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Abstract. In this paper we establish the existence of all higher logarithms as Deligne cohomology classes in a sense slightly weaker than that of [13, Sect. 12], but in a sense that should be strong enough for defining Chern classes on the algebraic K-theory of complex algebraic varieties. In particular, for each integer $p \geqslant 1$, we construct a multivalued holomorphic function on a Zariski open subset of the self dual grassmannian of p-planes in $\mathbb{C}^{2p}$ which satisfies a canonical $2p + 1$ term functional equation. The key new technical ingredient is the construction of a topology on the generic part of each Grassmannian which is coarser than the Zariski topology and where each open contains another which is both a $K(\pi, 1)$ and a rational $K(\pi, 1)$.

1. Introduction

Denote by G_q^p the Zariski open subset of the grassmannian of q-dimensional linear subspaces of $\mathbb{P}^{p+q}$ which are transverse to each coordinate hyperplane and each of their intersections. Intersecting elements of G_q^p with each of the $p+q+1$ coordinate hyperplanes of $\mathbb{P}^{p+q}$ defines $p + q + 1$ maps

$$A_i \colon G_q^p \to G_{q-1}^p, \quad 0 \leqslant i \leqslant p + q.$$

The spaces G_q^p with $0 \leqslant q \leqslant p$ and the face maps A_i form a truncated simplicial variety $G_\bullet^p$.

In [13, Sect. 12] (see also [3]) the pth higher logarithm is defined as a certain element of the 'multivalued Deligne cohomology' of $G_\bullet^p$. In that paper the existence of only the first three higher logarithms was established.

In this paper we establish the existence of all higher logarithms, but in a sense slightly weaker than that of [13, (12.4)] – we show that for each p, there is a Zariski open subset $U_\bullet^p$ of $G_\bullet^p$ on which the pth higher logarithm is defined as a multivalued (and ordinary) Deligne cohomology class. This should be sufficient to show that the pth Chern classes on the algebraic K-theory of a complex algebraic variety is represented by the pth higher logarithm (cf. [10, 14]). The key new technical ingredient is the construction of a topology on the generic part of each

Grassmannian which is coarser than the Zariski topology and where each open contains another which is both a $K(\pi, 1)$ and a rational $K(\pi, 1)$.

Hanamura and MacPherson [17] have a geometric construction of the part of each higher logarithms that lies in the multivalued de Rham complex of $G_\bullet^p$. The part of our higher logarithm that lies in the multivalued de Rham complex of $G_\bullet^p$ is only defined generically, so their result is stronger than ours in this respect (cf. Remark 7.6), but our result is stronger than theirs in that we construct higher logarithms as both multivalued and ordinary Deligne cohomology classes.

One part of the cocycle defining the pth higher logarithm is a multivalued function L_p defined on the Zariski open subset U_{p-1}^p of the self dual Grassmannian of p planes in $\mathbb{C}^{2p}$. The cocycle condition implies that this multivalued function satisfies the canonical $2p + 1$ term functional equation

$$\sum_{i=0}^{2p} (-1)^i A_i^* L_p = 0.$$

In the cases $p = 1, 2, 3$, the function L_p has a single valued cousin D_p which also satisfies the functional equation

$$\sum_{i=0}^{2p} (-1)^i A_i^* D_i = 0.$$

The first function D_1 is simply $\log |\ |$, the second is the Bloch-Wigner function, and the third, whose existence was established in [13, Sect. 11], can be expressed in terms of Ramakrishnan's single valued cousin of the classical trilogarithm, as was proved by Goncharov [9]. The functional equation implies that D_p $(p = 1, 2, 3)$ represents an element of $H^{2p-1}(\mathrm{GL}_p(\mathbb{C}), \mathbb{C}/\mathbb{R}(p))$. This class is known to be a non-zero rational multiple of the Borel element, the class used to define the Borel regulator ([4, 6, 9, 19], see also [12]). The single valued cousins of the higher logarithms constructed in this paper are constructed in [14] where it is shown that each represents a non-zero rational multiple of the class used to define the Beilinson Chern class on the part coming from GL_p of the K-theory of function fields of complex algebraic varieties.

We now discuss the content of this paper in more detail. As in [13], the algebra of multivalued differential forms on an algebraic manifold will be denoted by $\widetilde{\Omega}^\bullet(X)$. We will usually denote the ring of multivalued functions $\widetilde{\Omega}^0(X)$ by $\widetilde{\mathcal{O}}(X)$. There is a weight filtration $W_\bullet$ on $\widetilde{\Omega}^\bullet(X)$ which gives it the structure of a filtered d.g. algebra. The category of complex algebraic manifolds and regular maps between them will be denoted by $\mathcal{A}$. Since the pullback of a multivalued function under a regular map $X \to Y$ is not well defined, it is necessary to refine the category $\mathcal{A}$ in order that $\widetilde{\Omega}^\bullet$ becomes a well defined functor. Such a refinement $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ is defined

in [13, Sect. 2]. The objects of $\widetilde{\mathcal{A}}$ are universal coverings $\widetilde{X} \to X$ of objects of $\mathcal{A}$, and the morphisms are commutative squares

$$
\begin{array}{ccc}
\widetilde{X} & \longrightarrow & \widetilde{Y} \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

where the bottom arrow is a morphism of $\mathcal{A}$. The truncated simplicial variety $G_\bullet^p$ has a natural lift to a simplicial object of $\widetilde{\mathcal{A}}$, [13, (5.4)].

Denote the Deligne–Beilinson cohomology of a smooth (simplicial) variety X by $H_\mathcal{D}^\bullet(X, \mathbb{Q}(p))$. In [13, Sect. 12] the multivalued Deligne cohomology of a simplicial object $X_\bullet$ of $\widetilde{\mathcal{A}}$ was defined. It will be denoted by $H_{\mathcal{MD}}^\bullet(X_\bullet, \mathbb{Q}(p))$.

There are several equivalent ways to define rational $K(\pi, 1)$ spaces, but for our purposes in the introduction, perhaps the most pertinent comment is that a Zariski open subset U of a grassmannian is a rational $K(\pi, 1)$ if and only if

$$
H^\bullet(W_l \widetilde{\Omega}^\bullet(U)) = \mathbb{C}
$$

for all $l \geqslant 0$. Using this, we show in Section 9 that if $X_\bullet$ is a simplicial object of $\widetilde{\mathcal{A}}$ and each X_q is a rational $K(\pi, 1)$, then there is a natural isomorphism

$$
H_\mathcal{D}^\bullet(X_\bullet, \mathbb{Q}(p)) \approx H_{\mathcal{MD}}^\bullet(X_\bullet, \mathbb{Q}(p)).
$$

This was stated without proof in [13, (12.3)].

The main idea of this paper is to exploit this fact by replacing $G_\bullet^p$ by a Zariski open subset $U_\bullet^p$ where each U_q^p is a rational $K(\pi, 1)$. Once one has done this and established that $U_\bullet^p$ lifts to a simplicial object of $\widetilde{\mathcal{A}}$, the line of the proof of the existence of higher logarithms is relatively clear – there is a natural $\mathrm{GL}_p(\mathbb{C})$ bundle over $U_\bullet^p$ whose 'pth Chern class' (the existence of such a Chern class is not clear) is an element of

$$
H_\mathcal{D}^{2p}(U_\bullet^p, \mathbb{Q}(p)) \approx H_{\mathcal{MD}}^{2p}(U_\bullet^p, \mathbb{Q}(p)).
$$

The pth higher logarithm is a suitable rational multiple of this class.[*]

To recapitulate, one of the main obstacles to proving the existence of the pth higher logarithm is to establish the existence of such a Zariski open subset $U_\bullet^p$ of $G_\bullet^p$ where each U_q^p is a rational $K(\pi, 1)$. If the Zariski topology had the property that each open set contains another open which is a rational $K(\pi, 1)$, then one could easily find the sought after open subset $U_\bullet^p$ of $G_\bullet^p$. Unfortunately, this is not

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[*] If one only wants the multivalued function, or the higher logarithm in the sense of [13, (6.1)], then one can appeal directly to the analogue of [13, (8.9)] for $U_\bullet^p$ – cf. (7.4).

the case (cf. [11, (9.7)]). For this reason we introduce a coarser topology on the G_q^p, called the *constructible topology*, which does enjoy this property. This is done in Section 4.

We conclude the introduction with a brief description of the constructible topology and the idea behind the proof of the existence of $U_\bullet^p$. The first point is that each $\xi \in G_q^p$ determines an ordered configuration of $p + q + 1$ points in $\mathbb{P}^{p-1}$, no p of which lie on a hyperplane; the configuration is well defined up to projective equivalence. To see how this works, note that the set of $(q+1)$-dimensional planes in $\mathbb{P}^{p+q}$ which contain ξ comprise a projective space of dimension $p - 1$. The jth point of the configuration is the point of this projective space which corresponds to the join of the jth standard basis vector with ξ. Each such configuration determines a configuration of hyperplanes in $\mathbb{P}^{p-1}$ – the hyperplanes are those determined by the $(p - 1)$-element subsets of the points. The configuration corresponding to an element of G_2^3 and the corresponding arrangement of hyperplanes in $\mathbb{P}^2$ are illustrated in Fig. 1.

A configuration of hyperplanes in $\mathbb{P}^{p-1}$ corresponds to a central configuration of hyperplanes in $\mathbb{C}^p$. Denote the central configuration in $\mathbb{C}^p$ which corresponds to $\xi \in G_q^p$ by $\mathcal{C}(\xi)$.

One's natural instinct when trying to understand the topology of the G_q^p is to use the face maps $A_i \colon G_q^p \to G_{q-1}^p$ to study them inductively. The 'standard mistake' is to believe that all such face maps are fibrations. If they were, life would be easier, but less interesting. It is worthwhile to see how the face maps fail to be fibrations as it is relevant to the proof of the existence of $U_\bullet^p$. Observe that the fiber of the face map $A_i \colon G_q^p \to G_{q-1}^p$ over the point $\xi \in G_{q-1}^p$ is the complement of the arrangement $\mathcal{C}(\xi)$ in $\mathbb{C}^p$.

The simplest example where a face map is not a fibration is provided by any of the face maps $A_i \colon G_3^3 \to G_2^3$. The projectivization of the generic fiber is the complement of an arrangement determined by six points in $\mathbb{P}^2$, no three of which lie on a line, and where no three of the lines they determine are concurrent, except at one of the points $x_0, \ldots, x_5$. The complement of the arrangement on the right hand side of Fig. 1 is the projectivization of a special fiber of $A_6 \colon G_3^3 \to G_2^3$ as there is an exceptional triple point. Since the topology of the fiber of $A_6 \colon G_3^3 \to G_2^3$ is not constant, A_6 is not a fibration.

The basic closed subsets of G_q^p in the constructible topology are defined to be the closure of the set of points ξ where the combinatorics of $\mathcal{C}(\xi)$ is fixed. For example, the closure of the set of points in G_2^3 where the lines $x_0 x_2$, $x_1 x_3$ and $x_4 x_5$ intersect in a single point (as in Fig. 1) is a closed subset of G_2^3 in the constructible topology. The combinatorial objects which parameterize the closed sets are called *templates*.

