplication table and can therefore substantially simplify the recipe for this basis given in Section 4 of [12]. More precisely, given a cell intersection $[\mathcal{C} \cap \mathcal{R}]$, we use Theorem 5.1 of [12] and Theorem 4.1 above to write it as a direct sum of one-dimensional subspaces $\mathcal{S}$, each of which lies in a single subrepresentation ρ of $[\mathcal{C}]$. Given the subrepresentation ρ, we can decide which subspaces $\mathcal{S}$ lie in it as follows. Arguing as in the proof of [12, Theorem 4.3] by induction on the complexity of ρ, we can locate a left cell $\mathcal{C}'$ such that ρ appears in $\mathcal{C}'$ and a one-dimensional subspace $\mathcal{S}$ of $[\mathcal{C}' \cap \mathcal{R}]$ lying in ρ. Then we construct a chain of left cells from $\mathcal{C}'$ to $\mathcal{C}$ as in the proof of Theorem 2.2 above. Using Theorem 2.1 and the equivariance of the maps of Section 2, we use $\mathcal{S}$ to produce a subspace $\mathcal{S}'$ of $[\mathcal{C} \cap \mathcal{R}]$ lying in ρ, as desired. Repeating this procedure for every subrepresentation ρ and right cell $\mathcal{R}$ meeting $\mathcal{C}$, we produce bases of the desired type for every subrepresentation of $\mathcal{C}$.

There is one case in which it is very easy to write down a basis for a subrepresentation ρ of a left cell $[\mathcal{C}]$. This occurs when the finite group G corresponding to $[\mathcal{C}]$ is abelian and ρ is the special or Goldie rank representation (always occurring with multiplicity one). Here a basis for ρ is given by

$$\left\{ \sum_{w \in \mathcal{R} \cap \mathcal{C}} C_w \right\}_{\mathcal{R}}$$

as $\mathcal{R}$ runs over the all the right cells meeting $\mathcal{C}$. This recipe follows easily from either [11, Sec. 3] or the results in Section 2 of this paper. It also applies if the ordered pair (G, H) corresponding to $[\mathcal{C}]$ is $(S_3, 1)$, where as usual S_3 denotes the symmetric group on three letters.

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