Observe that $A_0 \colon G_3^3 \to G_2^3$ is a fibration over the constructible open subset of G_2^3 which consists of all ξ for which the projectivization of $\mathcal{C}(\xi)$ contains no exceptional triple points. By passing to a constructible open subset of G_3^3, one can arrange for the generic fiber of A_0 to be the complement of an arrangement of fiber
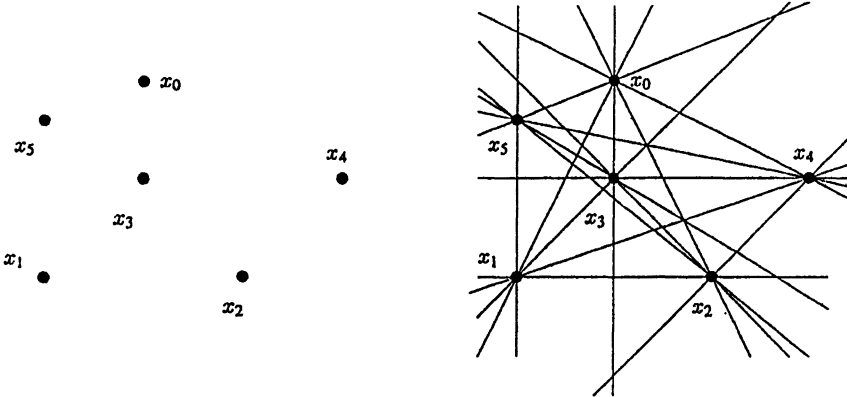
Fig. 1.

type, and by restricting A_0 to a smaller constructible open subset of G_2^3 we may assume that A_0 is a fibration whose fiber is the complement of an arrangement of fiber type. Since arrangements of fiber type are rational $K(\pi, 1)$s, this provides, via (5.2), the inductive step needed for finding the open subset $U_\bullet^p$ of $G_\bullet^p$ in which each U_q^p is a rational $K(\pi, 1)$.

It is assumed that the reader is familiar with [13].

Conventions. In this paper, all simplicial objects are strict – that is, they are functors from the category Δ of finite ordinals and *strictly* order preserving maps to, say, the category of algebraic varieties.

As is standard, the finite set $\{0, 1, \ldots, n\}$ with its natural ordering will be denoted by $[n]$. Let r and s be positive integers with $r \leqslant s$. Denote the full subcategory of Δ whose objects are the ordinals $[n]$ with $r \leqslant n \leqslant s$ by $\Delta[r, s]$. An (r, s) *truncated* simplicial object of a category $\mathcal{C}$ is a contravariant functor from $\Delta[r, s]$ to $\mathcal{C}$.

The word *simplicial* will be used generically to refer to both simplicial objects and truncated simplicial objects.

By Deligne cohomology, we shall mean Beilinson's refined version of Deligne cohomology as defined in [2]. It can be expressed as an extension

$$0 \to \mathrm{Ext}_{\mathcal{H}}^1(\mathbb{Q}, H^{k-1}(X, \mathbb{Q}(p)))$$

$$\to H_{\mathcal{D}}^k(X, \mathbb{Q}(p)) \to \mathrm{Hom}_{\mathcal{H}}(\mathbb{Q}, H^k(X, \mathbb{Q}(p))) \to 0,$$

where $\mathcal{H}$ denotes the category of $\mathbb{Q}$ mixed Hodge structures.

2. Constructible configurations and templates

Fix a ground field $\mathbb{F}$. Denote the projective space $\mathbb{P}^m(\mathbb{F})$ over $\mathbb{F}$ by $\mathbb{P}^m$. By a *configuration* of n points in $\mathbb{P}^m$, we shall mean an element $\mathbf{x}$ of $(\mathbb{P}^m)^n$. A
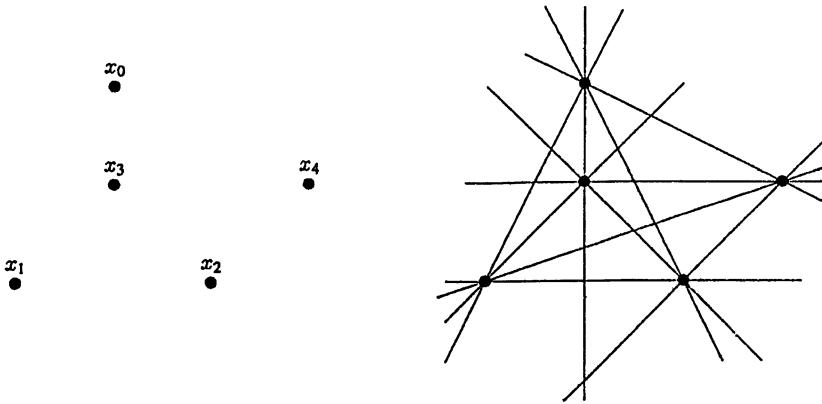
Fig. 2.

subconfiguration of $\mathbf{x}$ is any element of $(\mathbb{P}^m)^l$, $l \leqslant n$, obtained by deleting some of the components of $\mathbf{x}$.

A *linear configuration* in $\mathbb{P}^m$ is a finite collection of linear subspaces of $\mathbb{P}^m$. The *complete configuration* $\widehat{\mathcal{H}}$ associated to a linear configuration $\mathcal{H} = \{L_1, \ldots, L_l\}$ in $\mathbb{P}^m$ is the configuration consisting of the L_j and all of their non-empty intersections. A linear configuration is *complete* if it equals its completion.

The *join* of two linear subspaces L_1, L_2 of $\mathbb{P}^m$ is the smallest linear subspace of $\mathbb{P}^m$ which contains them both. It will be denoted by $L_1 * L_2$.

DEFINITION 2.1. The set $\mathcal{D}(\mathbf{x})$ of linear configurations in $\mathbb{P}^m$ *derived* from a particular configuration $\mathbf{x}$ of n points in $\mathbb{P}^m$ is the unique set of linear configurations in $\mathbb{P}^m$ which satisfies the following properties:

1. the completion of the configuration consisting of all hyperplanes in $\mathbb{P}^m$ that are spanned by a subconfiguration of $\mathbf{x}$ is in $\mathcal{D}(\mathbf{x})$;
2. every $\mathcal{H} \in \mathcal{D}(\mathbf{x})$ is complete;
3. if $\mathcal{H} \in \mathcal{D}(\mathbf{x})$, $L \in \mathcal{H}$ and $\mathcal{X}$ is a subconfiguration of $\mathbf{x}$ such that $L * \mathrm{span}\mathcal{X}$ is a hyperplane, then the completion of $\mathcal{H} \cup \{L * \mathrm{span}\mathcal{X}\}$ is also in $\mathcal{D}(\mathbf{x})$.

EXAMPLE 2.2. Let $\mathbf{x}$ be the configuration $(x_0, x_1, x_2, x_3, x_4)$ of 5 points in $\mathbb{P}^2(\mathbb{R})$ depicted in the left half of Fig. 2. The right half of Fig. 2 depicts the configuration defined in (1) of the definition of $\mathcal{D}(\mathbf{x})$. Every other linear configuration $\mathcal{H}$ in $\mathcal{D}(\mathbf{x})$ contains this configuration. The first linear configuration depicted in Fig. 3 is in $\mathcal{D}(\mathbf{x})$, the second is not.
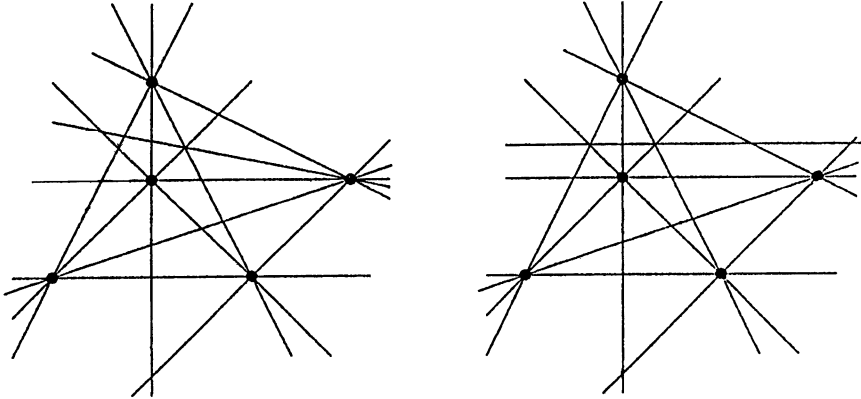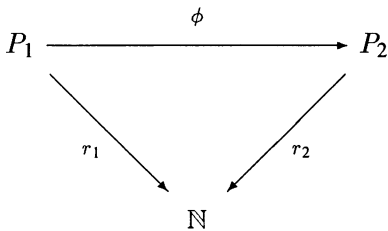
Fig. 3.

The class of all order preserving functions $r: P \to \mathbb{N}$ from a partially ordered set into $\mathbb{N}$ forms a category $\mathcal{P}$. A morphism Φ from $r_1: P_1 \to \mathbb{N}$ to $r_2: P_2 \to \mathbb{N}$ is an order preserving function $\phi: P_1 \to P_2$ such that the diagram

$$
\begin{array}{ccc}
P_1 & \xrightarrow{\phi} & P_2 \\
& {}_{r_1}\searrow \quad \swarrow{}_{r_2} & \\
& \mathbb{N} &
\end{array}
$$

commutes. We shall denote the k-*dimensional elements* $r^{-1}(k)$ of P by P_k.

DEFINITION 2.3. A *template* is an isomorphism class of objects of the category $\mathcal{P}$.

Templates are a generalization of matroids.

Each linear configuration $\mathcal{H}$ in $\mathbb{P}^m$ is a partially ordered set – the linear sub-spaces are ordered by inclusion. Define a rank function $r: \mathcal{H} \to \mathbb{N}$ by defining $r(L) = \dim L$ for each $L \in \mathcal{H}$. In this way we associate a template to each linear configuration.

The template of a linear configuration $\mathcal{H}$ derived from a configuration of points $\mathbf{x}$ in $\mathbb{P}^m$ has additional structure; namely, the marking of the points of $\mathbf{x}$. For this reason, we now define marked templates.

Denote the set $\{0, 1, \ldots, n\}$ by $[n]$. One can consider the class of triples (P, r, ψ), where P is a partially ordered set, $r: P \to \mathbb{N}$ is an order preserving function, and where $\psi: [n] \to P_0$ is a function. These form a category $\widetilde{\mathcal{P}}_n$; the morphisms are order preserving maps which preserve the rank functions r and the markings ψ.

DEFINITION 2.4. An n-*marked template* is an isomorphism class of objects of the category $\tilde{\mathcal{P}}_n$.

Each linear configuration derived from a configuration $\mathbf{x}$ of $n + 1$ points in $\mathbb{P}^m$ determines an n-marked template. If $\mathbf{x} = (x_0, x_1, \ldots, x_n)$, then the marking $\psi : [n] \to \mathcal{H}_0$ is defined by $\psi(j) = x_j$. We will view $\mathcal{D}(\mathbf{x})$ as a set of marked linear configurations. Denote the set of n-marked templates

$$\{T(\mathcal{H}) : \mathcal{H} \in \mathcal{D}(\mathbf{x})\}$$

associated to a configuration $\mathbf{x}$ of $n + 1$ points in $\mathbb{P}^m$ by $T(\mathbf{x})$. Taking $\mathcal{H}$ to $T(\mathcal{H})$ defines a bijection

$$\mathcal{D}(\mathbf{x}) \to T(\mathbf{x}).$$

We shall denote the element of $\mathcal{D}(\mathbf{x})$ which corresponds to $T \in T(\mathbf{x})$ by $\mathcal{H}_T$.

The group of projective equivalences PGL_{m+1} acts on the set of linear configurations in $\mathbb{P}^m$. Observe that if two linear configurations are projectively equivalent, they determine the same template. Consequently, $T(\mathbf{x})$ depends only on the projective equivalence class of $\mathbf{x}$.

3. Hyperplane arrangements of fiber type

We retain the notation of the previous section. We inductively define what it means for an arrangement of hyperplanes in $\mathbb{F}^n$ to be of *fiber type*. First, every arrangement of distinct points in $\mathbb{F}$ is of fiber type. An arrangement of hyperplanes $\mathcal{H}$ in $\mathbb{F}^n$ is of fiber type if there is a linear projection $\phi : \mathbb{F}^n \to \mathbb{F}^{n-1}$ and an arrangement of hyperplanes $\mathcal{A}$ in $\mathbb{F}^{n-1}$ of fiber type such that

  (a) the arrangement $\phi^{-1}\mathcal{A}$ is a sub-arrangement of $\mathcal{H}$;
  (b) the image under ϕ of each element of $\mathcal{H} - \phi^{-1}\mathcal{A}$ is all of $\mathbb{F}^{n-1}$;
  (c) for each $u \in \mathbb{F}^{n-1} - \cup\mathcal{A}$, the number of points in the induced arrangement of points $\phi^{-1}(u) \cap \mathcal{H}$ of $\phi^{-1}(u)$ by $\mathcal{H}$ is independent of u.

When $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, the conditions (a), (b) and (c) imply that the projection $\psi : \mathbb{C}^n - \bigcup\mathcal{H} \to \mathbb{C}^{n-1} - \bigcup\mathcal{A}$ is a topological fiber bundle.

PROPOSITION 3.1. *If $\mathbf{x}$ is a configuration of n points in $\mathbb{P}^m$, then, for each $T \in T(\mathbf{x})$, there is $T' \in T(\mathbf{x})$ such that $\mathcal{H}_T \subseteq \mathcal{H}_{T'}$ and such that $\mathcal{H}_{T'}$ is an arrangement of fiber type.*

Proof. We prove the result by induction on m. The result is trivially true when $m = 1$. Now suppose that $m \geqslant 1$. The image of $(x_0, \ldots, x_{n-1})$ under the linear projection

$$\phi : \mathbb{P}^m - \{x_n\} \to \mathbb{P}(T_{x_n}\mathbb{P}^m) \approx \mathbb{P}^{m-1}$$

is an $(n-1)$-marked configuration $\mathbf{z}$ of points in $\mathbb{P}^{m-1}$. The linear subspaces $L \in \mathcal{H}_T$ which contain x_0 induce a linear arrangement $\overline{\mathcal{H}}_T$ in $\mathbb{P}(T_{x_n}\mathbb{P}^m)$. It is easy to check that $\overline{\mathcal{H}}_T \in \mathcal{D}(\mathbf{z})$. By induction, there is a template $S \in \mathcal{T}(\mathbf{z})$ such that the arrangement $\mathcal{H}_S$ in $\mathbb{P}(T_{x_n}\mathbb{P}^m)$ is of fiber type and contains $\overline{\mathcal{H}}_T$. The inverse image of $\mathcal{H}_S$ under ϕ is an arrangement $\widetilde{\mathcal{H}}_S$ of hyperplanes in $\mathbb{P}^m$ each of whose hyperplanes contains x_n. The completion of the linear arrangement

$$\mathcal{H} := \widetilde{\mathcal{H}}_S \cup \mathcal{H}_T$$

is an element of $\mathcal{D}(\mathbf{x})$. The projection ϕ induces a linear projection

$$\psi \colon \mathbb{P}^m - \cup\mathcal{H} \to \mathbb{P}(T_{x_n}\mathbb{P}^m) - \cup\mathcal{H}_S$$

whose fibers are punctured lines. Adding to $\mathcal{H}$ the hyperplanes in $\mathbb{P}^m$ which are the join of x_n with a codimension 2 stratum of $\mathcal{H}$ we obtain a linear arrangement $\mathcal{H}'$ in $\mathbb{P}^m$ such that the restriction of ψ to $\mathbb{P}^m - \cup\mathcal{H}'$ is a linear map

$$\mathbb{P}^m - \cup\mathcal{H}' \to \mathbb{P}(T_{x_n}\mathbb{P}^m) - \cup\mathcal{H}_S$$

each of whose fibers is $\mathbb{P}^1$ minus the same number of points. That is, the arrangement $\mathcal{H}'$ is of fiber type. Let $T' \in \mathcal{T}(\mathbf{x})$ be the template which corresponds to the completion of $\mathcal{H}'$. Then $\cup\mathcal{H}_{T'} = \cup\mathcal{H}'$, and so $\mathcal{H}_{T'}$ is an arrangement of fiber type which contains $\mathcal{H}_T$. $\qquad\square$

4. The generic grassmannians

As in the previous sections, $\mathbb{F}$ will denote a fixed ground field. First recall that the grassmannian $G(q, \mathbb{P}^{p+q})$ of q-dimensional subspaces of $\mathbb{P}^{p+q}$ can be viewed as the orbit space

$$\left\{ (v_0, v_1, \ldots, v_{p+q}) \in (\mathbb{F}^p)^{p+q+1} : v_0, \ldots, v_{p+q} \text{ span } \mathbb{F}^p \right\} / \mathrm{GL}_p(\mathbb{F}),$$

where GL_p acts diagonally (cf. [13, Sect. 5]).

The generic part G_q^p of $G(q, \mathbb{P}^{p+q})$ is defined to be the set of those points in $G(q, \mathbb{P}^{p+q})$ which correspond to $(p+q+1)$-tuples of vectors $(v_0, \ldots, v_{p+q})$ in $\mathbb{F}^p$ where each p of the vectors are linearly independent.

The torus

$$(\mathbb{G}_m)^{p+q} \approx (\mathbb{G}_m)^{p+q+1} / \text{ diagonal}$$

acts on G_q^p via the action

$$(\lambda_0, \ldots, \lambda_{p+q}) : (v_0, \ldots, v_{p+q}) \mapsto (\lambda_0 v_0, \ldots, \lambda_{p+q} v_{p+q}).$$
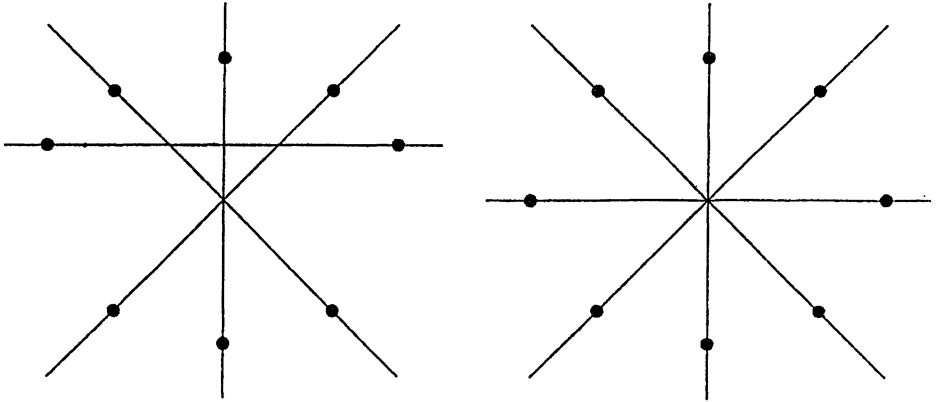
Fig. 4.

The quotient space is the variety

$$Y_q^p := \left\{ (x_0, \ldots, x_{p+q}) \in (\mathbb{P}^{p-1})^{p+q+1} : \text{each } p \text{ of the points span } \mathbb{P}^{p-1} \right\} / \mathrm{PGL}_p.$$

The morphism $\pi \colon G_q^p \to Y_q^p$ is a principal $(\mathbb{G}_m)^{p+q}$-bundle with a section [13, (5.9)]. Consequently,

$$G_q^p \approx Y_q^p \times (\mathbb{G}_m)^{p+q}.$$

Denote the point of $\mathbb{P}(V)$ which corresponds to $v \in V - \{0\}$ by $[v]$. The point v of G_q^p corresponding to the orbit of $(v_0, v_1, \ldots, v_{p+q})$ determines the point

$$\mathbf{x}(v) = ([v_0], [v_1], \ldots, [v_{p+q}])$$

of Y_q^p. We can therefore associate to each point of G_q^p the set $T(\mathbf{x}(v))$ of $(p+q)$-marked templates.

For each $(p+q)$-marked template T, define the subset $E_q^p(T)$ of G_q^p to be the Zariski closure of

$$\left\{ v \in G_q^p : T \in T(\mathbf{x}(v)) \right\}.$$

We will define two templates T_1 and T_2 to be (p, q)-*equivalent* if the subvarieties $E_q^p(T_1)$ and $E_q^p(T_2)$ of G_q^p are equal.

EXAMPLE 4.1. The two configurations in Fig. 4 determine templates T_1 and T_2, respectively. Both $E_4^3(T_1)$ and $E_4^3(T_2)$ are proper subvarieties of G_4^3, and $E_4^3(T_2)$ is a proper subvariety of $E_4^3(T_1)$.

Define $F_q^p(T) \subseteq Y_q^p$ to be the quotient of $E_q^p(T)$ by the torus action. Observe that:

PROPOSITION 4.2. *For each template T, the varieties $E_q^p(T)$ and $F_q^p(T) \times$* $(\mathbb{G}_m)^{p+q}$ *are isomorphic.*   □

DEFINITION 4.3. The *constructible topology* on G_q^p is the topology whose closed sets are finite unions of the sets $E_q^p(T)$. The *constructible topology* on Y_q^p is the topology whose closed sets are finite unions of the sets $F_q^p(T)$. The constructible topology on a subset of G_q^p or Y_q^p is the topology induced from the constructible topology on G_q^p or Y_q^p. In particular, the sets $E_q^p(T)$ and $F_q^p(T)$ have constructible topologies.

Evidently, the closed subsets of G_q^p are precisely the inverse images of closed subsets of Y_q^p under the projection $G_q^p \to Y_q^p$. Note that the constructible topology is coarser than the Zariski topology.

PROPOSITION 4.4. *For each (p, q)-marked template T,*

$$\left\{ v \in F_q^p(T) : T \in \mathcal{T}(\mathbf{x}(v)) \right\}$$

is a constructible open subset of $F_q^p(T)$.   □

5. Rational $K(\pi, 1)$ spaces

In this section we briefly review the definition and basic properties of rational $K(\pi, 1)$ spaces. Relevant references include [7, 11, 13 and 18].

As motivation, recall that if a topological space X is a $K(\pi, 1)$, then there is a natural isomorphism

$$H^\bullet(X, \mathbb{M}) \approx H^\bullet(\pi_1(M), M),$$

where M is a $\pi_1(M, *)$ module, and $\mathbb{M}$ denotes the corresponding local system over X.

One can define the continuous cohomology of a group π by

$$H_{cts}^\bullet(\pi; \mathbb{Q}) = \varinjlim_{\Gamma} H^\bullet(\Gamma, \mathbb{Q}),$$

where Γ ranges over all finitely generated nilpotent quotients of π. There is an evident map

$$H_{cts}^\bullet(\pi, \mathbb{Q}) \to H^\bullet(\pi, \mathbb{Q}).$$

A topological space X is defined to be a rational $K(\pi, 1)$ if the composition

$$H_{cts}^\bullet(\pi_1(X), \mathbb{Q}) \to H^\bullet(\pi_1(X), \mathbb{Q}) \to H^\bullet(X, \mathbb{Q})$$

is an isomorphism. Every nilmanifold is clearly both a $K(\pi, 1)$ and a rational $K(\pi, 1)$. In particular, the circle is both a $K(\pi, 1)$ and a rational $K(\pi, 1)$.

The following results will be used in Section 6. Proofs of them can be found in [11, Sect. 5].

THEOREM 5.1. *The one point union of two rational $K(\pi, 1)$s is a rational $K(\pi, 1)$. In particular, every Zariski open subset of $\mathbb{C}$ is both a $K(\pi, 1)$ and a rational $K(\pi, 1)$.*   □

THEOREM 5.2. *Suppose that $f: X \to Y$ is a fiber bundle with fiber F. If Y and F are rational $K(\pi, 1)$s, and if the natural action of $\pi_1(Y, y)$ on each cohomology group of F is unipotent, then X is a rational $K(\pi, 1)$.*   □

Since each Zariski open subset of $\mathbb{C}$ is both a $K(\pi, 1)$ and a rational $K(\pi, 1)$, and since the monodromy representations associated to a linear fibration is trivial [11, (5.12)] we obtain the following result:

COROLLARY 5.3. *The complement of an arrangement of hyperplanes in $\mathbb{C}^n$ which is of fiber type is both a $K(\pi, 1)$ and a rational $K(\pi, 1)$.*   □

6. The main theorem

In this section, we prove the following result.

THEOREM 6.1. *Each constructible open subset of $G_q^p(\mathbb{C})$ contains a constructible open subset which is both a $K(\pi, 1)$ and a rational $K(\pi, 1)$.*

Remark 6.2. It is easy to show that in the cases of G_0^p and G_1^p, the constructible topology is trivial. That is, the only constructible open sets in these spaces are the the empty set and the whole space. Thus Theorem 6.1 implies that $G_0^p(\mathbb{C})$ and $G_1^p(\mathbb{C})$ are $K(\pi, 1)$s and rational $K(\pi, 1)$s. This is clear in the case of G_0^p, and is proved directly in the case of G_1^p in [13, (8.6)].

The proof of Theorem 6.1 occupies the rest of this section. Since the classes of rational $K(\pi, 1)$s and $K(\pi, 1)$s are closed under products, and since each constructible open subset of G_q^p is a product of the corresponding constructible open subset of Y_q^p with $(\mathbb{C}^*)^{p+q}$, we need only prove that each constructible open subset of Y_q^p contains a constructible open set which is a both a $K(\pi, 1)$ and a rational $K(\pi, 1)$.

Suppose that $0 \leqslant i \leqslant p + q$. The ith *face map*,

$$A_i: Y_q^p \to Y_{q-1}^p$$

is defined by forgetting the ith point of a configuration of $p + q$ points in $\mathbb{P}^{p-1}$. The ith *dual face map*

$$B_i: Y_q^p \to Y_q^{p-1}$$

is obtained by projecting all but the ith point of a configuration of $p + q$ points in $\mathbb{P}^{p-1}$ onto a generic $\mathbb{P}^{p-2}$ using the ith point as the center of the projection.
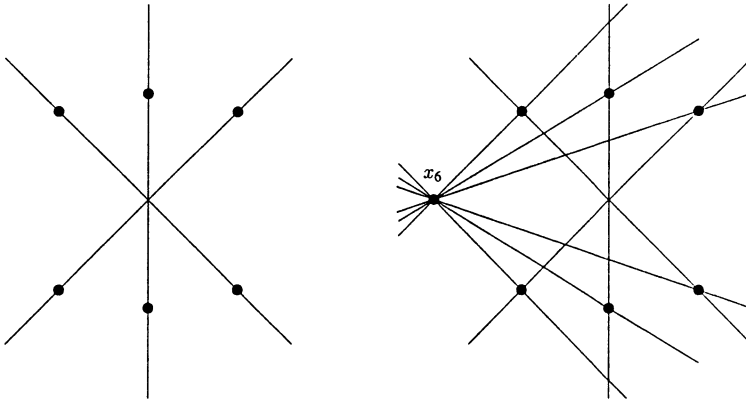
Fig. 5.

PROPOSITION 6.3. *If T is a $(p + q)$-marked template, then for each integer i satisfying $0 \leqslant i \leqslant p + q + 1$, there is a $(p + q + 1)$-marked template $A^i T$ (resp. $B^i T$) whose $(p, q+1)$-equivalence class (resp. $(p+1, q)$-equivalence class) depends only on the (p, q)-equivalence class of T. Moreover,*

$$A_i^{-1} F_q^p(T) = F_{q+1}^p(A^i T), \quad A_i^{-1} E_q^p(T) = E_{q+1}^p(A^i T)$$

and

$$B_i^{-1} F_q^p(T) = F_q^{p+1}(B^i T), \quad B^i E_q^p(T) = E_q^{p+1}(B^i T).$$

In particular, the face maps and dual face maps are continuous with respect to the constructible topology.

Proof. For simplicity of notation, we take $i = p + q + 1$. Suppose that T is a $(p + q)$-marked template. Represent it by the object (P, r, ψ) of the category $\widetilde{\mathcal{P}}_{p+q}$. Denote the marked elements $\psi(0), \ldots, \psi(p + q)$ of P_0 by $p_0, \ldots, p_{p+q}$. For each subset I of $\{0, \ldots, p + q\}$, denote the element of P which is the least upper bound of $\{p_i : i \in I\}$ by p_I. Set $r_I = r(p_I)$. Define $A^i T$ and $B^i T$ both to be the isomorphism class of the completion of the marked ordered set obtained from (P, r, ψ) by adding one extra element p_{p+q+1} to P_0, and elements $p_I * p_{p+q+1}$ to P_{1+r_I}. This is the 'smallest' template T' for which $A_i T' = T$.

For example, if T is the 6-marked template associated to the configuration on the left hand side of Fig. 5, then $A^6 T$ is the 7-marked template associated to the configuration on the right hand side of Fig. 5.

For $v \in Y_q^{p+1}$, it is clear that $T \in \mathcal{T}(\mathbf{x}(A_i v))$ if and only if $A^i T \in \mathcal{T}(\mathbf{x}(v))$. It follows that

$$A_i^{-1} F_q^p(T) = F_q^{p+1}(A^i T)$$

and that the $(p + 1, q)$-equivalence class of $A^i T$ depends only on the (p, q)-equivalence class of T.

The corresponding statement for $E_q^p(T)$ follows from (4.2). The statements with A replaced by B follow using the dual argument. $\qquad\qquad\qquad\square$

The following definition is an analogue of (2.1) for templates. It is used only in the proof of the next result.

DEFINITION 6.4. Suppose that (P, r, ψ) is an object of the category $\widetilde{\mathcal{P}}_n$. Suppose that $f: [m] \to [n]$ is an order preserving injection. Define the subset Q of P *generated* by f to be the smallest subset of P which contains $\{\psi \circ f(j) : j \in [m]\}$ and is closed under the following operations:

1. if $S \subseteq \mathrm{im} f$, then the least upper bound of S in P is in Q;
2. if $S \subseteq Q$, then the greatest lower bound of S in P is in Q;
3. if $v \in Q$ and $S \subseteq \mathrm{im} f$, then the greatest lower bound of S and v in P is in Q.

PROPOSITION 6.5. *If T is a $(p+q)$-marked template, then for each integer i satisfying $0 \leqslant i \leqslant p+q$, there is a $(p+q-1)$-marked template $A_i T$ (resp. $B_i T$) whose $(p, q-1)$-equivalence class (resp. $(p-1, q)$-equivalence class) depends only on the (p, q)-equivalence class of T. Moreover, $A_i F_q^p(T)$ is a constructible open subset of $F_{q-1}^p(A_i T)$, $A_i E_q^p(T)$ is a constructible open subset of $E_{q-1}^p(A_i T)$, $B_i F_q^p(T)$ is a constructible open subset of $F_q^{p-1}(B_i T)$ and $B_i E_q^p(T)$ is a constructible open subset of $E_q^{p-1}(B_i T)$.*

Proof. Suppose that T is a $(p + q)$-marked template. Let (P, r, ψ) be an object of the category $\widetilde{\mathcal{P}}_{p+q}$ which represents T. Let Q be the partially ordered subset of P generated by the ith face map $d_i: [p + q] \to [p + q + 1]$ – that is, the unique order preserving injection which omits the value i. Let $(Q, r, \psi \circ d_i)$ be the object of $\widetilde{\mathcal{P}}_{p+q-1}$ where r is the restriction of the rank function of P. Define $A_{p+q} T$ to be the $(p + q - 1)$-marked template which is represented by (Q, r, ψ).

For example, if T is the template associated to the configuration in the left hand side of Fig. 6, then $A_2 T$ is the template corresponding to the configuration on the right hand side of Fig. 6.

It is clear that $A_i F_q^p(T) \subseteq F_{q-1}^p(A_i T)$. That $A_i F_q^p(T)$ is a constructible open subset of $F_{q-1}^p(A_0 T)$ follows directly from (4.4).

The corresponding statements for $E_q^p(T)$ follows from (4.2). The statements with A replaced by B follow using the dual argument. $\qquad\qquad\qquad\square$

EXAMPLE 6.6. An example where $A_0 F_q^p(T)$ is a proper subset of $F_{q-1}^p(A_0 T)$ is given in Fig. 7. If T is the 13-marked template associated to the left hand figure, then the right hand configuration is an element of $F_8^3(A_0 T) - A_0(F_9^3(T))$.

COROLLARY 6.7. *If T is a $(p + q)$-marked template, then*

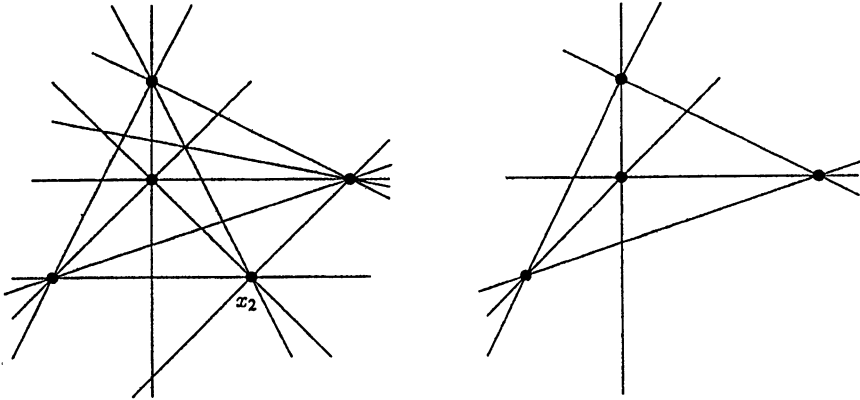$$F_q^p(T) \subseteq F_q^p(A^i A_i T) \quad and \quad E_q^p(T) \subseteq E_q^p(A^i A_i T)$$

Fig. 6.

Fig. 7.

for all integers i satisfying $0 \leqslant i \leqslant p + q$, and if T is a $(p + q)$-marked template, then

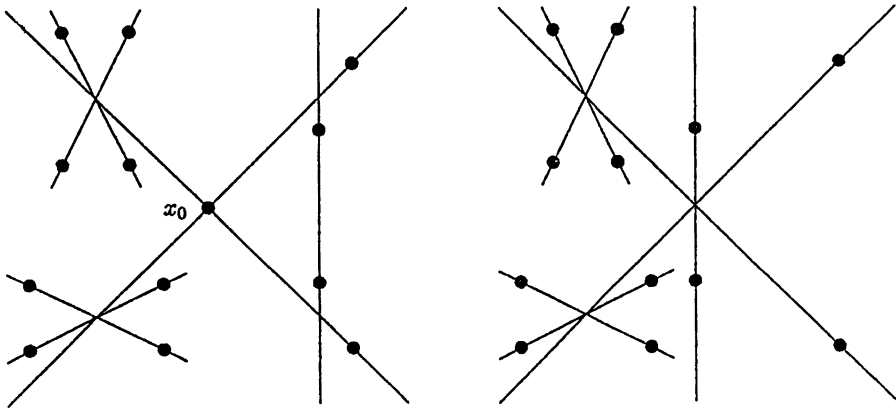$$F_q^p(T) \subseteq F_q^p(B^i B_i T) \quad and \quad E_q^p(T) \subseteq E_q^p(B^i B_i T)$$

for all integers i such that $0 \leqslant i \leqslant p + q + 1$.   □

We are now ready to prove Theorem 6.1. Throughout the remainder of this section, the ground field will be $\mathbb{C}$, unless explicitly stated to the contrary. Fix $p > 0$. The proof is by induction on q. When $q = 0$, Y_q^p is a point and the result is trivially true. Now suppose that $q > 0$ and that the result is true for Y_{q-1}^p. Suppose that U is a non-empty constructible open subset of Y_q^p. The idea behind the proof is to replace U by a smaller constructible open set L whose image under A_0 is a constructible open subset of Y_{q-1}^p and such that the map $L \to A_0 L$ is a fibration whose fibers are complements of arrangements of hyperplanes in $\mathbb{P}^{p-1}$ of fiber type

and whose monodromy representations are trivial. Using the inductive hypothesis, one then finds a constructible open subset L' of $A_0 L$ which is both a $K(\pi, 1)$ and a rational $K(\pi, 1)$. It will then follow from (5.2) that $A_0^{-1}(L')$ is the sought after constructible open subset of U. We now give the details.

Our first task is to find a constructible open subset W of Y_{q-1}^p such that the restriction of A_0 to $U \cap A_0^{-1} W$ is a family of hyperplane complements where each relative hyperplane is proper over the base. There are $(p + q)$-marked templates $T_1, \ldots, T_l$ such that

$$U = Y_q^p - \bigcup_{j=1}^{l} F_q^p(T_j).$$

For each j, either $F_{q-1}^p(A_0 T_j)$ is all of Y_{q-1}^p or is a proper closed subvariety. We may suppose that $F_{q-1}^p(A_0 T_j)$ is Y_{q-1}^p when $j \leqslant k$ and is a proper subvariety when $j > k$. When $j \leqslant k$, set

$$C_j = Y_{q-1}^p - A_0 F_q^p(T_j);$$

this is a constructible closed proper subset of Y_{q-1}^p by (4.4). Set

$$W = Y_{q-1}^p - \left(\bigcup_{j \leqslant k} C_j \cup \bigcup_{j > k} F_{q-1}^p(A_0 T_j) \right).$$

Then W is a non-empty constructible open subset of Y_{q-1}^p and the restriction of $A_0 \colon A_0^{-1} W \to W$ to $F_q^p(T_j)$ is proper and surjective when $j \leqslant k$.
    Now

$$A_0^{-1} W \cap U = A_0^{-1} W - \bigcup_{j \leqslant k} F_q^p(T_j).$$

is a constructible open subset of U. The fiber of

$$A_0 \colon A_0^{-1} W - \bigcup_{j \leqslant k} F_q^p(T_j) \to W$$

over $(x_1, \ldots, x_{p+q})$ is the complement of an arrangement of hyperplanes in $\mathbb{P}^{p-1}$ which is derived from the configuration $(x_1, \ldots, x_{p+q})$ and where each relative hyperplane is proper over W.

    Our next task is to replace W by a smaller constructible open set O such that the restriction of A_0 to $U \cap A_0^{-1} O$ is a fiber bundle over O.

    We say that two linear configurations in $\mathbb{C}^m$ have the *same combinatorics* if their associated partially ordered sets are isomorphic, or equivalently, if they determine the same template.

PROPOSITION 6.8. *There is a non-empty constructible open subset O of W such that the restriction of*

$$A_0 \colon A_0^{-1}W - \bigcup_{j \leqslant k} F_q^p(T_j) \to W$$

to $A_0^{-1}(O)$ has the property that each of its fibers is the complement of a linear configuration with the same combinatorics. Consequently,

$$A_0 \colon A^{-1}(O) \to O$$

is a fiber bundle where the action of $\pi_1(O, *)$ on the homology of the fibers is trivial.

Proof. Let $\mathbb{F}$ be the function field of Y_{q-1}^p. Then $\bigcup_{j \leqslant k} F_q^p(T_j)$ is a configuration of hyperplanes in $\mathbb{P}^{p-1}(\mathbb{F})$. For generic $v \in W$, the combinatorics of the restriction of this configuration to the fiber of A_0 over v has the same combinatorics as this configuration over the generic point of Y_{q-1}^p. The set of v for which the combinatorics is different is a closed constructible subset F of Y_{q-1}^p. The desired constructible open subset of Y_{q-1}^p is then $O = W - F$. $\qquad\square$

Next we further shrink both U and O to make the fibers of A_0 to $U \cap A_0^{-1}O$ complements of arrangements of hyperplanes of fiber type.

PROPOSITION 6.9. *There is a non-empty constructible open subset O' of O and a $(p + q)$-marked template T such that $A_0 F_q^p(T)$ contains O' and such that*

$$F_q^p(T) \supseteq \bigcup_{j \leqslant k} F_q^p(T_j)$$

and the map

$$A_0^{-1}O' - F_q^p(T) \to O'$$

induced by A_0 is a fiber bundle all of whose fibers are complements of arrangements of hyperplanes of fiber type.

Proof. As in the proof of the previous result, we shall denote the function field of Y_{q-1}^p by $\mathbb{F}$. The points $x_1, \ldots, x_{p+q}$ are defined over $\mathbb{F}$, and therefore may be regarded as a configuration $\mathbf{x}(\mathbb{F})$ of points in $\mathbb{P}^{p-1}(\mathbb{F})$. The set

$$\bigcup_{j \leqslant k} F_q^p(T_j),$$

is a configuration of hyperplanes defined over $\mathbb{F}$ and thus determines an element of $\mathcal{D}(\mathbf{x}(\mathbb{F}))$. Let $T' \in \mathcal{T}(\mathbf{x}(\mathbb{F}))$ be the corresponding template. By (3.1), there is a template $T \in \mathcal{T}(\mathbf{x}(\mathbb{F}))$ such that

$$\mathcal{H}_T \supseteq \bigcup_{j \leqslant k} F_q^p(T_j)$$

and such that $\mathcal{H}_T$ is an arrangement of fiber type. Since T is defined over the generic point of Y_{q-1}^p, it follows from (6.5) that $A_0 F_q^p(T)$ is a constructible open subset of Y_{q-1}^p. Moreover, the set of $v \in A_0 F_q^p(T)$ for which the combinatorics of the restriction of $\mathcal{H}(T)$ to the fiber of A_0 over v is given by T is a constructible open subset of $A_0 F_q^p(T)$. Let O' be the intersection of this open set with O. $\square$

By our inductive hypothesis, the constructible open set O' of Y_{q-1}^p contains a constructible open subset L which is a $K(\pi, 1)$ and a rational $K(\pi, 1)$. Since $A_0^{-1} L$ is a non-empty constructible open subset of Y_q^p,

$$V = A_0^{-1}(L) - \left(F_q^p(T) \cup \bigcup_{j \leqslant k} F_q^p(T_j) \right)$$

is also a non-empty constructible open subset of U. Further, the map

$$A_0 : V \to L$$

is a fibration each of whose fibers is the complement of an arrangement of hyperplanes in $\mathbb{P}^{p-1}$ of fiber type. It follows from (5.3) that the fibers are $K(\pi, 1)$s and rational $K(\pi, 1)$s. Since the base is a $K(\pi, 1)$ and a rational $K(\pi, 1)$, and since the monodromy is trivial (6.8), it follows from (5.2) that V is a $K(\pi, 1)$ and a rational $K(\pi, 1)$. This completes the proof of Theorem 6.1.

7. Existence and uniqueness of higher logarithms

In this section, we first establish the existence and uniqueness of the pth higher logarithm in the sense of [13, (6.1)], but with $G_\bullet^p$ replaced by a suitably chosen Zariski open subset $U_\bullet^p$. We then show how to construct the generalized p-logarithm, a multivalued Deligne cohomology class, in the sense of [13, (12.4)], but with $G_\bullet^p$ replaced by $U_\bullet^p$. We shall use the notation and definitions of [13].

We will say that a simplicial variety $U_\bullet$ is a *subvariety* of the simplicial variety $X_\bullet$ if each U_q is a subvariety of X_q, and if the inclusion $U_\bullet \hookrightarrow X_\bullet$ is a morphism of simplicial varieties. We will say that $U_\bullet$ is an *open* (resp. *closed, dense, constructible*) subset of $G_\bullet^p$ if each U_q is open (resp. closed, dense, constructible) in each X_q. There are analogous definitions with $G_\bullet^p$ replaced by $Y_\bullet^p$.

PROPOSITION 7.1. *For each positive integer p, each dense constructible open subset $V_\bullet^p$ of the truncated simplicial variety $G_\bullet^p$ contains a dense constructible open subset $U_\bullet^p$ where each U_q^p is a rational $K(\pi, 1)$. In particular, $G_\bullet^p$ contains a dense constructible open subset $U_\bullet^p$ where each U_q^p is a $K(\pi, 1)$ and a rational $K(\pi, 1)$.*

Proof. The only dense constructible open subset of G_0^p is G_0^p itself. So $V_0^p = G_0^p$, and we must take $U_0^p = G_0^p$. Suppose that $m > 0$ and that U_q^p has been constructed when $q < m$ such that each U_q^p is dense in G_q^p, $U_q^p \subseteq V_q^p$, and such that $A_i(U_q^p) \subseteq U_{q-1}^p$ whenever $0 < q < m$. Now,

$$V_m^p \cap \bigcap_{i=0}^{m} A_i^{-1} U_{m-1}^p$$

is a non-empty constructible open subset of G_m^p. So, by (6.1), it contains a non-empty, and therefore dense, constructible open subset V_m^p of G_m^p. The result now follows by induction. □

In order to apply the multivalued de Rham complex functor, we will need to know that such a constructible open subset $U_\bullet^p$ of $G_\bullet^p$ can be lifted to a truncated simplicial object in the category $\widetilde{A}$ defined in the introduction and in [13, Sect. 2].

THEOREM 7.2. *Each constructible open subset $U_\bullet^p$ of G_q^p can be lifted to a truncated simplicial object of the category $\widetilde{A}$.*

The lift is natural in the following sense: it comes with a lift $\tilde{\imath}$ of the inclusion $i: U_\bullet^p \hookrightarrow G_\bullet^p$ such that if $j: V_\bullet^p \hookrightarrow U_\bullet^p$ is an inclusion of constructible open subsets of $G_\bullet^p$, then $\tilde{\imath}\tilde{\jmath} = \widetilde{\imath\jmath}$.

As the proof of this theorem is technical; it is given in a separate section, Section 8.

Next, we show how to construct the pth higher logarithm in the sense of [13, (6.1)] defined on some constructible dense open subset of $G_\bullet^p$.

The following fact is is a direct consequence of [13, (7.8)] and [13, (8.2)(i)].

PROPOSITION 7.3. *If the complex algebraic variety X is a rational $K(\pi, 1)$ with $q(X) = 0$, then for all $l \geqslant 0$, the complex $W_l \widetilde{\Omega}^\bullet(X)$ is acyclic.*

The existence of the higher logarithms is now an immediate consequence of (7.1), (7.2), (7.3) and [13, (9.7)]:

THEOREM 7.4. *For each integer $p \geqslant 1$, there is a dense constructible open subset $U_\bullet^p$ of the simplicial variety $G_\bullet^p$ which has a lift to the category $\widetilde{A}$, and there is an element Z_p of the double complex $W_{2p} \widetilde{\Omega}^\bullet(U_\bullet^p)$, unique up to a coboundary, whose coboundary is the 'volume form'*

$$\frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_p}{x_p} \in \Omega^p(G_0^p).$$

Remark 7.5. With a little more care, one can arrange for each U_q^p to be invariant under the action of the symmetric group Σ_{p+q+1} on G_q^p and for the symbol (as defined in [13, p. 444]) of each component of Z_p to span a copy of the alternating representation. One should note, however, that it seems difficult to arrange for each U_q^p to be a rational $K(\pi, 1)$ and be preserved by the action of Σ_{p+q+1}.

Remark 7.6. Hanamura and MacPherson [17] give an explicit construction of all higher logarithms in the double complex $W_{2p}\tilde{\Omega}(G_\bullet^p)$. In particular, they show that it is not necessary to pass to a Zariski open subset of $G_\bullet^p$ as we did.

Next, we establish the existence of higher logarithms as Deligne cohomology classes. For this, we shall assume the reader is familiar with the definition of the multivalued Deligne cohomology functor $H_{\mathcal{MD}}^\bullet(\_, \mathbb{Q}(p))$ defined in [13, Sect. 12].

The key point here is the following result, a slightly stronger version of which was stated in [13, (12.3)], and which we will prove in Section 9. Recall from the introduction that $H_\mathcal{D}^\bullet$ denotes Beilinson's absolute Hodge cohomology.

THEOREM 7.7. *Suppose that $X_\bullet$ is a truncated simplicial variety with a lift to $\tilde{\mathcal{A}}$. If each X_q is a rational $K(\pi, 1)$, then for each integer p, there is a natural isomorphism*

$$H_{\mathcal{MD}}^\bullet(X_\bullet, \mathbb{Q}(p)) \approx H_\mathcal{D}^\bullet(X_\bullet, \mathbb{Q}(p)).$$

Granted this and results from the section on the descent of Chern classes in [14], the construction of the generalized pth higher logarithm as an element of $H_{\mathcal{MD}}^\bullet(U_\bullet^p, \mathbb{Q}(p))$ is relatively straightforward.

THEOREM 7.8. *If $U_\bullet^p$ is a dense subvariety of $G_\bullet^p$ where each U_q^p is a rational $K(\pi, 1)$, then there is an element of*

$$H_{\mathcal{MD}}^{2p}(U_\bullet^p, \mathbb{Q}(p))$$

whose restriction to G_0^p is the volume form.

Proof. Let V_m^p, be the subvariety of

$$\{(v_0, v_1, \ldots, v_m) : v_j \in \mathbb{C}^p\}$$

which consists of those $(m+1)$-tuples of vectors where each set of $\min(m+1, p)$ of the vectors is linearly independent. When $m \geqslant p$, there is a natural projection $V_m^p \to G_{m-p}^p$ which is a principal $\mathrm{GL}_p(\mathbb{C})$-bundle. Define face maps

$$A_i : V_m^p \to V_{m-1}^p$$

by omitting the ith vector. Denote the corresponding simplicial variety by $V_\bullet^p$. Denote the truncated simplicial space which consists only of those V_m^p with $p \leqslant m \leqslant 2p$ by $\widetilde{V}_\bullet^p$. There is a natural projection $\widetilde{V}_\bullet^p \to G_\bullet^p$ which is a principal $GL_p(\mathbb{C})$-bundle. We would like to say that this bundle has a Chern class

$$c_p \in H_{\mathcal{D}}^{2p}(G_\bullet^p, \mathbb{Q}(p)).$$

Since the variety $G_\bullet^p$ is truncated (it has no simplices in dimensions $< p$), the existence of such a Chern class is not immediate. Our next task is to establish the existence of this class. We do this using the Borel construction.

Let $E_\bullet$ be, say, the standard simplicial model for the universal bundle associated to $GL_p(\mathbb{C})$. What is important for us is that $E_\bullet$ is a simplicial variety with the homotopy type of a point and on which $GL_p(\mathbb{C})$ acts freely. Let $P_\bullet$ be the bisimplicial variety $V_\bullet^p \times E_\bullet$. It has the homotopy type of $V_\bullet^p$. Denote the quotient of $P_\bullet$ by the diagonal action of $GL_p(\mathbb{C})$ by $B_\bullet$. Since $GL_p(\mathbb{C})$ acts freely on $P_\bullet$, the quotient map

$$P_\bullet \to B_\bullet$$

is a principal $GL_p(\mathbb{C})$ bundle. By [1], this bundle has a Chern class

$$c_p \in H_{\mathcal{D}}^{2p}(B_\bullet, \mathbb{Q}(p)). \tag{1}$$

Denote the truncated simplicial variety consisting of those B_m with $p \leqslant m \leqslant 2p$ by $\widetilde{B}_\bullet$ and denote the restriction of the bundle $P_\bullet$ to $\widetilde{B}_\bullet$ by $\widetilde{P}_\bullet$. It is proven in the section on descent of Chern classes in [14] that there is a canonical class

$$\tilde{c}_p \in sH_{\mathcal{D}}^{2p}(\widetilde{B}, \mathbb{Q}(p))$$

whose image under the natural map

$$H_{\mathcal{D}}^{2p}(\widetilde{B}_\bullet, \mathbb{Q}(p)) \to H_{\mathcal{D}}^{2p}(B_\bullet, \mathbb{Q}(p))$$

is the alternating part of c_p.

There is a commutative diagram

$$
\begin{array}{ccc}
\widetilde{P}_\bullet & \longrightarrow & \widetilde{V}_\bullet^p \\
\downarrow & & \downarrow \\
\widetilde{B}_\bullet & \longrightarrow & G_\bullet^p
\end{array}
$$

of principal $GL_p(\mathbb{C})$ bundles obtained by collapsing out $E_\bullet$. Since the action of $GL_p(\mathbb{C})$ on $\widetilde{V}^p$ is free, the bottom arrow is a homotopy equivalence of simplicial varieties, and therefore induces an isomorphism on Deligne cohomology.

We can therefore restrict the Chern class (1) to $G_\bullet^p$ to obtain a class in

$$H_\mathcal{D}^{2p}(G_\bullet^p, \mathbb{Q}(p)).$$

It follows from (7.7) that we can restrict this class to $U_\bullet^p$ to obtain a class C_p in

$$H_{\mathcal{MD}}^{2p}(U_\bullet^p, \mathbb{Q}(p))$$

provided that each U_q^p is a rational $K(\pi, 1)$.

Finally, to prove Theorem 7.8, we have to show that the restriction of C_p to $U_0^p = G_0^p$ is a non-zero multiple of the volume form in $H^p(G_0^p)$. It is proved in [14] that the restriction of C_p to G_0^p is $(p-1)!$vol. It follows that $C_p/(p-1)!$ is a generalized p-logarithm. This completes the proof of Theorem 7.8.   □

8. Proof of Theorem 7.2

In the proof we shall need the following construction. Let

$$X_q^p = \mathbb{C}^{p+q+1} - \Delta,$$

where Δ denotes the fat diagonal – that is, the locus of points in $\mathbb{C}^{p+q+1}$ where the coordinates are not all distinct. Define

$$A_i: X_q^p \to X_{q-1}^p$$

by deleting the ith coordinate:

$$A_i: (t_0, \ldots, t_{p+q}) \mapsto (t_0, \ldots, \hat{t_i}, \ldots, t_{p+q}).$$

Denote the truncated simplicial variety consisting of those X_q^p with $0 \leqslant q \leqslant p$ by $X_\bullet^p$. We can define a morphism $\phi: X_\bullet^p \to G_\bullet^p$ by taking $(t_0, \ldots, t_{p+q})$ to the $\mathrm{GL}_p(\mathbb{C})$ orbit of the $(p + q + 1)$-tuple of vectors

$$\begin{pmatrix} 1 \\ t_0 \\ t_0^2 \\ \vdots \\ t_0^{p-1} \end{pmatrix}, \begin{pmatrix} 1 \\ t_1 \\ t_1^2 \\ \vdots \\ t_1^{p-1} \end{pmatrix}, \ldots \begin{pmatrix} 1 \\ t_{p+q} \\ t_{p+q}^2 \\ \vdots \\ t_{p+q}^{p-1} \end{pmatrix}.$$

This map is easily seen to be a well defined morphism of simplicial varieties. It induces a morphism $\overline{\phi}: X_\bullet^p \to Y_\bullet^p$.

LEMMA 8.1. *The image of X_q^p in G_q^p is dense in G_q^p in the constructible topology.*

Proof. In view of (4.2), we need only prove that the image of X_q^p in Y_q^p is dense in Y_q^p in the constructible topology. We do this by induction on q. Since Y_0^p is a point, the result is trivially true when $q = 0$. Suppose that $q > 0$. Denote the constructible closure of the image of X_q^p in Y_q^p by C_q^p. By induction, $C_{q-1}^p = Y_{q-1}^p$. The intersection of C_q^p with each fiber of $A_0: Y_q^p \to Y_{q-1}^p$ is a constructible closed subset of the fiber. Note that the fiber of $A_0: Y_q^p \to Y_{q-1}^p$ is the complement of a linear arrangement in $\mathbb{P}^{p-1}$ and that each of its constructible closed subsets is the intersection of the fiber with a finite union of linear subspaces of $\mathbb{P}^{p-1}$. Since the intersection of C_q^p with the fiber is an open subset of a rational normal curve in $\mathbb{P}^{p-1}$, and since each rational normal curve is non-degenerate, it follows that the fiber of $A_0: C_q^p \to Y_{q-1}^p$ equals the fiber of $A_0: Y_q^p \to Y_{q-1}^p$. It follows that $C_q^p = Y_q^p$. $\qquad\square$

Denote the topological analogue of the category $\widetilde{\mathcal{A}}$ by $\widetilde{\text{Top}}$. Observe that a simplicial object of $\mathcal{A}$ has a lift to the category $\widetilde{\mathcal{A}}$ if and only if it has a lift to the category $\widetilde{\text{Top}}$.

PROPOSITION 8.2. *Suppose that $Y_\bullet$ and $Z_\bullet$ are simplicial topological spaces where each Y_n and Z_n is path connected. If $f: Y_\bullet \to Z_\bullet$ is a morphism of simplicial spaces, and if $Y_\bullet$ has a lift to $\widetilde{\text{Top}}$, then both $Z_\bullet$ and f have lifts to $\widetilde{\text{Top}}$.*

Proof. We use the equivalence of the category $\widetilde{\mathcal{A}}$ with the category $\mathcal{A}_*$ which is constructed in [13, Sect. 2]. We first construct a simplicial object of $\mathcal{A}_*$ which corresponds to the lift of $Y_\bullet$ to a simplicial object of $\widetilde{\mathcal{A}}$.

Let $\widetilde{Y}_\bullet$ be the simplicial object of $\widetilde{\mathcal{A}}$ which is the lift of $Y_\bullet$. Choose a base point y_n' of $\widetilde{Y}_n$ for each n, and let y_n be its image in Y_n. Each strictly order preserving map $\phi: [m] \to [n]$ induces a morphism $A_\phi': \widetilde{Y}_n \to \widetilde{Y}_m$ of $\widetilde{\mathcal{A}}$ which covers the face map $A_\phi: Y_n \to Y_m$. Since each $\widetilde{Y}_n$ is connected and simply connected, there is a unique homotopy class of paths in $\widetilde{Y}_m$ from y_m' to $A_\phi'(y_n')$. Its image in Y_m is a distinguished homotopy class of paths γ_ϕ in Y_m from y_m to $A_\phi(y_n)$. The pair (A_ϕ, γ_ϕ) is a morphism $(Y_n, y_n) \to (Y_m, y_m)$ in the category $\mathcal{A}_*$ and the collection (Y_n, y_n) of pointed spaces together with the maps (A_ϕ, γ_ϕ) is a simplicial object of $\mathcal{A}_*$.

We now use this to construct a lift of $X_\bullet$ to $\mathcal{A}_*$. Let $x_n = f(y_n)$. For each order preserving injection $\phi: [m] \to [n]$, let μ_ϕ be the homotopy class $f \cdot \gamma_\phi$ of paths in X_m from x_m to $A_\phi(x_n)$. The collection of pointed spaces (X_n, x_n) together with the pairs (A_ϕ, μ_ϕ) is easily seen to be a simplicial object of $\mathcal{A}_*$. Take $\widetilde{X}_n$ to be the standard model of the universal covering space of (X_n, x_n) – it consists of homotopy classes ρ of paths that emanate from x_n. The face maps A_ϕ lift to face maps A_ϕ' by defining $A_\phi'(\rho)$ to be the homotopy class of paths $\mu_\phi \cdot \rho$ in $\widetilde{X}_m$. This is a simplicial object of $\widetilde{\mathcal{A}}$ which lifts $X_\bullet$. $\qquad\square$

COROLLARY 8.3. *Suppose that $Y_\bullet$ and $Z_\bullet$ are simplicial topological spaces where each Y_n and Z_n is path connected. If $f: Y_\bullet \to Z_\bullet$ is a morphism of simplicial spaces, and if each simplex Y_n of $Y_\bullet$ is simply connected, then $Z_\bullet$ has a canonical lift to $\widetilde{\text{Top}}$ such that f is a morphism of $\widetilde{\text{Top}}$.*   □

The following result is needed in the proof of the theorem.

LEMMA 8.4. *Suppose that $f \in \mathbb{R}[t_1, \ldots, t_n]$. If $f \neq 0$, there is a real number $K > 1$ such that f is bounded away from zero in the region*

$$D_n(K) := \left\{ (t_1, \ldots, t_n) : t_1 \geqslant K, t_2 \geqslant K e^{t_1}, \ldots, t_n \geqslant K\, e^{t_{n-1}} \right\}.$$

Proof. The proof is by induction on the number of variables n. The result is trivially true when $n = 1$. Now suppose that $n > 1$ and that the result has been proved for polynomials with fewer than n variables. Set $x = (t_1, \ldots, t_{n-1})$ and $y = t_n$. We can write

$$f = a_d(x)y^d + a_{d-1}(x)y^{d-1} + \cdots + a_1(x)y + a_0(x) \tag{2}$$

where each $a_j(x) \in \mathbb{R}[t_1, \ldots, t_{n-1}]$ and $a_d \neq 0$. If $d = 0$, then we are in the previous case and the result holds by induction. So assume that $d > 0$. By induction, there exist real constants $C > 0$ and $K > 1$ such that $|a_d(x)| \geqslant C$ for all $x \in D_{n-1}(K)$. By a standard estimate, the roots $\theta(x)$ of the polynomial (2) satisfy

$$|\theta(x)| \leqslant 1 + \max_{0 \leqslant j < d} \left| \frac{a_j(x)}{a_d(x)} \right| \leqslant 1 + \max_{0 \leqslant j < d} \frac{|a_j(x)|}{C} \leqslant \|x\|^l$$

for some positive integer l and for each $x \in D_{n-1}(K)$, provided K is sufficiently large.

Observe that if $(t_1, \ldots, t_{n-1}) \in D_{n-1}(K)$, then

$$1 < K \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{n-1}$$

so that $\|x\| \leqslant \sqrt{n-1}\, t_{n-1}$ when $x \in D_{n-1}(K)$. It follows from the previous inequality that

$$1 + |\theta(x)| < e^{t_{n-1}} \leqslant y$$

provided that t_{n-1} is sufficiently large, which can be arranged by increasing K if necessary. Since

$$f(x, y) = \pm a_d(x) \prod_{j=1}^{d} (y - \theta_j(x)),$$

it follows that if $(x, y) \in D_n(K)$, then $|f(x, y)| \geqslant C$.   □

Proof of Theorem 7.2. We first give a brief proof of (7.2) in the case when $U^p_\bullet = G^p_\bullet$. For each q, the subset

$$\Delta^p_q := \{(t_0, \ldots, t_{p+q}) : t_i \in \mathbb{R}, 0 < t_0 < t_1 < \cdots < t_{p+q}\}$$

of $X^p_q(\mathbb{R})$ is contractible. Moreover, each of the face maps A_i maps Δ^p_q into Δ^p_{q-1}. It follows that we have morphisms

$$\Delta^p_\bullet \hookrightarrow X^p_\bullet \to G^p_\bullet$$

of truncated simplicial spaces. Since each Δ^p_q is contractible, $\Delta^p_\bullet$ has a unique lift to a simplicial object of the category $\widetilde{\mathrm{Top}}$. If follows from (8.3) that both $X^p_\bullet$ and $G^p_\bullet$ have lifts to $\widetilde{\mathrm{Top}}$, and therefore to $\widetilde{\mathcal{A}}$.

The strategy in the general case is similar. We seek a simplicial space $D_\bullet$, each of whose simplices is contractible, which maps to $U^p_\bullet$. It follows from (8.1) that the pullback of U^p_q to X^p_q is a proper open subvariety V^p_q of X^p_q. By standard arguments, there is a non zero polynomial $f_q \in \mathbb{R}[t_0, \ldots, t_{p+q}]$ such that

$$X^p_q - f^{-1}_q(0) \subseteq V^p_q.$$

It follows from (8.4) that there is a real number $K_q > 1$ such that

$$D^p_q(K_q) := \left\{ (t_0, \ldots, t_{p+q}) \in X^p_q(\mathbb{R}) : t_0 \geqslant K_q, t_j \geqslant K_q \, e^{t_{j-1}} \text{ when } j \geqslant 1 \right\} \subset V^p_q.$$

Let

$$K = \max_{0 \leqslant q \leqslant p} K_q.$$

Then $D^p_q(K) \subset V^p_q$. It is not difficult to show that $A_i(D^p_q(K)) \subseteq D^p_{q-1}$ for each i. It follows that the $D^p_q(K)$, with $0 \leqslant q \leqslant p$, form a truncated simplicial space $D^p_\bullet(K)$ which maps to $V^p_\bullet$, and therefore to $U^p_\bullet$. It is not difficult to show that each $D^p_q(K)$ is contractible. It follows from (8.3) that U^p_q lifts to a simplicial object of $\widetilde{\mathcal{A}}$.   □

9. Proof of 7.7

We only give a detailed sketch of the proof. First we prove the result when $X_\bullet$ is replaced by a single space.

Denote the Malcev Lie algebra associated to the pointed space (Y, y) by $\mathfrak{p}(Y, y)$. Now suppose that Y is a complex algebraic manifold. Recall from [13, p. 470]

that the multivalued Deligne cohomology of the object (Y, y) of the category $\widetilde{\mathcal{A}}$ is defined to be the cohomology of the complex*

$$MD(Y, \mathbb{Q}(p)) := \text{Cone}\Big(W_{2p}\text{Hom}_{\mathbb{Q}}^{cts}(\Lambda^{\bullet}\mathfrak{p}(Y, y), \mathbb{Q})$$

$$\oplus W_{2p}\text{Hom}_{\mathbb{C}}^{cts}(\Lambda^{\bullet}\mathfrak{p}(Y, y), F^{p}\widetilde{\Omega}^{\bullet}(Y, y))$$

$$\rightarrow W_{2p}\text{Hom}_{\mathbb{C}}^{cts}(\Lambda^{\bullet}\mathfrak{p}(Y, y), \widetilde{\Omega}^{\bullet}(Y, y))\Big)[-1].$$

We shall need a $\mathbb{Q}$ analogue of $\text{Hom}_{\mathbb{C}}^{cts}(\Lambda^{\bullet}\mathfrak{p}(Y, y), \widetilde{\mathcal{O}}(Y))$. This will be constructed using continuous cohomology of certain path spaces.

The space of paths in a topological space Y which go from $y \in Y$ to $z \in Y$ will be denoted by $P_{y,z}Y$. The homology group $H_0(P_{y,z}Y, \mathbb{Q})$ has a natural topology which agrees with the filtration of $H_0(P_{y,y}Y, \mathbb{Q}) \approx \mathbb{Q}\pi_1(Y, y)$ by powers of its augmentation ideal when $y = z$ (cf. [15, Sect. 3]). Denote the continuous dual of $H_0(P_{y,z}Y, \mathbb{Q})$ by $H_{cts}^0(P_{y,z}Y, \mathbb{Q})$. These groups fit together to form a local system over $Y \times Y$ whose fiber over (y, z) is $H_{cts}^0(P_{y,z}Y, \mathbb{Q})$. It is a direct limit of unipotent local systems over $Y \times Y$ and a direct limit of unipotent variations of mixed Hodge structure when Y is a smooth algebraic variety [15].

There are two natural inclusions of $\text{Hom}_{\mathbb{Q}}^{cts}(\Lambda^{\bullet}\mathfrak{p}(Y, a), \mathbb{Q})$, $(a = y, z)$, into

$$\text{Hom}_{\mathbb{Q}}^{cts}(\Lambda^{\bullet}\mathfrak{p}(Y, y) \otimes_{\mathbb{Q}} \Lambda^{\bullet}\mathfrak{p}(Y, z), H_{cts}^0(P_{y,z}Y)).$$

They are induced by the two projections of $(Y, y) \times (Y, z)$ onto (Y, a) and by the inclusion of the constants into $H_{cts}^0(P_{y,z}Y)$. We shall denote them by ϕ_1 and ϕ_2, respectively.

PROPOSITION 9.1. *If $H_1(Y, \mathbb{Q})$ is finite dimensional, then ϕ_1 and ϕ_2 are both quasi-isomorphisms.*

Proof. We prove the result for ϕ_2, the other case being similar. By a standard spectral sequence argument, it suffices to show that

$$\text{Hom}_{\mathbb{Q}}^{cts}(\Lambda^{\bullet}\mathfrak{p}(Y, y), H_{cts}^0(P_{y,z}Y))$$

is acyclic. We may, without loss of generality, take $y = z$. Because $H_1(Y)$ is finite dimensional, each graded quotient of the topology on $H_0(P_{y,y}Y, \mathbb{Q})$ is finite dimensional, and it follows that the dual of $H_{cts}^0(P_{y,y}Y), \mathbb{Q})$ is isomorphic to the completion of $H_0(P_{y,y}Y, \mathbb{Q}) \approx \mathbb{Q}\pi_1(Y, y)$. This, in turn, is isomorphic to the completion of $U\mathfrak{p}(Y, y)$. So there is a natural isomorphism

$$\text{Hom}_{\mathbb{Q}}^{cts}(\Lambda^{\bullet}\mathfrak{p}(Y, y), H_{cts}^0(P_{y,y}Y)) \approx \text{Hom}_{\mathbb{Q}}^{cts}(\Lambda^{\bullet}\mathfrak{p}(Y, y)\widehat{\otimes}U\mathfrak{p}(Y, y), \mathbb{Q})$$

---

* Note that there is a typo in the definition of $MD(X, \mathbb{Q}(p))$ in [13, p. 470] – one should quotient out by $F^{p}C(\mathfrak{g}, \widetilde{\Omega}^{\bullet})$ as defined on op cit. p. 469 and not just by $F^{p}\widetilde{\Omega}^{\bullet}$.

of chain complexes. This last complex is acyclic, as it is the continuous dual of an acyclic complex (cf. [11, (3.9)]).   □

The following result is a straightforward refinement of the previous result.

PROPOSITION 9.2. *If Y is a smooth algebraic variety, then each of the complexes in the previous result is a complex of mixed Hodge structures, and the two natural inclusions ϕ_1 and ϕ_2 of* $\mathrm{Hom}^{cts}_{\mathbb{Q}}(\Lambda^\bullet\mathfrak{p}(Y,a),\mathbb{Q})$ *$(a = y, z)$ into*

$$\mathrm{Hom}^{cts}_{\mathbb{Q}}(\Lambda^\bullet\mathfrak{p}(Y,y) \otimes_{\mathbb{Q}} \Lambda^\bullet\mathfrak{p}(Y,z), H^0_{cts}(P_{y,z}Y))$$

are quasi-isomorphisms in the category of complexes of mixed Hodge structures.   □

Next, observe that each $F \in \widetilde{\mathcal{O}}(Y, y)$ induces a linear map

$$H_0(P_{y,z}Y) \to \mathbb{C}$$

by taking the path γ to the difference $F(z) - F(y)$ where the branch of F at z is obtained by analytically continuing F along γ. It follows from standard properties of iterated integrals that this map is continuous. Consequently, we obtain a linear map

$$\widetilde{\mathcal{O}}(Y, y) \to H^0_{cts}(P_{y,z}Y, \mathbb{C}).$$

It follows from [13, Sect. 3] and Chen's de Rham Theorem for the fundamental group that when $q(Y) = 0$, this map is an isomorphism of $W_\bullet$ filtered vector spaces. This isomorphism is $\pi_1(Y, y)$-equivariant with respect to the standard actions of $\pi_1(Y, y)$ on $\widetilde{\mathcal{O}}(Y, y)$ and $H^0_{cts}(P_{y,z}Y, \mathbb{C})$.
    Recall that there is a natural homomorphism

$$\theta \colon \mathrm{Hom}^{cts}_{\mathbb{C}}(\Lambda^\bullet\mathfrak{p}(Y,y), \mathbb{C}) \to \Omega^\bullet(Y)$$

of $W_\bullet$ filtered d.g. algebras [13, (7.7)].
    Fix a point z of Y. Consider the complex

$$\mathrm{Cone}\Big(W_{2p}\mathrm{Hom}^{cts}_{\mathbb{Q}}\big(\Lambda^\bullet\mathfrak{p}(Y,y) \otimes_{\mathbb{Q}} \Lambda^\bullet\mathfrak{p}(Y,z), H^0_{cts}(P_{y,z}Y,\mathbb{Q})\big)$$
$$\oplus W_{2p}\mathrm{Hom}^{cts}_{\mathbb{C}}(\Lambda^\bullet\mathfrak{p}(Y,y), F^p\widetilde{\Omega}^\bullet(Y,y))$$
$$\to W_{2p}\mathrm{Hom}^{cts}_{\mathbb{C}}(\Lambda^\bullet\mathfrak{p}(Y,y), \widetilde{\Omega}^\bullet(Y,y))\Big)[-1].$$

Here

$$\mathrm{Hom}^{cts}_{\mathbb{Q}}(\Lambda^\bullet\mathfrak{p}(Y,y) \otimes_{\mathbb{Q}} \Lambda^\bullet\mathfrak{p}(Y,y), H^0_{cts}(P_{y,z}Y,\mathbb{Q}))$$

is mapped into

$$W_{2p}\mathrm{Hom}_{\mathbb{C}}^{cts}(\Lambda^{\bullet}\mathfrak{p}(Y,y),\widetilde{\mathcal{O}}(Y,y)\otimes\Omega^{\bullet}(Y))$$

using the identification of $H_{cts}^{0}(P_{y,z}Y,\mathbb{C}))$ with $\widetilde{\mathcal{O}}(Y,y)$ and the map θ in the second factor. It is straightforward to check it is a chain map.

Define a map from this complex to $MD(Y,\mathbb{Q}(p))$ by defining it to be ϕ_1 on the first factor and the identity on the other two factors. Since ϕ_1 is a $W_{\bullet}$ filtered quasi-isomorphism, this map is a quasi-isomorphism.

Next, Define $MD'(X,\mathbb{Q}(p))$ to be the complex

$$\mathrm{Cone}\left(W_{2p}\mathrm{Hom}_{\mathbb{Q}}^{cts}(\Lambda^{\bullet}\mathfrak{p}(Y,y),\mathbb{Q})\oplus F^{p}W_{2p}\Omega^{\bullet}(Y)\rightarrow W_{2p}\Omega^{\bullet}(Y)\right)[-1],$$

where the map $\mathrm{Hom}_{\mathbb{Q}}^{cts}(\Lambda^{\bullet}\mathfrak{p}(Y,y),\mathbb{Q})\rightarrow\Omega^{\bullet}(Y)$ is induced by θ. It can be mapped to the previous complex using ϕ_2 on the first factor and the obvious inclusions on the other two factors. Since

$$\mathrm{Hom}_{\mathbb{C}}^{cts}(\Lambda^{\bullet}\mathfrak{p}(Y,y),\widetilde{\mathcal{O}}^{\bullet}(Y,y))$$

is an acyclic complex of mixed Hodge structures, it follows that this map is also a quasi-isomorphism. That is, we can equally well compute $H_{\mathcal{MD}}^{\bullet}(Y,\mathbb{Q}(p))$ using the complex $MD'(Y,\mathbb{Q}(p))$.

A map of $MD'(Y,\mathbb{Q}(p))$ into a standard complex that computes $H_{\mathcal{D}}^{\bullet}(Y,\mathbb{Q}(p))$ can now be constructed using the techniques of the proof of [5, (11.7)]. Taking homology, we obtain a map

$$\psi\colon H_{\mathcal{MD}}^{\bullet}(Y,\mathbb{Q}(p))\rightarrow H_{\mathcal{D}}^{\bullet}(Y,\mathbb{Q}(p))$$

for all smooth varieties.

PROPOSITION 9.3. *If Y is a rational $K(\pi,1)$, then ψ is an isomorphism.*

Proof. The homology of the complex $\mathrm{Hom}_{\mathbb{Q}}^{cts}(\Lambda^{\bullet}\mathfrak{p}(Y,y),\mathbb{Q})$ is the continuous cohomology $H_{cts}^{\bullet}(\mathfrak{p}(Y,y))$ of the Lie algebra $\mathfrak{p}(Y,y)$. The natural map

$$H_{cts}^{\bullet}(\mathfrak{p}(Y,y))\rightarrow H^{\bullet}(Y,\mathbb{Q}),$$

is a morphism of mixed Hodge structures [5, (11.7)]. Since the multivalued Deligne cohomology is constructed as a cone, we have a long exact sequence

$$\cdots\rightarrow W_{2p}H_{cts}^{k-1}(Y,\mathbb{Q})\oplus F^{p}W_{2p}\Omega^{k-1}(Y)$$
$$\xrightarrow{\theta-i}W_{2p}\Omega^{k-1}(Y)\rightarrow H_{\mathcal{MD}}^{k}(Y,\mathbb{Q}(p))\rightarrow\cdots,$$

where i denotes the inclusion of $F^p\Omega^\bullet$ into $\Omega^\bullet$. The map ψ induces a map from this long exact sequence into the standard long exact sequence

$$\cdots \to W_{2p}H^{k-1}(Y,\mathbb{Q}) \oplus F^p W_{2p}H^{k-1}(Y)$$
$$\to W_{2p}H^{k-1}(Y,\mathbb{C}) \to H_{\mathcal{D}}^k(Y,\mathbb{Q}(p)) \to \cdots.$$

When Y is a rational $K(\pi,1)$, each of the maps

$$H^\bullet_{cts}(Y,\mathbb{Q}) \xrightarrow{\theta} \Omega^\bullet(Y) \to H^\bullet(Y,\mathbb{C})$$

is a $(W_\bullet, F^\bullet)$ bifiltered quasi-isomorphism [13, (8.2)(i),(iii)]. The result now follows using the 5-lemma.   □

One can take $y = z$ in each of the chain maps above. If one does this, the assignment of each of these complexes to an object of the category $\mathcal{A}_*$ defined in [13, Sect. 2] is a functor.

Suppose that $X_\bullet$ is a simplicial object of the category $\widetilde{\mathcal{A}}$. Choose a base point x_n of each X_n; $X_\bullet$ now determines a simplicial object of the category $\mathcal{A}_*$, and we may apply any of the functors above to $X_\bullet$ to obtain a double complex. Using standard arguments, we see that the total complex associated to the double complex $MD'(X_\bullet,\mathbb{Q}(p))$ computed the multivalued Deligne cohomology of $X_\bullet$ and that there is a map

$$\Psi\colon H^\bullet_{\mathcal{MD}}(X_\bullet,\mathbb{Q}(p)) \to H^\bullet_{\mathcal{D}}(X_\bullet,\mathbb{Q}(p)).$$

When each X_n is a rational $K(\pi,1)$, it is not difficult to show, using an argument similar to the proof of (9.3) and the skeleton filtration, that Ψ is an isomorphism.

10. Higher logarithms and extensions of Tate variations

The higher logarithms we have constructed generically on $G^p_\bullet$ are related to extensions of (Tate) variations of mixed Hodge structures. Indeed, by [5, (12.1)] and [11, (8.6)], if a space X is a rational $K(\pi,1)$ with $q(X) = 0$, then there are natural isomorphisms

$$H^\bullet_{\mathcal{D}}(X,\mathbb{Q}(p)) \approx \mathrm{Ext}^\bullet_{\mathcal{H}(X)}(\mathbb{Q},\mathbb{Q}(p)) \approx \mathrm{Ext}^\bullet_{\mathcal{T}(X)}(\mathbb{Q},\mathbb{Q}(p)),$$

where $\mathcal{H}(X)$ and $\mathcal{T}(X)$ denote the categories of unipotent variations of mixed Hodge structure over X and Tate variations of mixed Hodge structure over X, respectively. Thus if $X_\bullet$ is a simplicial variety where each X_n is a rational $K(\pi,1)$ with $q(X) = 0$, then, in some sense, we may identify $H^\bullet_{\mathcal{MD}}(X_\bullet,\mathbb{Q}(p))$ with the 'hyper-ext' group of extensions of $\mathbb{Q}$ by $\mathbb{Q}(p)$ associated to $X_\bullet$ (cf. [11, Sect. 10]).

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References

1. Beilinson, A.: Higher regulators and values of L-functions of curves, *Func. Anal. and its Appl.* 14 (1980), 116–118.
2. Beilinson, A.: *Notes on absolute Hodge cohomology* in Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory, *Contemporary Math. 55*, part I, *Amer. Math. Soc.*, Providence, 1986, 35–68.
3. Beilinson, A., MacPherson, R., and Schechtman, V.: Notes on motivic cohomology, *Duke Math. J.* 54 (1987), 679–710.
4. Bloch, S.: Higher regulators, Algebraic K-theory, and zeta functions of elliptic curves, unpublished manuscript, 1978.
5. Carlson, J. and Hain, R.: Extensions of Variations of Mixed Hodge Structure, *Théorie de Hodge, Luminy, Juin, 1987*, Astérisque no. 179–180, 39–65.
6. Dupont, J.: The dilogarithm as a characteristic class for flat bundles, *J. Pure and App. Alg.* 44 (1987), 137–164.
7. Falk, M. and Randell, R.: The lower central series of a fiber-type arrangement, *Invent. Math.* 82 (1985), 77–88.
8. Gelfand, I., Goresky, M., MacPherson, R., and Serganova, V.: Geometries, convex polyhedra and Schubert cells, *Advances in Math.* 63 (1987), 301–316.
9. Goncharov, G.: Geometry of configurations, polylogarithms and motivic cohomology, *Advances in Math.* 114 (1995), 197–318.
10. Goncharov, A.: Explicit construction of characteristic classes, *Advances in Soviet Math.* 16 (1993), 169–210.
11. Hain, R.: Algebraic cycles and extensions of variations of mixed Hodge structure, *Proc. Symp. Pure Math.* 53 (1993), 175–221.
12. Hain, R.: Classical polylogarithms, in Motives, *Proc. Symp. Pure Math.*, to appear.
13. Hain, R. and MacPherson, R.: Higher Logarithms, *Ill. J. Math.* 34 (1990), 392–475.
14. Hain, R. and Yang, J.: Real Grassmann polylogarithms and Chern classes, *Math. Annalen*, to appear.
15. Hain, R. and Zucker, S.: Unipotent variations of mixed Hodge structure, *Invent. Math.* 88 (1987), 83–124.
16. Hanamura, M. and MacPherson, R.: Geometric construction of polylogarithms, *Duke Math. J.* 70 (1993), to appear.
17. Hanamura, M. and MacPherson, R.: Geometric construction of polylogarithms, II, preprint, 1993.
18. Kohno, T.: Séries de Poincaré-Kozul associée aux groupes de tresse pure, *Invent. Math.* 82 (1985), 57–75.
19. Yang, J.: Algebraic K-groups of number fields and the Hain-MacPherson trilogarithm, *Ph.D. Thesis*, University of Washington, 1991.